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► **To cite this version:**

Andrea Bondesan, Marc Briant. PERTURBATIVE CAUCHY THEORY FOR A FLUX-INCOMPRESSIBLE MAXWELL-STEFAN SYSTEM. 2019. hal-02307937v2

HAL Id: hal-02307937

<https://hal.science/hal-02307937v2>

Preprint submitted on 9 Oct 2019 (v2), last revised 6 Aug 2020 (v3)

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**PERTURBATIVE CAUCHY THEORY
FOR A FLUX-INCOMPRESSIBLE MAXWELL-STEFAN SYSTEM
IN A NON-EQUIMOLAR REGIME**

ANDREA BONDESAN AND MARC BRIANT

ABSTRACT. We establish a quantitative Cauchy theory in Sobolev spaces for the Maxwell-Stefan equations with an incompressibility condition on the total flux. More precisely, we prove existence, uniqueness in a weak sense and exponential trend to equilibrium of solutions in a perturbative regime around any macroscopic equilibrium state of the mixture, not necessarily constant. In particular, an orthogonal viewpoint that we found specific to the incompressible setting, combined with the use of a suitable anisotropic norm, allows us to get rid of the usual closure assumption of equimolar diffusion.

Keywords: Fluid mixtures, incompressible Maxwell-Stefan, perturbative Cauchy theory.

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1. INTRODUCTION

We consider a chemically non-reacting ideal gaseous mixture composed of $N \geq 2$ different species, having atomic masses $(m_i)_{1 \leq i \leq N}$ and evolving in the 3-dimensional torus \mathbb{T}^3 . We assume isothermal and isobaric conditions, in order to avoid any convective effect and to focus our attention on a purely diffusive setting. For any $1 \leq i \leq N$, the balance of mass links the number of particles $c_i = c_i(t, x)$ of the i -th species to its flux $\mathcal{F}_i = \mathcal{F}_i(t, x)$ via the continuity equation

$$(1.1) \quad \partial_t c_i + \nabla_x \cdot \mathcal{F}_i = 0 \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^3.$$

Let $c = \sum_i c_i$ denote the total number of particles in the mixture and set $n_i = c_i/c$, the mole fraction of the i -th species. The Maxwell-Stefan equations then give

The authors would like to thank Laurent Boudin and B erence Grec for fruitful discussions on the theory of gaseous and fluid mixtures.

relations between the fluxes and the mole fractions and read, for any $1 \leq i \leq N$,

$$(1.2) \quad -c \nabla_x n_i = \sum_{j=1}^N \frac{n_j \mathcal{F}_i - n_i \mathcal{F}_j}{D_{ij}} \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^3,$$

where $D_{ij} = D_{ji} > 0$ are the effective diffusion coefficients between species i and j .

Independently introduced in the 19th century by Maxwell [20] for dilute gases and Stefan [22] for fluids, equations (1.2) describe the cross-diffusive interactions inside a mixture and therefore lie in the class of the so-called cross-diffusion models [21, 19, 18, 10, 15]. In particular, the system (1.2) gives a generalization [17] of Fick's law of mono-species diffusion [12], making it of core importance for applications in physics and medicine, where it can be used for example to model the propagation of polluting particles in the air or to characterize the gas exchanges in the lower generations of the human lung [23, 7, 2]. Besides, the Maxwell-Stefan equations also raise a great theoretical interest, as their mathematical analysis appears to be very challenging. The difficulties come from the fact that summing over i the relations (1.2), we obtain a linear dependence on the mole fractions' gradients which imposes to introduce a further condition in order to close the system and provide a satisfactory Cauchy theory to (1.1)–(1.2). To our knowledge, the existing mathematical results that deal with the problem of existence and uniqueness of solutions to the sole system (1.1)–(1.2) are all tied up to the assumption that the mixture is subject to a transient equimolar diffusion [17], namely the total diffusive flux satisfies

$$(1.3) \quad \sum_{i=1}^N \mathcal{F}_i(t, x) = 0, \quad t \geq 0, \quad x \in \mathbb{T}^3.$$

In [13] Giovangigli proved existence, uniqueness and trend to equilibrium for perturbative solutions in Sobolev spaces on \mathbb{R}^3 , when the initial datum is sufficiently close to a constant stationary state of the mixture. Working in a bounded domain Ω , Bothe exploited classical results from the theory of quasi-linear parabolic equations in order to show [1] local-in-time existence and uniqueness for solutions in $L^p(\Omega)$, starting from a general initial datum. Boudin *et al.* investigated in [4] the particular case of a 3-species mixture, when two effective diffusion coefficients are equal: the authors were able to establish global existence and uniqueness in $L^\infty(\Omega)$, as well as a long-time convergence of solutions towards the corresponding constant equilibrium state. By passing to entropy variables, Jüngel and Stelzer were then able to obtain [16] global-in-time existence of weak solutions in $H^1(\Omega)$ as well as an exponential decay to the homogeneous steady state of the mixture, for arbitrary diffusion coefficients and for general initial data. These techniques have been later on applied for extending the same kind of results to situations where convective effects are considered [8] (the Maxwell-Stefan system is here coupled with the incompressible Navier-Stokes equations, which prescribe the evolution of the incompressible molar average mixture velocity) or chemical reactions are taken into account [9].

The present article aims at studying the problem of existence and uniqueness of perturbative solutions to an incompressible variant of the Maxwell-Stefan system (1.1)–(1.2)–(1.3), which is written for any $1 \leq i \leq N$ on $\mathbb{R}_+ \times \mathbb{T}^3$ in terms of the

bulk velocities $(u_i)_{1 \leq i \leq N}$ as

$$(1.4) \quad \partial_t c_i + \nabla_x \cdot (c_i u_i) = 0,$$

$$(1.5) \quad -\nabla_x c_i = \sum_{j=1}^N c_i c_j \frac{u_i - u_j}{\Delta_{ij}},$$

$$(1.6) \quad \nabla_x \cdot \left(\sum_{i=1}^N c_i u_i \right) = 0,$$

where the closure relation (1.3) is replaced by the incompressibility-like condition (1.6). Note in particular that the model (1.1)–(1.2) can be easily recovered *a priori* from (1.4)–(1.5)–(1.6) by defining $\mathcal{F}_i = c_i u_i$ for any $1 \leq i \leq N$ and by supposing that $\sum_i c_i^{\text{in}}(x) = \text{const.}$ on \mathbb{T}^3 . Indeed, thanks to this hypothesis, the total number number of particles c remains constant over time on \mathbb{T}^3 , since both $\partial_t c = 0$ and $\nabla_x c = 0$ are obtained from (1.4)–(1.5)–(1.6). The quantities Δ_{ij} are then linked to the effective diffusion coefficients through the relations $D_{ij} = \Delta_{ij}/c$.

The above Maxwell-Stefan-type system is of peculiar significance, as recent works [5, 14, 3] managed to formally derive it starting from the kinetic equations. Providing a Cauchy theory for (1.4)–(1.5)–(1.6) thus becomes crucial if one wants to deal with its rigorous hydrodynamical derivation and, by this, show the physical coherence between the mesoscopic and the macroscopic descriptions. Moreover, we underline that the condition (1.6), naturally appearing in these formal asymptotics from the kinetic level, constitutes a generalization of the closure assumption (1.3), so that our work allows to treat the Maxwell-Stefan equations in a non-equimolar regime. In particular, we stress the fact that this kind of incompressibility condition distinguishes from the one considered in [8], since we here concentrate on a purely diffusive setting where the convection effects are neglected (thus assuming the absence of viscous flow).

As usually done in the literature [13, 1, 16], we begin by introducing the matrix

$$(1.7) \quad A(\mathbf{c}) = \left(\frac{c_i c_j}{\Delta_{ij}} - \left(\sum_{k=1}^N \frac{c_i c_k}{\Delta_{ik}} \right) \delta_{ij} \right)_{1 \leq i, j \leq N},$$

which depends on $(c_i)_{1 \leq i \leq N}$. In this way, the system of equations (1.4)–(1.5) can be initially rewritten in a more convenient vectorial form as

$$\partial_t \mathbf{c} + \nabla_x \cdot (\mathbf{c} \mathbf{u}) = 0,$$

$$\nabla_x \mathbf{c} = A(\mathbf{c}) \mathbf{u},$$

where bold letters will denote N -vectors referring to the species of the mixture, so that in this case $\mathbf{c} = (c_1, \dots, c_N)$ and $\mathbf{u} = (u_1, \dots, u_N)$, and the product $\mathbf{c} \mathbf{u}$ has to be understood componentwise. A natural idea for tackling our problem would then be to invert the gradient relation in order to express \mathbf{u} in terms of \mathbf{c} and obtain an evolution equation for the sole unknown \mathbf{c} , by replacing $\mathbf{u} = A(\mathbf{c})^{-1} \nabla_x \mathbf{c}$ into the continuity equation. Unfortunately, it is possible to prove [13, 1, 16] that the matrix A is only positive semi-definite, with $\text{Ker} A = \text{Span}(\mathbf{1})$. Therefore, any existing Cauchy theory for the Maxwell-Stefan equations is based on the possibility of explicitly computing the pseudoinverse of A , which is defined on the space $(\text{Span}(\mathbf{1}))^\perp$.

This can be achieved for example [1, 16, 8, 9] using the Perron-Frobenius theory for quasi-positive matrices. However, a major drawback of this strategy is that the computations giving the explicit form of A^{-1} are extremely intricate and do not offer a neat understanding of the action of A on the velocities \mathbf{u} . As already pointed out, since one cannot see the part of \mathbf{u} that evolves in $\ker A$, a closure assumption of type (1.3) is needed in order to compensate this lack of information.

In this work we propose another approach which takes inspiration from the micro-macro decomposition techniques commonly used in the kinetic theory of gases. More precisely, by defining the orthogonal projection π_A onto $\text{Span}(\mathbf{1})$, associated to the non-injective operator A , we split $\mathbf{u} = \pi_A(\mathbf{u}) + \mathbf{U}$ into a part projected onto $\text{Span}(\mathbf{1})$ and an orthogonal part \mathbf{U} which is projected onto $(\text{Ker}A)^\perp$. Thanks to the incompressibility condition we are in particular able to get rid of the closure assumption (1.3) and construct a new system of equations, equivalent to (1.4)–(1.6) for full velocities \mathbf{u} , in which the Maxwell-Stefan matrix only acts on \mathbf{U} . In such a way, we are able to deal with the pseudoinverse of A (which is always well-defined on $(\ker A)^\perp$), without the need of computing its explicit structure. Moreover, our alternative strategy provides an original point of view in which we exhibit a clear separation between $\pi_A(\mathbf{u})$ and \mathbf{U} , allowing to show explicitly the actual action of A on the sole vector \mathbf{U} . We shall then prove that the orthogonal reformulation of the system (1.4)–(1.5)–(1.6) in terms of the couple (\mathbf{c}, \mathbf{U}) is fully closed and exhibits a quasilinear parabolic structure. With the use of a suitable Sobolev anisotropic norm we shall subsequently establish a negative feedback coming from the Maxwell-Stefan operator (1.7). This fact will allow to derive the *a priori* energy estimates leading to global-in-time existence and uniqueness (in a perturbative sense) for the couple (\mathbf{c}, \mathbf{U}) with orthogonal velocities and, eventually, the same result will hold for the couple (\mathbf{c}, \mathbf{u}) with full velocities, solution of the original system (1.4)–(1.5)–(1.6). Note in particular that even if we do not recover a strong uniqueness property, we shall still get the general, though not optimal, description of all the possible perturbative solutions (\mathbf{c}, \mathbf{u}) which are close enough to some macroscopic equilibrium state (not necessarily constant, as opposed to [13]).

In the next section we present all the notations and we state our main theorem. Section 3 is then dedicated to the investigation of the fundamental properties (spectral gap and some Sobolev estimates) of the Maxwell-Stefan matrix $A(\mathbf{c})$. At last, in Section 4 we shall prove our main result.

2. MAIN RESULT

2.1. Notations and conventions. Let us first introduce the main notations that we use throughout the paper. Vectors and vector-valued operators in \mathbb{R}^N will be denoted by a bold symbol, whereas their components will be denoted by the same indexed symbol. For instance, \mathbf{w} represents the vector or vector-valued operator (w_1, \dots, w_N) . In particular, we shall use the symbol $\mathbf{1}$ to name the specific vector $(1, \dots, 1)$. Henceforth, the multiplication of N -vectors has to be understood in a component by component way, so that for any $\mathbf{w}, \mathbf{W} \in \mathbb{R}^N$ and any $q \in \mathbb{Q}$ we have

$$\mathbf{w}\mathbf{W} = (w_i W_i)_{1 \leq i \leq N}, \quad \mathbf{w}^q = (w_i^q)_{1 \leq i \leq N}.$$

Moreover, we introduce the Euclidean scalar product in \mathbb{R}^N weighted by a vector $\mathbf{w} \in (\mathbb{R}_+^*)^N$, which is defined as

$$\langle \mathbf{c}, \mathbf{d} \rangle_{\mathbf{w}} = \sum_{i=1}^N c_i d_i w_i,$$

and induces the norm $\|\mathbf{c}\|_{\mathbf{w}}^2 = \langle \mathbf{c}, \mathbf{c} \rangle_{\mathbf{w}}$. When $\mathbf{w} = \mathbf{1}$, the index $\mathbf{1}$ will be dropped in both the notations for the scalar product and the norm.

The convention we choose for the functional spaces is to index the space by the name of the concerned variable. For p in $[1, +\infty]$ we have

$$L_t^p = L^p(0, +\infty), \quad L_x^p = L^p(\mathbb{T}^3), \quad L_{t,x}^p = L^p(\mathbb{R}^+ \times \mathbb{T}^3).$$

To any positive measurable function $\mathbf{w} : \mathbb{T}^3 \rightarrow (\mathbb{R}_+^*)^N$ in the variable x , we associate the weighted Hilbert space $L^2(\mathbb{T}^3, \mathbf{w})$, which is defined by the scalar product and norm

$$\langle \mathbf{c}, \mathbf{d} \rangle_{L_x^2(\mathbf{w})} = \sum_{i=1}^N \langle c_i, d_i \rangle_{L_x^2(w_i)} = \sum_{i=1}^N \int_{\mathbb{T}^3} c_i d_i w_i^2 dx,$$

$$\|\mathbf{c}\|_{L_x^2(\mathbf{w})}^2 = \sum_{i=1}^N \|c_i\|_{L_x^2(w_i)}^2 = \sum_{i=1}^N \int_{\mathbb{T}^3} c_i^2 w_i^2 dx.$$

Finally, in the same way we can introduce the corresponding weighted Sobolev spaces. Consider a multi-index $\alpha \in \mathbb{N}^3$ of length $|\alpha| = \sum_{k=1}^3 \alpha_k$. For any $s \in \mathbb{N}$ and any vector-valued function $\mathbf{c} \in H^s(\mathbb{T}^3, \mathbf{w})$, we define the norm

$$\|\mathbf{c}\|_{H_x^s(\mathbf{w})} = \left(\sum_{i=1}^N \sum_{|\alpha| \leq s} \|\partial_x^\alpha c_i\|_{L_x^2(w_i)}^2 \right)^{1/2}.$$

2.2. Main theorem. We build up a Cauchy theory for the incompressible Maxwell-Stefan system (1.4)–(1.5)–(1.6) perturbed around any macroscopic equilibrium state of the form $(\bar{\mathbf{c}}, \bar{\mathbf{u}})$, where $\bar{\mathbf{c}} \in (\mathbb{R}_+^*)^N$ is a positive constant N -vector and $\bar{\mathbf{u}} = (\bar{u}, \dots, \bar{u})$ is such that the velocity vector $\bar{\mathbf{u}} : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}^3$ is common to all the species and satisfies $\nabla_x \cdot \bar{\mathbf{u}} = 0$. We thus look at solutions of type $(\mathbf{c}, \mathbf{u}) = (\bar{\mathbf{c}} + \varepsilon \tilde{\mathbf{c}}, \bar{\mathbf{u}} + \varepsilon \tilde{\mathbf{u}})$, with $\varepsilon \in (0, 1]$ being the small parameter of the perturbation. The following theorem gathers the main properties that we are able to prove.

Theorem 2.1. *Let $s > 3$ be an integer, $\bar{\mathbf{u}} : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}^3$ be in $L^\infty(\mathbb{R}_+; H^s(\mathbb{T}^3))$ with $\nabla_x \cdot \bar{\mathbf{u}} = 0$, and consider $\bar{\mathbf{c}} > 0$. There exist $\delta_{MS}, C_{MS}, C'_{MS}, \lambda_{MS} > 0$ such that for all $\varepsilon \in (0, 1]$ and for any initial datum $(\tilde{\mathbf{c}}^{\text{in}}, \tilde{\mathbf{u}}^{\text{in}}) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$ satisfying, for almost any $x \in \mathbb{T}^3$ and for any $1 \leq i \leq N$,*

- (i) **Mass compatibility:** $\sum_{i=1}^N \tilde{c}_i^{\text{in}}(x) = 0$ and $\int_{\mathbb{T}^3} \tilde{c}_i^{\text{in}}(x) dx = 0$,
- (ii) **Mass positivity:** $\bar{c}_i + \varepsilon \tilde{c}_i^{\text{in}}(x) > 0$,
- (iii) **Moment compatibility:** $\nabla_x \tilde{c}_i^{\text{in}} = \sum_{j \neq i} \frac{c_i^{\text{in}} c_j^{\text{in}}}{\Delta_{ij}} (\tilde{u}_j^{\text{in}} - \tilde{u}_i^{\text{in}})$,

(iv) **Smallness assumptions:** $\|\tilde{\mathbf{c}}^{\text{in}}\|_{H_x^s} \leq \delta_{MS}$ and $\|\bar{\mathbf{u}}\|_{L_t^\infty H_x^s} \leq \delta_{MS}$,

there exists a unique weak solution

$$(\mathbf{c}, \mathbf{u}) = (\bar{\mathbf{c}} + \varepsilon \tilde{\mathbf{c}}, \bar{\mathbf{u}} + \varepsilon \tilde{\mathbf{u}})$$

in $L^\infty(\mathbb{R}_+; H^s(\mathbb{T}^3)) \times L^\infty(\mathbb{R}_+; H^{s-1}(\mathbb{T}^3))$ to the incompressible Maxwell-Stefan system (1.4)–(1.5)–(1.6), such that initially $(\tilde{\mathbf{c}}, \tilde{\mathbf{u}})|_{t=0} = (\tilde{\mathbf{c}}^{\text{in}}, \tilde{\mathbf{u}}^{\text{in}})$ a.e. on \mathbb{T}^3 .

Moreover, \mathbf{c} is positive and the following equimolar diffusion-like relation holds a.e. on $\mathbb{R}_+ \times \mathbb{T}^3$:

$$(2.1) \quad \langle \mathbf{c}, \tilde{\mathbf{u}} \rangle = \sum_{i=1}^N c_i(t, x) \tilde{u}_i(t, x) = 0.$$

Finally, for almost any time $t \geq 0$

$$\|\tilde{\mathbf{c}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})} \leq e^{-t\lambda_{MS}} \|\tilde{\mathbf{c}}^{\text{in}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})},$$

$$\|\tilde{\mathbf{u}}\|_{H_x^{s-1}} \leq C_{MS} e^{-t\lambda_{MS}} \|\tilde{\mathbf{c}}^{\text{in}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})},$$

$$\int_0^t e^{2(t-\tau)\lambda_{MS}} \|\tilde{\mathbf{u}}(\tau)\|_{H_x^s}^2 d\tau \leq C'_{MS} \|\tilde{\mathbf{c}}^{\text{in}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})}^2.$$

The constants δ_{MS} , λ_{MS} , C_{MS} and C'_{MS} are constructive and only depend on s , the number of species N , the diffusion coefficients $(\Delta_{ij})_{1 \leq i, j \leq N}$ and the constant vector $\bar{\mathbf{c}}$. In particular, they are independent of the parameter ε .

Remark 2.2. Let us make a few comments about the above theorem.

- (1) Our result establishes a Cauchy theory and trend to equilibrium around the macroscopic state $(\bar{\mathbf{c}}, \bar{\mathbf{u}})$, which extends the previous works where either a closure assumption was made or the equilibrium considered was $(\bar{\mathbf{c}}, \mathbf{0})$. In particular, we give here a complete description of all possible studies (at order ε) around an equilibrium state that has a constant mass vector $\bar{\mathbf{c}}$.
- (2) The “mass compatibility” and the “moment compatibility” assumptions are not closure hypotheses, they actually exactly come from the system of equations (1.4)–(1.5) applied at time $t = 0$. We impose these conditions at the beginning, so that our initial datum is compatible with the Maxwell-Stefan system.
- (3) We emphasize again that we do not prove strong uniqueness for the solutions. Indeed, we can construct infinitely many solutions to the Maxwell-Stefan system, by considering different constant masses $\bar{\mathbf{c}}$ and incompressible momenta $\bar{\mathbf{u}}$. However, these are all the possible solutions in a perturbative setting, and the uniqueness property has to be understood in this perturbative sense: as soon as a macroscopic equilibrium $(\bar{\mathbf{c}}, \bar{\mathbf{u}})$ is fixed, we recover strong uniqueness around this specific state.
- (4) The solution we construct has actually more regularity with respect to t , provided that $s > 4$ and $\bar{\mathbf{u}} \in C^0(\mathbb{R}_+; H^s(\mathbb{T}^3))$. Indeed, we point out that, in this case, the couple (\mathbf{c}, \mathbf{u}) also belongs to $C^0(\mathbb{R}_+; H^{s-1}(\mathbb{T}^3)) \times C^0(\mathbb{R}_+; H^{s-2}(\mathbb{T}^3))$, allowing in particular to properly define the initial value problem.

- (5) The constants δ_{MS} , λ_{MS} and C_{MS} are not explicitly computed, but their values can be determined respectively from formulae (4.22), (4.24) and (4.25).

3. PROPERTIES OF THE MAXWELL-STEFAN MATRIX

We prove some properties of the Maxwell-Stefan matrix A , as well as some estimates on its derivatives. We conclude with properties and estimates on the pseudo-inverse of A on its image.

Proposition 3.1. *For any $\mathbf{c} \geq 0$ the matrix $A(\mathbf{c})$ is nonpositive, in the sense that there exist two positive constants λ_A and μ_A such that, for any $\mathbf{X} \in \mathbb{R}^N$,*

$$\|A(\mathbf{c})\mathbf{X}\| \leq \mu_A \langle \mathbf{c}, \mathbf{1} \rangle^2 \|\mathbf{X}\|,$$

$$\langle \mathbf{X}, A(\mathbf{c})\mathbf{X} \rangle \leq -\lambda_A \left(\min_{1 \leq i \leq N} c_i \right)^2 \left[\|\mathbf{X}\|^2 - \langle \mathbf{X}, \mathbf{1} \rangle^2 \right] \leq 0.$$

Proof of Proposition 3.1. Let us consider two N -vectors, $\mathbf{c} \geq 0$ and \mathbf{X} . The boundness of $A(\mathbf{c})$ can be showed in the supremum norm, since all norms are equivalent in \mathbb{R}^N . It is straightforward that, for any $1 \leq i \leq N$,

$$\left| \sum_{j=1}^N \frac{c_i c_j}{\Delta_{ij}} (X_j - X_i) \right| \leq \frac{2 \max_{1 \leq i \leq N} c_i}{\min_{1 \leq i, j \leq N} \Delta_{ij}} \left(\sum_{j=1}^N c_j \right) \max_{1 \leq j \leq N} |X_j|,$$

which raises the first inequality, since $\max_{1 \leq i \leq N} c_i \leq \sum_{j=1}^N c_j$ and we can thus choose $\mu_A = 2 / \min_{1 \leq i, j \leq N} \Delta_{ij}$.

We then compute

$$\langle \mathbf{X}, A(\mathbf{c})\mathbf{X} \rangle = - \sum_{i=1}^N \sum_{j=1}^N \frac{c_i c_j}{\Delta_{ij}} (X_i - X_j) X_i = - \sum_{i=1}^N \sum_{j=1}^N \frac{c_i c_j}{\Delta_{ij}} (X_i - X_j)^2 \leq 0.$$

Note in particular that, since $c_i \geq 0$ and $\Delta_{ij} > 0$ for any $1 \leq i, j \leq N$, the relation $A(\mathbf{c})\mathbf{X} = 0$ implies $X_i = X_j$ for all i and j , and so $\ker A = \text{Span}(\mathbf{1})$. If we now set $\lambda_A = 2 / \max_{1 \leq i, j \leq N} \Delta_{ij}$, we can deduce the bound

$$\langle \mathbf{X}, A(\mathbf{c})\mathbf{X} \rangle \leq -\frac{\lambda_A}{2} \left(\min_{1 \leq i \leq N} c_i \right)^2 \sum_{i=1}^N \sum_{j=1}^N (X_i - X_j)^2,$$

and conclude the proof. \square

As we shall need controls in Sobolev spaces, we then give below some estimates on the x -derivatives of the Maxwell-Stefan matrix.

Proposition 3.2. *Consider a multi-index $\alpha \in \mathbb{N}^3$ and let $\mathbf{c}, \tilde{\mathbf{U}} \in H^{|\alpha|}(\mathbb{T}^3)$, with $\mathbf{c} \geq 0$. Then, for any $\mathbf{X} \in \mathbb{R}^N$,*

$$\begin{aligned} \langle \partial_x^\alpha [A(\mathbf{c})\mathbf{U}], \mathbf{X} \rangle &\leq \langle A(\mathbf{c}) (\partial_x^\alpha \mathbf{U}), \mathbf{X} \rangle + 2N^2 \mu_A \langle \mathbf{c}, \mathbf{1} \rangle \|\mathbf{X}\| \sum_{\substack{\alpha_1 + \alpha_3 = \alpha \\ |\alpha_1| \geq 1}} \|\partial_x^{\alpha_1} \mathbf{c}\| \|\partial_x^{\alpha_3} \mathbf{U}\| \\ &\quad + N^2 \mu_A \|\mathbf{X}\| \sum_{\substack{\alpha_1 + \alpha_2 + \alpha_3 = \alpha \\ |\alpha_1|, |\alpha_2| \geq 1}} \|\partial_x^{\alpha_1} \mathbf{c}\| \|\partial_x^{\alpha_2} \mathbf{c}\| \|\partial_x^{\alpha_3} \mathbf{U}\|, \end{aligned}$$

where μ_A is defined in Proposition 3.1.

Proof of Proposition 3.2. Let \mathbf{X} be in \mathbb{R}^N . We can explicitly compute

$$\begin{aligned} \langle \partial_x^\alpha [A(\mathbf{c})\mathbf{U}], \mathbf{X} \rangle &= \sum_{i=1}^N \partial_x^\alpha \left(\sum_{j=1}^N \frac{c_i c_j}{\Delta_{ij}} (U_j - U_i) \right) X_i \\ &= \sum_{1 \leq i, j \leq N} X_i \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \alpha} \frac{\partial_x^{\alpha_1} c_i \partial_x^{\alpha_2} c_j}{\Delta_{ij}} (\partial_x^{\alpha_3} U_j - \partial_x^{\alpha_3} U_i) \\ &= \langle A(\mathbf{c}) \partial_x^\alpha \mathbf{U}, \mathbf{X} \rangle \\ &\quad + \sum_{1 \leq i, j \leq N} X_i \sum_{\substack{\alpha_1 + \alpha_2 + \alpha_3 = \alpha \\ |\alpha_1|, |\alpha_2| \geq 1}} \frac{\partial_x^{\alpha_1} c_i \partial_x^{\alpha_2} c_j}{\Delta_{ij}} (\partial_x^{\alpha_3} U_j - \partial_x^{\alpha_3} U_i) \\ &\quad + \sum_{1 \leq i, j \leq N} X_i \sum_{\substack{\alpha_2 + \alpha_3 = \alpha \\ |\alpha_2| \geq 1}} \frac{c_i \partial_x^{\alpha_2} c_j}{\Delta_{ij}} (\partial_x^{\alpha_3} U_j - \partial_x^{\alpha_3} U_i) \\ &\quad + \sum_{1 \leq i, j \leq N} X_i \sum_{\substack{\alpha_1 + \alpha_3 = \alpha \\ |\alpha_1| \geq 1}} \frac{c_j \partial_x^{\alpha_1} c_i}{\Delta_{ij}} (\partial_x^{\alpha_3} U_j - \partial_x^{\alpha_3} U_i). \end{aligned}$$

We then use Cauchy-Schwarz inequality and the fact that $\Delta = \min_{i,j} \Delta_{ij} > 0$, together with $0 \leq c_i \leq \sum_{j=1}^N c_j$ and $|c_i| \leq \|\mathbf{c}\|$, to finally get

$$\begin{aligned} \langle \partial_x^\alpha [A(\mathbf{c})\mathbf{U}], \mathbf{X} \rangle &\leq \langle A(\mathbf{c}) \partial_x^\alpha \mathbf{U}, \mathbf{X} \rangle + \frac{4N^2}{\Delta} \left[\left(\sum_{j=1}^N c_j \right) \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ |\alpha_1| \geq 1}} \|\partial_x^{\alpha_1} \mathbf{c}\| \|\partial_x^{\alpha_2} \mathbf{U}\| \right] \|\mathbf{X}\| \\ &\quad + \frac{2N^2}{\Delta} \left[\sum_{\substack{\alpha_1 + \alpha_2 + \alpha_3 = \alpha \\ |\alpha_1|, |\alpha_2| \geq 1}} \|\partial_x^{\alpha_1} \mathbf{c}\| \|\partial_x^{\alpha_2} \mathbf{c}\| \|\partial_x^{\alpha_3} \mathbf{U}\| \right] \|\mathbf{X}\|, \end{aligned}$$

which is the expected result. \square

We conclude the present section with a control on the pseudoinverse of $A(\mathbf{c})$, which is defined on $(\text{Span}(\mathbf{1}))^\perp$.

Proposition 3.3. For any $\mathbf{c} \in (\mathbb{R}_+^*)^N$ and any $\mathbf{U} \in (\text{Span}(\mathbf{1}))^\perp$, the following estimates hold:

$$\|A(\mathbf{c})^{-1}\mathbf{U}\| \leq \frac{1}{\lambda_A \left(\min_{1 \leq i \leq N} c_i \right)^2} \|\mathbf{U}\|,$$

$$\langle A(\mathbf{c})^{-1}\mathbf{U}, \mathbf{U} \rangle \leq -\frac{\lambda_A \left(\min_{1 \leq i \leq N} c_i \right)^2}{\mu_A^2 \langle \mathbf{c}, \mathbf{1} \rangle^4} \|\mathbf{U}\|^2,$$

where λ_A and μ_A are defined in Proposition 3.1.

Proof of Proposition 3.3. The proof is a direct application of Proposition 3.1. Indeed, Cauchy-Schwarz inequality yields, for any $\mathbf{X} \in (\text{Span}(\mathbf{1}))^\perp$,

$$-\|\mathbf{X}\| \|A(\mathbf{c})\mathbf{X}\| \leq -\lambda_A \left(\min_{1 \leq i \leq N} c_i \right)^2 \|\mathbf{X}\|^2,$$

so that

$$\|\mathbf{X}\| \leq \frac{1}{\lambda_A \left(\min_{1 \leq i \leq N} c_i \right)^2} \|A(\mathbf{c})\mathbf{X}\|,$$

which proves the first estimate by simply taking $\mathbf{X} = A(\mathbf{c})^{-1}\mathbf{U}$.

The spectral gap property comes from the boundedness of $A(\mathbf{c})$, given by Proposition 3.1 for $\mathbf{X} = A(\mathbf{c})^{-1}\mathbf{U}$, which translates into a coercivity estimate

$$\|\mathbf{U}\|^2 \leq \left(\mu_A \langle \mathbf{c}, \mathbf{1} \rangle^2 \right)^2 \|A(\mathbf{c})^{-1}(\mathbf{U})\|^2,$$

that we plug into the spectral gap inequality satisfied by $A(\mathbf{c})$. \square

4. PERTURBATIVE CAUCHY THEORY FOR THE MAXWELL-STEFAN SYSTEM

We recall the vectorial form of Maxwell-Stefan system (1.4)–(1.6):

$$(4.1) \quad \partial_t \mathbf{c} + \nabla_x \cdot (\mathbf{c}\mathbf{u}) = 0,$$

$$(4.2) \quad \nabla_x \mathbf{c} = A(\mathbf{c})\mathbf{u},$$

$$(4.3) \quad \nabla_x \cdot \langle \mathbf{c}, \mathbf{u} \rangle = 0.$$

where

$$A(\mathbf{c}) = \left(\frac{c_i c_j}{\Delta_{ij}} - \left(\sum_{k=1}^N \frac{c_i c_k}{\Delta_{ik}} \right) \delta_{ij} \right)_{1 \leq i, j \leq N}.$$

The Cauchy theory we build offers an explicit description of all the solutions (\mathbf{c}, \mathbf{u}) which are perturbed around a global macroscopic equilibrium state. We point out in particular that, because of the incompressibility condition (4.3), any macroscopic stationary state has the form $(\bar{c}_i, \bar{u})_{1 \leq i \leq N}$, where each \bar{c}_i is a positive constant and the velocity $\bar{u} : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}^3$, common to all the species, satisfies $\nabla_x \cdot \bar{u}(t, x) = 0$

for any $t \geq 0$ and $x \in \mathbb{T}^3$. For a sake of clarity, throughout the present section any perturbative vector-valued function $\mathbf{w} = (w_1, \dots, w_N)$ shall be written under the specific form $\mathbf{w} = \overline{\mathbf{w}} + \varepsilon \tilde{\mathbf{w}}$, where the component $\overline{\mathbf{w}}$ with the overbar symbol always refers to some (macroscopic) stationary state of the mixture and the component $\tilde{\mathbf{w}}$ overlined by a tilde refers to the fluctuation around the corresponding equilibrium state. Moreover, note that for simplicity the specific quantity $\overline{\mathbf{u}}$ will always denote an N -vector where all the components are given by a common incompressible velocity \overline{u} .

The present section is divided into two parts. In the first one, we show how to derive the new orthogonal system equivalent to (4.1)–(4.2)–(4.3), and state the counterpart of Theorem 2.1 in terms of this new reformulation for the unknowns \mathbf{c} and \mathbf{U} , the orthogonal part of \mathbf{u} . In the second part we prove all the required properties (existence and uniqueness, positivity and exponential decay to equilibrium) for the couple (\mathbf{c}, \mathbf{U}) , properties that will be also satisfied by the original unknowns (\mathbf{c}, \mathbf{u}) .

4.1. An orthogonal incompressible Maxwell-Stefan system. Here we present the equivalent orthogonal reformulation of (4.1)–(4.2)–(4.3), which allows to transfer the study of existence and uniqueness for solutions (\mathbf{c}, \mathbf{u}) to the development of a Cauchy theory for the new unknowns (\mathbf{c}, \mathbf{U}) , where we denote with $\mathbf{U} = \mathbf{u} - \pi_A(\mathbf{u})$ the part of \mathbf{u} that is projected onto $(\ker A)^\perp$, π_A being the orthogonal projection onto $\ker A = \text{Span}(\mathbf{1})$.

Before stating our result, we introduce a useful notation that allows to preserve the vectorial structure of the Maxwell-Stefan system. We suppose that, for some $V \in \mathbb{R}^3$ and some N -vector $\mathbf{w} \in (\mathbb{R}^3)^N$ whose components lie in \mathbb{R}^3 , the standard notation of the scalar product in \mathbb{R}^3 is extended to any multiplication of type $V \cdot \mathbf{w}$ in the following sense

$$V \cdot \mathbf{w} = (V \cdot w_i)_{1 \leq i \leq N}.$$

Proposition 4.1. *Let $s \in \mathbb{N}^*$, $C_0 > 0$, and consider two functions $\mathbf{c}^{\text{in}} \in H^s(\mathbb{T}^3)$ and $\mathbf{u}^{\text{in}} \in H^{s-1}(\mathbb{T}^3)$ verifying, for almost any $x \in \mathbb{T}^3$,*

$$\mathbf{c}^{\text{in}}(x) \geq 0 \quad \text{and} \quad \sum_{i=1}^N c_i^{\text{in}}(x) = C_0.$$

Then, $(\mathbf{c}, \mathbf{u}) \in L^\infty(\mathbb{R}_+; H^s(\mathbb{T}^3)) \times L^\infty(\mathbb{R}_+; H^{s-1}(\mathbb{T}^3))$ is a solution to the incompressible Maxwell-Stefan system (4.1)–(4.2)–(4.3), associated to the initial datum $(\mathbf{c}^{\text{in}}, \mathbf{u}^{\text{in}})$, if and only if there exist two functions $\mathbf{U} : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}^{3N}$ and $\overline{u} : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}$ in $L^\infty(\mathbb{R}_+; H^{s-1}(\mathbb{T}^3))$ such that, for almost any $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^3$,

$$(4.4) \quad \mathbf{U}(t, x) \in (\ker A)^\perp \quad \text{and} \quad \nabla_x \cdot \overline{u}(t, x) = 0,$$

$$(4.5) \quad \mathbf{U}(t, x) + \overline{u}(t, x) - \frac{1}{C_0} \langle \mathbf{c}, \mathbf{U} \rangle \mathbf{1} = \mathbf{u}(t, x),$$

$$(4.6) \quad \begin{cases} \partial_t \mathbf{c} + \nabla_x \cdot \left(\mathbf{c} \left(\mathbf{U} - \frac{\langle \mathbf{c}, \mathbf{U} \rangle}{\langle \mathbf{c}, \mathbf{1} \rangle} \mathbf{1} \right) \right) + \overline{u} \cdot \nabla_x \mathbf{c} = 0, \\ \nabla_x \mathbf{c} = A(\mathbf{c}) \mathbf{U}. \end{cases}$$

Remark 4.2. *The above result is not difficult to prove but we underline again that it is of great importance, since it turns the incompressible Maxwell-Stefan system (4.1)–(4.2)–(4.3) with full velocity vectors \mathbf{u} into a system only depending on their orthogonal component $\mathbf{U} \in (\ker A)^\perp$, while the projection onto $\ker A$ raises a simple transport term in the continuity equation (4.1). Notice in particular that we differentiate between $C_0 = \langle \mathbf{c}^{\text{in}}, \mathbf{1} \rangle$ in (4.5) and $\langle \mathbf{c}, \mathbf{1} \rangle$ in (4.6). As we shall see, in both equations it will turn out that these two quantities are equal, but keeping the notation $\langle \mathbf{c}, \mathbf{1} \rangle$ offers a fully closed system.*

Moreover, note that Proposition 4.1 actually shows that all perturbative solutions of the Maxwell-Stefan system (1.4)–(1.5)–(1.6) are of the form described by Theorem 2.1, that is $\bar{\mathbf{c}} + \varepsilon \tilde{\mathbf{c}}$ and $\bar{\mathbf{u}} + \varepsilon \tilde{\mathbf{u}}$. Note however that Theorem 2.1 is not optimal, since we require $\bar{\mathbf{u}}$ to be more regular than the perturbation $\tilde{\mathbf{u}}$.

Proof of Proposition 4.1. Let π_A be the orthogonal projection operator onto $\ker A$ and consider a solution (\mathbf{c}, \mathbf{u}) of the Maxwell-Stefan system (4.1)–(4.2)–(4.3). The first implication directly follows from the decomposition

$$\mathbf{u} = \pi_A(\mathbf{u}) + (\mathbf{u} - \pi_A(\mathbf{u})) = \frac{\langle \mathbf{u}, \mathbf{1} \rangle}{\|\mathbf{1}\|} \mathbf{1} + \pi_A^\perp(\mathbf{u}),$$

where we recall that $\|\cdot\|$ defines the Euclidean norm induced by the scalar product $\langle \cdot, \cdot \rangle$ in \mathbb{R}^N , weighted by the vector $\mathbf{1}$.

First of all, observe that summing over the continuity equations (4.1) and using the incompressibility condition (4.3), it follows that $\partial_t \langle \mathbf{c}, \mathbf{1} \rangle = 0$. Moreover, if we sum the gradient relations (4.2), we also get

$$\nabla_x \left(\sum_{i=1}^N c_i \right) = \sum_{1 \leq i, j \leq N} \frac{c_i c_j}{\Delta_{ij}} (u_j - u_i) = 0.$$

Therefore, the quantity $\langle \mathbf{c}, \mathbf{1} \rangle$ is independent of t and x , allowing to initially deduce that

$$(4.7) \quad \sum_{i=1}^N c_i(t, x) = \sum_{i=1}^N c_i^{\text{in}}(x) = C_0 \quad \text{a.e. on } \mathbb{R}_+ \times \mathbb{T}^3.$$

Now, defining $\mathbf{U} = \pi_A^\perp(\mathbf{u})$ and $W = \frac{\langle \mathbf{u}, \mathbf{1} \rangle}{\|\mathbf{1}\|}$, we easily recover (4.4)–(4.5). The transport equation (4.2) can then be rewritten in terms of \mathbf{U} and W as

$$(4.8) \quad \partial_t \mathbf{c} + \nabla_x \cdot (\mathbf{c} \mathbf{U}) + \mathbf{c} \nabla_x \cdot W + W \cdot \nabla_x \mathbf{c} = 0.$$

In a similar way, the incompressibility condition (4.3) in these new unknowns reads

$$\begin{aligned} 0 &= \sum_{i=1}^N \nabla_x \cdot (c_i (U_i + W)) = \sum_{i=1}^N \nabla_x \cdot (c_i U_i) + \nabla_x \cdot \left(\sum_{i=1}^N c_i \right) \cdot W + (\nabla_x \cdot W) \sum_{i=1}^N c_i \\ &= \nabla_x \cdot \langle \mathbf{c}, \mathbf{U} \rangle + C_0 \nabla_x \cdot W, \end{aligned}$$

where we have used (4.7). We thus infer the existence of a divergence-free function $\bar{u} : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}^3$ such that, for almost any $(t, x) \in \mathbb{T}^3 \times \mathbb{R}^3$,

$$\begin{cases} \nabla_x \cdot \bar{u}(t, x) = 0, \\ W(t, x) = -\frac{1}{C_0} \langle \mathbf{c}, \mathbf{U} \rangle(t, x) + \bar{u}(t, x). \end{cases}$$

Plugging the above relation into (4.8) and replacing C_0 by its value $\langle \mathbf{c}, \mathbf{1} \rangle$, we recover the first equation of (4.6). Finally, the decomposition (4.1) also yields the second relation of (4.6), since $\pi_A(\mathbf{u}) \in \ker A$ and thus

$$A(\mathbf{c})\mathbf{u} = A(\mathbf{c})\mathbf{U},$$

proving that $(\mathbf{c}, \mathbf{U}, \bar{u})$ is a solution of the orthogonal reformulation (4.4)–(4.5)–(4.6).

Consider now a triple $(\mathbf{c}, \mathbf{U}, \bar{u})$ satisfying conditions (4.4)–(4.5)–(4.6). The reverse implication then follows by defining

$$(4.9) \quad \mathbf{u} = \mathbf{U} + \left(-\frac{1}{C_0} \langle \mathbf{c}, \mathbf{U} \rangle + \bar{u} \right) \mathbf{1}.$$

Indeed, summing over $1 \leq i \leq N$ the gradient relations of (4.6), we get

$$\nabla_x \left(\sum_{i=1}^N c_i \right) = 0,$$

which is used when one also sums over $1 \leq i \leq N$ the transport equations of (4.6), to deduce

$$0 = \partial_t \left(\sum_{i=1}^N c_i \right) + \nabla_x \cdot \left(\sum_{i=1}^N c_i U_i \right) - \nabla_x \cdot \left(\langle \mathbf{c}, \mathbf{U} \rangle \frac{\left(\sum_{i=1}^N c_i \right)}{\langle \mathbf{c}, \mathbf{1} \rangle} \right) = \partial_t \left(\sum_{i=1}^N c_i \right),$$

since $\langle \mathbf{c}, \mathbf{1} \rangle = \sum_{i=1}^N c_i$ by definition. Thus, the quantity $\langle \mathbf{c}, \mathbf{1} \rangle$ is independent of (t, x) , allowing to infer that

$$\sum_{i=1}^N c_i(t, x) = \langle \mathbf{c}^{\text{in}}, \mathbf{1} \rangle = C_0, \quad \text{a.e. on } \mathbb{R}_+ \times \mathbb{T}^3.$$

This recovery of (4.7) not only implies the incompressibility condition (4.3) but also, with the divergence free property of \bar{u} , that

$$\begin{aligned} \nabla_x \cdot (\mathbf{c}\mathbf{u}) &= \nabla_x \cdot \left(\mathbf{c} \left(\mathbf{U} - \frac{\langle \mathbf{c}, \mathbf{U} \rangle}{C_0} \mathbf{1} \right) \right) + \bar{u} \cdot \nabla_x \mathbf{c} \\ &= \nabla_x \cdot \left(\mathbf{c} \left(\mathbf{U} - \frac{\langle \mathbf{c}, \mathbf{U} \rangle}{\langle \mathbf{c}, \mathbf{1} \rangle} \mathbf{1} \right) \right) + \bar{u} \cdot \nabla_x \mathbf{c}. \end{aligned}$$

Therefore, the first equation of (4.6) rewrites

$$\partial_t \mathbf{c} + \nabla_x \cdot (\mathbf{c}\mathbf{u}) = 0,$$

and, thanks again to the fact that $\ker A = \text{Span}(\mathbf{1})$, one finally sees that

$$\nabla_x \mathbf{c} = A(\mathbf{c})\mathbf{U} = A(\mathbf{c})\mathbf{u}.$$

This ensures that (\mathbf{c}, \mathbf{u}) , with \mathbf{u} defined by (4.9), solves the Maxwell-Stefan system (4.1)–(4.2)–(4.3), thus concluding the proof. \square

By means of this orthogonal reformulation, we can now prove our main result.

4.2. Proof of Theorem 2.1. This last part is devoted to showing the validity of Theorem 2.1. We shall divide the proof into several steps which help in enlightening the basic ideas behind our strategy. We first restate our result about solutions (\mathbf{c}, \mathbf{u}) in terms of the orthogonal reformulation (4.4)–(4.5)–(4.6), about solutions $(\mathbf{c}, \mathbf{U}, \bar{u})$. Thanks to preliminary lemmata describing the main properties of the matrix A and of its pseudoinverse obtained in Section 3, we then derive uniform (in ε) *a priori* energy estimates for the solution (\mathbf{c}, \mathbf{U}) , which provide the exponential relaxation towards the global equilibrium $(\bar{\mathbf{c}}, \bar{\mathbf{u}})$. Starting from this we are thus able to recover the positivity of \mathbf{c} , and to prove global existence and uniqueness for solutions to (4.6) having the specific perturbative forms $\mathbf{c} = \bar{\mathbf{c}} + \varepsilon \tilde{\mathbf{c}}$ and $\mathbf{U} = \varepsilon \tilde{\mathbf{U}}$. The combination of these results will eventually allow to deduce global existence, uniqueness and exponential decay for the couple (\mathbf{c}, \mathbf{u}) , using the reconstruction condition (4.5).

Step 1 – Reformulation in terms of orthogonal velocities. Let us begin with a simple lemma needed in order to understand the shape of the velocities \mathbf{U} and \bar{u} , when they are associated to a constant state $\bar{\mathbf{c}}$.

Lemma 4.3. *Let $s \in \mathbb{N}^*$ and let $\bar{\mathbf{c}}$ be a positive constant N -vector. For any functions \mathbf{U}, \bar{u} in $L^\infty(\mathbb{R}_+; H^{s-1}(\mathbb{T}^3))$ such that $\mathbf{U} \in (\text{Span}(\mathbf{1}))^\perp$ and $\nabla_x \cdot \bar{u} = 0$, a triple $(\bar{\mathbf{c}}, \mathbf{U}, \bar{u})$ is solution to the system of equations (4.4)–(4.5)–(4.6) if and only if*

$$\mathbf{U}(t, x) = 0 \quad \text{a.e. on } \mathbb{R}_+ \times \mathbb{T}^3.$$

Proof of Lemma 4.3. The proof is very simple. Because $\bar{\mathbf{c}}$ is constant, the gradient equation of (4.6) reads $A(\bar{\mathbf{c}})\mathbf{U} = 0$. But \mathbf{U} belongs to $(\ker A)^\perp$, which means that the pseudoinverse A^{-1} remains well-defined. Consequently, for almost any $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^3$, $\mathbf{U}(t, x) = 0$.

The reverse implication is direct. □

We are now interested in building a Cauchy theory for the orthogonal form of the Maxwell-Stefan system, around the stationary solutions given by Proposition 4.3. More precisely, we want to prove existence and uniqueness for perturbative solutions to (4.4)–(4.5)–(4.6) of the form

$$\begin{cases} \mathbf{c}(t, x) = \bar{\mathbf{c}} + \varepsilon \tilde{\mathbf{c}}, \\ \mathbf{U} = \varepsilon \tilde{\mathbf{U}}. \end{cases}$$

In terms of these particular solutions, the system (4.4)–(4.5)–(4.6) translates into

$$(4.10) \quad \partial_t \tilde{\mathbf{c}} + \bar{\mathbf{c}} \nabla_x \cdot \mathbf{V}_{\tilde{\mathbf{U}}} + \bar{u} \cdot \nabla_x \tilde{\mathbf{c}} + \varepsilon \nabla_x \cdot (\tilde{\mathbf{c}} \mathbf{V}_{\tilde{\mathbf{U}}}) = 0,$$

$$(4.11) \quad \nabla_x \tilde{\mathbf{c}} = A(\bar{\mathbf{c}} + \varepsilon \tilde{\mathbf{c}}) \tilde{\mathbf{U}},$$

with the notation $\mathbf{V}_{\tilde{\mathbf{U}}} = \tilde{\mathbf{U}} - \frac{\langle \mathbf{c}, \tilde{\mathbf{U}} \rangle}{\langle \mathbf{c}, \mathbf{1} \rangle} \mathbf{1}$. The orthogonal reformulation of Theorem 2.1 then writes in the following way.

Theorem 4.4. *Let $s > 3$ be an integer, $\bar{u} : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}^3$ be in $L^\infty(\mathbb{R}_+; H^s(\mathbb{T}^3))$ with $\nabla_x \cdot \bar{u} = 0$ and consider a constant N -vector $\bar{\mathbf{c}} > 0$. There exist $\delta_s, C_s, \lambda_s > 0$ such that for all $\varepsilon \in (0, 1]$ and for any $(\tilde{\mathbf{c}}^{\text{in}}, \tilde{\mathbf{U}}^{\text{in}}) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$ satisfying, for almost any $x \in \mathbb{T}^3$ and for any $1 \leq i \leq N$,*

$$(i) \text{ **Mass compatibility:** } \sum_{i=1}^N \tilde{c}_i^{\text{in}}(x) = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} \tilde{c}_i^{\text{in}}(x) dx = 0,$$

$$(ii) \text{ **Mass positivity:** } \bar{c}_i + \varepsilon \tilde{c}_i^{\text{in}}(x) > 0,$$

$$(iii) \text{ **Moment compatibility:** } \tilde{\mathbf{U}}^{\text{in}}(x) = A (\bar{\mathbf{c}} + \varepsilon \tilde{\mathbf{c}}^{\text{in}}(x))^{-1} \nabla_x \tilde{\mathbf{c}}^{\text{in}}(x),$$

$$(iv) \text{ **Smallness assumptions:** } \|\tilde{\mathbf{c}}^{\text{in}}\|_{H_x^s} \leq \delta_s \quad \text{and} \quad \|\bar{u}\|_{L_t^\infty H_x^s} \leq \delta_s,$$

there exists a unique weak solution $(\tilde{\mathbf{c}}, \tilde{\mathbf{U}}) \in L^\infty(\mathbb{R}_+; H^s(\mathbb{T}^3)) \times L^\infty(\mathbb{R}_+; H^{s-1}(\mathbb{T}^3))$ to the system of equations (4.10)–(4.11), having $(\tilde{\mathbf{c}}^{\text{in}}, \tilde{\mathbf{U}}^{\text{in}})$ as initial datum. In particular, for almost any $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^3$, the vector $\mathbf{c}(t, x) = \bar{\mathbf{c}} + \varepsilon \tilde{\mathbf{c}}(t, x)$ is positive and $\tilde{\mathbf{U}}(t, x)$ belongs to $(\ker A)^\perp$.

Moreover, the following estimates hold for almost any $t \geq 0$

$$\|\tilde{\mathbf{c}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})} \leq e^{-\lambda_s t} \|\tilde{\mathbf{c}}^{\text{in}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})},$$

$$\|\tilde{\mathbf{U}}\|_{H_x^{s-1}} \leq C_s e^{-\lambda_s t} \|\tilde{\mathbf{c}}^{\text{in}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})},$$

$$\int_0^t e^{2\lambda_s(t-\tau)} \|\tilde{\mathbf{U}}(\tau)\|_{H_x^s}^2 d\tau \leq C'_s \|\tilde{\mathbf{c}}^{\text{in}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})}^2.$$

The constants δ_s , λ_s and C_s are constructive and are given respectively by (4.22), (4.24) and (4.25).

Remark 4.5. We underline that the reconstruction condition (4.5) does not involve the vector \mathbf{c} , but only the velocity vector. This means in particular that the solution \mathbf{c} provided by the above theorem coincides exactly with the one given by Theorem 2.1. However, there is a real difference when one reformulates Theorem 2.1 in terms of the orthogonal velocities. Indeed, it is important to notice that the velocity vectors $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{U}}$ are not the same, since (4.5) tells us that actually $\tilde{\mathbf{u}} = \mathbf{V}_{\tilde{\mathbf{U}}}$, from which we only deduce the relation $\langle \mathbf{c}, \tilde{\mathbf{u}} \rangle = 0$, and no more the orthogonality $\langle \tilde{\mathbf{u}}, \mathbf{1} \rangle = 0$, satisfied instead by the vector $\tilde{\mathbf{U}}$.

Step 2 – A priori energy estimates and positivity. The two *a priori* results (exponential decay and positivity of \mathbf{c}) that we now derive are of crucial importance, as they will allow us to exhibit existence and uniqueness for the couple $(\tilde{\mathbf{c}}, \tilde{\mathbf{U}})$ in the next section.

Before we start, we present a simple result which establishes two relevant properties satisfied by the solution of (4.10)–(4.11). We show in particular that $\tilde{\mathbf{c}}$ has zero mean on the torus, a feature that will let us exploit Poincaré inequality in the proof of the *a priori* estimates.

Lemma 4.6. Let $\bar{\mathbf{c}}, C_0 > 0$ be such that $\langle \bar{\mathbf{c}}, \mathbf{1} \rangle = C_0$, and consider a triple $(\tilde{\mathbf{c}}^{\text{in}}, \tilde{\mathbf{U}}^{\text{in}}, \bar{u})$ satisfying the hypotheses of Theorem 4.4. If $(\tilde{\mathbf{c}}, \tilde{\mathbf{U}})$ is a weak solution of (4.10)–(4.11) with initial datum $(\tilde{\mathbf{c}}^{\text{in}}, \tilde{\mathbf{U}}^{\text{in}})$, then, for almost any $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^3$ and for

any $1 \leq i \leq N$,

$$(4.12) \quad \sum_{i=1}^N \tilde{c}_i(t, x) = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} \tilde{c}_i(t, x) dx = 0.$$

In particular, the conservation of the total mass $\langle \mathbf{c}, \mathbf{1} \rangle = C_0$ holds almost everywhere on $\mathbb{R}_+ \times \mathbb{T}^3$.

Proof of Lemma 4.6. We have already showed how to recover the preservation of the total mass inside the proof of Proposition 4.1.

The second property follows directly from the fact that $\bar{\mathbf{c}}$ is a constant N -vector and \bar{u} is divergence-free. Indeed, using these two assumptions the mass equation (4.10) can be written under a divergent form as

$$\partial_t \tilde{\mathbf{c}} + \nabla_x \cdot (\mathbf{c} \mathbf{V}_{\tilde{U}} + \tilde{\mathbf{c}} \bar{\mathbf{u}}) = 0.$$

Integrating over the torus we thus obtain

$$\frac{d}{dt} \int_{\mathbb{T}^3} \tilde{\mathbf{c}}(t, x) dx = 0 \quad \text{for a.e. } t \geq 0,$$

which gives the expected result since $\tilde{\mathbf{c}}^{\text{in}}$ has zero mean on the torus. \square

The result providing the *a priori* energy estimates is then the following.

Proposition 4.7. *Let $s > 3$ be an integer. There exist $\delta_s, \lambda_s, C_s, C'_s > 0$ such that, under the assumptions of Theorem 4.4, if $(\tilde{\mathbf{c}}, \tilde{\mathbf{U}}, \bar{u})$ is a solution of the perturbed orthogonal system (4.10)–(4.11) satisfying the initial controls*

$$\|\tilde{\mathbf{c}}^{\text{in}}\|_{H_x^s} \leq \delta_s \quad \text{and} \quad \|\bar{u}\|_{L_t^\infty H_x^s} \leq \delta_s,$$

then, for almost any $t \geq 0$,

$$\|\tilde{\mathbf{c}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})} \leq e^{-\lambda_s t} \|\tilde{\mathbf{c}}^{\text{in}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})},$$

$$\|\tilde{\mathbf{U}}\|_{H_x^{s-1}} \leq C_s e^{-\lambda_s t} \|\tilde{\mathbf{c}}^{\text{in}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})},$$

$$\int_0^t e^{2\lambda_s(t-\tau)} \|\tilde{\mathbf{U}}(\tau)\|_{H_x^s}^2 d\tau \leq C'_s \|\tilde{\mathbf{c}}^{\text{in}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})}^2.$$

The constants δ_s, λ_s, C_s and C'_s are explicit and only depend on s , the number of species N , the diffusion coefficients $(\Delta_{ij})_{1 \leq i, j \leq N}$ and the constant vector $\bar{\mathbf{c}}$. In particular, they are independent of the parameter ε .

Proof of Proposition 4.7. We fix a multi-index $\alpha \in \mathbb{N}^3$ such that $|\alpha| \leq s$. Recall that we have defined

$$\mathbf{V}_{\tilde{U}} = \tilde{\mathbf{U}} - \frac{\langle \mathbf{c}, \tilde{\mathbf{U}} \rangle}{\langle \mathbf{c}, \mathbf{1} \rangle} \mathbf{1}.$$

We successively apply the α -derivative to the transport equation (4.10), take the scalar product with the vector $\left(\frac{1}{\tilde{c}_i} \partial_x^\alpha \tilde{c}_i\right)_{1 \leq i \leq N}$, and integrate over \mathbb{T}^3 . This yields, after integrating by parts,

$$(4.13) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_x^\alpha \tilde{\mathbf{c}}\|_{L_x^2(\tilde{c}^{-\frac{1}{2}})}^2 &= \int_{\mathbb{T}^3} \langle \nabla_x \partial_x^\alpha \tilde{\mathbf{c}}, \partial_x^\alpha \mathbf{V}_{\tilde{U}} \rangle dx + \int_{\mathbb{T}^3} \langle \nabla_x \partial_x^\alpha \tilde{\mathbf{c}}, \partial_x^\alpha (\tilde{\mathbf{c}} \tilde{\mathbf{u}}) \rangle_{\tilde{c}^{-1}} dx \\ &+ \varepsilon \int_{\mathbb{T}^3} \langle \nabla_x \partial_x^\alpha \tilde{\mathbf{c}}, \partial_x^\alpha (\tilde{\mathbf{c}} \mathbf{V}_{\tilde{U}}) \rangle_{\tilde{c}^{-1}} dx. \end{aligned}$$

We estimate these three terms separately. We first notice that summing over i the gradient equations (4.11) we obtain

$$\sum_{i=1}^N \nabla_x \tilde{c}_i(t, x) = 0 \quad \text{a.e. on } \mathbb{R}_+ \times \mathbb{T}^3,$$

which means that $\nabla_x \tilde{\mathbf{c}}$ belongs to $(\text{Span}(\mathbf{1}))^\perp$. Applying the α -derivative to both sides of this relation then gives

$$\sum_{i=1}^N \nabla_x \partial_x^\alpha \tilde{c}_i(t, x) = 0 \quad \text{a.e. on } \mathbb{R}_+ \times \mathbb{T}^3,$$

from which we deduce that also

$$(4.14) \quad \nabla_x \partial_x^\alpha \tilde{\mathbf{c}} \in (\text{Span}(\mathbf{1}))^\perp \quad \text{a.e. on } \mathbb{R}_+ \times \mathbb{T}^3.$$

Thanks to the orthogonality (4.14) of the higher derivatives and using the gradient relation (4.11), the first term on the right-hand side of (4.13) becomes

$$\begin{aligned} \int_{\mathbb{T}^3} \langle \nabla_x \partial_x^\alpha \tilde{\mathbf{c}}, \partial_x^\alpha \mathbf{V}_{\tilde{U}} \rangle dx &= \int_{\mathbb{T}^3} \langle \partial_x^\alpha \nabla_x \tilde{\mathbf{c}}, \partial_x^\alpha \tilde{\mathbf{U}} \rangle dx - \frac{1}{C_0} \partial_x^\alpha \langle \mathbf{c}, \tilde{\mathbf{U}} \rangle \int_{\mathbb{T}^3} \langle \nabla_x \partial_x^\alpha \tilde{\mathbf{c}}, \mathbf{1} \rangle dx \\ &= \int_{\mathbb{T}^3} \langle \partial_x^\alpha [A(\mathbf{c}) \tilde{\mathbf{U}}], \partial_x^\alpha \tilde{\mathbf{U}} \rangle dx. \end{aligned}$$

We can now apply Proposition 3.2 with $\mathbf{X} = \partial_x^\alpha \tilde{\mathbf{U}}$ and use the mass conservation given in Lemma 4.6, in combination with the spectral gap of $A(\mathbf{c})$ from Proposition

3.1, to recover the initial bound

$$\begin{aligned}
& \int_{\mathbb{T}^3} \langle \nabla_x \partial_x^\alpha \tilde{\mathbf{c}}, \partial_x^\alpha \mathbf{V}_{\tilde{U}} \rangle dx \leq \int_{\mathbb{T}^3} \langle A(\mathbf{c}) \partial_x^\alpha \tilde{\mathbf{U}}, \partial_x^\alpha \tilde{\mathbf{U}} \rangle dx \\
& \quad + 2N^2 \mu_A C_0 \int_{\mathbb{T}^3} \left\| \partial_x^\alpha \tilde{\mathbf{U}} \right\| \sum_{\substack{\alpha_1 + \alpha_3 = \alpha \\ |\alpha_1| \geq 1}} \left\| \partial_x^{\alpha_1} \mathbf{c} \right\| \left\| \partial_x^{\alpha_3} \tilde{\mathbf{U}} \right\| dx \\
& \quad + N \mu_A \int_{\mathbb{T}^3} \left\| \partial_x^\alpha \tilde{\mathbf{U}} \right\| \sum_{\substack{\alpha_1 + \alpha_2 + \alpha_3 = \alpha \\ \alpha_1, \alpha_2 \geq 1}} \left\| \partial_x^{\alpha_1} \mathbf{c} \right\| \left\| \partial_x^{\alpha_2} \mathbf{c} \right\| \left\| \partial_x^{\alpha_3} \tilde{\mathbf{U}} \right\| dx \\
& \leq -\lambda_A \left(\min_{1 \leq i \leq N} c_i \right)^2 \left\| \partial_x^\alpha \tilde{\mathbf{U}} \right\|_{L_x^2}^2 \\
& \quad + 2\varepsilon N^2 \mu_A C_0 \left\| \tilde{\mathbf{U}} \right\|_{H_x^s} \left[\int_{\mathbb{T}^3} \left(\sum_{\substack{\alpha_1 + \alpha_3 = \alpha \\ |\alpha_1| \geq 1}} \left\| \partial_x^{\alpha_1} \tilde{\mathbf{c}} \right\| \left\| \partial_x^{\alpha_3} \tilde{\mathbf{U}} \right\| \right)^2 dx \right]^{\frac{1}{2}} \\
& \quad + \varepsilon^2 N^2 \mu_A \left\| \tilde{\mathbf{U}} \right\|_{H_x^s} \left[\int_{\mathbb{T}^3} \left(\sum_{\substack{\alpha_1 + \alpha_2 + \alpha_3 = \alpha \\ |\alpha_1|, |\alpha_2| \geq 1}} \left\| \partial_x^{\alpha_1} \tilde{\mathbf{c}} \right\| \left\| \partial_x^{\alpha_2} \tilde{\mathbf{c}} \right\| \left\| \partial_x^{\alpha_3} \tilde{\mathbf{U}} \right\| \right)^2 dx \right]^{\frac{1}{2}},
\end{aligned}$$

where we have also used the Cauchy-Schwarz inequality and the fact that the L_x^2 norm of $\partial_x^\alpha \tilde{\mathbf{U}}$ is controlled by the H_x^s norm of $\tilde{\mathbf{U}}$.

Recalling our choice $s > 3$, in order to control the bi and tri-norm terms inside the integrals we use the continuous embedding of $H_x^{s/2}$ in L_x^∞ , which holds as soon as $s/2 > 3/2$. We detail our procedure for the tri-norm term, the bi-norm term being treated in the same way. Since $\alpha_1 + \alpha_2 + \alpha_3 = \alpha$, at most one of the $|\alpha_i|$ can be strictly larger than $|\alpha|/2$. Hence, at least two $|\alpha_i|$ are lower or equal to $|\alpha|/2 \leq s/2$.

We therefore split the tri-norm term into three sums as

$$\begin{aligned}
& \int_{\mathbb{T}^3} \left(\sum_{\substack{\alpha_1 + \alpha_2 + \alpha_3 = \alpha \\ |\alpha_1|, |\alpha_2| \geq 1}} \|\partial_x^{\alpha_1} \tilde{\mathbf{c}}\| \|\partial_x^{\alpha_2} \tilde{\mathbf{c}}\| \|\partial_x^{\alpha_3} \tilde{\mathbf{U}}\| \right)^2 dx \\
& \leq \sum_{\substack{\alpha_1 + \alpha_2 + \alpha_3 = \alpha \\ |\alpha_1|, |\alpha_2| \geq 1 \\ |\alpha_1|, |\alpha_2| \leq \frac{s}{2}}} \int_{\mathbb{T}^3} \|\partial_x^{\alpha_1} \tilde{\mathbf{c}}\|^2 \|\partial_x^{\alpha_2} \tilde{\mathbf{c}}\|^2 \|\partial_x^{\alpha_3} \tilde{\mathbf{U}}\|^2 dx \\
& \quad + \sum_{\substack{\alpha_1 + \alpha_2 + \alpha_3 = \alpha \\ |\alpha_1|, |\alpha_2| \geq 1 \\ |\alpha_1|, |\alpha_3| \leq \frac{s}{2}}} \int_{\mathbb{T}^3} \|\partial_x^{\alpha_1} \tilde{\mathbf{c}}\|^2 \|\partial_x^{\alpha_2} \tilde{\mathbf{c}}\|^2 \|\partial_x^{\alpha_3} \tilde{\mathbf{U}}\|^2 dx \\
& \quad + \sum_{\substack{\alpha_1 + \alpha_2 + \alpha_3 = \alpha \\ |\alpha_1|, |\alpha_2| \geq 1 \\ |\alpha_2|, |\alpha_3| \leq \frac{s}{2}}} \int_{\mathbb{T}^3} \|\partial_x^{\alpha_1} \tilde{\mathbf{c}}\|^2 \|\partial_x^{\alpha_2} \tilde{\mathbf{c}}\|^2 \|\partial_x^{\alpha_3} \tilde{\mathbf{U}}\|^2 dx
\end{aligned}$$

For any α_k -derivative such that $|\alpha_k| \leq s/2$, we bound the corresponding factor by its L_x^∞ norm and we then exploit the mentioned embedding of $H_x^{s/2}$ in L_x^∞ in order to recover the correct Sobolev norm. In the sequel C_{sob} will refer to any positive constant that appears when using the Sobolev embeddings. The first sum produces

$$\begin{aligned}
& \sum_{\substack{\alpha_1 + \alpha_2 + \alpha_3 = \alpha \\ |\alpha_1|, |\alpha_2| \geq 1 \\ |\alpha_1|, |\alpha_2| \leq \frac{s}{2}}} \int_{\mathbb{T}^3} \|\partial_x^{\alpha_1} \tilde{\mathbf{c}}\|^2 \|\partial_x^{\alpha_2} \tilde{\mathbf{c}}\|^2 \|\partial_x^{\alpha_3} \tilde{\mathbf{U}}\|^2 dx \\
& \leq \sum_{\substack{\alpha_1 + \alpha_2 + \alpha_3 = \alpha \\ |\alpha_1|, |\alpha_2| \geq 1 \\ |\alpha_1|, |\alpha_2| \leq \frac{s}{2}}} \|\partial_x^{\alpha_1} \tilde{\mathbf{c}}\|_{L_x^\infty}^2 \|\partial_x^{\alpha_2} \tilde{\mathbf{c}}\|_{L_x^\infty}^2 \|\partial_x^{\alpha_3} \tilde{\mathbf{U}}\|_{L_x^2}^2 \\
& \leq C_{\text{sob}} \sum_{\substack{\alpha_1 + \alpha_2 + \alpha_3 = \alpha \\ |\alpha_1|, |\alpha_2| \geq 1 \\ |\alpha_1|, |\alpha_2| \leq \frac{s}{2}}} \|\partial_x^{\alpha_1} \tilde{\mathbf{c}}\|_{H_x^{s/2}}^2 \|\partial_x^{\alpha_2} \tilde{\mathbf{c}}\|_{H_x^{s/2}}^2 \|\partial_x^{\alpha_3} \tilde{\mathbf{U}}\|_{L_x^2}^2 \\
& \leq s^3 C_{\text{sob}} \|\tilde{\mathbf{c}}\|_{H_x^s}^4 \|\tilde{\mathbf{U}}\|_{H_x^s}^2,
\end{aligned}$$

and the two others are dealt with in the same way. Consequently, the tri-norm term can be estimated as

$$(4.15) \quad \int_{\mathbb{T}^3} \left(\sum_{\substack{\alpha_1 + \alpha_2 + \alpha_3 = \alpha \\ |\alpha_1|, |\alpha_2| \geq 1}} \|\partial_x^{\alpha_1} \tilde{\mathbf{c}}\| \|\partial_x^{\alpha_2} \tilde{\mathbf{c}}\| \|\partial_x^{\alpha_3} \tilde{\mathbf{U}}\| \right)^2 dx \leq 3s^3 C_{\text{sob}} \|\tilde{\mathbf{c}}\|_{H_x^s}^4 \|\tilde{\mathbf{U}}\|_{H_x^s}^2.$$

Moreover, the previous Sobolev embedding also yields, for any $1 \leq i \leq N$,

$$(4.16) \quad c_i(t, x) \geq \bar{c}_i - \varepsilon \|\tilde{\mathbf{c}}\|_{L_x^\infty} \geq \min_{1 \leq i \leq N} \bar{c}_i - \varepsilon C_{\text{sob}} \|\tilde{\mathbf{c}}\|_{H_x^s} \quad \text{a.e. on } \mathbb{R}_+ \times \mathbb{T}^3.$$

We thus infer the first upper bound

$$(4.17) \quad \int_{\mathbb{T}^3} \langle \nabla_x \partial_x^\alpha \tilde{\mathbf{c}}, \partial_x^\alpha \mathbf{V}_{\tilde{U}} \rangle dx \leq -\lambda_A \left(\min_{1 \leq i \leq N} \bar{c}_i - \varepsilon C_{\text{sob}} \|\tilde{\mathbf{c}}\|_{H_x^s} \right)^2 \left\| \partial_x^\alpha \tilde{\mathbf{U}} \right\|_{L_x^2}^2 \\ + \varepsilon s^2 N^2 \mu_A \left(4C_0 + 6\varepsilon \|\tilde{\mathbf{c}}\|_{H_x^s} \right) \|\tilde{\mathbf{c}}\|_{H_x^s} \left\| \tilde{\mathbf{U}} \right\|_{H_x^s}^2.$$

The second and third term on the right-hand side of (4.13) are handled more easily. As we did for establishing (4.15), we apply the Leibniz derivation rule and the Cauchy-Schwarz inequality, together with the Sobolev embedding that allows to distribute the H_x^s norm to each factor of the products. In this way we obtain the estimates

$$\int_{\mathbb{T}^3} \langle \nabla_x \partial_x^\alpha \tilde{\mathbf{c}}, \partial_x^\alpha (\tilde{\mathbf{c}} \bar{\mathbf{u}}) \rangle_{\bar{\mathbf{c}}^{-1}} dx \leq \frac{C_{\text{sob}}}{\min_{1 \leq i \leq N} \bar{c}_i} \|\nabla_x \partial_x^\alpha \tilde{\mathbf{c}}\|_{L_x^2} \|\tilde{\mathbf{c}}\|_{H_x^s} \|\bar{\mathbf{u}}\|_{H_x^s}, \\ \int_{\mathbb{T}^3} \langle \nabla_x \partial_x^\alpha \tilde{\mathbf{c}}, \partial_x^\alpha (\tilde{\mathbf{c}} \mathbf{V}_{\tilde{U}}) \rangle_{\bar{\mathbf{c}}^{-1}} dx \leq \frac{C_{\text{sob}}}{\min_{1 \leq i \leq N} \bar{c}_i} \|\nabla_x \partial_x^\alpha \tilde{\mathbf{c}}\|_{L_x^2} \|\tilde{\mathbf{c}}\|_{H_x^s} \left(1 + \frac{1}{C_0} \|\mathbf{c}\|_{H_x^s} \right) \left\| \tilde{\mathbf{U}} \right\|_{H_x^s},$$

where we have used that $\frac{1}{\bar{c}_i} \leq \frac{1}{\min_i \bar{c}_i}$ for any $1 \leq i \leq N$. In order to control the L_x^2 norm of $\nabla_x \partial_x^\alpha \tilde{\mathbf{c}}$, we exploit the gradient relation (4.11). By similar computations to the ones providing the estimate of Proposition 3.2, and thanks to the continuous Sobolev embedding $H_x^{s/2} \hookrightarrow L_x^\infty$, one infers

$$(4.18) \quad \|\nabla_x \partial_x^\alpha \tilde{\mathbf{c}}\|_{L_x^2} \leq \|\nabla_x \tilde{\mathbf{c}}\|_{H_x^s} = \left\| A(\mathbf{c}) \tilde{\mathbf{U}} \right\|_{H_x^s} \\ \leq C_{\text{sob}} (s^2 + C_0 s) \|\mathbf{c}\|_{H_x^s}^2 \left\| \tilde{\mathbf{U}} \right\|_{H_x^s} \\ \leq 2C_{\text{sob}} (s^2 + C_0 s) \left(C_0^2 |\mathbb{T}^3| + \varepsilon^2 \|\tilde{\mathbf{c}}\|_{H_x^s}^2 \right) \left\| \tilde{\mathbf{U}} \right\|_{H_x^s} \\ \leq C_s \left(1 + \varepsilon^2 \|\tilde{\mathbf{c}}\|_{H_x^s}^2 \right) \left\| \tilde{\mathbf{U}} \right\|_{H_x^s},$$

where we have also used that

$$\int_{\mathbb{T}^3} (\bar{c}_i + \varepsilon \tilde{c}_i)^2 dx \leq 2 \left(\int_{\mathbb{T}^3} \bar{c}_i^2 dx + \varepsilon^2 \int_{\mathbb{T}^3} \tilde{c}_i^2 dx \right), \\ 0 \leq \bar{c}_i \leq \sum_{j=1}^N \bar{c}_j = C_0.$$

Since $\varepsilon \leq 1$ and $\|\bar{\mathbf{u}}\|_{H_x^s} \leq \sqrt{N} \|\bar{u}\|_{H_x^s}$, we finally deduce the upper bounds

$$(4.19) \quad \int_{\mathbb{T}^3} \langle \nabla_x \partial_x^\alpha \tilde{\mathbf{c}}, \partial_x^\alpha (\tilde{\mathbf{c}}\bar{\mathbf{u}}) \rangle_{\bar{\mathbf{c}}^{-1}} dx \leq \frac{\sqrt{N} C_s}{\min_{1 \leq i \leq N} \bar{c}_i} \|\tilde{\mathbf{c}}\|_{H_x^s} \left(1 + \|\tilde{\mathbf{c}}\|_{H_x^s}\right)^2 \|\bar{u}\|_{H_x^s} \|\tilde{\mathbf{U}}\|_{H_x^s},$$

$$(4.20) \quad \int_{\mathbb{T}^3} \langle \nabla_x \partial_x^\alpha \tilde{\mathbf{c}}, \partial_x^\alpha (\tilde{\mathbf{c}}\mathbf{V}_{\tilde{U}}) \rangle_{\bar{\mathbf{c}}^{-1}} dx \leq \frac{C_s}{\min_{1 \leq i \leq N} \bar{c}_i} \|\tilde{\mathbf{c}}\|_{H_x^s} \left(1 + \|\tilde{\mathbf{c}}\|_{H_x^s}\right)^3 \|\tilde{\mathbf{U}}\|_{H_x^s}^2,$$

by accordingly increasing the value of the constant C_s .

To conclude, we gather (4.13) with the estimates (4.17), (4.19) and (4.20), and we sum over all $|\alpha| \leq s$. In this way, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tilde{\mathbf{c}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})}^2 &\leq -\lambda_A \left(\min_{1 \leq i \leq N} \bar{c}_i \right)^2 \|\tilde{\mathbf{U}}\|_{H_x^s}^2 \\ &\quad + \frac{s^3 \sqrt{N} C_s}{\min_{1 \leq i \leq N} \bar{c}_i} \|\tilde{\mathbf{c}}\|_{H_x^s} \|\bar{u}\|_{H_x^s} \left(1 + \|\tilde{\mathbf{c}}\|_{H_x^s}\right)^2 \|\tilde{\mathbf{U}}\|_{H_x^s} \\ &\quad + \varepsilon \left(2\lambda_A C_{\text{sob}} \min_{1 \leq i \leq N} \bar{c}_i + \varepsilon \lambda_A C_{\text{sob}}^2 + \frac{s^3 C_s}{\min_{1 \leq i \leq N} \bar{c}_i} \right. \\ &\quad \left. + s^5 C_{\text{sob}} N^2 \mu_A (4C_0 + 6\varepsilon) \right) \|\tilde{\mathbf{c}}\|_{H_x^s} \left(1 + \|\tilde{\mathbf{c}}\|_{H_x^s}\right)^3 \|\tilde{\mathbf{U}}\|_{H_x^s}^2. \end{aligned}$$

In order to close the estimate above, since $\bar{\mathbf{c}}$ is constant we first easily check that

$$\|\tilde{\mathbf{c}}\|_{H_x^s} \leq \max_{1 \leq i \leq N} \bar{c}_i \|\tilde{\mathbf{c}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})} \leq C_0 \|\tilde{\mathbf{c}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})}.$$

Moreover, recalling Lemma 4.6, we can apply the Poincaré inequality to $\tilde{\mathbf{c}}$, which has zero mean on the torus. Denoting $C_{\mathbb{T}^3} > 0$ the Poincaré constant, we can thus compute

$$(4.21) \quad \|\tilde{\mathbf{c}}\|_{H_x^s} \leq C_{\mathbb{T}^3} \|\nabla_x \tilde{\mathbf{c}}\|_{H_x^s} \leq C_{\mathbb{T}^3} C_s \left(1 + \varepsilon^2 \|\tilde{\mathbf{c}}\|_{H_x^s}^2\right) \|\tilde{\mathbf{U}}\|_{H_x^s},$$

where we have also used (4.18).

We denote by C_s any positive constant that only depends on s , N , λ_A , μ_A , $\bar{\mathbf{c}}$, C_{sob} , C_0 and $C_{\mathbb{T}^3}$. Thanks to the above estimates, we can consequently infer the validity of the bound

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tilde{\mathbf{c}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})}^2 &\leq - \left[\lambda_A \left(\min_{1 \leq i \leq N} \bar{c}_i \right)^2 - C_s \|\bar{u}\|_{H_x^s} \left(1 + \|\tilde{\mathbf{c}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})}\right)^4 \right. \\ &\quad \left. - C_s \|\tilde{\mathbf{c}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})} \left(1 + \|\tilde{\mathbf{c}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})}\right)^3 \right] \|\tilde{\mathbf{U}}\|_{H_x^s}^2. \end{aligned}$$

Therefore, if $\|\tilde{\mathbf{c}}^{\text{in}}\|_{H_x^s} \leq \delta_s$ and $\|\bar{u}\|_{H_x^s} \leq \delta_s$ for almost any $t \geq 0$, where $\delta_s > 0$ is chosen such that

$$(4.22) \quad C_s \delta_s \left((1 + \delta_s)^4 + (1 + \delta_s)^3 \right) \leq \frac{\lambda_A \left(\min_{1 \leq i \leq N} \bar{c}_i \right)^2}{2},$$

we ensure that the $H_x^s(\bar{c}^{-\frac{1}{2}})$ norm of $\tilde{\mathbf{c}}$ keeps diminishing and satisfies

$$(4.23) \quad \frac{d}{dt} \|\tilde{\mathbf{c}}\|_{H_x^s(\bar{c}^{-\frac{1}{2}})}^2 \leq -\frac{\lambda_A \left(\min_{1 \leq i \leq N} \bar{c}_i \right)^2}{2} \|\tilde{\mathbf{U}}\|_{H_x^s}^2 \quad \text{for a.e. } t \geq 0.$$

Moreover, since the Poincaré inequality (4.21) tells us that the norm of $\tilde{\mathbf{U}}$ controls the one of $\tilde{\mathbf{c}}$, we recover the estimate

$$\frac{d}{dt} \|\tilde{\mathbf{c}}\|_{H_x^s(\bar{c}^{-\frac{1}{2}})}^2 \leq -\frac{\lambda_A \left(\min_{1 \leq i \leq N} \bar{c}_i \right)^2}{2C_{\mathbb{T}^3}^2 C_s^2 (1 + \delta_s^2)^2} \|\tilde{\mathbf{c}}\|_{H_x^s}^2 \leq -\frac{\lambda_A \left(\min_{1 \leq i \leq N} \bar{c}_i \right)^3}{2C_{\mathbb{T}^3}^2 C_s^2 (1 + \delta_s^2)^2} \|\tilde{\mathbf{c}}\|_{H_x^s(\bar{c}^{-\frac{1}{2}})}^2.$$

Setting

$$(4.24) \quad \lambda_s = \frac{\lambda_A \left(\min_{1 \leq i \leq N} \bar{c}_i \right)^3}{4C_{\mathbb{T}^3}^2 C_s^2 (1 + \delta_s^2)^2},$$

Grönwall's lemma finally tells us that for a.e. $t \geq 0$

$$\|\tilde{\mathbf{c}}\|_{H_x^s(\bar{c}^{-\frac{1}{2}})} \leq e^{-\lambda_s t} \|\tilde{\mathbf{c}}^{\text{in}}\|_{H_x^s(\bar{c}^{-\frac{1}{2}})},$$

and we also recover

$$(4.25) \quad \|\tilde{\mathbf{U}}\|_{H_x^{s-1}} = \|A(\mathbf{c})^{-1} \nabla_x \tilde{\mathbf{c}}\|_{H_x^{s-1}} \leq \tilde{C}_s \|\nabla_x \tilde{\mathbf{c}}\|_{H_x^{s-1}} \leq \tilde{C}_s \|\tilde{\mathbf{c}}\|_{H_x^s} \leq C_s \|\tilde{\mathbf{c}}\|_{H_x^s(\bar{c}^{-\frac{1}{2}})},$$

by simply adjusting the value of C_s . The constant $C_s = C_s(C_0, \lambda_A, \mu_A, s, \delta_s, \bar{\mathbf{c}}) > 0$ is obtained by inverting $A(\mathbf{c})$ and repeating the previous computations, *via* the continuous Sobolev embedding already mentioned. In particular, note that for our choice of δ_s one sees from (4.16) that \mathbf{c} does not vanish anywhere and there is therefore no singularity in $A(\mathbf{c})^{-1}$.

The last estimate on the integral of $\|\tilde{\mathbf{U}}\|_{H_x^s}^2$ is a direct application of Grönwall's lemma from (4.23). This concludes the proof. \square

Before going into details in the proofs of existence and uniqueness, we present here another result which establishes that the positivity of \mathbf{c} is obtained *a priori*. This will help the reader in clarifying the last statement we gave in the previous proof, about the invertibility of $A(\mathbf{c})$. Moreover, note that ensuring the positivity of \mathbf{c} *a priori* is crucial, since it will leave us free on the choice of the iterative scheme to be used in the next section, when constructing the solution of the system (4.10)–(4.11).

Lemma 4.8. *Consider an initial datum $(\tilde{\mathbf{c}}^{\text{in}}, \tilde{\mathbf{U}}^{\text{in}})$ satisfying the assumptions of Theorem 4.4. If $(\tilde{\mathbf{c}}, \tilde{\mathbf{U}})$ is a solution of (4.10)–(4.11) with initial datum $(\tilde{\mathbf{c}}^{\text{in}}, \tilde{\mathbf{U}}^{\text{in}})$, then, for almost any $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^3$ the vector $\bar{\mathbf{c}} + \varepsilon \tilde{\mathbf{c}}(t, x)$ is positive.*

Proof of Lemma 4.8. The proof is rather simple. Since we start from a positive initial datum $\bar{\mathbf{c}} + \varepsilon \tilde{\mathbf{c}}^{\text{in}}$, by inverting the gradient relation (4.11) as

$$\tilde{\mathbf{U}} = \frac{1}{\varepsilon} A(\mathbf{c})^{-1} \nabla_x \mathbf{c},$$

and plugging this identity into (4.10), we recover an elliptic equation for $\mathbf{c} = \bar{\mathbf{c}} + \varepsilon \tilde{\mathbf{c}}$. Standard elliptic weak minimum principles then apply to \mathbf{c} , ensuring its nonnegativity at least until a maximal finite time which we call T_0 . But the previous *a priori* estimate obtained on $\tilde{\mathbf{c}}$ tells us that in fact the positivity of \mathbf{c} is satisfied as long as δ_s is also chosen in (4.16) to be strictly smaller than $\min_{1 \leq i \leq N} \bar{c}_i$. This guarantees that $T_0 = +\infty$ and thus $\mathbf{c}(t, x)$ is positive a.e. on $\mathbb{R}_+ \times \mathbb{T}^3$. \square

Step 3 – Existence and uniqueness of the couple $(\tilde{\mathbf{c}}, \tilde{\mathbf{U}})$. We now have all the tools needed in order to construct our Cauchy theory for the couple $(\tilde{\mathbf{c}}, \tilde{\mathbf{U}})$. We shall first present the existence result and then prove the uniqueness of the constructed solution.

Proposition 4.9. *Let $s > 3$ be an integer and consider a triple $(\tilde{\mathbf{c}}^{\text{in}}, \tilde{\mathbf{U}}^{\text{in}}, \bar{u})$ satisfying the assumptions of Theorem 4.4. There exists $\delta_s > 0$ such that, for all $\varepsilon \in (0, 1]$, there exists a global weak solution $(\tilde{\mathbf{c}}, \tilde{\mathbf{U}}) \in L^\infty(\mathbb{R}_+; H^s(\mathbb{T}^3)) \times L^\infty(\mathbb{R}_+; H^{s-1}(\mathbb{T}^3))$ of the system (4.10)–(4.11), with initial datum $(\tilde{\mathbf{c}}^{\text{in}}, \tilde{\mathbf{U}}^{\text{in}})$.*

Proof of Proposition 4.9. The proof is standard and is based on an iterative scheme, where we first construct a solution on a well-chosen time interval $[0, T_0]$, and we then show that this interval can be extended to $[0, +\infty)$. Note however that one has to be careful with the estimates, since the conservation of the exact exponential decay rate is crucial. The underlying mechanism lies on the fact that our problem is actually quasilinear parabolic for small initial data. Indeed, noticing that

$$\tilde{\mathbf{U}} = A(\mathbf{c})^{-1} \nabla_x \tilde{\mathbf{c}},$$

we solely have to solve

$$\begin{aligned} \partial_t \tilde{\mathbf{c}} + \bar{\mathbf{c}} \nabla_x \cdot \left(A(\mathbf{c})^{-1} \nabla_x \tilde{\mathbf{c}} - \frac{\langle \mathbf{c}, A(\mathbf{c})^{-1} \nabla_x \tilde{\mathbf{c}} \rangle \mathbf{1}}{\langle \mathbf{c}, \mathbf{1} \rangle} \right) + \bar{u} \cdot \nabla_x \tilde{\mathbf{c}} \\ + \varepsilon \nabla_x \cdot \left(\tilde{\mathbf{c}} \left(A(\mathbf{c})^{-1} \nabla_x \tilde{\mathbf{c}} - \frac{\langle \mathbf{c}, A(\mathbf{c})^{-1} \nabla_x \tilde{\mathbf{c}} \rangle \mathbf{1}}{\langle \mathbf{c}, \mathbf{1} \rangle} \right) \right) = 0. \end{aligned}$$

From Proposition 3.3 we see that the higher order term is of order 2, symmetric and negative for $\mathbf{c} > 0$, which makes this equation quasilinear parabolic.

We initially set

$$(4.26) \quad \tilde{\mathbf{c}}^{(0)} = \tilde{\mathbf{c}}^{\text{in}}, \quad T_0 = 3 \frac{C_{\mathbb{T}^3} \min_{1 \leq i \leq N} \bar{c}_i}{4C_0 C_s (1 + 4\delta_s^2) \delta_s},$$

where δ_s , C_s and $C_{\mathbb{T}^3}$ respectively come from (4.22), (4.25) and (4.21).

Suppose that an N -vector function $\tilde{\mathbf{c}}^{(n)} \in L^\infty(0, T_0; H^s(\mathbb{T}^3))$ is given, satisfying

$$(4.27) \quad \|\tilde{\mathbf{c}}^{(n)}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})} \leq 2\delta_s e^{-\lambda_s t}, \quad \sum_{i=1}^N \tilde{c}_i^{(n)}(t, x) = 0 \quad \text{a.e. on } (0, T_0) \times \mathbb{T}^3.$$

For $s > 3$, the Sobolev embedding $H_x^s \hookrightarrow L_x^\infty$ makes applicable standard parabolic methods on the torus (see for instance [11, Section 7.1]) which raise the existence of a solution $\tilde{\mathbf{c}}^{(n+1)} \in L^2(0, T_0; H^1(\mathbb{T}^3))$ to the following linear equation

$$(4.28) \quad \partial_t \tilde{\mathbf{c}}^{(n+1)} + \bar{u} \cdot \nabla_x \tilde{\mathbf{c}}^{(n)} + \nabla_x \cdot \left(\mathbf{c}^{(n)} \left(A(\mathbf{c}^{(n)})^{-1} \nabla_x \tilde{\mathbf{c}}^{(n+1)} - \frac{\langle \mathbf{c}^{(n)}, A(\mathbf{c}^{(n)})^{-1} \nabla_x \tilde{\mathbf{c}}^{(n+1)} \rangle}{\langle \mathbf{c}^{(n)}, \mathbf{1} \rangle} \mathbf{1} \right) \right) = 0,$$

with initial datum $\tilde{\mathbf{c}}^{\text{in}}$. Note that summing (4.28) over $1 \leq i \leq N$ yields

$$\sum_{i=1}^N \tilde{c}_i^{(n+1)}(t, x) = \sum_{i=1}^N \tilde{c}_i^{(n+1)}(0, x) = 0 \quad \text{a.e. on } (0, T_0) \times \mathbb{T}^3,$$

which shows, thanks to Proposition 3.3, that $(\text{Span}(\mathbf{1}))^\perp$ is stable for (4.28), implying that $A(\mathbf{c}^{(n)})^{-1} \nabla_x \tilde{\mathbf{c}}^{(n+1)}$ is well-defined at almost every time $t \in (0, T_0)$.

The same computations carried out to derive the *a priori* estimates in Proposition 4.7 give (see in particular (4.19) for the term containing \bar{u})

$$\begin{aligned} & \frac{d}{dt} \|\tilde{\mathbf{c}}^{(n+1)}\|_{H_x^s(\bar{c}^{-\frac{1}{2}})}^2 \\ & \leq - \left(C_{\mathbb{T}^3} \min_{1 \leq i \leq N} \bar{c}_i \right) \left(\frac{2\lambda_A}{C_{\mathbb{T}^3}^2 C_s^2 (1 + \delta_s^2)^2} + C_s (1 + 4\varepsilon^2 \delta_s^2) \|\bar{u}\|_{H_x^s} \right) \|\nabla_x \tilde{\mathbf{c}}^{(n+1)}\|_{H_x^s}^2 \\ & \quad + C_s (1 + 4\varepsilon^2 \delta_s^2) \|\nabla_x \tilde{\mathbf{c}}^{(n+1)}\|_{H_x^s} \|\bar{u}\|_{H_x^s} \|\tilde{\mathbf{c}}^{(n)}\|_{H_x^s(\bar{c}^{-\frac{1}{2}})}, \end{aligned}$$

where we used that $\|\tilde{\mathbf{c}}^{(n)}\|_{H_x^s} \leq 2\delta_s$. Note that $C_s (1 + \varepsilon^2 \delta_s^2) \|\bar{u}\|_{H_x^s}$ inside the negative term comes from the absence of $\nabla_x \cdot (\tilde{\mathbf{c}}^{(n+1)} \bar{\mathbf{u}})$ in (4.28), whereas the multiplicative constant in front of it originates from the definition of λ_s . We now use Young's inequality to get

$$\begin{aligned} & \frac{d}{dt} \|\tilde{\mathbf{c}}^{(n+1)}\|_{H_x^s(\bar{c}^{-\frac{1}{2}})}^2 \\ & \leq - \left(C_{\mathbb{T}^3} \min_{1 \leq i \leq N} \bar{c}_i \right) \left(\frac{2\lambda_s}{\left(\min_{1 \leq i \leq N} \bar{c}_i \right)^3} + C_s (1 + 4\varepsilon^2 \delta_s^2) \|\bar{u}\|_{H_x^s} - \frac{\eta}{C_{\mathbb{T}^3} \min_{1 \leq i \leq N} \bar{c}_i} \right) \\ & \quad \times \|\nabla_x \tilde{\mathbf{c}}^{(n+1)}\|_{H_x^s}^2 \\ & \quad + \frac{C_s^2 (1 + 4\varepsilon^2 \delta_s^2)^2 \|\bar{u}\|_{H_x^s}^2}{\eta} \|\tilde{\mathbf{c}}^{(n)}\|_{H_x^s}^2, \end{aligned}$$

for any $\eta > 0$. Therefore, if we choose

$$\eta = C_s (1 + 4\varepsilon^2 \delta_s^2) \|\bar{u}\|_{H_x^s} C_{\mathbb{T}^3} \min_{1 \leq i \leq N} \bar{c}_i,$$

thanks to Poincaré inequality (4.21) and to the assumption $\|\bar{u}\|_{H_x^s} \leq \delta_s$, we obtain

$$\frac{d}{dt} \|\tilde{\mathbf{c}}^{(n+1)}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})}^2 \leq -2\lambda_s \|\tilde{\mathbf{c}}^{(n+1)}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})}^2 + \frac{C_0 C_s (1 + 4\delta_s^2) \delta_s}{C_{\mathbb{T}^3} \min_{1 \leq i \leq N} \bar{c}_i} \|\tilde{\mathbf{c}}^{(n)}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})}^2.$$

Eventually, we apply Grönwall's lemma using the exponential decay of $\|\tilde{\mathbf{c}}^{(n)}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})}^2$ given by the iterative assumption (4.27), and we successively get, for almost every time $t \in (0, T_0)$,

$$\begin{aligned} \|\tilde{\mathbf{c}}^{(n+1)}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})}^2 &\leq \left(\|\tilde{\mathbf{c}}^{\text{in}}\|_{H_x^s(\bar{\mathbf{c}}^{-\frac{1}{2}})}^2 + 4\delta_s^2 \frac{C_0 C_s (1 + 4\delta_s^2) \delta_s}{C_{\mathbb{T}^3} \min_{1 \leq i \leq N} \bar{c}_i} t \right) e^{-2\lambda_s t} \\ &\leq \delta_s^2 \left(1 + 4 \frac{C_0 C_s (1 + 4\delta_s^2) \delta_s}{C_{\mathbb{T}^3} \min_{1 \leq i \leq N} \bar{c}_i} T_0 \right) e^{-2\lambda_s t} \\ &\leq 4\delta_s^2 e^{-2\lambda_s t}, \end{aligned}$$

thanks to the definition of T_0 given in (4.26). This proves that $\tilde{\mathbf{c}}^{(n+1)}$ belongs to $L^\infty(0, T_0; H^s(\mathbb{T}^3))$ and satisfies the iterative assumptions (4.27).

By induction, we thus construct a sequence $(\tilde{\mathbf{c}}^{(n)})_{n \in \mathbb{N}}$ defined a.e. on $(0, T_0) \times \mathbb{T}^3$, belonging to $(\text{Span}(\mathbf{1}))^\perp$, and bounded by $2\delta_s$ in $L^\infty(0, T_0; H^s(\mathbb{T}^3))$. Moreover, the iterative equation (4.28) gives an explicit formula for $\partial_t \tilde{\mathbf{c}}^{(n+1)}$ in terms of $\tilde{\mathbf{c}}^{(n)}$, $\tilde{\mathbf{c}}^{(n+1)}$ and \bar{u} . Again, the continuous Sobolev embedding $H_x^s \hookrightarrow L_x^\infty$ for $s > 3/2$ and Proposition 3.3 raise the existence of a polynomial P in two variables, with coefficients only depending on s , $\bar{\mathbf{c}}$, $\|\bar{u}\|_{H_x^s}$, λ_A and μ_A , such that

$$\|\partial_t \tilde{\mathbf{c}}^{(n+1)}\|_{L_x^2} \leq P \left(\|\tilde{\mathbf{c}}^{(n)}\|_{H_x^s}, \|\tilde{\mathbf{c}}^{(n+1)}\|_{H_x^s} \right) \leq P(2\delta_s, 2\delta_s) \quad \text{for a.e. } t \geq 0.$$

This shows that $(\partial_t \tilde{\mathbf{c}}^{(n)})_{n \in \mathbb{N}}$ is bounded in $L^\infty(0, T_0; L^2(\mathbb{T}^3))$, uniformly with respect to $n \in \mathbb{N}$.

Therefore, choosing $0 < s' < s - 2$, by Sobolev embeddings, there exists an N -vector function $\tilde{\mathbf{c}}^\infty \in L^\infty(0, T_0; H^s(\mathbb{T}^3))$ such that, up to a subsequence,

- (1) $(\tilde{\mathbf{c}}^{(n)})_{n \in \mathbb{N}}$ converges, weakly-* in $L^\infty(0, T_0)$ and weakly in H_x^s , to $\tilde{\mathbf{c}}^\infty$,
- (2) $(\tilde{\mathbf{c}}^{(n)})_{n \in \mathbb{N}}$, $(\nabla_x \tilde{\mathbf{c}}^{(n)})_{n \in \mathbb{N}}$ and $(\nabla_x \nabla_x \tilde{\mathbf{c}}^{(n)})_{n \in \mathbb{N}}$ converge weakly-* in $L^\infty(0, T_0)$ and strongly in $H_x^{s'}$,
- (3) $(\partial_t \tilde{\mathbf{c}}^{(n)})_{n \in \mathbb{N}}$ converges weakly-* in $L^\infty(0, T_0)$ and weakly in L_x^2 .

Integrating our scheme (4.28) against test functions, we can then pass to the limit as n goes to $+\infty$ (the nonlinear terms being bounded and dealt with thanks to the strong convergences in $H_x^{s'}$), and we see that $\tilde{\mathbf{c}}^\infty$ is a weak solution to

$$\begin{aligned} \partial_t \tilde{\mathbf{c}}^\infty + \bar{\mathbf{c}} \nabla_x \cdot \left(A(\mathbf{c}^\infty)^{-1} \nabla_x \tilde{\mathbf{c}}^\infty - \frac{\langle \mathbf{c}^\infty, A(\mathbf{c}^\infty)^{-1} \nabla_x \tilde{\mathbf{c}}^\infty \rangle}{\langle \mathbf{c}^\infty, \mathbf{1} \rangle} \mathbf{1} \right) + \bar{u} \cdot \nabla_x \tilde{\mathbf{c}}^\infty \\ + \varepsilon \nabla_x \cdot \left(\tilde{\mathbf{c}}^\infty \left(A(\mathbf{c}^\infty)^{-1} \nabla_x \tilde{\mathbf{c}}^\infty - \frac{\langle \mathbf{c}^\infty, A(\mathbf{c}^\infty)^{-1} \nabla_x \tilde{\mathbf{c}}^\infty \rangle}{\langle \mathbf{c}^\infty, \mathbf{1} \rangle} \mathbf{1} \right) \right) = 0. \end{aligned}$$

Denoting $\tilde{\mathbf{U}}^\infty = A(\mathbf{c}^\infty)^{-1} \nabla_x \tilde{\mathbf{c}}^\infty$, this proves that $(\tilde{\mathbf{c}}^\infty, \tilde{\mathbf{U}}^\infty)$ is a weak solution to the system (4.10)–(4.11), belonging to $L^\infty(0, T_0; H^s(\mathbb{T}^3)) \times L^\infty(0, T_0; H^{s-1}(\mathbb{T}^3))$. In particular, looking at equations (4.10)–(4.11), by means of the continuous embedding of $H_x^{s/2}$ in L_x^∞ one easily checks that $(\partial_t \tilde{\mathbf{c}}^\infty, \partial_t \tilde{\mathbf{U}}^\infty)$ belongs to $L^\infty(0, T_0; L^2(\mathbb{T}^3)) \times L^\infty(0, T_0; L^2(\mathbb{T}^3))$ as soon as $s > 4$. Applying the Aubin-Lions-Simon theorem (see for example [6, Theorem II.5.16]), we thus also ensure that $(\tilde{\mathbf{c}}^\infty, \tilde{\mathbf{U}}^\infty)$ belongs to $C^0([0, T_0]; H^{s-1}(\mathbb{T}^3)) \times C^0([0, T_0]; H^{s-2}(\mathbb{T}^3))$ for any $s > 4$.

Therefore, using the continuity of $\tilde{\mathbf{c}}^\infty$, we can finally conclude thanks to the *a priori* estimates established in Proposition 4.7, which state that $\|\tilde{\mathbf{c}}^\infty(T_0)\|_{H_x^s} \leq \delta_s$. Indeed, we can restart our scheme at T_0 from this initial condition and we can obtain a solution on the time interval $[T_0, 2T_0]$. Again, using the continuity of $\tilde{\mathbf{c}}^\infty$ with respect to $t \in [T_0, 2T_0]$ and Proposition 4.7, the corresponding sequence will be bounded by δ_s at $2T_0$, and by induction we can construct a weak solution of (4.10)–(4.11) on $[0, +\infty)$. \square

In the next result we conclude by recovering the uniqueness of the solution to the orthogonal system (4.10)–(4.11). We remind the reader that this property has to be understood in a perturbative sense, since we are only able to prove the uniqueness of the fluctuations $(\tilde{\mathbf{c}}, \tilde{\mathbf{U}})$ around the macroscopic equilibrium state $(\bar{\mathbf{c}}, \mathbf{0})$.

Proposition 4.10. *Let $s > 3$ be an integer, and consider a couple $(\tilde{\mathbf{c}}^{\text{in}}, \tilde{\mathbf{U}}^{\text{in}})$ satisfying the assumptions of Theorem 4.4. There exists $\delta_s > 0$ such that, if $(\tilde{\mathbf{c}}, \tilde{\mathbf{U}})$ and $(\tilde{\mathbf{d}}, \tilde{\mathbf{W}})$ are two solutions of (4.10)–(4.11) having the same initial datum $(\tilde{\mathbf{c}}^{\text{in}}, \tilde{\mathbf{U}}^{\text{in}})$, then $\tilde{\mathbf{c}} = \tilde{\mathbf{d}}$ and $\tilde{\mathbf{U}} = \tilde{\mathbf{W}}$.*

Proof of Proposition 4.10. Subtracting the two sets of equations satisfied by $(\tilde{\mathbf{c}}, \tilde{\mathbf{U}})$ and $(\tilde{\mathbf{d}}, \tilde{\mathbf{W}})$, and denoting $\tilde{\mathbf{h}} = \tilde{\mathbf{c}} - \tilde{\mathbf{d}}$ and $\tilde{\mathbf{R}} = \tilde{\mathbf{U}} - \tilde{\mathbf{W}}$, we initially establish the relations

$$(4.29) \quad \partial_t \tilde{\mathbf{h}} + \bar{\mathbf{c}} \nabla_x \cdot \mathbf{V}_{\tilde{R}} + \bar{\mathbf{u}} \cdot \nabla_x \tilde{\mathbf{h}} + \varepsilon \nabla_x \cdot (\tilde{\mathbf{h}} \mathbf{V}_{\tilde{U}}) + \varepsilon \nabla_x \cdot (\tilde{\mathbf{d}} \mathbf{V}_{\tilde{R}}) = 0,$$

$$(4.30) \quad \nabla_x \tilde{\mathbf{h}} = A(\mathbf{c}) \tilde{\mathbf{R}} + [A(\mathbf{c}) - A(\mathbf{d})] \tilde{\mathbf{W}},$$

with an obvious meaning for the shorthand $\mathbf{V}_{\tilde{R}}$.

We shall give similar computations to the ones derived for the *a priori* estimates, except that we here restrict our investigation to the sole L_x^2 setting, since it will prove itself to be sufficient in order to deduce uniqueness. However, we still need the solutions to be in H_x^s for some $s > 3$, in order to again take advantage of the Sobolev embedding $H_x^{s/2} \hookrightarrow L_x^\infty$. We compute the scalar product between $\bar{\mathbf{c}}^{-1} \tilde{\mathbf{h}}$ and the equation (4.29), and we integrate over the torus. As in the proof of Proposition 4.7, we use the gradient equation (4.30) and its orthogonal properties to recover

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\| \tilde{\mathbf{h}} \right\|_{L_x^2(\bar{\mathbf{c}}^{-\frac{1}{2}})}^2 &\leq \int_{\mathbb{T}^3} \langle A(\mathbf{c}) \tilde{\mathbf{R}}, \tilde{\mathbf{R}} \rangle dx + \int_{\mathbb{T}^3} \langle [A(\mathbf{c}) - A(\mathbf{d})] \tilde{\mathbf{W}}, \tilde{\mathbf{R}} \rangle_{\bar{\mathbf{c}}^{-1}} dx \\ &\quad + \int_{\mathbb{T}^3} \langle \nabla_x \tilde{\mathbf{h}}, \tilde{\mathbf{h}} (\bar{\mathbf{u}} + \varepsilon \mathbf{V}_{\tilde{R}}) \rangle_{\bar{\mathbf{c}}^{-1}} dx + \varepsilon \int_{\mathbb{T}^3} \langle \nabla_x \tilde{\mathbf{h}}, \tilde{\mathbf{d}} \mathbf{V}_{\tilde{R}} \rangle_{\bar{\mathbf{c}}^{-1}} dx. \end{aligned}$$

We use the spectral gap of $A(\mathbf{c})$ for the first term on the right-hand side, while the remaining terms are dealt with thanks to the *a priori* estimates derived in

Proposition 4.7 and the usual Sobolev embedding, in the following way:

$$|\tilde{\mathbf{c}}(t, x)| \leq \|\mathbf{c}\|_{H_x^s} \leq \delta_s, \quad |\bar{u}(t, x)| \leq \delta_s, \quad \left| \tilde{\mathbf{d}}(t, x) \right| \leq \delta_s, \quad \text{a.e. on } \mathbb{R}_+ \times \mathbb{T}^3.$$

This initially gives

$$(4.31) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \left\| \tilde{\mathbf{h}} \right\|_{L_x^2(\bar{\mathbf{c}}^{-\frac{1}{2}})}^2 &\leq -\frac{\lambda_A}{2} \left\| \tilde{\mathbf{R}} \right\|_{L_x^2}^2 + \frac{\sum_{1 \leq i, j \leq N} \int_{\mathbb{T}^3} \left| \tilde{R}_i \right| |c_i c_j - d_i d_j| \left| \tilde{W}_j - \tilde{W}_i \right| dx}{\min_{1 \leq i \leq N} \bar{c}_i \min_{1 \leq i, j \leq N} \Delta_{ij}} \\ &+ \left[\delta_s + \varepsilon \delta_s \left(1 + \frac{N \delta_s^2}{C_0} \right) \right] \left\| \tilde{\mathbf{h}} \right\|_{L_x^2} \left\| \nabla_x \tilde{\mathbf{h}} \right\|_{L_x^2} + \varepsilon \delta_s \left\| \nabla_x \tilde{\mathbf{h}} \right\|_{L_x^2} \left\| \tilde{\mathbf{R}} \right\|_{L_x^2}. \end{aligned}$$

Then, the algebraic manipulation

$$|c_i c_j - d_i d_j| = \left| \frac{1}{2} (c_i - d_i) (c_j + d_j) + \frac{1}{2} (c_i + d_i) (c_j - d_j) \right| \leq \varepsilon \frac{\delta_s}{2} (|h_i| + |h_j|)$$

and the Cauchy-Schwarz inequality yield the control

$$(4.32) \quad \sum_{1 \leq i, j \leq N} \int_{\mathbb{T}^3} \left| \tilde{R}_i \right| |c_i c_j - d_i d_j| \left| \tilde{W}_j - \tilde{W}_i \right| dx \leq 2\varepsilon \delta_s^2 \left\| \tilde{\mathbf{R}} \right\|_{L_x^2} \left\| \tilde{\mathbf{h}} \right\|_{L_x^2}.$$

From the gradient relation (4.30) and from the Poincaré inequality (4.21), we also deduce the existence of a constant $C_s > 0$ such that

$$(4.33) \quad \left\| \tilde{\mathbf{R}} \right\|_{L_x^2} \geq C_s \left\| \nabla_x \tilde{\mathbf{h}} \right\| - \varepsilon \delta_s^2 \left\| \tilde{\mathbf{h}} \right\|_{L_x^2} \geq (C_s - \varepsilon \delta_s^2) \left\| \tilde{\mathbf{h}} \right\|_{L_x^2}.$$

We now use (4.32), (4.33) and the fact that $0 < \varepsilon \leq 1$ inside (4.31) to finally infer the upper bound

$$\frac{1}{2} \frac{d}{dt} \left\| \tilde{\mathbf{h}} \right\|_{L_x^2(\bar{\mathbf{c}}^{-\frac{1}{2}})}^2 \leq \left(-\frac{\lambda_A}{2} + \delta_s K(\delta_s) \right) \left\| \tilde{\mathbf{R}} \right\|_{L_x^2}^2$$

where $K(\delta_s) > 0$ is a polynomial in δ_s whose coefficients only depend on $\bar{\mathbf{c}}$ and on the number of species N . By choosing δ_s small enough so that both Proposition 4.7 holds and the inequality $-\frac{\lambda_A}{2} + \delta_s K(\delta_s) \leq 0$ is satisfied, we conclude that $\left\| \tilde{\mathbf{h}} \right\|_{L_x^2(\bar{\mathbf{c}}^{-\frac{1}{2}})}$

decreases over time. Therefore, since initially $\tilde{\mathbf{h}}^{\text{in}} = 0$, we deduce that $\tilde{\mathbf{h}} = 0$ at any time $t \geq 0$.

This implies that $\tilde{\mathbf{c}} = \tilde{\mathbf{d}}$, from which we also deduce that the gradient relation (4.30) becomes

$$A(\mathbf{c}) \tilde{\mathbf{R}} = 0.$$

We thus infer that $\tilde{\mathbf{R}} = 0$, since $\tilde{\mathbf{R}} \in (\ker A)^\perp$. Consequently, $\tilde{\mathbf{U}} = \tilde{\mathbf{V}}$ and the uniqueness is established. \square

Step 5 – Conclusion. We are finally able to end our study of the incompressible Maxwell-Stefan system (4.1)–(4.2)–(4.3). Theorem 4.4 is a direct gathering of Proposition 4.7, Lemmata 4.6 and 4.8, and Propositions 4.9–4.10.

Our main Theorem 2.1 then directly follows from Theorem 4.4 with the unique orthogonal writing (4.5) established in Proposition 4.1. In fact, as soon as the unique solution $(\bar{\mathbf{c}} + \varepsilon \tilde{\mathbf{c}}, \bar{\mathbf{u}} + \varepsilon \tilde{\mathbf{U}})$ of the orthogonal system (4.10)–(4.11) is established, the

corresponding unique perturbative solution of the Maxwell-Stefan system (4.1)–(4.2) with incompressibility condition (4.3) is given by $(\bar{\mathbf{c}} + \varepsilon\tilde{\mathbf{c}}, \bar{\mathbf{u}} + \varepsilon\tilde{\mathbf{u}})$, where

$$\tilde{\mathbf{u}} = \tilde{\mathbf{U}} - \frac{1}{C_0} \langle \mathbf{c}, \tilde{\mathbf{U}} \rangle \mathbf{1}$$

satisfies $\langle \mathbf{c}, \tilde{\mathbf{u}} \rangle = 0$. In particular, the exponential decay of $\tilde{\mathbf{u}}$ directly follows from the exponential decays of $\tilde{\mathbf{c}}$ and $\tilde{\mathbf{U}}$.

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