



HAL
open science

Conservativity and weak consistency of a class of staggered finite volume methods for the euler equations

R. Herbin, J.-C. Latché, S. Minjeaud, N. Therme

► To cite this version:

R. Herbin, J.-C. Latché, S. Minjeaud, N. Therme. Conservativity and weak consistency of a class of staggered finite volume methods for the euler equations. *Mathematics of Computation*, 2021, 90 (329), pp.1155-1177. 10.1090/mcom/3575 . hal-02308023v3

HAL Id: hal-02308023

<https://hal.science/hal-02308023v3>

Submitted on 22 Jun 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

CONSERVATIVITY AND WEAK CONSISTENCY OF A CLASS OF STAGGERED FINITE VOLUME METHODS FOR THE EULER EQUATIONS

R. HERBIN¹, J.-C. LATCHÉ², S. MINJEAUD³ AND N. THERME⁴

Abstract. We address here a class of staggered schemes for the compressible Euler equations ; this scheme was introduced in recent papers and possesses the following features: upwinding is performed with respect to the material velocity only and the internal energy balance is solved, with a correction term designed on consistency arguments. These schemes have been shown in previous works to preserve the convex of admissible states and have been extensively tested numerically. The aim of the present paper is twofold: we derive a local total energy equation satisfied by the solutions, so that the schemes are in fact conservative, and we prove that they are consistent in the Lax-Wendroff sense.

June 22, 2020.

1. INTRODUCTION

The aim of this paper is to prove some conservativity and consistency properties of a class of numerical schemes for the compressible Euler equations which model the flow of an ideal gas and read:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1a}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, \tag{1b}$$

$$\partial_t(\rho E) + \operatorname{div}(\rho E \mathbf{u}) + \operatorname{div}(p \mathbf{u}) = 0, \tag{1c}$$

$$p = (\gamma - 1) \rho e, \quad E = E_k + e, \quad E_k = \frac{1}{2} |\mathbf{u}|^2, \tag{1d}$$

where t is the time, ρ , \mathbf{u} , p , E , e and E_k denote the density, velocity, pressure, total energy, internal energy and kinetic energy respectively, and $\gamma > 1$ is the ratio of specific heats. The notation $|\cdot|$ is used both for the Euclidean norm of a vector of \mathbb{R}^d or for the absolute value of a real number. The space–time domain is denoted by $\Omega \times (0, T)$, where Ω is an open bounded connected subset of \mathbb{R}^d , $1 \leq d \leq 3$, and $T < +\infty$. System (1) is supplemented by the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ *a.e.* on $\partial\Omega$, \mathbf{n} denoting the unit normal vector to the boundary, and by initial conditions for ρ , e and \mathbf{u} , denoted by ρ_0 , e_0 and \mathbf{u}_0 respectively, with $\rho_0 \in L^\infty(\Omega)$, $\rho_0 > 0$ and $e_0 \in L^\infty(\Omega)$, $e_0 > 0$.

The use of staggered space discretization for compressible flows is classical: indeed, the well-known Marker-And-Cell (MAC) scheme of [12, 13], was followed by numerous works, see e.g. [3, 18–20, 23, 25, 27, 29] or [28] for a textbook); a salient feature of staggered discretisations is their native stability for the numerical simulation of incompressible flows, thus yielding asymptotic preserving schemes in the low Mach number regime. The decoupling of the fluxes between material velocity and pressure–like part has been implemented in industrial hydrodynamics codes for quite some time for instance in Lagrange-remap schemes, see e.g. [2, 5, 26]. Following this line of thought, some numerical schemes for the Euler equations on general staggered grids have been recently developed [10, 11, 14, 15] (see also [6] for a similar approach in the Lagrange-projection framework), with share the following features:

- a special attention is paid to the discretization of the momentum convection in Equation (1b) to ensure that discrete solutions satisfy a discrete kinetic energy balance, *i.e.* a discrete version of

$$\partial_t(\rho E_k) + \operatorname{div}(\rho E_k \mathbf{u}) + \mathbf{u} \cdot \nabla p = -\mathcal{R}, \quad \mathcal{R} \geq 0, \tag{2}$$

WARNING: — Give at least one key words —

¹ I2M, CNRS and Université d’Aix-Marseille, France, raphaele.herbin@univ-amu.fr

² Institut de Radioprotection et de Sûreté Nucléaire (IRSN), PSN-RES/SA2I, Cadarache, St-Paul-lez-Durance, 13115, France, jean-claude.latche@irsn.fr

³ Laboratoire Jean Dieudonné, Université de Nice-Sophia Antipolis, France, sebastian.minjeaud@unice.fr

⁴ CEA/CESTA 33116, Le Barp, France, nicolas.therme@cea.fr

where the expression of the discrete remainder \mathcal{R} is established from the discrete mass and momentum balance equations (see Section 5. This quantity \mathcal{R} may be seen as the dissipation associated to the numerical diffusion.

- the scheme solves the internal energy balance, *i.e.* the relation which may be formally derived at the continuous level by subtracting (2) from (1c):

$$\partial_t(\rho e) + \operatorname{div}(\rho e \mathbf{u}) + p \operatorname{div} \mathbf{u} = \mathcal{R}. \quad (3)$$

These features are also at the heart of a recently developed collocated finite volume scheme [17], and yield an essential outcome: a suitable discretization of (3) ensures that $e \geq 0$ and that an entropy inequality can be derived [9], with a remarkably unsophisticated definition of the convection fluxes: indeed, upwinding is performed equation per equation, with respect to the material velocity only, so without any exact or approximate solution of Riemann problems at interfaces, thus yielding a natural extension to various equations of state and to more complex problems, such as the reactive Euler equations. Here, the final goal is to use rather a staggered arrangement of the unknowns to obtain a (class of) schemes that are natively able to cope with all Mach flows [11, 16]; however, passing from the collocated to the staggered framework renders the application of the above-described strategy much more intricate: indeed, the discrete kinetic energy balance is posed on the dual mesh while the internal energy balance is posed on the primal one, and a simple addition of the discrete analogues of (2) and (3) is no more possible. A crucial question when designing the various steps of the scheme is whether the resulting scheme is conservative and consistent; here in particular, special attention must be paid to the convection operator for the velocity, which is designed in order to respect stability properties only (*i.e.*, as already mentioned, so as to allow to derive a discrete kinetic energy balance).

The purpose of this paper is to obtain (positive) answers to these questions: we show that a discrete convection operator on the primal mesh may be derived from the staggered operator for a "cell average" of the face variables (*i.e.*, in this context, either the velocity components or the kinetic energy), and that this convection operator satisfies a Lax-Wendroff consistency property. To fix ideas, we will work with specific choices for the space and time discretizations, namely an explicit scheme based, with velocity unknowns belonging to the so-called Rannacher-Turek low-order finite element space [24]; however, the arguments invoked below are rather general. The extension to simplicial meshes, with non conforming P1 velocity unknowns is straightforward. A similar construction is also possible for the so-called Marker And Cell (MAC) scheme; in particular, the derivation of a convection operator on primal meshes from its staggered analogue is performed, with this latter discretization, in [22].

The schemes studied here are implemented in the open-source software CALIF³S [4] developed at the French Institut de Radioprotection et de Sûreté Nucléaire (IRSN); they have been extensively tested numerically in [10, 11, 14, 15] and are routinely used in nuclear safety applications. We thus do not reproduce these numerical tests here. Note that the proof of consistency given below applies in particular to the MUSCL-like scheme which was introduced in [10].

2. MESHES AND UNKNOWNNS

Let Ω be an open bounded polyhedral set of \mathbb{R}^d , $d \geq 1$. We suppose given a mesh \mathcal{M} of Ω , *i.e.* a finite collection of compact connected sets $K \in \mathcal{M}$, called cells, such that

$$\overline{\Omega} = \bigcup_{K \in \mathcal{M}} K \quad \text{and} \quad \overset{\circ}{K} \cap \overset{\circ}{L} = \emptyset, \quad \forall K \in \mathcal{M}, \forall L \in \mathcal{M}, K \neq L.$$

The cells are assumed to be intervals (if $d = 1$), quadrangles (if $d = 2$), or hexahedra (if $d = 3$). In the multidimensional case, the boundary of each cell $K \in \mathcal{M}$ is a union of 4 (if $d = 2$) or 6 (if $d = 3$) parts of hyperplanes of \mathbb{R}^d , which are called faces for short in the following, for $d = 2$ as for $d = 3$; the boundary ∂K of K reads $\partial K = \cup_{\sigma \in \mathcal{E}_K} \sigma$ where \mathcal{E}_K is the set of the faces of K . We denote by \mathcal{E} the set of all faces, namely $\mathcal{E} = \cup_{K \in \mathcal{M}} \mathcal{E}_K$. We suppose that the mesh is conforming in the usual sense, *i.e.* that any cells K and L of the mesh is either disjoint (possibly up to common vertex or edge) or share a whole face. We denote by \mathcal{E}_{int} the set of elements σ of \mathcal{E} such that there exist K and L in \mathcal{M} ($K \neq L$) such that $\sigma \in \mathcal{E}_K \cap \mathcal{E}_L$; such a face σ is denoted by $\sigma = K|L$. The set of faces located on the boundary of Ω , *i.e.* $\mathcal{E} \setminus \mathcal{E}_{\text{int}}$, is denoted by \mathcal{E}_{ext} . For $K \in \mathcal{M}$ we denote by h_K the diameter of K . The size of the mesh \mathcal{M} is $h_{\mathcal{M}} = \max\{h_K, K \in \mathcal{M}\}$. For $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we denote by $\mathbf{n}_{K,\sigma}$ the unit normal vector to σ outward K . If A is a measurable set of \mathbb{R}^d or, for $d > 1$, of \mathbb{R}^{d-1} , we denote by $|A|$ the Lebesgue measure of A .

We also introduce now a dual mesh, that is a new partition of Ω indexed by the elements of \mathcal{E} , namely $\Omega = \cup_{\sigma \in \mathcal{E}} D_\sigma$. For an internal face $\sigma = K|L$, the set D_σ is supposed to be a subset of $K \cup L$ and we define $D_{K,\sigma} = D_\sigma \cap K$, so that $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$ (see Figure 1); for an external face σ of a cell K , D_σ is a subset of K , and $D_\sigma = D_{K,\sigma}$. The cells $(D_\sigma)_{\sigma \in \mathcal{E}}$ are referred to as the dual or diamond cells, and $D_{K,\sigma}$ as half dual cells or half diamond cells. For a rectangular or a rectangular paralleliped, we define $D_{K,\sigma}$ as the cone having the mass center of K as vertex and the face σ as basis; this definition is extended to general primal meshes by supposing that $|D_{K,\sigma}|$ is still equal to $|K|$ divided by the number of the faces of K , denoted by ζ :

$$|D_{K,\sigma}| = \frac{|K|}{\zeta}, \quad (4)$$

and that the sub-cells connectivities (*i.e.* the way the half-dual cells share a common face) is left unchanged. In one space dimension, $D_{K,\sigma}$ is half of the cell K adjacent to σ . The faces of the dual cells are referred to as dual faces; we denote by $\mathcal{E}(D_\sigma)$ and $\mathcal{E}(D_{K,\sigma})$ the set of faces of D_σ and $D_{K,\sigma}$ respectively (so $\sigma \in \mathcal{E}(D_{K,\sigma})$) and by $\epsilon = \sigma|\tau$ the dual face separating D_σ and D_τ . Note that the actual geometry of the dual cells or dual faces does not need to be specified (and a dual cell may not be a polytope, a dual face being possibly not included in an hyperplane of \mathbb{R}^d), and we will see in the following that it is not required for the definition of the scheme.

The scalar unknowns are associated to the cells of the mesh, and read $(\rho_K)_{K \in \mathcal{M}}$, $(p_K)_{K \in \mathcal{M}}$ and $(e_K)_{K \in \mathcal{M}}$ for the density, the pressure and the internal energy respectively. The velocity unknowns are associated to the faces and read $(\mathbf{u}_\sigma)_{\sigma \in \mathcal{E}}$, with $\mathbf{u}_\sigma = (u_{1,\sigma}, \dots, u_{d,\sigma})^\dagger$ the velocity vector associated to σ .

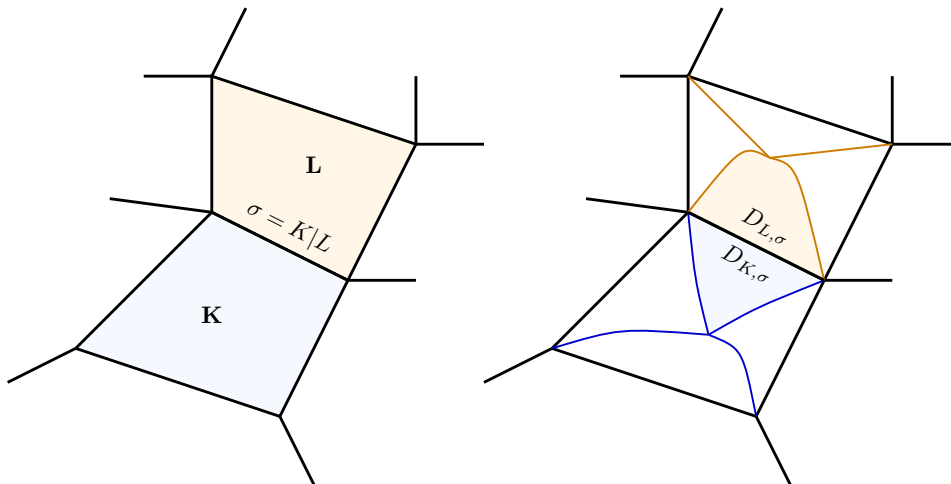


FIGURE 1. Mesh and associated notations.

3. GENERAL FORM OF THE SCHEME

For the sake of simplicity, we focus here on an explicit-in-time form of the scheme. Let $(t_n)_{0 \leq n \leq N}$, with $0 = t_0 < t_1 < \dots < t_N = T$, define a partition of the time interval $(0, T)$, which we suppose uniform for the sake of simplicity, and let $\delta t = t_{n+1} - t_n$ for $0 \leq n \leq N - 1$ be the (constant) time step. We consider a fractional step scheme, which involves only explicit steps and reads in its fully discrete form, for $0 \leq n \leq N - 1$:

$$\forall K \in \mathcal{M}, \frac{|K|}{\delta t} (\rho_K^{n+1} - \rho_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^n = 0, \quad (5a)$$

$$\forall K \in \mathcal{M}, \frac{|K|}{\delta t} (\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^n e_\sigma^n + |K| p_K^n (\operatorname{div} \mathbf{u})_K^n = S_K^n, \quad (5b)$$

$$\forall K \in \mathcal{M}, p_K^{n+1} = (\gamma - 1) \rho_K^{n+1} e_K^{n+1}, \quad (5c)$$

For $1 \leq i \leq d$, $\forall \sigma \in \mathcal{E}$,

$$\frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^{n+1} u_{\sigma,i}^{n+1} - \rho_{D_\sigma}^n u_{\sigma,i}^n) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^n u_{\epsilon,i}^n + |D_\sigma| (\nabla p)_{\sigma,i}^{n+1} = 0, \quad (5d)$$

where the terms introduced for each discrete equation are now defined. The mass flux $F_{K,\sigma}^n$ (equations (5a) and (5b)) vanishes on external edges to comply with the boundary conditions and, for $\sigma \in \mathcal{E}_{\text{int}}$, $\sigma = K|L$, we have

$$F_{K,\sigma}^n = |\sigma| \rho_\sigma^n \mathbf{u}_\sigma^n \cdot \mathbf{n}_{K,\sigma}, \quad (6)$$

where ρ_σ^n is an approximation of the density at the face which is a convex combination of ρ_K^n and ρ_L^n , which does not need to be specified further for the matter at hand in this paper. Similarly, the internal energy e_σ^n at the face $\sigma = K|L$ in Equation (5b) is supposed to be a convex combination of e_K^n and e_L^n . Note that this assumption allows us to deal both with the upwind scheme or with the quasi 2nd-order scheme using a MUSCL-like choice introduced in [10].

In the same relation, the quantity $(\text{div} \mathbf{u})_K^n$ reads:

$$(\text{div} \mathbf{u})_K^n = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \mathbf{u}_\sigma^n \cdot \mathbf{n}_{K,\sigma}.$$

The pressure gradient in the momentum balance equation (5d) is obtained by transposition of the divergence operator with respect to the L^2 inner product, and reads for $\sigma = K|L$

$$(\nabla p)_\sigma^{n+1} = \frac{1}{|D_\sigma|} (p_L^{n+1} - p_K^{n+1}) \mathbf{n}_{K,\sigma},$$

and $(\nabla p)_{\sigma,i}^{n+1}$ stands for the i^{th} component of $(\nabla p)_\sigma^{n+1}$. The momentum convection operator (*i.e.* the first two terms of Equation (5d)) are detailed in Section 4. The term S_K^n at the right-hand side of Equation (5b) is a corrective term introduced for consistency, with the aim to compensate the discrete dissipation due to the numerical diffusion in the discrete momentum balance equation (5d); its precise expression is given in Section 5. A numerical diffusion term, not featured here, may be introduced in applications in the momentum balance equation to stabilize the scheme. The initial approximations for ρ , e and \mathbf{u} are given by the average of the initial conditions ρ_0 and e_0 on the primal cells and of \mathbf{u}_0 on the dual cells:

$$\begin{aligned} \forall K \in \mathcal{M}, \quad \rho_K^0 &= \frac{1}{|K|} \int_K \rho_0(\mathbf{x}) \, d\mathbf{x}, \quad \text{and} \quad e_K^0 = \frac{1}{|K|} \int_K e_0(\mathbf{x}) \, d\mathbf{x}, \\ \forall \sigma \in \mathcal{E}, \quad \mathbf{u}_\sigma^0 &= \frac{1}{|D_\sigma|} \int_{D_\sigma} \mathbf{u}_0(\mathbf{x}) \, d\mathbf{x}. \end{aligned} \quad (7)$$

4. THE DISCRETE CONVECTION ON THE DUAL MESH

Let z be a variable associated to the faces of the mesh, *i.e.* to the degrees of freedom $(z_\sigma^n)_{\sigma \in \mathcal{E}, 0 \leq n \leq N}$. The discrete convection operator for z (*i.e.* the discretization of $\partial_t(\rho z) + \text{div}(\rho z \mathbf{u})$) is defined by:

$$\mathcal{C}_\sigma^n z = \frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^{n+1} z_\sigma^{n+1} - \rho_{D_\sigma}^n z_\sigma^n) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^n z_\epsilon^n, \quad (8)$$

where, for $\epsilon = \sigma|\tau$, z_ϵ^n is supposed to be a convex combination of z_σ^n and z_τ^n . The density $\rho_{D_\sigma}^n$ at the face σ is a weighted combination of the density in the neighbouring primal cells, given by, for $0 \leq n \leq N$:

$$\begin{aligned} \rho_{D_\sigma}^n &= \frac{1}{|D_\sigma|} (|D_{K,\sigma}| \rho_K^n + |D_{L,\sigma}| \rho_L^n), \quad \text{for } \sigma = K|L \in \mathcal{E}_{\text{int}} \\ \rho_{D_\sigma}^n &= \rho_K^n, \quad \text{for } \sigma \in \mathcal{E}_{\text{ext}}, \sigma \in \mathcal{E}(K). \end{aligned} \quad (9)$$

We suppose that the mass fluxes through the dual edges ϵ are obtained thanks to a mass balance over the half-diamond cells, *i.e.* satisfy, for $\sigma \in \mathcal{E}(K)$:

$$\sum_{\epsilon \in \mathcal{E}(D_{K,\sigma})} F_{\sigma,\epsilon}^n = \frac{1}{\zeta} \sum_{\tau \in \mathcal{E}(K)} F_{K,\tau}^n, \quad (10)$$

where we recall that ζ stands for the number of faces of K , so $\zeta = 2$ in one space dimension, $\zeta = 4$ in two space dimensions and $\zeta = 6$ in three space dimensions. In addition, they are also supposed to satisfy, for $K \in \mathcal{M}$, $\sigma \in \mathcal{E}(K)$ and $\epsilon \subset K$, $\epsilon \in \mathcal{E}(D_\sigma)$:

$$F_{\sigma,\epsilon} = \sum_{\tau \in \mathcal{E}(K)} \xi_{\epsilon,\tau} F_{K,\tau}, \quad (11)$$

where the coefficients $\xi_{\epsilon,\tau}$ are fixed real numbers, *i.e.* independent of the cell K and the mesh \mathcal{M} . Finally, for diamond cells adjacent to a boundary (which are in fact reduced to a half-diamond cell), one dual face (at least) is an external face, and coincides with a primal face, let us say σ ; the mass flux through this face is supposed to vanish (*i.e.* to take the same value as the primal mass flux $F_{K,\sigma}$, with K the cell adjacent to σ). The expression of the coefficients ($\xi_{\epsilon,\tau}$) may be found in [1] for quadrangles in two space dimensions and in [21] for various meshes (including hexahedra in three space dimensions).

The essential outcome of the construction of the dual mass fluxes lies in the fact that (9) and (10), together with the assumption (4) imply that a mass balance holds over the dual cells:

$$\forall \sigma \in \mathcal{E}, \quad \frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^{n+1} - \rho_{D_\sigma}^n) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^n = 0. \quad (12)$$

5. DISCRETE KINETIC ENERGY BALANCE

Thanks to this relation, the scheme satisfies a discrete kinetic energy balance, which is stated in the following lemma (see [15] for a proof).

Lemma 5.1 (Discrete kinetic energy balance). *A solution to the system (5) satisfies the following equality, for $1 \leq i \leq d$, $\sigma \in \mathcal{E}$ and $0 \leq n \leq N - 1$:*

$$\begin{aligned} \frac{1}{2} \frac{|D_\sigma|}{\delta t} \left[\rho_{D_\sigma}^{n+1} (u_{\sigma,i}^{n+1})^2 - \rho_{D_\sigma}^n (u_{\sigma,i}^n)^2 \right] + \frac{1}{2} \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^n (u_{\epsilon,i}^n)^2 \\ + |D_\sigma| (\nabla p)_{\sigma,i}^{n+1} u_{\sigma,i}^{n+1} = -R_{\sigma,i}^{n+1}, \end{aligned} \quad (13)$$

with:

$$\begin{aligned} R_{\sigma,i}^{n+1} = \frac{1}{2} \frac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^{n+1} (u_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2 - \frac{1}{2} \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^n (u_{\epsilon,i}^n - u_{\sigma,i}^n)^2 \\ + (u_{\sigma,i}^{n+1} - u_{\sigma,i}^n) \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^n (u_{\epsilon,i}^n - u_{\sigma,i}^n). \end{aligned} \quad (14)$$

The corrective term in the internal energy balance compensates this remainder term:

$$\forall K \in \mathcal{M}, \quad S_K^{n+1} = \frac{1}{2} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}(K)} R_{\sigma,i}^{n+1}. \quad (15)$$

At the first time step, this corrective term is simply set to zero:

$$S_K^0 = 0, \quad \forall K \in \mathcal{M}.$$

With the upwind choice for the velocity at the dual face, the remainder term $R_{\sigma,i}^{n+1}$ may be proven to be non-negative under a CFL condition. In this case, S_K^{n+1} is also non-negative, and with a suitable choice of the internal energy at the face (upwind or MUSCL choice with respect to the material velocity), the internal energy remains non-negative by construction of the scheme [10].

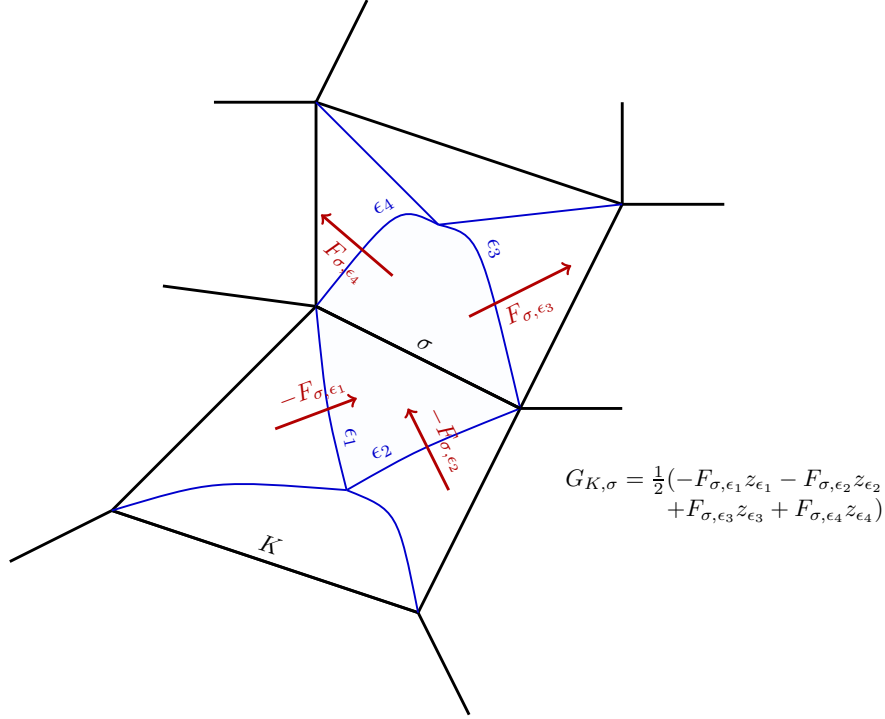


FIGURE 2. Definition of the convection flux at a primal face.

6. RETURNING TO THE PRIMAL MESH

For $K \in \mathcal{M}$, let us define a reconstructed convection operator \mathcal{C}_K^n from the face convection operators \mathcal{C}_σ^n defined by (8) as follows. For a given scalar field $(z_\sigma)_{\sigma \in \mathcal{E}_{\text{int}}}$, we set:

$$\begin{aligned} \mathcal{C}_K^n z &= \frac{1}{2} \sum_{\sigma \in \mathcal{E}(K)} \mathcal{C}_\sigma^n z \\ &= \frac{1}{\delta t} \sum_{\sigma \in \mathcal{E}(K)} \frac{|D_\sigma|}{2} (\rho_{D_\sigma}^{n+1} z_\sigma^{n+1} - \rho_{D_\sigma}^n z_\sigma^n) + \frac{1}{2} \sum_{\sigma \in \mathcal{E}(K)} \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma, \epsilon}^n z_\epsilon^n. \end{aligned} \quad (16)$$

We now define the following quantities (see Figure 2):

$$|K| (\rho z)_K^n = \sum_{\sigma \in \mathcal{E}(K)} \frac{|D_\sigma|}{2} \rho_{D_\sigma}^n z_\sigma^n, \quad (17)$$

$$G_{K, \sigma}^n = -\frac{1}{2} \sum_{\epsilon \in \mathcal{E}(D_\sigma), \epsilon \subset K} F_{\sigma, \epsilon}^n z_\epsilon^n + \frac{1}{2} \sum_{\epsilon \in \mathcal{E}(D_\sigma), \epsilon \not\subset K} F_{\sigma, \epsilon}^n z_\epsilon^n. \quad (18)$$

We easily check that the fluxes $G_{K, \sigma}^n$ are conservative, in the sense that, for $\sigma = K|L$, $G_{K, \sigma}^n = -G_{L, \sigma}^n$. A simple reordering of the summations in (16) using the conservativity of the mass fluxes through the dual edges yields:

$$\mathcal{C}_K^n z = \frac{|K|}{\delta t} ((\rho z)_K^{n+1} - (\rho z)_K^n) + \sum_{\sigma \in \mathcal{E}(K)} G_{K, \sigma}^n. \quad (19)$$

7. LAX-WENDROFF CONSISTENCY OF THE PRIMAL MESH CONVECTION OPERATOR

The aim of this section is to check the consistency, in the Lax-Wendroff sense, of the discrete operator \mathcal{C}_K^n .

We suppose given a sequence of meshes $(\mathcal{M}^{(m)})_{m \in \mathbb{N}}$ and time steps $(\delta t^{(m)})_{m \in \mathbb{N}}$, with $h_{\mathcal{M}^{(m)}}$ and $\delta t^{(m)}$ tending to zero as m tends to $+\infty$. For $m \in \mathbb{N}$, let us denote by $\rho^{(m)}$, $\mathbf{u}^{(m)}$ and $z^{(m)}$ the discrete functions corresponding to the approximation on the mesh $\mathcal{M}^{(m)}$ and with the time step $\delta t^{(m)}$ of ρ , \mathbf{u} and z respectively, defined by:

$$\rho^{(m)}(\mathbf{x}, t) = \sum_{n=0}^{N^{(m)}-1} \sum_{K \in \mathcal{M}^{(m)}} \rho_K^n \mathcal{X}_K \mathcal{X}_{[t_n, t_{n+1})}, \quad (20)$$

$$\mathbf{u}^{(m)}(\mathbf{x}, t) = \sum_{n=0}^{N^{(m)}-1} \sum_{\sigma \in \mathcal{E}^{(m)}} \mathbf{u}_\sigma^n \mathcal{X}_{D_\sigma} \mathcal{X}_{[t_n, t_{n+1})}, \quad (21)$$

$$z^{(m)}(\mathbf{x}, t) = \sum_{n=0}^{N^{(m)}-1} \sum_{\sigma \in \mathcal{E}^{(m)}} z_\sigma^n \mathcal{X}_{D_\sigma} \mathcal{X}_{[t_n, t_{n+1})}, \quad (22)$$

where \mathcal{X}_K , \mathcal{X}_{D_σ} and $\mathcal{X}_{[t_n, t_{n+1})}$ stand for the characteristic function of K , D_σ and the interval $[t_n, t_{n+1})$ respectively.

We denote by $\theta_{\mathcal{M},1}$ the following measure of the regularity of a mesh:

$$\theta_{\mathcal{M},1} = \max \left\{ \frac{|K|}{|L|}, \frac{|L|}{|K|} \text{ for } \sigma = K|L \in \mathcal{E}_{\text{int}}, \right\}, \quad (23)$$

Note that, since $|D_{K,\sigma}| = |K|/\zeta$ (recall that ζ is the number of faces of K), we have

$$|D_\sigma| = \frac{1}{\zeta} (|K| + |L|) \leq \frac{1}{\zeta} (1 + \theta_{\mathcal{M},1}) |K|. \quad (24)$$

Thanks to (24), the proof of the following lemma is a straightforward consequence of [8, Lemma 4.3].

Lemma 7.1 (Limit of the difference of unknowns translates). *Let $\theta > 0$ and $(\mathcal{M}^{(m)})_{m \in \mathbb{N}}$ be a sequence of meshes such that $\theta_{\mathcal{M}^{(m)},1} \leq \theta$ for all $m \in \mathbb{N}$ and $\lim_{m \rightarrow +\infty} h_{\mathcal{M}^{(m)}} = 0$. We also assume that $\delta t^{(m)}$ tends to zero.*

Let $(\varrho^{(m)})_{m \in \mathbb{N}}$ and $(z^{(m)})_{m \in \mathbb{N}}$ be a sequence of discrete functions associated to the sequence of meshes and time steps, that are piecewise constant over respectively the primal cells and the dual cells. Let us suppose that $(\varrho^{(m)})_{m \in \mathbb{N}}$ and $(z^{(m)})_{m \in \mathbb{N}}$ converge in $L^1(\Omega \times (0, T))$ to $\bar{\varrho}$ and \bar{z} respectively. Then

$$\lim_{m \rightarrow +\infty} \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}^{(m)} \\ \sigma = K|L}} |D_\sigma| |\varrho_K^n - \varrho_L^n| = 0, \quad (25)$$

and

$$\lim_{m \rightarrow +\infty} \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} |K| \sum_{\sigma, \tau \in \mathcal{E}(K)} |z_\sigma^n - z_\tau^n| = 0. \quad (26)$$

Let $\varphi \in C_c^\infty(\Omega \times [0, T])$ and let us define φ_K^n by

$$\varphi_K^n = \varphi(\mathbf{x}_K, t_n), \text{ for } K \in \mathcal{M}^{(m)} \text{ and } 0 \leq n \leq N^{(m)}, \quad (27)$$

where \mathbf{x}_K stands for an arbitrary point of K . For $\sigma \in \mathcal{E}_{\text{int}}$, $\sigma = K|L$, we denote by d_σ the distance $d_\sigma = |\mathbf{x}_K - \mathbf{x}_L|$, and we introduce the following additional measure of the mesh regularity:

$$\theta_{\mathcal{M},2} = \max_{K \in \mathcal{M}} \left\{ \frac{\sum_{\tau \in \mathcal{E}(K)} |\tau|}{|K|} \max_{\sigma = K|L \in \mathcal{E}(K)} \{h_K + h_L\} \right\}. \quad (28)$$

Note that for $\sigma \in \mathcal{E}_{\text{int}}$, $\sigma = K|L$, we have

$$|\sigma| d_\sigma \leq |\sigma| (h_K + h_L) \leq \theta_{\mathcal{M},2} \min(|K|, |L|) \leq \frac{2\theta_{\mathcal{M},2}}{\zeta} |D_\sigma|. \quad (29)$$

Also note that the parameter $\theta_{\mathcal{M},2}$ defined in (28) is bounded independently of the mesh size if there exists $C \in \mathbb{R}_+$ such that $|K| \geq Ch$ and $|\sigma| \leq Ch^{d-1}$, for all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}$.

Let the piecewise vector function $\nabla_{\mathcal{E},\mathcal{T}}\varphi$ be defined by:

$$\nabla_{\mathcal{E},\mathcal{T}}\varphi = \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}} (\nabla_{\mathcal{E},\mathcal{T}}\varphi)_{\sigma}^n \mathcal{X}_{D_{\sigma}}(\mathbf{x}) \mathcal{X}_{[t_n, t_{n+1}]}(t), \text{ with}$$

$$(\nabla_{\mathcal{E},\mathcal{T}}\varphi)_{\sigma}^n = \begin{cases} \frac{|\sigma|}{|D_{\sigma}|} (\varphi_L^n - \varphi_K^n) \mathbf{n}_{K,\sigma} & \text{if } \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L, \\ 0 & \text{if } \sigma \in \mathcal{E}_{\text{ext}}. \end{cases} \quad (30)$$

This vector function may be seen as a non-consistent discrete gradient which seems to appear first in [7], where its convergence properties are shown for specific meshes and norms. The following weak convergence result holds [8, Theorem 3.2].

Lemma 7.2. *Let $(\mathcal{M}^{(m)})_{m \in \mathbb{N}}$ be a sequence of meshes such that the mesh step $h_{\mathcal{M}^{(m)}}$ tends to zero when m tends to $+\infty$. We suppose that there exists a real number θ such that $\theta_{\mathcal{M}^{(m)},2} \leq \theta$ for $m \in \mathbb{N}$. Let $(\delta t^{(m)})_{m \in \mathbb{N}}$ be a sequence of time steps such that $\delta t^{(m)}$ tends to zero when m tends to $+\infty$. Let $\varphi \in C_c^{\infty}(\Omega \times [0, T])$ and, for $m \in \mathbb{N}$, $\nabla_{\mathcal{E}^{(m)},\mathcal{T}^{(m)}}\varphi \in L^{\infty}(\Omega \times (0, T))^d$ be defined by (30).*

Then the sequence $(\nabla_{\mathcal{E}^{(m)},\mathcal{T}^{(m)}}\varphi)_{m \in \mathbb{N}}$ is bounded in $L^{\infty}(\Omega \times (0, T))^d$ uniformly with respect to m and converges to $\nabla\varphi$ in $L^{\infty}(\Omega \times (0, T))^d$ weak \star .

Multiplying (17) by φ_K^n defined by (27), summing over $K \in \mathcal{M}^{(m)}$ and $n = 0, \dots, N^{(m)} - 1$ yields

$$\sum_{n=0}^{N^{(m)}-1} \delta t \sum_{K \in \mathcal{M}^{(m)}} \mathcal{C}_K^n z \varphi_K^n = T_{\partial t}^{(m)} + T_{\text{div}}^{(m)} \quad (31)$$

with

$$T_{\partial t}^{(m)} = \sum_{n=0}^{N^{(m)}-1} \sum_{K \in \mathcal{M}^{(m)}} |K| ((\rho z)_K^{n+1} - (\rho z)_K^n) \varphi_K^n, \quad (32)$$

$$T_{\text{div}}^{(m)} = \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} \varphi_K^n \sum_{\sigma \in \mathcal{E}(K)} G_{K,\sigma}^n. \quad (33)$$

We are now in position to state the following consistency results.

Lemma 7.3 (Consistency of the time derivative). *Let $(\mathcal{M}^{(m)})_{m \in \mathbb{N}}$ and $(\delta t^{(m)})_{m \in \mathbb{N}}$ be a sequence of meshes and time steps such that the space and time steps, $h_{\mathcal{M}^{(m)}}$ and $\delta t^{(m)}$, tend to zero when m tends to $+\infty$. Let $(\rho^{(m)})_{m \in \mathbb{N}}$ and $(z^{(m)})_{m \in \mathbb{N}}$ be a sequence of discrete functions associated to $(\mathcal{M}^{(m)})_{m \in \mathbb{N}}$ and $(\delta t^{(m)})_{m \in \mathbb{N}}$. Let us suppose that $(\rho^{(m)})_{m \in \mathbb{N}}$ and $(z^{(m)})_{m \in \mathbb{N}}$ converge in $L^1(\Omega \times (0, T))$ to $\bar{\rho}$ and \bar{z} respectively, and that both sequences are bounded in $L^{\infty}(\Omega \times (0, T))$. Finally, let $\varphi \in C_c^{\infty}(\Omega \times [0, T])$ and let $T_{\partial t}^{(m)}$ be defined by (32). Then*

$$\lim_{m \rightarrow +\infty} T_{\partial t}^{(m)} = - \int_0^T \int_{\Omega} \bar{\rho} \bar{z} \partial_t \varphi \, d\mathbf{x} \, dt - \int_{\Omega} \rho_0 z_0 \varphi(\mathbf{x}, 0) \, d\mathbf{x}.$$

Proof. Let $(\rho z)^{(m)}$ be the function associated to $((\rho z)_K^n)_{K \in \mathcal{M}, 0 \leq n < N}$ that is piecewise constant on the primal cells and time intervals. Since in general, $|D_{K,\sigma}| \neq \frac{1}{2}|D_{\sigma}|$ (so that $|K| \neq \frac{1}{2} \sum_{\sigma \in \mathcal{E}(K)} |D_{\sigma}|$), the quantity $(\rho z)_K$ given by Equation (17) is not a convex combination of the value of the density and z at the primal faces, so that it is not clear that $(\rho z)^{(m)}$ converges to $\bar{\rho} \bar{z}$ in $L^1(\Omega \times (0, T))$ as $m \rightarrow +\infty$. However, since $\{D_{\sigma}, \sigma \in \mathcal{E}\}$ is a partition of Ω , we may expect that $(\rho z)^{(m)}$ weakly converges to $\bar{\rho} \bar{z}$; we prove this result in a first preliminary step and prove the consistency of the time derivative term in a second step.

Step 1. Weak convergence of $(\rho z)^{(m)}$. Let $\psi \in C_c^\infty(\Omega \times [0, T])$, and, for $K \in \mathcal{M}^{(m)}$ and $0 \leq n \leq N^{(m)}$, let us define $\tilde{\psi}_K^n$ as the mean value of ψ over $K \times (t_n, t_{n+1})$. Let us write

$$I^{(m)} = \int_0^T \int_\Omega (\rho z)^{(m)} \psi \, d\mathbf{x} \, dt = \sum_{n=0}^{N^{(m)}-1} \delta t \sum_{K \in \mathcal{M}^{(m)}} |K| (\rho z)_K^n \tilde{\psi}_K^n.$$

By the definitions (17) and (9), we get:

$$I^{(m)} = \sum_{n=0}^{N^{(m)}-1} \delta t \sum_{K \in \mathcal{M}^{(m)}} \sum_{\sigma \in \mathcal{E}(K)} \frac{1}{2} (|D_{K,\sigma}| \rho_K^n + |D_{L,\sigma}| \rho_L^n) z_\sigma^n \tilde{\psi}_K^n.$$

We now decompose $I^{(m)} = \mathcal{T}_1^{(m)} + R_1^{(m)}$ with

$$\mathcal{T}_1^{(m)} = \sum_{n=0}^{N^{(m)}-1} \delta t \sum_{K \in \mathcal{M}^{(m)}} \sum_{\sigma \in \mathcal{E}(K)} |D_{K,\sigma}| \rho_K^n z_\sigma^n \tilde{\psi}_K^n.$$

Then

$$\mathcal{T}_1^{(m)} = \int_0^T \int_\Omega \rho^{(m)} z^{(m)} \psi \, d\mathbf{x} \, dt.$$

The sequences $(\rho^{(m)})_{m \in \mathbb{N}}$ and $(z^{(m)})_{m \in \mathbb{N}}$ converge to $\bar{\rho}$ and \bar{z} in $L^p(\Omega \times (0, T))$ for $1 \leq p < +\infty$ (since, by assumption, these sequences converge in $L^1(\Omega \times (0, T))$ and are uniformly bounded). We thus have:

$$\lim_{m \rightarrow +\infty} \mathcal{T}_1^{(m)} = \int_0^T \int_\Omega \bar{\rho} \bar{z} \psi \, d\mathbf{x} \, dt.$$

The remainder term $R_2^{(m)}$ reads

$$R_1^{(m)} = \sum_{n=0}^{N^{(m)}-1} \delta t \sum_{K \in \mathcal{M}^{(m)}} \sum_{\sigma \in \mathcal{E}(K)} \frac{1}{2} (|D_{L,\sigma}| \rho_L^n - |D_{K,\sigma}| \rho_K^n) z_\sigma^n \tilde{\psi}_K^n.$$

For m large enough, since the support in space of ψ is compact in Ω , ψ vanishes in the cells having a face included in the boundary, and a reordering of the sums yields

$$\begin{aligned} R_1^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t \sum_{K \in \mathcal{M}^{(m)}} \sum_{\substack{\sigma \in \mathcal{E}(K) \\ \sigma=K/L}} \frac{1}{2} |D_{K,\sigma}| \rho_K^n z_\sigma^n (\tilde{\psi}_L^n - \tilde{\psi}_K^n), \\ &\leq C_\psi T |\Omega| \|\rho^{(m)}\|_{L^\infty(\Omega \times (0, T))} \|z^{(m)}\|_{L^\infty(\Omega \times (0, T))} h_{\mathcal{M}^{(m)}} \end{aligned}$$

where $C_\psi \in \mathbb{R}_+$ depends only on ψ ; therefore, $R_1^{(m)}$ tends to zero when m tends to $+\infty$. Finally, we observe that the sequence $(\rho z)_{m \in \mathbb{N}}$ is bounded in $L^\infty(\Omega \times (0, T))$, so that, by density, we obtain the weak convergence of this sequence in $L^p(\Omega \times (0, T))$, for $1 \leq p < +\infty$ (note that weak convergence for a limited range of indexes p would have been obtained under weaker assumptions on the unknowns, namely a control in L^q with a given finite q , only under regularity assumptions for the mesh; uniform boundedness is the only case where no such assumption is needed). A similar proof shows that the function of space only $(\rho z)^0$ obtained from the initial conditions (7) weakly converges to $\rho_0 z_0$ in $L^p(\Omega)$, for $1 \leq p < +\infty$.

Step 2. Weak consistency of the time term. Let us now turn to $T_{\partial t}^{(m)}$. A discrete integration by parts with respect to time yields $T_{\partial t}^{(m)} = T_{\partial t,1}^{(m)} + T_{\partial t,2}^{(m)}$ with

$$\begin{aligned} T_{\partial t,1}^{(m)} &= - \sum_{n=0}^{N^{(m)}-1} \delta t \sum_{K \in \mathcal{M}^{(m)}} |K| (\rho z)_K^{n+1} \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t}, \\ T_{\partial t,2}^{(m)} &= - \sum_{K \in \mathcal{M}^{(m)}} |K| (\rho z)_K^0 \varphi_K^0. \end{aligned}$$

Let us denote by $\bar{\partial}_t^{(m)}\varphi$ the following time discrete derivative of φ :

$$\bar{\partial}_t^{(m)}\varphi = \sum_{n=0}^{N^{(m)}-1} \sum_{K \in \mathcal{M}^{(m)}} \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t} \mathcal{X}_K \mathcal{X}_{[t_n, t_{n+1})}.$$

With this notation, we get:

$$T_{\partial t, 1}^{(m)} = - \int_{\delta t}^T \int_{\Omega} (\rho z)^{(m)} \bar{\partial}_t^{(m)}\varphi(\cdot, t - \delta t) \, d\mathbf{x} \, dt.$$

Thanks to the regularity of φ , $\bar{\partial}_t^{(m)}\varphi$ tends to $\partial_t\varphi$ when m tends to $+\infty$ in $L^\infty(\Omega \times (0, T))$, so that, thanks to the weak convergence of the sequence $((\rho z)^{(m)})_{m \in \mathbb{N}}$,

$$\lim_{m \rightarrow +\infty} T_{\partial t, 1}^{(m)} = - \int_0^T \int_{\Omega} \bar{\rho} \bar{z} \partial_t \varphi \, d\mathbf{x} \, dt.$$

Similarly, the sequence of piecewise functions of space $((\varphi^{(m)})^0)_{m \in \mathbb{N}}$ converges to $\varphi(\cdot, 0)$ in $L^\infty(\Omega)$, so that

$$\lim_{m \rightarrow +\infty} T_{\partial t, 2}^{(m)} = - \int_{\Omega} \rho_0 z_0 \varphi(\mathbf{x}, 0) \, d\mathbf{x},$$

which concludes the proof. \square

We now turn to the second term of (31), namely the divergence term $T_{\text{div}}^{(m)}$.

Lemma 7.4 (Consistency of the divergence term). *Let $\theta > 0$ and $(\mathcal{M}^{(m)})_{m \in \mathbb{N}}$ be a sequence of meshes such that $\max(\theta_{\mathcal{M}^{(m)}, 1}, \theta_{\mathcal{M}^{(m)}, 2}) \leq \theta$ for all $m \in \mathbb{N}$ and $\lim_{m \rightarrow +\infty} h_{\mathcal{M}^{(m)}} = 0$. Let $(\delta t^{(m)})_{m \in \mathbb{N}}$ be a sequence of time steps such that $\delta t^{(m)}$ tends to zero when m tends to $+\infty$. Let $(\rho^{(m)})_{m \in \mathbb{N}}$, $(\mathbf{u}^{(m)})_{m \in \mathbb{N}}$ and $(z^{(m)})_{m \in \mathbb{N}}$ be a sequence of discrete functions associated to $(\mathcal{M}^{(m)})_{m \in \mathbb{N}}$ and $(\delta t^{(m)})_{m \in \mathbb{N}}$. Let us suppose that $(\rho^{(m)})_{m \in \mathbb{N}}$ and $(z^{(m)})_{m \in \mathbb{N}}$ converge in $L^1(\Omega \times (0, T))$ to $\bar{\rho}$ and \bar{z} respectively, and that both sequences are bounded in $L^\infty(\Omega \times (0, T))$. We assume in addition that $(\mathbf{u}^{(m)})_{m \in \mathbb{N}}$ converges to $\bar{\mathbf{u}}$ in $L^1(\Omega \times (0, T))^d$, and is bounded in $L^\infty(\Omega \times (0, T))^d$. Finally, let $\varphi \in C_c^\infty(\Omega \times [0, T])$ and let $T_{\text{div}}^{(m)}$ be defined by (33). Then*

$$\lim_{m \rightarrow +\infty} T_{\text{div}}^{(m)} = - \int_0^T \int_{\Omega} \bar{\rho} \bar{z} \bar{\mathbf{u}} \cdot \nabla \varphi \, d\mathbf{x} \, dt.$$

Proof. Since the support in space of φ is compact in Ω , for m large enough, φ_K^n vanishes for $0 \leq n \leq N^{(m)} - 1$ for every cell K having one of its face on the boundary. For such an index m , reordering the summations, we get:

$$T_{\text{div}}^{(m)} = \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(m)} \\ \sigma = K|L}} G_{K, \sigma}^n (\varphi_K^n - \varphi_L^n). \quad (34)$$

Let $\sigma \in \mathcal{E}_{\text{int}}$, $\sigma = K|L$. By the property (10) of the mass fluxes through the dual cells, we get

$$\begin{aligned} - \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset K}} F_{\sigma, \epsilon}^n z_\epsilon^n &= -z_\sigma^n \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset K}} F_{\sigma, \epsilon}^n + R_{\sigma, 1}^n \\ &= z_\sigma^n \left(F_{K, \sigma}^n - \frac{1}{\zeta} \sum_{\tau \in \mathcal{E}(K)} F_{K, \tau}^n \right) + R_{\sigma, 1}^n, \end{aligned}$$

with

$$R_{\sigma, 1}^n = - \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset K}} F_{\sigma, \epsilon}^n (z_\epsilon^n - z_\sigma^n).$$

Similarly:

$$\sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \not\subset K}} F_{\sigma, \epsilon}^n z_\epsilon^n = z_\sigma^n \left(F_{K, \sigma}^n + \frac{1}{\zeta} \sum_{\tau \in \mathcal{E}(L)} F_{L, \tau}^n \right) + R_{\sigma, 2}^n,$$

with

$$R_{\sigma,2}^n = \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \not\subset K}} F_{\sigma,\epsilon}^n (z_\epsilon^n - z_\sigma^n).$$

Finally, we get

$$G_{K,\sigma}^n = F_{K,\sigma}^n z_\sigma^n + \frac{1}{2} R_{\sigma,1}^n + \frac{1}{2} R_{\sigma,2}^n + R_{\sigma,3}^n, \quad (35)$$

with

$$R_{\sigma,3}^n = \frac{1}{2\zeta} z_\sigma^n \left(\sum_{\tau \in \mathcal{E}(L)} F_{L,\tau}^n - \sum_{\tau \in \mathcal{E}(K)} F_{K,\tau}^n \right)$$

Using the expression (35) in (34), we get $T_{\text{div}}^{(m)} = T_0^{(m)} + T_2^{(m)} + T_3^{(m)}$, with

$$\begin{aligned} T_0^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(m)} \\ \sigma = K|L}} F_{K,\sigma}^n z_\sigma^n (\varphi_K^n - \varphi_L^n), \\ T_2^{(m)} &= \frac{1}{2} \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(m)} \\ \sigma = K|L}} (R_{\sigma,1}^n + R_{\sigma,2}^n) (\varphi_K^n - \varphi_L^n), \\ T_3^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(m)} \\ \sigma = K|L}} R_{\sigma,3}^n (\varphi_K^n - \varphi_L^n). \end{aligned}$$

We begin with the term $T_0^{(m)}$, which we decompose as $T_0^{(m)} = T_1^{(m)} + R_0^{(m)}$ with:

$$\begin{aligned} T_1^{(m)} &= - \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(m)} \\ \sigma = K|L}} (|D_{K,\sigma}| \rho_K^n + |D_{L,\sigma}| \rho_L^n) z_\sigma^n \mathbf{u}_\sigma^n \cdot \frac{|\sigma|}{|D_\sigma|} (\varphi_L^n - \varphi_K^n) \mathbf{n}_{K,\sigma} \\ &= - \int_0^T \int_\Omega \rho^{(m)} z^{(m)} \mathbf{u}^{(m)} \cdot \nabla_{\mathcal{E}^{(m)}, \mathcal{T}^{(m)}} \varphi \, d\mathbf{x} \, dt; \end{aligned}$$

again using the fact that the convergence in $L^1(\Omega \times (0, T))$ and the boundedness in $L^\infty(\Omega \times (0, T))$ implies the convergence in any $L^p(\Omega \times (0, T))$, $1 \leq p < +\infty$, we get by Lemma 7.2 that

$$\lim_{m \rightarrow +\infty} T_1^{(m)} = - \int_0^T \int_\Omega \bar{\rho} \bar{z} \bar{\mathbf{u}} \cdot \nabla \varphi \, d\mathbf{x} \, dt.$$

The residual term $R_0^{(m)}$ reads:

$$\begin{aligned} R_0^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(m)} \\ \sigma = K|L}} |\sigma| \left[\frac{|D_{K,\sigma}|}{|D_\sigma|} (\rho_\sigma^n - \rho_K^n) + \frac{|D_{L,\sigma}|}{|D_\sigma|} (\rho_\sigma^n - \rho_L^n) \right] \\ &\quad z_\sigma^n \mathbf{u}_\sigma^n \cdot \mathbf{n}_{K,\sigma} (\varphi_K^n - \varphi_L^n), \end{aligned}$$

The regularity of φ implies that, for $\sigma \in \mathcal{E}$, $\sigma = K|L$ and $0 \leq n \leq N-1$, $|\varphi_K^n - \varphi_L^n| \leq C_\varphi d_\sigma$. Hence, since ρ_σ^n is a convex combination of ρ_K^n and ρ_L^n , there exists $C > 0$, depending on the L^∞ bound of the unknowns and on φ , such that

$$|R_0^{(m)}| \leq C \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(m)} \\ \sigma = K|L}} |\sigma| d_\sigma |\rho_K^n - \rho_L^n|,$$

and thus $R_0^{(m)}$ tends to zero when m tends to $+\infty$ thanks to the assumed regularity of the sequence of meshes and Lemma 7.1.

The term $T_2^{(m)}$ reads:

$$T_2^{(m)} = \frac{1}{2} \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}^{(m)} \\ \sigma=K|L}} \left[\sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset L}} F_{\sigma,\epsilon}^n (z_\epsilon^n - z_\sigma^n) - \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset K}} F_{\sigma,\epsilon}^n (z_\epsilon^n - z_\sigma^n) \right] (\varphi_K^n - \varphi_L^n).$$

For $\epsilon = \sigma|\tau$, z_ϵ is supposed to be a convex combination of z_σ and z_τ , so that $|z_\epsilon - z_\sigma| \leq |z_\tau - z_\sigma|$. We thus get, reordering the sums, with the same constant C_φ as previously:

$$|T_2^{(m)}| \leq C_\varphi \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} \sum_{\substack{\epsilon \subset K \\ \epsilon = \sigma|\tau}} (d_\sigma + d_\tau) |F_{\sigma,\epsilon}^n| |z_\sigma^n - z_\tau^n|$$

Using now the expressions (11) and (6), and introducing $\xi = \max_{\tau \in \mathcal{E}(K)} \xi_{\epsilon,\tau}$ (recall that the coefficients $\xi_{\epsilon,\tau}$ do not depend on the cell K), we get:

$$|T_2^{(m)}| \leq C_\varphi \xi \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} |K| \sum_{\substack{\epsilon \subset K \\ \epsilon = \sigma|\tau}} \frac{1}{|K|} (d_\sigma + d_\tau) \left(\sum_{\omega \in \mathcal{E}(K)} |\omega| \rho_\omega^n |\mathbf{u}_\omega^n| \right) |z_\sigma^n - z_\tau^n|.$$

Since the unknowns are assumed to be bounded, by the assumed regularity of the sequence of meshes (see (28) for the definition of the regularity parameter $\theta_{\mathcal{M},2}$), Lemma 7.1 thus implies that $T_2^{(m)}$ tends to zero when m tends to $+\infty$.

Let us turn to the term $T_3^{(m)}$; using the fact that, for any cell K ,

$$\sum_{\sigma \in \mathcal{E}(K)} |\sigma| \mathbf{n}_{K,\sigma} = 0,$$

and denoting by \mathbf{u}_K^n a convex combination of the face velocities $(\mathbf{u}_\sigma)_{\sigma \in \mathcal{E}(K)}$, we get:

$$\begin{aligned} T_3^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(m)} \\ \sigma=K|L}} \frac{1}{2\zeta} z_\sigma^n \left(\sum_{\tau \in \mathcal{E}(L)} F_{L,\tau}^n - \sum_{\tau \in \mathcal{E}(K)} F_{K,\tau}^n \right) (\varphi_K^n - \varphi_L^n) \\ &= \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(m)} \\ \sigma=K|L}} \frac{1}{2\zeta} z_\sigma^n \left(\sum_{\tau \in \mathcal{E}(L)} |\tau| (\rho_\tau^n \mathbf{u}_\tau^n - \rho_L^n \mathbf{u}_L^n) \cdot \mathbf{n}_{L,\tau} \right. \\ &\quad \left. - \sum_{\tau \in \mathcal{E}(K)} |\tau| (\rho_\tau^n \mathbf{u}_\tau^n - \rho_K^n \mathbf{u}_K^n) \cdot \mathbf{n}_{K,\tau} \right) (\varphi_K^n - \varphi_L^n). \end{aligned}$$

Using the regularity of φ and the fact that the unknowns are assumed to be bounded, we get by a reordering of the summations that there exists C_φ only depending on φ such that

$$|T_3^{(m)}| \leq C_\varphi \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}^{(m)} \\ \sigma=K|L}} |V_{\sigma,K}| |\rho_\sigma^n \mathbf{u}_\sigma^n - \rho_K^n \mathbf{u}_K^n| + |V_{\sigma,L}| |\rho_\sigma^n \mathbf{u}_\sigma^n - \rho_L^n \mathbf{u}_L^n|,$$

where the quantity $|V_{\sigma,K}|$ is homogeneous to a volume, reads

$$|V_{\sigma,K}| = |\sigma| \sum_{\tau \in \mathcal{E}(K)} d_\tau.$$

and satisfies

$$|V_{\sigma,K}| < \sum_{\sigma \in \mathcal{E}(K)} |V_{\sigma,K}| = \sum_{\sigma, \tau \in \mathcal{E}(K)} |\sigma| d_\tau = \sum_{\tau \in \mathcal{E}(K)} d_\tau \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \leq \zeta \theta_{\mathcal{M}^{(m)},2} |K|.$$

By the regularity assumption of the sequence of meshes (see the Definition (23) of the regularity parameter $\theta_{\mathcal{M},1}$), we also have $|V_{\sigma,K}| < C|D_\sigma|$ with C independent of the mesh. Developing the difference of products in this relation by the identity $2(ab - cd) = (a+c)(b-d) + (a-c)(b+d)$ for $(a, b, c, d) \in \mathbb{R}^4$, invoking the assumed boundedness of the unknowns and Lemma 7.1 thus yields that $T_3^{(m)}$ tends to zero when m tends to $+\infty$. \square

8. LAX-WENDROFF CONSISTENCY OF THE SCHEME

The consistency of the mass balance equation easily follows by the arguments developed in [8], and we focus here on the equations involving the velocity convection operator, namely the momentum and the total energy balance equations.

8.1. The momentum balance equation

We show in this section that the limit of a convergent sequence of discrete solutions satisfies a weak form of the momentum balance equation. To this purpose, we derive a discrete momentum balance equation posed over the primal cells, involving the reconstructed convection operator defined by (19), and pass to the limit in the resulting relation. The consistency of the convection operator is already treated in the previous section, and we essentially have to check the consistency of the obtained (primal cell) discrete pressure gradient to obtain the following result.

Theorem 8.1 (Consistency of the primal cell momentum balance equation). *Let $\theta > 0$ and $(\mathcal{M}^{(m)})_{m \in \mathbb{N}}$ be a sequence of meshes such that $\max(\theta_{\mathcal{M}^{(m)},1}, \theta_{\mathcal{M}^{(m)},2}) \leq \theta$ for all $m \in \mathbb{N}$ and $\lim_{m \rightarrow +\infty} h_{\mathcal{M}^{(m)}} = 0$. We also assume that $\delta t^{(m)}$ tends to zero.*

Let us suppose that $(\rho^{(m)})_{m \in \mathbb{N}}$ and $(e^{(m)})_{m \in \mathbb{N}}$ converge in $L^1(\Omega \times (0, T))$ to $\bar{\rho}$ and \bar{e} respectively, and are bounded in $L^\infty(\Omega \times (0, T))$; let the sequence $(\mathbf{u}^{(m)})_{m \in \mathbb{N}}$ converge in $L^1(\Omega \times (0, T))^d$ to $\bar{\mathbf{u}}$ and be bounded in $L^\infty(\Omega \times (0, T))^d$.

Then the sequence $(p^{(m)})_{m \in \mathbb{N}}$ defined by $p^{(m)} = (\gamma - 1)\rho^{(m)}e^{(m)}$ for $m \in \mathbb{N}$ converges in $L^1(\Omega \times (0, T))$ to $\bar{p} = (\gamma - 1)\bar{\rho}\bar{e}$ and the limits $\bar{\rho}$, $\bar{\mathbf{u}}$ and \bar{p} satisfy the following weak form of the momentum balance equation which reads:

$$\int_0^T \int_\Omega \left(\bar{\rho} \bar{u}_i \partial_t \varphi + \bar{\rho} \bar{u}_i \bar{\mathbf{u}} \cdot \nabla \varphi + \bar{p} \partial_i \varphi \right) dx dt + \int_\Omega \rho_0 (\mathbf{u}_0)_i \varphi(\mathbf{x}, 0) dx = 0, \quad (36)$$

for any function $\varphi \in C_c^\infty(\Omega \times [0, T])$ and for $1 \leq i \leq d$.

Proof. Since the sequences $(\rho^{(m)})_{m \in \mathbb{N}}$ and $(e^{(m)})_{m \in \mathbb{N}}$ are uniformly bounded, their convergence in $L^1(\Omega \times (0, T))$ implies their convergence in $L^p(\Omega \times (0, T))$ for any finite $p > 1$, which yields the convergence of $(p^{(m)})_{m \in \mathbb{N}}$ to \bar{p} . Let us now, for $m \in \mathbb{N}$ and for $K \in \mathcal{M}^{(m)}$, sum the discrete momentum balance (5d) over the faces of K and divide by 2. We obtain, for $K \in \mathcal{M}^{(m)}$, $0 \leq n < N^{(m)}$ and $1 \leq i \leq d$, dropping some superscripts $^{(m)}$ for short:

$$C_K^n u_i + \frac{1}{2} \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{\sigma = K|L} (p_L^{n+1} - p_K^{n+1}) n_{K,\sigma,i} = 0. \quad (37)$$

Let $\varphi \in C_c^\infty(\Omega \times [0, T])$ and $\varphi_K^n = \varphi(\mathbf{x}_K, t_n)$, for $K \in \mathcal{M}^{(m)}$ and $0 \leq n \leq N^{(m)}$, with $\mathbf{x}_K \in K$. Multiplying (37) by $\delta t^{(m)} \varphi_K^n$ and summing over the cells and time levels, we obtain

$$\begin{aligned} T_1^{(m)} + T_2^{(m)} &= 0, \text{ with} \\ T_1^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} C_K^n u_i \varphi_K^n, \\ T_2^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} \frac{1}{2} \varphi_K^n \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{\sigma = K|L} (p_L^{n+1} - p_K^{n+1}) n_{K,\sigma,i}. \end{aligned}$$

By lemmas 7.3 and 7.4, we get:

$$\lim_{m \rightarrow +\infty} T_1^{(m)} = - \int_0^T \int_\Omega \left(\bar{\rho} \bar{u}_i \partial_t \varphi + \bar{\rho} \bar{u}_i \bar{\mathbf{u}} \cdot \nabla \varphi \right) dx dt - \int_\Omega \rho_0 (u_0)_i \varphi(\mathbf{x}, 0) dx.$$

Reordering the sums in $T_2^{(m)}$ yields

$$\begin{aligned} T_2^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(m)} \\ \sigma=K|L}} |\sigma| (p_L^{n+1} - p_K^{n+1}) \frac{\varphi_K^n + \varphi_L^n}{2} n_{K,\sigma,i} \\ &= - \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} p_K^{n+1} \sum_{\substack{\sigma \in \mathcal{E}^{(K)} \\ \sigma=K|L}} |\sigma| \frac{\varphi_K^n + \varphi_L^n}{2} n_{K,\sigma,i}. \end{aligned}$$

For $\sigma \in \mathcal{E}_{\text{int}}^{(m)}$ and $0 \leq n < N^{(m)}$, let

$$\varphi_\sigma^n = \frac{1}{\delta t^{(m)} |\sigma|} \int_{t_n}^{t_{n+1}} \int_\sigma \varphi \, d\gamma \, dt, \quad \delta \varphi_\sigma^n = \frac{\varphi_K^n + \varphi_L^n}{2} - \varphi_\sigma^n.$$

By regularity of φ , there exists a real number C_φ such that $|\delta \varphi_\sigma^n| \leq C_\varphi (h_K + h_L)$. With the notation

$$(\partial_i \varphi)_K^n = \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}^{(K)} \\ \sigma=K|L}} |\sigma| n_{K,\sigma,i} \varphi_\sigma^n = \int_{t_n}^{t_{n+1}} \int_K \partial_i \varphi \, d\mathbf{x} \, dt,$$

we remark that

$$|(\partial_i \varphi)_K^n| \leq C_\varphi \delta t^{(m)} |K|. \quad (38)$$

We now split $T_2^{(m)}$ as $T_2^{(m)} = -T_{2,1}^{(m)} - T_{2,2}^{(m)} - T_{2,3}^{(m)}$ with

$$\begin{aligned} T_{2,1}^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} p_K^n (\partial_i \varphi)_K^n, \\ T_{2,2}^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} (p_K^{n+1} - p_K^n) (\partial_i \varphi)_K^n, \\ T_{2,3}^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} p_K^{n+1} \sum_{\substack{\sigma \in \mathcal{E}^{(K)} \\ \sigma=K|L}} |\sigma| \delta \varphi_\sigma^n n_{K,\sigma,i}. \end{aligned}$$

The first term reads

$$T_{2,1}^{(m)} = \int_0^T \int_\Omega p^{(m)} \partial_i \varphi \, d\mathbf{x} \, dt,$$

so that, since $p^{(m)} \rightarrow \bar{p}$ in $L^1(\Omega \times (0, T))$ as $m \rightarrow +\infty$,

$$\lim_{m \rightarrow +\infty} T_{2,1}^{(m)} = \int_0^T \int_\Omega \bar{p} \partial_i \varphi \, d\mathbf{x} \, dt.$$

Thanks to (38), we get for $T_{2,2}^{(m)}$:

$$|T_{2,2}^{(m)}| \leq C_\varphi \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} |K| |p_K^{n+1} - p_K^n|,$$

so $T_{2,2}^{(m)}$ tends to zero when m tends to $+\infty$ by [8, Lemma 4.3]. Reordering the sums in $T_{2,3}^{(m)}$, we get

$$T_{2,3}^{(m)} = \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(m)} \\ \sigma=K|L}} |\sigma| \delta \varphi_\sigma^n (p_K^{n+1} - p_L^{n+1}) n_{K,\sigma,i}.$$

Thanks to the estimate of $|\delta\varphi_\sigma^n|$ and the regularity assumption on the sequence of meshes which implies that $|\sigma| (h_K + h_L) \leq C |D_\sigma|$ (see Inequality (29)), we obtain:

$$|T_{2,3}^{(m)}| \leq C_\varphi \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(m)} \\ \sigma=K|L}} |D_\sigma| |p_K^{n+1} - p_L^{n+1}|,$$

and $T_{2,3}^{(m)}$ tends to zero when m tends to $+\infty$ once again by [8, Lemma 4.3], which concludes the proof. \square

8.2. The energy balance equation

We now derive a kinetic energy balance equation associated to the primal mesh and add to the internal energy balance equation to obtain a discrete total energy balance, which shows that the scheme is, in some sense, locally conservative with respect to the total energy.

Lemma 8.2 (Local discrete total energy balance). *A solution to the scheme (5) satisfies the following total energy balance equation, for $0 \leq n \leq N - 1$ and $K \in \mathcal{M}$:*

$$\begin{aligned} \frac{|K|}{\delta t} \left[(\rho_K^{n+2} e_K^{n+2} + (\rho E_k)_K^{n+1}) - (\rho_K^{n+1} e_K^{n+1} + (\rho E_k)_K^n) \right] \\ + \sum_{\sigma=K|L}^{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^{n+1} e_\sigma^{n+1} + (G_k)_\sigma^n + \sum_{\sigma=K|L}^{\sigma \in \mathcal{E}(K)} |\sigma| \frac{p_K^{n+1} + p_L^{n+1}}{2} \mathbf{u}_\sigma^{n+1} \cdot \mathbf{n}_{K,\sigma} = 0, \end{aligned} \quad (39)$$

where the kinetic energy E_k and its associated flux G_k result from the construction of the convection operator on the primal mesh from the convection on the dual mesh described in Section 6:

$$\begin{aligned} |K| (\rho E_k)_K^n &= \frac{1}{4} \sum_{\sigma \in \mathcal{E}(K)} |D_\sigma| \rho_{D_\sigma}^n |\mathbf{u}_\sigma^n|^2, \\ (G_k)_\sigma^n &= -\frac{1}{2} \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset K}} F_{\sigma,\epsilon}^n |\mathbf{u}_\epsilon^n|^2 + \frac{1}{2} \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \not\subset K}} F_{\sigma,\epsilon}^n |\mathbf{u}_\epsilon^n|^2. \end{aligned} \quad (40)$$

Proof. For $K \in \mathcal{M}$, summing over $\sigma \in \mathcal{E}(K)$ and $i = 1, \dots, d$ the kinetic energy balance (13) divided by two and adding the internal energy balance (5b) of the scheme written at the time step $n + 2$ yields Equation (39). Thanks to its *ad hoc* definition (15), the corrective term S_K^{n+1} exactly compensates with the sum of the residual terms R_σ^{n+1} defined by (14) to yield a conservative equation. \square

We are now in position to prove the following weak consistency result by passing to the limit in the scheme.

Theorem 8.3 (Consistency of the total energy balance equation). *Let $\theta > 0$ and $(\mathcal{M}^{(m)})_{m \in \mathbb{N}}$ be a sequence of meshes such that $\max(\theta_{\mathcal{M}^{(m)},1}, \theta_{\mathcal{M}^{(m)},2}) \leq \theta$ for all $m \in \mathbb{N}$ and $\lim_{m \rightarrow +\infty} h_{\mathcal{M}^{(m)}} = 0$. We also assume that $\delta t^{(m)}$ tends to zero.*

Let us suppose that $(\rho^{(m)})_{m \in \mathbb{N}}$ and $(e^{(m)})_{m \in \mathbb{N}}$ converge in $L^1(\Omega \times (0, T))$ to $\bar{\rho}$ and \bar{e} respectively, and are bounded in $L^\infty(\Omega \times (0, T))$; let the sequence $(\mathbf{u}^{(m)})_{m \in \mathbb{N}}$ converge in $L^1(\Omega \times (0, T))^d$ to $\bar{\mathbf{u}}$ and be bounded in $L^\infty(\Omega \times (0, T))^d$. We suppose in addition that ρ_0 and e_0 belong to $W^{1,1}(\Omega)$ and \mathbf{u}_0 belongs to $W^{1,1}(\Omega)^d$.

Then the sequence $(p^{(m)})_{m \in \mathbb{N}}$ defined by $p^{(m)} = (\gamma - 1) \rho^{(m)} e^{(m)}$ for $m \in \mathbb{N}$ converges in $L^1(\Omega \times (0, T))$ to $\bar{p} = (\gamma - 1) \bar{\rho} \bar{e}$ and the limits $\bar{\rho}$, $\bar{\mathbf{u}}$, \bar{p} and \bar{e} satisfy the following weak form of the total energy balance equation:

$$\int_0^T \int_\Omega \left(\bar{\rho} \bar{E} \partial_t \varphi + (\bar{\rho} \bar{E} + \bar{p}) \bar{\mathbf{u}} \cdot \nabla \varphi \right) d\mathbf{x} dt + \int_\Omega \rho_0 (E)_0 \varphi(\mathbf{x}, 0) d\mathbf{x} = 0, \\ \text{with } \bar{E} = \frac{1}{2} |\bar{\mathbf{u}}|^2 + \bar{e}, (E)_0 = \frac{1}{2} |\mathbf{u}_0|^2 + e_0 \quad (41)$$

and for any function $\varphi \in C_c^\infty(\Omega \times [0, T])$.

Proof. The convergence of $(p^{(m)})_{m \in \mathbb{N}}$ to \bar{p} is shown in the proof of Theorem 8.1. We now pass to the limit in a weak form of Equation (39). Let $\varphi \in C_c^\infty(\Omega \times [0, T])$, $m \in \mathbb{N}$ and $\varphi_K^n = \varphi(\mathbf{x}_K, t_n)$, for $K \in \mathcal{M}^{(m)}$ and $0 \leq n \leq N^{(m)}$, with $\mathbf{x}_K \in K$. Multiplying (39) by $\delta t^{(m)} \varphi_K^n$ and summing over the cells and time levels, we obtain, dropping the superscripts $^{(m)}$ for short whenever it does not hinder comprehension,

$$\begin{aligned} T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + T_4^{(m)} + T_5^{(m)} &= 0, \text{ with} \\ T_1^{(m)} &= \sum_{n=0}^{N^{(m)}-2} \sum_{K \in \mathcal{M}^{(m)}} |K| (\rho_K^{n+2} e_K^{n+2} - \rho_K^{n+1} e_K^{n+1}) \varphi_K^n, \\ T_2^{(m)} &= \sum_{n=0}^{N^{(m)}-2} \sum_{K \in \mathcal{M}^{(m)}} |K| \left((\rho E_k)_K^{n+1} - (\rho E_k)_K^n \right) \varphi_K^n, \\ T_3^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} \varphi_K^n \sum_{\sigma \in \mathcal{E}(K)} |\sigma| F_{K,\sigma}^{n+1} e_\sigma^{n+1}, \\ T_4^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} \varphi_K^n \sum_{\sigma \in \mathcal{E}(K)} (G_k)_\sigma^n, \\ T_5^{(m)} &= \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} \varphi_K^n \sum_{\sigma=K|L} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \frac{p_K^{n+1} + p_L^{n+1}}{2} \mathbf{u}_\sigma^{n+1} \cdot \mathbf{n}_{K,\sigma}. \end{aligned}$$

Reordering the sums in $T_1^{(m)}$ yields $T_1^{(m)} = T_{1,1}^{(m)} + T_{1,2}^{(m)}$ with

$$\begin{aligned} T_{1,1}^{(m)} &= - \sum_{n=0}^{N^{(m)}-2} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} |K| \rho_K^{n+2} e_K^{n+2} \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t^{(m)}}, \\ T_{1,2}^{(m)} &= - \sum_{K \in \mathcal{M}^{(m)}} |K| \rho_K^1 e_K^1 \varphi_K^0. \end{aligned}$$

By a proof similar to that of Step 2 in Lemma 7.3, we get:

$$\lim_{m \rightarrow +\infty} T_{1,1}^{(m)} = - \int_0^T \int_\Omega \bar{\rho} \bar{e} \partial_t \varphi \, d\mathbf{x} \, dt.$$

The treatment of the term $T_{1,2}^{(m)}$ is more intricate, since ρ_K^1 and e_K^1 are not directly related to the initial conditions; this is the reason why the $W^{1,1}$ regularity assumption on the initial conditions is needed. Indeed, using the discrete internal energy balance (5b), we get:

$$\begin{aligned} \|\rho^1 e^1 - \rho^0 e^0\|_{L^1(\Omega)} &\leq \delta t^{(m)} (\|\operatorname{div}_{\mathcal{M}}(\rho^0 e^0 \mathbf{u}^0)\| + \|p^0 \operatorname{div}(\mathbf{u}^0)\|), \text{ with} \\ \|\operatorname{div}_{\mathcal{M}}(\rho^0 e^0 \mathbf{u}^0)\| &= \sum_{K \in \mathcal{M}^{(m)}} \left| \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^n e_\sigma^n \right|, \\ \|p^0 \operatorname{div}(\mathbf{u}^0)\| &= \sum_{K \in \mathcal{M}^{(m)}} |K| |p_K^n (\operatorname{div} \mathbf{u})_K^n|. \end{aligned}$$

These last two terms are bounded if $\rho_0, e_0 \in W^{1,1}(\Omega)$ and $\mathbf{u}_0 \in W^{1,1}(\Omega)^d$ and thus, when m converges to $+\infty$, $\rho^1 e^1$ tends to $\rho^0 e^0$ in $L^1(\Omega)$ and

$$\lim_{m \rightarrow +\infty} T_{1,2}^{(m)} = - \int_\Omega \rho_0 e_0 \varphi(\mathbf{x}, 0) \, d\mathbf{x}.$$

The convergence of the sequence of discrete velocities in $L^1(\Omega \times (0, T))^d$ and its boundedness in $L^\infty(\Omega \times (0, T))^d$ yield that the sequence of discrete functions associated to the kinetic energy also converges in $L^1(\Omega \times (0, T))$ and is uniformly bounded, and that the limit is equal to $\frac{1}{2} |\bar{u}|^2$. In addition, thanks to the fact that the initial

velocity \mathbf{u}_0 belongs to $L^\infty(\Omega)^d$, the sequence $(\frac{1}{2} |(\mathbf{u}^{(m)})^0|^2)_{m \in \mathbb{N}}$ defined by (7) converges to $\frac{1}{2} |\mathbf{u}_0|^2$. Lemma 7.3, with $z = E_k$, thus shows that

$$\lim_{m \rightarrow +\infty} T_2^{(m)} = - \int_0^T \int_\Omega \frac{1}{2} \bar{\rho} |\bar{\mathbf{u}}|^2 \partial_t \varphi \, d\mathbf{x} \, dt - \int_\Omega \rho_0 |\mathbf{u}_0|^2 \varphi(\mathbf{x}, 0) \, d\mathbf{x}.$$

Standard arguments combining Lemmas 7.1 and 7.2 show that

$$\lim_{m \rightarrow +\infty} T_3^{(m)} = - \int_0^T \int_\Omega \bar{\rho} \bar{\mathbf{e}} \cdot \bar{\mathbf{u}} \cdot \nabla \varphi \, d\mathbf{x} \, dt, \quad \lim_{m \rightarrow +\infty} T_5^{(m)} = - \int_0^T \int_\Omega \bar{p} \bar{\mathbf{u}} \cdot \nabla \varphi \, d\mathbf{x} \, dt.$$

Let $(\widetilde{G}_k)_\sigma^n$ be defined by:

$$(\widetilde{G}_k)_\sigma^n = - \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset K}} F_{\sigma, \epsilon}^n (E_k)_\epsilon^n + \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \not\subset K}} F_{\sigma, \epsilon}^n (E_k)_\epsilon^n,$$

where, for $\epsilon = \sigma|\tau$, the quantity $(E_k)_\epsilon^n$ may be any convex combination of $(E_k)_\sigma^n$ and $(E_k)_\tau^n$, for instance $(E_k)_\sigma^n$ with D_σ the upwind cell of ϵ with respect to $F_{\sigma, \epsilon}^n$. Let us define $\widetilde{T}_4^{(m)}$ by:

$$\widetilde{T}_4^{(m)} = \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} \varphi_K^n \sum_{\sigma \in \mathcal{E}(K)} (\widetilde{G}_k)_\sigma^n,$$

Then, the convergence of the sequence of discrete kinetic energies already invoked for the term $T_2^{(m)}$ implies that, thanks to Lemma 7.4,

$$\lim_{m \rightarrow +\infty} \widetilde{T}_4^{(m)} = - \int_0^T \int_\Omega \frac{1}{2} |\bar{\mathbf{u}}|^2 \bar{\mathbf{u}} \cdot \nabla \varphi \, d\mathbf{x} \, dt.$$

Let us write $T_4^{(m)} = \widetilde{T}_4^{(m)} + R_4^{(m)}$ with

$$R_4^{(m)} = \sum_{n=0}^{N^{(m)}-1} \delta t^{(m)} \sum_{K \in \mathcal{M}^{(m)}} \varphi_K^n \sum_{\sigma=K|L} \left[(G_k)_\sigma^n - (\widetilde{G}_k)_\sigma^n \right].$$

The fact that $R_4^{(m)}$ tends to zero when m tends to $+\infty$ follows from the assumed boundedness of the unknowns and Lemma 7.1 and concludes the proof. \square

Remark 8.1 (On the $W^{1,1}$ assumption on the initial data). *Note that the $W^{1,1}$ assumption on the initial data is needed because of the time shift on the pressure terms, and because we have taken into account the initial condition. However, if we take the test function $\varphi \in C_c(\Omega \times (0, T))$ instead of $\varphi \in C_c(\Omega \times [0, T])$, then this assumption is not needed. In particular, the $W^{1,1}$ assumption on the initial data is not needed to recover the Rankine-Hugoniot condition.*

REFERENCES

1. G. Ansanay-Alex, F. Babik, J.-C. Latché, and D. Vola, *An L^2 -stable approximation of the Navier-Stokes convection operator for low-order non-conforming finite elements*, International Journal for Numerical Methods in Fluids **66** (2011), 555–580.
2. D.J. Benson, *Computational methods in lagrangian and eulerian hydrocodes*, Comput. Methods Appl. Mech. Eng. **99** (1992), 235–394.
3. H. Bijl and P. Wesseling, *A unified method for computing incompressible and compressible flows in boundary-fitted coordinates*, Journal of Computational Physics **141** (1998), 153–173.
4. CALIF³S, *A software components library for the computation of reactive turbulent flows*, <https://gforge.irsn.fr/gf/project/isis>.
5. E.J. Caramana, D.E. Burton, M.J. Shashkov, and P.P. Whalen, *The construction of compatible hydrodynamics algorithms utilizing conservation of total energy*, Journal of Computational Physics **146** (1998), no. 1, 227 – 262.
6. G. Dakin, B. Després, and S. Jaouen, *High-order staggered schemes for compressible hydrodynamics. Weak consistency and numerical validation*, Journal of Computational Physics **376** (2019), 339–364.
7. R. Eymard and T. Gallouët, *H-convergence and numerical schemes for elliptic equations*, SIAM Journal on Numerical Analysis **41** (2000), 539–562.

8. T. Gallouët, R. Herbin, and J.-C. Latché, *On the weak consistency of finite volume schemes for conservation laws on general meshes*, SeMA Journal **76** (2019), no. 4, 581–594.
9. T. Gallouët, R. Herbin, J.-C. Latché, and N. Therme, *Consistent internal energy based schemes for the compressible Euler equations*, Numerical Simulation in Physics and Engineering: Trends and Applications, Lecture Notes of the XVIII Jacques-Louis Lions Spanish-French School, Springer, SEMA/SIMAI Series, 2020.
10. L. Gastaldo, R. Herbin, J.-C. Latché, and N. Therme, *A MUSCL-type segregated - explicit staggered scheme for the Euler equations.*, Computer and Fluids **175** (2018), 91–110.
11. D. Grapsas, R. Herbin, W. Kheriji, and J.-C. Latché, *An unconditionally stable staggered pressure correction scheme for the compressible Navier-Stokes equations*, SMAI-Journal of Computational Mathematics **2** (2016), 51–97.
12. F.H. Harlow and A.A. Amsden, *A numerical fluid dynamics calculation method for all flow speeds*, Journal of Computational Physics **8** (1971), 197–213.
13. F.H. Harlow and J.E. Welsh, *Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface*, Physics of Fluids **8** (1965), 2182–2189.
14. R. Herbin, W. Kheriji, and J.-C. Latché, *On some implicit and semi-implicit staggered schemes for the shallow water and Euler equations*, Mathematical Modelling and Numerical Analysis **48** (2014), 1807–1857.
15. R. Herbin, J.-C. Latché, and T.T. Nguyen, *Consistent segregated staggered schemes with explicit steps for the isentropic and full Euler equations.*, Mathematical Modelling and Numerical Analysis **52** (2018), 893–944.
16. R. Herbin, J.-C. Latché, and K. Saleh, *Low Mach number limit of some staggered schemes for compressible barotropic flows*, under revision (2020), <https://arxiv.org/abs/1803.09568>.
17. R. Herbin, J.-C. Latché, and C. Zaza, *A cell-centred pressure-correction scheme for the compressible Euler equations*, IMA Journal of Numerical Analysis (2019).
18. R.I. Issa, *Solution of the implicitly discretised fluid flow equations by operator splitting*, Journal of Computational Physics **62** (1985), 40–65.
19. K.C. Karki and S.V. Patankar, *Pressure based calculation procedure for viscous flows at all speeds in arbitrary configurations*, AIAA Journal **27** (1989), 1167–1174.
20. N. Kwatra, J. Su, J.T. Grétarsson, and R. Fedkiw, *A method for avoiding the acoustic time step restriction in compressible flow*, Journal of Computational Physics **228** (2009), 4146–4161.
21. J.-C. Latché, B. Piar, and K. Saleh, *A discrete kinetic energy preserving convection operator for variable density flows on locally refined staggered meshes.*, in preparation (2019).
22. J. Llobell, *Schémas volumes finis à mailles décalées pour la dynamique des gaz*, Ph.D. thesis, Université Côte d’Azur, 2018.
23. J.J. McGuirk and G.J. Page, *Shock capturing using a pressure-correction method*, AIAA Journal **28** (1990), 1751–1757.
24. R. Rannacher and S. Turek, *Simple nonconforming quadrilateral Stokes element*, Numerical Methods for Partial Differential Equations **8** (1992), 97–111.
25. R.W.C.P. Verstappen and A.E.P. Velman, *Symmetry-preserving discretization of turbulent flow*, Journal of Computational Physics **187** (2003), 343–368.
26. J. VonNeumann and R. D. Richtmyer, *A method for the numerical calculation of hydrodynamic shocks*, Journal of Applied Physics **21** (1950), no. 3, 232–237.
27. C. Wall, C.D. Pierce, and P. Moin, *A semi-implicit method for resolution of acoustic waves in low Mach number flows*, Journal of Computational Physics **181** (2002), 545–563.
28. P. Wesseling, *Principles of computational fluid dynamics*, Springer Series in Computational Mathematics, vol. 29, Springer, 2001.
29. S.Y. Yoon and T. Yabe, *The unified simulation for incompressible and compressible flow by the predictor-corrector scheme based on the CIP method*, Computer Physics Communications **119** (1999), 149–158.