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# A transference inequality for rational approximation to points in geometric progression

Jérémy Champagne and Damien Roy

*Dedicated to the memory of Alan Baker*

**Abstract.** We establish a transference inequality conjectured by Badziahin and Bugeaud relating exponents of rational approximation of points in geometric progression.

**Keywords.** exponents of Diophantine approximation, Hausdorff dimension, polynomials, transference inequalities.

**2010 Mathematics Subject Classification.** Primary 11J13; Secondary 11J82, 11J83.

Let  $\xi \in \mathbb{R}$ . For each integer  $n \geq 1$ , one defines  $\omega_n(\xi)$  as the supremum of all  $\omega \in \mathbb{R}$  for which there exist infinitely many non-zero polynomials  $P(x) \in \mathbb{Z}[x]$  of degree at most  $n$  with

$$|P(\xi)| \leq \|P\|^{-\omega},$$

where  $\|P\|$  stands for the largest absolute value of the coefficients of  $P$ . These quantities form the basis of Mahler's classification of numbers [Mah32]. Dually, one defines  $\lambda_n(\xi)$  as the supremum of all  $\lambda \in \mathbb{R}$  for which there exist infinitely many non-zero points  $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$  such that

$$\max_{1 \leq m \leq n} |x_0 \xi^m - x_m| \leq \|\mathbf{x}\|^{-\lambda},$$

where  $\|\mathbf{x}\|$  stands for the maximum norm of  $\mathbf{x}$ .

Suppose now that  $\xi$  is transcendental over  $\mathbb{Q}$ , i.e. that  $\xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}$ . In [BaBu19], D. Badziahin and Y. Bugeaud prove that for any integers  $k, n$  with  $2 \leq k \leq n$ , we have

$$\lambda_n(\xi) \geq \frac{\omega_k^{\text{lead}}(\xi) - n + k}{(k-1)\omega_k^{\text{lead}}(\xi) + n}, \quad (1)$$

where  $\omega_k^{\text{lead}}(\xi)$  is defined as  $\omega_k(\xi)$  but by restricting to polynomials  $P(x) \in \mathbb{Z}[x]$  of degree  $k$  whose coefficient  $c_k(P)$  of  $x^k$  has largest absolute value  $|c_k(P)| = \|P\|$ . They conjecture that this inequality remains true if  $\omega_k^{\text{lead}}(\xi)$  is replaced by  $\omega_k(\xi)$  and they prove this is indeed the case if  $k = 2$  or  $k = n - 1$ . Their proof for  $k = 2$  is based on the fact that  $\lambda_n(\xi) = \lambda_n(1/\xi)$ . The purpose of this note is to prove this conjecture based on their inequality (1), the invariance of  $\lambda_n$  and  $\omega_k$  by general fractional linear transformations with rational coefficients, and the following observation.

**Theorem.** *Let  $k \geq 1$  be an integer and let  $r_0, \dots, r_k$  be distinct integers. There is an integer  $M \geq 1$  such that, for each  $\xi \in \mathbb{R}$ , there exists at least one index  $i \in \{0, 1, \dots, k\}$ , with  $r_i \neq \xi$ , for which the point  $\xi_i = 1/(M(\xi - r_i))$  satisfies  $\omega_k^{\text{lead}}(\xi_i) = \omega_k(\xi_i) = \omega_k(\xi)$ .*

*Proof.* We first note that there exist positive constants  $C_1$  and  $C_2$  such that

$$\|P\| \leq C_1 \max\{|P(r_i)|; 0 \leq i \leq k\} \quad \text{and} \quad \max\{\|P(x + r_i)\|; 0 \leq i \leq k\} \leq C_2 \|P\|$$

for any polynomial  $P \in \mathbb{R}[x]$  of degree at most  $k$ . Let  $\xi \in \mathbb{R}$ . Choose an integer  $M$  with  $M \geq C_1 C_2$  and a sequence of polynomials  $(P_j)_{j \geq 1}$  in  $\mathbb{Z}[x]$  of degree at most  $k$  with strictly increasing norms such

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that

$$\lim_{j \rightarrow \infty} -\frac{\log |P_j(\xi)|}{\log \|P_j\|} = \omega_k(\xi). \tag{2}$$

Then, choose  $i \in \{0, 1, \dots, k\}$  such that  $\|P_j\| \leq C_1 |P_j(r_i)|$  for an infinite set  $S$  of positive integers  $j$ , and set

$$Q_j(x) = (Mx)^k P_j\left(\frac{1}{Mx} + r_i\right) \in \mathbb{Z}[x]$$

for each  $j \in S$ . For those values of  $j$ , the absolute value of the coefficient of  $x^k$  in  $Q_j(x)$  is  $|c_k(Q_j)| = M^k |P_j(r_i)|$  while its other coefficients have absolute value at most

$$M^{k-1} \|P_j(x + r_i)\| \leq C_2 M^{k-1} \|P_j\| \leq C_1 C_2 M^{k-1} |P_j(r_i)| \leq M^k |P_j(r_i)|,$$

thus  $|c_k(Q_j)| = \|Q_j\|$ . We also have  $r_i \neq \xi$  and

$$|Q_j(\xi_i)| = (M|\xi_i|)^k |P_j(\xi)|$$

where  $\xi_i = 1/(M(\xi - r_i))$ . As the ratio  $\|Q_j\|/\|P_j\|$  is bounded from above and from below by positive constants, we deduce from (2) that  $-\log |Q_j(\xi_i)|/\log \|Q_j\|$  converges to  $\omega_k(\xi)$  as  $j$  goes to infinity in  $S$ . Altogether, this means that  $\omega_k^{\text{lead}}(\xi_i) \geq \omega_k(\xi)$ . However, it is well known (and easy to prove) that  $\omega_k(\xi) = \omega_k(\xi_i)$  because  $\xi_i$  is the image of  $\xi$  by a linear fractional transformation with rational coefficients. The conclusion follows because  $\omega_k(\xi_i) \geq \omega_k^{\text{lead}}(\xi_i)$  by the very definition of  $\omega_k^{\text{lead}}$ .

Applying the formula (1) with  $\xi$  replaced by  $\xi_i$  and using the fact that  $\lambda_n(\xi_i) = \lambda_n(\xi)$ , we reach the desired inequality.

**Corollary 1.**  $\lambda_n(\xi) \geq \frac{\omega_k(\xi) - n + k}{(k - 1)\omega_k(\xi) + n}$  for any  $\xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}$  and any integers  $2 \leq k \leq n$ .

We thank Yann Bugeaud and Victor Beresnevich for pointing out that our theorem formalizes principles that are implicit in Lemmas 3 and 4 of [Bak76] as well as on pages 25–26 of [Spr69] (for the choice of  $r_i = i$ ). The next corollary is an application to metrical theory that was suggested by Yann Bugeaud.

**Corollary 2.** Let  $k \geq 1$  be an integer and let  $w \in \mathbb{R}$ . The sets  $S = \{\xi \in \mathbb{R}; \omega_k(\xi) \geq w\}$  and  $S^{\text{lead}} = \{\xi \in \mathbb{R}; \omega_k^{\text{lead}}(\xi) \geq w\}$  have the same Hausdorff dimension. This remains true if we replace the large inequalities  $\geq$  by strict inequalities  $>$  in the definitions of  $S$  and  $S^{\text{lead}}$ .

*Proof.* We have  $S^{\text{lead}} \subset S$  and, for an appropriate choice of integers  $r_0, \dots, r_k$  and  $M$ , the theorem gives  $S \subseteq \bigcup_{i=0}^k \tau_i^{-1}(S^{\text{lead}})$  where  $\tau_i(\xi) = 1/(M(\xi - r_i))$  for each  $\xi \in \mathbb{R} \setminus \{r_i\}$  ( $0 \leq i \leq k$ ). The conclusion follows by the invariance of the Hausdorff dimension under fractional linear transformations.

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