



A transference inequality for rational approximation to points in geometric progression

Jérémy Champagne, Damien Roy

► To cite this version:

Jérémy Champagne, Damien Roy. A transference inequality for rational approximation to points in geometric progression. 2019. hal-02307784v2

HAL Id: hal-02307784

<https://hal.science/hal-02307784v2>

Preprint submitted on 24 Oct 2019 (v2), last revised 14 May 2020 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A TRANSFERENCE INEQUALITY FOR RATIONAL APPROXIMATION TO POINTS IN GEOMETRIC PROGRESSION

JÉRÉMY CHAMPAGNE AND DAMIEN ROY

Dedicated to the memory of Alan Baker.

ABSTRACT. We establish a transference inequality conjectured by Badziahin and Bugeaud relating exponents of rational approximation of points in geometric progression.

Let $\xi \in \mathbb{R}$. For each integer $n \geq 1$, one defines $\omega_n(\xi)$ as the supremum of all $\omega \in \mathbb{R}$ for which there exist infinitely many non-zero polynomials $P(x) \in \mathbb{Z}[x]$ of degree at most n with

$$|P(\xi)| \leq \|P\|^{-\omega},$$

where $\|P\|$ stands for the largest absolute value of the coefficients of P . These quantities form the basis of Mahler's classification of numbers [3]. Dually, one defines $\lambda_n(\xi)$ as the supremum of all $\lambda \in \mathbb{R}$ for which there exist infinitely many non-zero points $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$ such that

$$\max_{1 \leq m \leq n} |x_0 \xi^m - x_m| \leq \|\mathbf{x}\|^{-\lambda},$$

where $\|\mathbf{x}\|$ stands for the maximum norm of \mathbf{x} .

Suppose now that ξ is transcendental over \mathbb{Q} , i.e. that $\xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}$. In [1], D. Badziahin and Y. Bugeaud prove that for any integers k, n with $2 \leq k \leq n$, we have

$$(1) \quad \lambda_n(\xi) \geq \frac{\omega_k^{\text{lead}}(\xi) - n + k}{(k-1)\omega_k^{\text{lead}}(\xi) + n},$$

where $\omega_k^{\text{lead}}(\xi)$ is defined as $\omega_k(\xi)$ but by restricting to polynomials $P(x) \in \mathbb{Z}[x]$ of degree k whose coefficient $c_k(P)$ of x^k has largest absolute value $|c_k(P)| = \|P\|$. They conjecture that this inequality remains true if $\omega_k^{\text{lead}}(\xi)$ is replaced by $\omega_k(\xi)$ and they prove this is indeed the case if $k = 2$ or $k = n-1$. Their proof for $k = 2$ is based on the fact that $\lambda_n(\xi) = \lambda_n(1/\xi)$. The purpose of this note is to prove this conjecture based on their inequality (1), the invariance of λ_n and ω_k by general fractional linear transformations with rational coefficients, and the following observation.

Theorem. *Let $k \geq 1$ be an integer and let r_0, \dots, r_k be distinct integers. There is an integer $M \geq 1$ such that, for each $\xi \in \mathbb{R}$, there exists at least one index $i \in \{0, 1, \dots, k\}$, with $r_i \neq \xi$, for which the point $\xi_i = 1/(M(\xi - r_i))$ satisfies $\omega_k^{\text{lead}}(\xi_i) = \omega_k(\xi_i) = \omega_k(\xi)$.*

Proof. We first note that there exist positive constants C_1 and C_2 such that

$$\|P\| \leq C_1 \max\{|P(r_i)|; 0 \leq i \leq k\} \quad \text{and} \quad \max\{\|P(x + r_i)\|; 0 \leq i \leq k\} \leq C_2 \|P\|$$

2010 *Mathematics Subject Classification.* Primary 11J13; Secondary 11J82, 11J83.

Key words and phrases. exponents of Diophantine approximation, Hausdorff dimension, polynomials, transference inequalities.

Work of the authors partially supported by NSERC.

for any polynomial $P \in \mathbb{R}[x]$ of degree at most k . Let $\xi \in \mathbb{R}$. Choose an integer M with $M \geq C_1 C_2$ and a sequence of polynomials $(P_j)_{j \geq 1}$ in $\mathbb{Z}[x]$ of degree at most k with strictly increasing norms such that

$$(2) \quad \lim_{j \rightarrow \infty} -\frac{\log |P_j(\xi)|}{\log \|P_j\|} = \omega_k(\xi).$$

Then, choose $i \in \{0, 1, \dots, k\}$ such that $\|P_j\| \leq C_1 |P_j(r_i)|$ for an infinite set S of positive integers j , and set

$$Q_j(x) = (Mx)^k P_j\left(\frac{1}{Mx} + r_i\right) \in \mathbb{Z}[x]$$

for each $j \in S$. For those values of j , the absolute value of the coefficient of x^k in $Q_j(x)$ is $|c_k(Q_j)| = M^k |P_j(r_i)|$ while its other coefficients have absolute value at most

$$M^{k-1} \|P_j(x + r_i)\| \leq C_2 M^{k-1} \|P_j\| \leq C_1 C_2 M^{k-1} |P_j(r_i)| \leq M^k |P_j(r_i)|,$$

thus $|c_k(Q_j)| = \|Q_j\|$. We also have $r_i \neq \xi$ and

$$|Q_j(\xi_i)| = (M|\xi_i|)^k |P_j(\xi)|$$

where $\xi_i = 1/(M(\xi - r_i))$. As the ratio $\|Q_j\|/\|P_j\|$ is bounded from above and from below by positive constants, we deduce from (2) that $-\log |Q_j(\xi_i)|/\log \|Q_j\|$ converges to $\omega_k(\xi)$ as j goes to infinity in S . Altogether, this means that $\omega_k^{\text{lead}}(\xi_i) \geq \omega_k(\xi)$. However, it is well known (and easy to prove) that $\omega_k(\xi) = \omega_k(\xi_i)$ because ξ_i is the image of ξ by a linear fractional transformation with rational coefficients. The conclusion follows because $\omega_k(\xi_i) \geq \omega_k^{\text{lead}}(\xi_i)$ by the very definition of ω_k^{lead} . \square

Applying the formula (1) with ξ replaced by ξ_i and using the fact that $\lambda_n(\xi_i) = \lambda_n(\xi)$, we reach the desired inequality.

Corollary 1. $\lambda_n(\xi) \geq \frac{\omega_k(\xi) - n + k}{(k-1)\omega_k(\xi) + n}$ for any $\xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}$ and any integers $2 \leq k \leq n$.

We thank Yann Bugeaud and Victor Beresnevich for pointing out that our theorem formalizes principles that are implicit in Lemmas 3 and 4 of [2] as well as on pages 25–26 of [4] (for the choice of $r_i = i$). The next corollary is an application to metrical theory that was suggested by Yann Bugeaud.

Corollary 2. Let $k \geq 1$ be an integer and let $w \in \mathbb{R}$. The sets $S = \{\xi \in \mathbb{R}; \omega_k(\xi) \geq w\}$ and $S^{\text{lead}} = \{\xi \in \mathbb{R}; \omega_k^{\text{lead}}(\xi) \geq w\}$ have the same Hausdorff dimension. This remains true if we replace the large inequalities \geq by strict inequalities $>$ in the definitions of S and S^{lead} .

Proof. We have $S^{\text{lead}} \subset S$ and, for an appropriate choice of integers r_0, \dots, r_k and M , the theorem gives $S \subseteq \bigcup_{i=0}^k \tau_i^{-1}(S^{\text{lead}})$ where $\tau_i(\xi) = 1/(M(\xi - r_i))$ for each $\xi \in \mathbb{R} \setminus \{r_i\}$ ($0 \leq i \leq k$). The conclusion follows by the invariance of the Hausdorff dimension under fractional linear transformations. \square

REFERENCES

- [1] D. Badziahin and Y. Bugeaud, On simultaneous rational approximation to a real number and its integral powers II, preprint, 2019, 17 pages; arXiv:1906.05508 [math.NT].
- [2] R. C. Baker, Sprindžuk's theorem and Hausdorff dimension, *Mathematika* **23** (1976), 184–197.
- [3] K. Mahler, Zur Approximation der Exponentialfunktion und des Logarithmus. Teil I, *J. Reine Angew. Math.* **166** (1932), 118–136.

- [4] V. G. Sprindžuk, *Mahler's problem in metric number theory*, Translations of Mathematical Monographs, Vol. 25, American Mathematical Society, Providence, R.I. 1969, vii+192 pp.

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ D'OTTAWA, 150 LOUIS PASTEUR, OTTAWA, ONTARIO K1N 6N5, CANADA

E-mail address: jcham016@uottawa.ca

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ D'OTTAWA, 150 LOUIS PASTEUR, OTTAWA, ONTARIO K1N 6N5, CANADA

E-mail address: droy@uottawa.ca