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► **To cite this version:**

Jérémy Champagne, Damien Roy. A transference inequality for rational approximation to points in geometric progression. 2019. hal-02307784v2

HAL Id: hal-02307784

<https://hal.science/hal-02307784v2>

Preprint submitted on 24 Oct 2019 (v2), last revised 14 May 2020 (v3)

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A TRANSFERENCE INEQUALITY FOR RATIONAL APPROXIMATION TO POINTS IN GEOMETRIC PROGRESSION

JÉRÉMY CHAMPAGNE AND DAMIEN ROY

Dedicated to the memory of Alan Baker.

ABSTRACT. We establish a transference inequality conjectured by Badziahin and Bugeaud relating exponents of rational approximation of points in geometric progression.

Let $\xi \in \mathbb{R}$. For each integer $n \geq 1$, one defines $\omega_n(\xi)$ as the supremum of all $\omega \in \mathbb{R}$ for which there exist infinitely many non-zero polynomials $P(x) \in \mathbb{Z}[x]$ of degree at most n with

$$|P(\xi)| \leq \|P\|^{-\omega},$$

where $\|P\|$ stands for the largest absolute value of the coefficients of P . These quantities form the basis of Mahler's classification of numbers [3]. Dually, one defines $\lambda_n(\xi)$ as the supremum of all $\lambda \in \mathbb{R}$ for which there exist infinitely many non-zero points $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$ such that

$$\max_{1 \leq m \leq n} |x_0 \xi^m - x_m| \leq \|\mathbf{x}\|^{-\lambda},$$

where $\|\mathbf{x}\|$ stands for the maximum norm of \mathbf{x} .

Suppose now that ξ is transcendental over \mathbb{Q} , i.e. that $\xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}$. In [1], D. Badziahin and Y. Bugeaud prove that for any integers k, n with $2 \leq k \leq n$, we have

$$(1) \quad \lambda_n(\xi) \geq \frac{\omega_k^{\text{lead}}(\xi) - n + k}{(k-1)\omega_k^{\text{lead}}(\xi) + n},$$

where $\omega_k^{\text{lead}}(\xi)$ is defined as $\omega_k(\xi)$ but by restricting to polynomials $P(x) \in \mathbb{Z}[x]$ of degree k whose coefficient $c_k(P)$ of x^k has largest absolute value $|c_k(P)| = \|P\|$. They conjecture that this inequality remains true if $\omega_k^{\text{lead}}(\xi)$ is replaced by $\omega_k(\xi)$ and they prove this is indeed the case if $k = 2$ or $k = n-1$. Their proof for $k = 2$ is based on the fact that $\lambda_n(\xi) = \lambda_n(1/\xi)$. The purpose of this note is to prove this conjecture based on their inequality (1), the invariance of λ_n and ω_k by general fractional linear transformations with rational coefficients, and the following observation.

Theorem. *Let $k \geq 1$ be an integer and let r_0, \dots, r_k be distinct integers. There is an integer $M \geq 1$ such that, for each $\xi \in \mathbb{R}$, there exists at least one index $i \in \{0, 1, \dots, k\}$, with $r_i \neq \xi$, for which the point $\xi_i = 1/(M(\xi - r_i))$ satisfies $\omega_k^{\text{lead}}(\xi_i) = \omega_k(\xi_i) = \omega_k(\xi)$.*

Proof. We first note that there exist positive constants C_1 and C_2 such that

$$\|P\| \leq C_1 \max\{|P(r_i)|; 0 \leq i \leq k\} \quad \text{and} \quad \max\{\|P(x + r_i)\|; 0 \leq i \leq k\} \leq C_2 \|P\|$$

2010 *Mathematics Subject Classification.* Primary 11J13; Secondary 11J82, 11J83.

Key words and phrases. exponents of Diophantine approximation, Hausdorff dimension, polynomials, transference inequalities.

Work of the authors partially supported by NSERC.

for any polynomial $P \in \mathbb{R}[x]$ of degree at most k . Let $\xi \in \mathbb{R}$. Choose an integer M with $M \geq C_1 C_2$ and a sequence of polynomials $(P_j)_{j \geq 1}$ in $\mathbb{Z}[x]$ of degree at most k with strictly increasing norms such that

$$(2) \quad \lim_{j \rightarrow \infty} -\frac{\log |P_j(\xi)|}{\log \|P_j\|} = \omega_k(\xi).$$

Then, choose $i \in \{0, 1, \dots, k\}$ such that $\|P_j\| \leq C_1 |P_j(r_i)|$ for an infinite set S of positive integers j , and set

$$Q_j(x) = (Mx)^k P_j\left(\frac{1}{Mx} + r_i\right) \in \mathbb{Z}[x]$$

for each $j \in S$. For those values of j , the absolute value of the coefficient of x^k in $Q_j(x)$ is $|c_k(Q_j)| = M^k |P_j(r_i)|$ while its other coefficients have absolute value at most

$$M^{k-1} \|P_j(x + r_i)\| \leq C_2 M^{k-1} \|P_j\| \leq C_1 C_2 M^{k-1} |P_j(r_i)| \leq M^k |P_j(r_i)|,$$

thus $|c_k(Q_j)| = \|Q_j\|$. We also have $r_i \neq \xi$ and

$$|Q_j(\xi_i)| = (M|\xi_i|)^k |P_j(\xi)|$$

where $\xi_i = 1/(M(\xi - r_i))$. As the ratio $\|Q_j\|/\|P_j\|$ is bounded from above and from below by positive constants, we deduce from (2) that $-\log |Q_j(\xi_i)|/\log \|Q_j\|$ converges to $\omega_k(\xi)$ as j goes to infinity in S . Altogether, this means that $\omega_k^{\text{lead}}(\xi_i) \geq \omega_k(\xi)$. However, it is well known (and easy to prove) that $\omega_k(\xi) = \omega_k(\xi_i)$ because ξ_i is the image of ξ by a linear fractional transformation with rational coefficients. The conclusion follows because $\omega_k(\xi_i) \geq \omega_k^{\text{lead}}(\xi_i)$ by the very definition of ω_k^{lead} . \square

Applying the formula (1) with ξ replaced by ξ_i and using the fact that $\lambda_n(\xi_i) = \lambda_n(\xi)$, we reach the desired inequality.

Corollary 1. $\lambda_n(\xi) \geq \frac{\omega_k(\xi) - n + k}{(k-1)\omega_k(\xi) + n}$ for any $\xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}$ and any integers $2 \leq k \leq n$.

We thank Yann Bugeaud and Victor Beresnevich for pointing out that our theorem formalizes principles that are implicit in Lemmas 3 and 4 of [2] as well as on pages 25–26 of [4] (for the choice of $r_i = i$). The next corollary is an application to metrical theory that was suggested by Yann Bugeaud.

Corollary 2. Let $k \geq 1$ be an integer and let $w \in \mathbb{R}$. The sets $S = \{\xi \in \mathbb{R}; \omega_k(\xi) \geq w\}$ and $S^{\text{lead}} = \{\xi \in \mathbb{R}; \omega_k^{\text{lead}}(\xi) \geq w\}$ have the same Hausdorff dimension. This remains true if we replace the large inequalities \geq by strict inequalities $>$ in the definitions of S and S^{lead} .

Proof. We have $S^{\text{lead}} \subset S$ and, for an appropriate choice of integers r_0, \dots, r_k and M , the theorem gives $S \subseteq \bigcup_{i=0}^k \tau_i^{-1}(S^{\text{lead}})$ where $\tau_i(\xi) = 1/(M(\xi - r_i))$ for each $\xi \in \mathbb{R} \setminus \{r_i\}$ ($0 \leq i \leq k$). The conclusion follows by the invariance of the Hausdorff dimension under fractional linear transformations. \square

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