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# VECTOR FIELDS AND GENUS IN DIMENSION 3

PIERRE DEHORNOY AND ANA RECHTMAN

ABSTRACT. Given a flow on a 3-dimensional integral homology sphere, we give a formula for the Euler characteristic of its transverse surfaces, in terms of boundary data only. We illustrate the formula with several examples, in particular with surfaces of low genus. As an application, we show that for a right-handed flow with an ergodic invariant measure, the genus is an asymptotic invariant of order 2 proportional to the helicity.

## INTRODUCTION

In this paper, we are interested in topological properties of non-singular vector fields on 3-manifolds. In this context, the notion of **transverse surface** is useful: it is an embedded surface that is everywhere transverse to the vector field. When the surface is closed and intersects all orbits of the induced flow, one speaks of **global cross section**. Up to changing the time-parameter (*i.e.* multiplying the vector field by a strictly positive function), the dynamics of the flow is then described by the dynamics of the first-return map on the global cross section. The description of all global cross sections to a vector field is given by Schwartzman-Fuller-Sullivan-Fried Theory, which is purely homological [Sch57, Ful65, Sul76, Fri82]. Moreover, Thurston and Fried give formulas for computing the genus of global cross sections [Thu86, Fri79].

Unfortunately global cross sections exist in only few cases, to begin with the ambient manifold has to fiber over the circle. In particular they never exist when the underlying manifold is a homology sphere. The notion of **Birkhoff section** is then useful: it is similar to a cross section, but it may have finitely many boundary components, provided these are tangent to the vector field. If the surface does not intersect all the orbits of the flow, one just speaks of **Birkhoff surface**. Schwartzman-Fuller-Sullivan-Fried Theory may be extended to this context, but this has only been partially done [Fri82, Ghy09, Hry19]. What we do here is to push a bit further Fried's and Ghys' ideas. In particular we provide a formula for the genus of transverse surfaces with boundary, using a specific notion of self-linking of orbits.

**Theorem A.** *Assume that  $M$  is a 3-dimensional integral homology sphere and that  $X$  is a non-singular vector field on  $M$ . Let  $\{\gamma_i\}_{1 \leq i \leq m}$  be a finite collection of periodic orbits of  $X$  and  $\{n_i\}_{1 \leq i \leq m}$  a collection of integers. If  $S$  is a transverse surface to  $X$  with oriented boundary  $\cup n_i \gamma_i$ , then the Euler characteristic of  $S$  is given by*

$$\chi(S) = - \sum_{1 \leq i < j \leq m} (n_i + n_j) \text{Lk}(\gamma_i, \gamma_j) - \sum_{1 \leq i \leq m} n_i \text{Slk}^{\zeta_X}(\gamma_i),$$

where  $\zeta_X$  denotes any vector field everywhere transverse to  $X$  and  $\text{Slk}^{\zeta_X}$  the self-linking given by the framing  $\zeta_X$  (see Definition 1.3).

It is likely that Theorem A holds in the case of  $M$  a rational homology sphere. The difference is that the group  $H^2(M, \partial M; \mathbb{Q})$  may have torsion so that  $\zeta_X$  is not uniquely defined. It is also likely that Theorem A may be adapted to an arbitrary 3-manifold  $M$ . In this case, one would have to adapt the definition of linking number which is not anymore well-defined.

The role of linking numbers in this theory was first underlined by Ghys, who introduced the notion of right-handed vector field [Ghy09]: it is a vector field on a homology sphere all of whose invariant positive measures link positively. Under this hypothesis, every periodic orbit bounds a transverse surface  $S$  that is a Birkhoff section for the vector field. This in turn gives a fibration of the complement of the periodic orbit

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over  $\mathbb{S}^1$ , implying that the genus  $S$  is the minimal genus of all the surfaces whose boundary is the periodic orbit (see [Thu86]). The genus of  $S$  is called the **genus** of the periodic orbit, in other words it is the genus of the knot given by the periodic orbit. For right-handed vector fields, we can use Theorem A to prove that the genus of periodic orbits has a prescribed behaviour when the length of the orbits tends to infinity.

**Corollary B.** *Let  $M$  be a 3-manifold that is an integral homology sphere,  $X$  a non-singular right-handed vector field on  $M$  and  $\mu$  a  $X$ -invariant measure. If  $(\gamma_n)_{n \in \mathbb{N}}$  is a sequence of periodic orbits whose lengths  $(t_n)_{n \in \mathbb{N}}$  tend to infinity and such that  $(\gamma_n)_{n \in \mathbb{N}}$  tends to  $\mu$  in the weak-sense, then the sequence  $(\frac{1}{t_n}g(\gamma_n))_{n \in \mathbb{N}}$  tends to half the absolute value of the helicity of  $(X, \mu)$ .*

In Section 1 we give a proof of Theorem A. We then illustrate Theorem A in Section 2 first with the example of the Hopf vector field, where we easily compute the genus of a Hopf link with  $n$  components. We also illustrate it with some specific Anosov flows, in these examples it is relevant to find Birkhoff sections of low genus. Corollary B is proved in Section 3.

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## 1. THE EULER CHARACTERISTIC OF A TRANSVERSE SURFACE

The aim of this section is to give a proof of Theorem A, which is done in the last part 1.e. We first recall some classical results on global cross sections in 1.a and in 1.b we explain how the genus of such a section can be obtained. Sections 1.c and 1.d describe the elements used in the proof of the theorem.

1.a. **Schwartzman-Fuller-Sullivan-Fried Theory.** We recall classical results, but we state them in the more general context of a 3-manifold  $M$  with toric boundary and a non-singular vector field  $X$  tangent to  $\partial M$ . The original proofs extend easily to this case. These assumptions on  $M$  are used in the next parts of this section.

First we recall the definition of **asymptotic cycles**. The original one is in terms of almost-periodic orbits [Sch57]: for  $p \in M$  and  $t > 0$ , we denote by  $k_X(p, t)$  the closed curve obtained by connecting the arc of orbit  $\phi_X^{[0, t]}(p)$  with a segment of bounded length (recall  $M$  is compact). An asymptotic cycle is a positive linear combination of accumulation points of sequences of the form  $(\frac{1}{t_n}[k_X(p_n, t_n)])_{n \in \mathbb{N}}$ , with  $t_n \rightarrow \infty$ .

Alternatively, for  $\mu$  a  $X$ -invariant positive measure, one can consider the 1-current  $c_\mu : \Omega^1(M) \rightarrow \mathbb{R}$  which maps a 1-form  $f$  to  $\int_M f(X(p)) d\mu(p)$ . The invariance of  $\mu$  implies that  $c_\mu$  is closed, hence it determines a 1-cycle  $[c_\mu] \in H_1(M; \mathbb{R})$  in the sense of De Rham. The two notions actually coincide: if  $\mu$  is ergodic,  $c_\mu$  can be obtained as one of the accumulation points in the previous paragraph [Sul76].

One denotes by  $\mathcal{S}_X$  the set of all asymptotic cycles of  $X$ ; it is a convex cone in  $H_1(M; \mathbb{R})$ . Schwartzman's criterion is the following. Recall that Poincaré duality identifies  $H^1(M; \mathbb{R})$  with  $H_2(M, \partial M; \mathbb{R})$ .

**Theorem 1.1** (Schwartzman). *A class  $\sigma \in H^1(M; \mathbb{Z})$  is dual to a global cross section if, and only if, for every asymptotic cycle  $c \in \mathcal{S}_X$  one has  $c(\sigma) > 0$ .*

1.b. **Genus of global cross sections.** In the context of the previous part, a standard argument shows that if two global cross sections to a vector field are homologous, then they are isotopic along the flow [Thu86]. Actually it shows more: a global cross section minimizes the genus in its homology class. So one may wonder how to compute this genus. Thurston and Fried give a satisfying answer [Fri79]. Denote by  $X^\perp$  the normal bundle to  $X$  (it is the 2-dimensional bundle  $TM/\mathbb{R}X$ ), and by  $e(X^\perp) \in H^2(M, \partial M; \mathbb{Z})$  its Euler class.

**Theorem 1.2** (Fried-Thurston). *Assume that  $S$  is a surface transverse to  $X$ . Then one has  $\chi(S) = e(X^\perp)([S])$ .*

The argument is short: since  $S$  is transverse to  $X$ , the restricted bundle  $X^\perp|_S$  is isomorphic to the tangent bundle  $TS$ . In particular one has  $e(X^\perp)([S]) = e(TS)([S]) = \chi(S)$ .

Said differently, if  $\zeta$  is a vector field in generic position with respect to  $X$ , the set  $L_{\zeta, X}$  where  $\zeta$  is tangent to  $X$  is a 1-manifold. In order to consider its homology class, one has to orient  $L_{\zeta, X}$  and to equip it with multiplicities. Here is how one can do it: one takes a small disc  $D$  positively transverse to  $X$  and transverse

to  $L_{\zeta, X}$ . The projection of  $\zeta$  on  $D$  along  $X$  defines a vector field  $\zeta_D$  on  $D$  with a singularity at the center. The index of the singularity may be positive or negative (it cannot be 0 for in this case  $X$  and  $\zeta$  would not be in generic position) and thus the product of its sign with the orientation of  $D$  induces a new orientation on  $D$ . Since  $D$  is transverse to  $L_{\zeta, X}$ , this new orientation induces an orientation of  $L_{\zeta, X}$ . The multiplicity then comes from the absolute value of the index of the singularity. Observe that the multiplicity is locally constant by continuity and hence constant on each connected component of  $L_{\zeta, X}$ . The point is that these choices are independent of  $D$ . If one perturbs  $D$  by keeping it transverse to  $X$  and  $L_{\zeta, X}$ , by continuity and discreteness the multiplicity and orientation do not change. If one perturbs  $D$  by keeping it transverse to  $X$  but change the relative position to respect to  $TL_{\zeta, X}$ , the induced orientation on  $L_{\zeta, X}$  changes, but the index is also changed by its opposite. Hence the product is constant (see Figure 1).

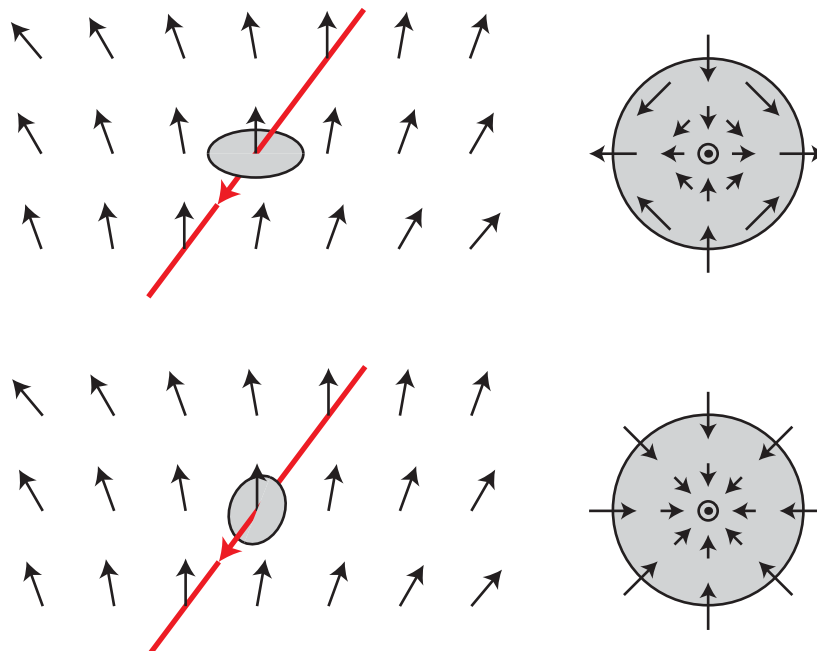


FIGURE 1. Orientation of the dual  $L_{\zeta, X}$  of the Euler class  $e(X^\perp)$ . On this picture the vector field  $X$  is locally vertical. A vector field  $\zeta$  in general position with respect to  $X$  is shown. It is tangent to  $X$  (*i.e.*, vertical) on a dimension 1-submanifold  $L_{\zeta, X} := \{X \parallel \zeta\}$ . We consider an arbitrary disc  $D$  transverse to both  $X$  and  $L_{\zeta, X}$ . Projecting  $\zeta$  on  $D$  along  $X$  we get a vector field  $\zeta_D$  with one singularity at the center. The index of this vector field gives a multiplicity to the disc, and together with the orientation of  $D$  an orientation and a multiplicity to  $L_{\zeta, X}$ . We check that changing  $D$  may change the index by its opposite (bottom), but also the orientation, so that their product is unchanged.

The class  $[L_{\zeta, X}] \in H_1(M)$  is then Poincaré dual of  $e(X^\perp)$ , see [BoT82, Prop. 12.8]. Given a surface  $S$  transverse to  $X$ , projecting  $\zeta$  on  $S$  along  $X$  yields a vector field  $\zeta_S$  on  $S$ , which vanishes exactly when  $S$  intersects  $L_{\zeta, X}$ . The Euler characteristic of  $S$  can be computed with the Poincaré-Hopf formula for  $\zeta_S$ . The crucial point [Thu86, Fri79] is that, thanks to the orientation and multiplicity of  $L_{\zeta, X}$ , each intersection point contributes with the right sign to the sum.

**1.c. Linking and self-linking on homology spheres.** We now assume that  $M$  is a rational homology sphere. Given to links  $L_1, L_2$ , their **linking number**  $\text{Lk}(L_1, L_2)$  is defined as  $\langle L_1, S_2 \rangle$ , where  $S_2$  is a rational 2-chain bounded by  $L_2$ . The existence of  $S_2$  is guaranteed by the vanishing of  $[L_2]$ , whereas the vanishing of  $[L_1]$  implies that the linking number is independent of the choice of  $S_2$ . Linking number is symmetric, although this is not obvious from the definition we just gave.

A **framing**  $f$  of a link  $L$  is a section of its unit normal bundle. It induces an isotopy class  $L^f$  in  $M \setminus L$  obtained by pushing  $L$  off itself in the direction of  $f$ .

**Definition 1.3.** Given a link  $L$  and a framing  $f$ , the **self-linking**  $\text{Slk}^f(L)$  is defined as the linking number of  $L$  and  $L^f$ .

Note that under our assumption that  $M$  is a homology sphere, every individual curve has a preferred framing corresponding to a self-linking number zero. By definition, this **zero-framing** is induced by any surface bounded by the considered curve. Therefore one can see the self-linking number with respect to a framing  $f$  as the algebraic intersection number between the framing and a surface bounded by the considered curve.

1.d. **Boundary slope.** We assume now that  $M$  is an integral homology sphere for this section. Fix a link  $L = K_1 \cup \dots \cup K_m$  in  $M$ . The boundary operator  $\partial$  realizes an isomorphism  $\text{H}_2(M \setminus L, L; \mathbb{R}) \simeq \text{H}_1(L; \mathbb{R})$ , as can be seen by writing the long exact sequence. Therefore a class in  $\text{H}_2(M \setminus L, L; \mathbb{R})$  is determined by its boundary. For  $\sigma$  in  $\text{H}_2(M \setminus L, L; \mathbb{R})$ , denote by  $n_i(\sigma)$  its **longitudinal boundary coordinates**, that is, the real numbers such that  $\partial\sigma = \sum n_i(\sigma)[K_i]$ .

In this context, we denote by  $M_L$  the **normal compactification** of  $M \setminus L$ , that is, the manifold obtained from  $M$  by replacing every point of  $L$  by the circle of those half-planes bounded by  $TL$ . The boundary  $\partial M_L$  is then isomorphic to  $L \times \mathbb{S}^1$ : it is a disjoint union of tori. The manifold  $M_L$  is actually isomorphic to  $M \setminus \nu(L)$ , where  $\nu(L)$  is an open tubular neighbourhood of  $L$ .

By the excision theorem, there is an isomorphism  $\text{H}_2(M \setminus L, L; \mathbb{R}) \simeq \text{H}_2(M_L, \partial M_L; \mathbb{R})$ , so that we can also see the class  $\sigma$  as an element of  $\text{H}_2(M_L, \partial M_L; \mathbb{R})$ . There, its boundary is an element of  $\text{H}_1(\partial M_L; \mathbb{R})$ , whose dimension is higher than the dimension of  $\text{H}_1(L; \mathbb{R})$ . For distinction we then write  $\partial^\bullet : \text{H}_2(M \setminus L, L; \mathbb{R}) \rightarrow \text{H}_1(L; \mathbb{R})$  and  $\partial^\circ : \text{H}_2(M_L, \partial M_L; \mathbb{R}) \rightarrow \text{H}_1(\partial M_L; \mathbb{R})$  for the two operators (see Figure 2). As we said the first one is an isomorphism, while the second one is only injective.

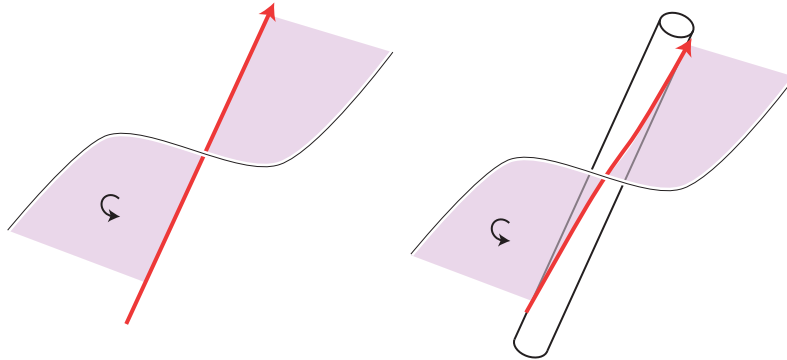


FIGURE 2. On the left, a link  $L$  (red) in a 3-manifold  $M$  and a surface  $S$  (purple) representing a class  $\sigma$  with  $\partial^\bullet\sigma = L$ . On the right, the corresponding manifold  $M_L$  with boundary  $L \times \mathbb{S}^1$  and the corresponding surface  $S$  whose boundary  $\partial^\circ S$  sits in  $L \times \mathbb{S}^1$ . The additional information given by the meridian coordinate of  $\partial^\circ S$  tells how many times  $S$  wraps around  $L$ .

Let  $S$  be a surface representing the class  $\sigma$  above. In order to understand the image of  $\partial^\circ$ , we have to understand the framing induced by  $S$  along every boundary component (that is, the slope of  $S \cap (K_i \times \mathbb{S}^1)$  for every component  $K_i$  of  $L$ ).

**Lemma 1.4.** *If  $M$  is an integral homology sphere and  $S$  is a surface with  $\partial^\bullet S = \sum_i n_i K_i$ , then the coordinates of  $\partial^\circ S$  along  $K_i$  in the (meridian, 0-longitude)-basis are  $\left( -\sum_{j \neq i} n_j \text{Lk}(K_i, K_j), n_i \right)$ .*

*Proof.* Since  $M$  is a homology sphere, all surfaces with the same boundary induce the same framing on the boundary. A surface realizing  $[S]$  is obtained by desingularizing the union  $\cup_i n_i(\sigma)S_i$  where  $S_i$  is a surface in  $M$  with boundary  $K_i$ . Observe that the intersection of  $S_i$  and  $S_j$ , for  $i \neq j$ , can be made either empty or transverse. If the intersection is non-empty, the desingularization is obtained by removing one meridian to  $K_i$

everytime  $S_j$  intersects  $K_i$ . So the total meridian contribution of all surfaces on  $K_i$  is  $-\sum_{j \neq i} n_j \text{Lk}(K_i, K_j)$ . On the other hand the longitudinal coordinate is unchanged in this desingularisation process, thus it is  $n_i$ , hence the result.  $\square$

1.e. **The Euler class of  $X_\Gamma^\perp$ .** Assume that  $M$  is an integral homology sphere with empty boundary. We are given a non-singular vector field  $X$  on  $M$ , a finite collection  $\Gamma = \gamma_1 \cup \dots \cup \gamma_m$  of periodic orbits of  $X$  and multiplicities  $n_1, \dots, n_m$  which are integers. In this context the existence of a Birkhoff section for  $X$  bounded by  $\cup_{i=1}^m n_i \gamma_i$  is the same as the existence of a cross section  $(S, \partial S)$  for the extension of  $X$  to the manifold  $M_\Gamma$  such that the longitudinal coordinates of  $\partial^\circ S$  are  $(n_1, \dots, n_m)$ .

Denote by  $X_\Gamma$  the extension of  $X$  to  $M_\Gamma$ . In order to understand the genus of the cross section, one wonders whether the class  $e(X_\Gamma^\perp)$  may be easily represented.

Since the Euler class of  $X^\perp$  vanishes, there exists non-singular vector fields on  $M$  everywhere transverse to  $X$ . Since  $M$  is an integral homology sphere, two such vector fields are homotopic through vector fields that are everywhere transverse to  $X$ . Indeed the first vector field gives an origin to the normal sphere bundle, so that the second vector field yields a function on the circle. Since  $M$  is a homology sphere, this function is homotopic to a constant function.

Denote by  $\zeta_X$  a vector field transverse to  $X$ . As in 1.b, we use  $\zeta_X$  to realise the Euler class  $e(X_\Gamma^\perp)$ . Indeed, the Euler class of a vector bundle is represented by the intersection with a generic section (see [BoT82, Prop. 12.8]). Here one has to take into account the fact that  $M_\Gamma$  has boundary.

*Proof of Theorem A.* The vector field  $\zeta_X$  is transverse to  $X$  on  $M$  but not tangent to  $\partial M_\Gamma$ . In order to make it tangent to the boundary of the manifold, we have to “rotate”  $\zeta_X$  towards  $X$  around each component  $\gamma_i$ . This can be achieved by combining, near  $\gamma_i$ , the two vector fields  $\zeta_X$  and  $X_\Gamma$ .

We obtain a vector field  $\zeta_{X,\Gamma}$  which is transverse to  $X_\Gamma$  on  $M_\Gamma$ , except at the boundary components, where it is tangent to  $X_\Gamma$  along two curves that correspond to the framings given by  $\zeta_X$  and  $-\zeta_X$  (see Figure 3). In particular the set  $L_{\zeta_X,\Gamma} := \{p \in M_\Gamma \mid X_\Gamma(p) \parallel \zeta_{X,\Gamma}(p)\}$  is a collection of two longitudes  $\gamma_i^{\text{in}}, \gamma_i^{\text{out}}$  for every boundary component  $\gamma_i \times \mathbb{S}^1$  of  $\partial M_\Gamma$ , given by the framing induced by  $\zeta_X$ .

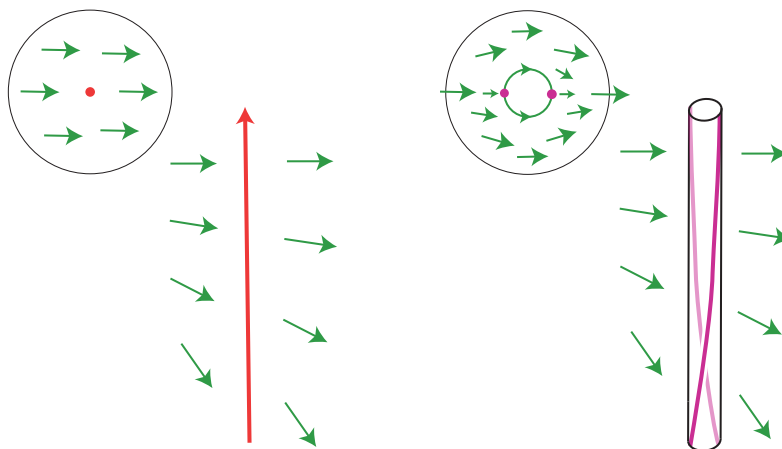


FIGURE 3. On the left, the vector field  $\zeta_X$  (green) on  $M$ . Since it is transverse to  $X$ , it is transverse to a link  $\Gamma$  (red) made of periodic orbits of  $X$ . Seen from above,  $\Gamma$  is a point and  $\zeta_X$  is a non-vanishing vector field. On the right, the modification of  $\zeta_X$  into  $\zeta_{X,\Gamma}$ . Seen from above, one has to slow down  $\zeta_X$  so that it has (transversal) speed 0 on  $\Gamma$ . The set  $L_{\zeta_X,\Gamma}$  (pink) then consists of two longitudes per component of  $\Gamma$ .

Orienting  $L_{\zeta_X,\Gamma}$  so that its class is dual to  $e(X_\Gamma^\perp)$  can be done as in 1.b. Considering a component  $\gamma_i$  of  $\Gamma$ , one has to take a small disc  $D$  in  $M$  transverse to  $\gamma_i$ . The corresponding annulus  $D_\Gamma$  in  $M_\Gamma$  is automatically transverse to  $L_{\zeta_X,\Gamma}$  which is a collection of longitudes. The projection of  $\zeta_{X,\Gamma}$  on  $D_\Gamma$  exhibits

two singularities of index  $-\frac{1}{2}$  (see the right-hand disc in Figure 3). Therefore, if  $\gamma_i^{\text{in}}$  and  $\gamma_i^{\text{out}}$  are oriented in the direction opposite to  $X$ , they both have multiplicity  $\frac{1}{2}$ .

Let  $\sigma \in H_2(M_\Gamma, \partial M_\Gamma; \mathbb{Z})$  be the class of the transverse surface  $S$  in Theorem A, then  $e(X_\Gamma^\perp)(\sigma) = \langle L_{\zeta_X, \Gamma}, \sigma \rangle$ . This intersection equals

$$\sum_{i=1}^m \langle \gamma_i^{\text{in}} + \gamma_i^{\text{out}}, (\partial^\circ \sigma)_i \rangle,$$

where  $(\partial^\circ \sigma)_i$  denotes the part of  $\partial^\circ \sigma$  on the component  $\gamma_i \times \mathbb{S}^1$  of  $\partial M_\Gamma$ .

Now the algebraic intersection of two curves on a 2-torus is the determinant of their coordinates in homology. Since  $\gamma_i^{\text{in}}$  and  $\gamma_i^{\text{out}}$  both have coordinates  $(-\text{Slk}^{\zeta_X}(\gamma_i), -1)$  in the (meridian, 0-longitude)-basis of  $\gamma_i \times \mathbb{S}^1$ , and since every intersection point contributes to  $\frac{1}{2}$  to the intersection, thanks to Lemma 1.4, we have

$$\langle \gamma_i^{\text{in}} + \gamma_i^{\text{out}}, (\partial^\circ \sigma)_i \rangle = -n_i \text{Slk}^{\zeta_X}(\gamma_i) - \sum_{j \neq i} n_j \text{Lk}(K_i, K_j).$$

Summing over all boundary components, we get the desired formula.  $\square$

## 2. EXAMPLES

**2.a. Birkhoff sections for the Hopf vector field.** Let  $X_{\text{Hopf}}$  denote the Hopf vector field on  $\mathbb{S}^3$  and  $\phi_{\text{Hopf}}$  denote the associated flow. Every orbit  $\gamma$  of  $\phi_{\text{Hopf}}$  is periodic and bounds a disc  $D_\gamma$  transverse to  $X_{\text{Hopf}}$ . Since any two orbits have linking number  $+1$ , the disc  $D_\gamma$  is a Birkhoff section for  $\phi_{\text{Hopf}}$ . More generally, if  $\cup n_i \gamma_i$  and  $\cup n'_i \gamma'_i$  are two collections of disjoint periodic orbits with multiplicities, their linking number is given by  $(\sum n_i)(\sum n'_i)$ .

On the other hand, a vector field transverse to  $X_{\text{Hopf}}$  can be easily found by taking another Hopf fibration  $\zeta_{\text{Hopf}}$  orthogonal to  $X_{\text{Hopf}}$ . One checks that for every periodic orbit  $\gamma$  of  $\phi_{\text{Hopf}}$ , one has  $\text{Slk}^{\zeta_{\text{Hopf}}}(\gamma) = -1$ .

For a Birkhoff section that is a disc  $D_\gamma$  with one boundary component  $\gamma$ , Theorem A then yields

$$\chi(D_\gamma) = -(-1) = +1,$$

as expected.

Consider now a collection  $\gamma_1, \dots, \gamma_m$  of  $m$  periodic orbits. The link  $\Gamma := \sum_i \gamma_i$  links positively with any positive invariant measure. Hence it bounds a Birkhoff section, denoted by  $S_\Gamma$ . This section can be obtained from the union  $D_{\gamma_1} \cup \dots \cup D_{\gamma_m}$  by desingularizing along the segments where two discs intersect (these intersections can be assumed transverse). The Euler characteristic of the resulting surface can be computed by hand, but Theorem A directly yields

$$\chi(S_\Gamma) = -m(m-1) - (-m) = -m(m-2).$$

Since  $S_\Gamma$  has  $m$  boundary components, we obtain

$$g(S_\Gamma) = 1 - \frac{\chi(S_\Gamma) + m}{2} = 1 + \frac{m(m-3)}{2}$$

which is the genus of a Hopf link with  $m$  components. One can generalize a bit more by considering a collection  $\cup_i n_i \gamma_i$ , where the  $n_i$  are integers. The condition for bounding a Birkhoff section then becomes  $\sum_i n_i > 0$ . Denoting by  $S_{\cup_i n_i \gamma_i}$  such a Birkhoff section, Theorem A yields

$$\chi(S_{\cup_i n_i \gamma_i}) = - \sum_{1 \leq i < j \leq m} (n_i + n_j) + \sum_{1 \leq i \leq m} n_i = \sum_{1 \leq i \leq m} (1-m)n_i + \sum_{1 \leq i \leq m} n_i = (2-m) \sum_{1 \leq i \leq m} n_i.$$

**2.b. Birkhoff sections for the geodesic flow on a triangular orbifold.** Consider a hyperbolic triangular orbifold  $\mathcal{O}_{p,q,r}$ : the quotient of  $\mathbb{H}^2$  by the group  $G_{p,q,r}$  of orientation-preserving isometries that preserve a tiling of  $\mathbb{H}^2$  by triangles of angles  $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$  (it is then necessary that  $p, q, r$  are positive integers with  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ ).

Consider the 3-manifold  $T^1 \mathcal{O}_{p,q,r} = T^1 \mathbb{H}^2 / G_{p,q,r}$ : it is a Seifert-fibered manifold whose singular fibers correspond to the fibers of the vertices of the triangle. The geodesic flow on  $T^1 \mathbb{H}^2$  is simply defined by

lifting the geodesics of  $\mathbb{H}^2$ , it descends to a well-defined flow on  $T^1\mathcal{O}_{p,q,r}$ , which also lifts the geodesics of  $\mathcal{O}_{p,q,r}$ .

The manifold  $T^1\mathcal{O}_{p,q,r}$  is a rational homology sphere in general, and an integral homology sphere for  $(p, q, r) = (2, 3, 7)$ . It is known that the geodesic flow  $\phi_{\text{geod}}$  on  $T^1\mathcal{O}_{p,q,r}$  is left-handed [Deh16], meaning that all linking numbers between distinct invariant measures are negative. This implies that every link  $\Gamma = \cup_i n_i \gamma_i$  formed of periodic orbits  $\gamma_i$ 's with negative multiplicities  $n_i$ 's bounds a Birkhoff section for  $\phi_{\text{geod}}$ .

Also  $\phi_{\text{geod}}$  is an Anosov flow, so that a vector field  $\zeta$  transverse to  $X_{\text{geod}}$  is easy to construct: one can take a vector field tangent to the stable direction at every point.

Considering a link  $\Gamma = \cup_i n_i \gamma_i$  made of periodic orbits with negative multiplicities, Theorem A gives a way to compute the Euler characteristic of the Birkhoff section bounded by  $\Gamma$ . Using the tools developed in [Deh16], one can check that the only Birkhoff sections with one boundary component and of Euler characteristic  $-1$  are those exhibited in [Deh15]. One can interpret this observation by saying that these flows have one preferred Birkhoff section.

On the other hand, there is no need to restrict to links whose components have only negative multiplicities. Considering a link of the form  $\Gamma = \cup_i n_i \gamma_i$ , where the  $n_i$ 's are integers,  $\Gamma$  bounds a Birkhoff section if and only if it links positively with any invariant measure. Theorem A then yields the Euler characteristic. Numerical computations show that there are infinitely many genus one Birkhoff sections. This confirms the main result of [DeS19] which implies that for every hyperbolic automorphism  $A$  of  $\mathbb{T}^2$  and every hyperbolic triangular orbifold  $\mathcal{O}_{p,q,r}$ , there is a Birkhoff section for the geodesic flow on  $\mathcal{O}_{p,q,r}$  whose induced first-return is given by  $A$ .

### 3. GENUS AND THE RUELLE INVARIANT

We now turn to Corollary B.

In the context of  $M$  a homology sphere and  $X$  a non-singular vector field on  $M$ , for every periodic orbit  $\gamma$  of  $X$ , we have presented two preferred framings, namely the zero-framing determined by a spanning surface, and the framing given by a vector field  $\zeta_X$  everywhere transverse to  $X$ . By definition, the difference of these two framings is  $\pm \text{Slk}^{\zeta_X}(\gamma)$ .

If  $\gamma$  is the unique boundary component of a Birkhoff section, Theorem A says that the genus of  $\gamma$  (as a knot in  $M$ ) is  $1 - (\text{Slk}^{\zeta_X}(\gamma) - 1)/2$ . One wonders whether this quantity has an asymptotic behaviour when  $\gamma$  tends to fill  $M$ . Two related quantities are known to have one, and we present them now. Both rely on a third framing on  $\gamma$ , given by the differential of the flow.

**3.a. Three framings on  $\gamma \times \mathbb{S}^1$ .** Recall that  $M_\gamma$  is obtained from  $M$  by replacing every point of  $\gamma$  by its sphere normal bundle  $S(TM/\mathbb{R}X)$  which is topologically a circle. If  $X$  is at least  $C^1$ , we can then extend  $X$  to  $\gamma \times \mathbb{S}^1$  using the differential of the flow, and obtain a non-singular vector field  $X_\gamma$  on  $M_\gamma$ . Now we have three framings on  $\gamma \times \mathbb{S}^1$ , two integral ones (the zero-framing and the one induced by  $\zeta_X$ ) and one real (induced by  $DX$ ).

The restriction of  $X_\gamma$  to  $\partial M_\gamma$  is a vector field on a torus whose first coordinate (in the  $X$ -direction) can be made constant. Hence it has a well-defined translation number: the **Ruelle invariant**  $R^X(\gamma)$  is defined as the translation number of  $X_\gamma|_{\gamma \times \mathbb{S}^1}$  with respect to the framing  $\zeta_X$  [Rue85, GaG97]. On the other hand the rotation number of  $X_\gamma|_{\gamma \times \mathbb{S}^1}$  with respect to the zero-framing is given by  $\text{Slk}^{DX}(\gamma)$ . Both these numbers are real (and not necessarily integers).

Since the quantities  $R^X(\gamma)$ ,  $\text{Slk}^{\zeta_X}(\gamma)$ , and  $\text{Slk}^{DX}(\gamma)$  denote the respective difference between the three possible pairs of framings, we have

$$(1) \quad \text{Slk}^{\zeta_X}(\gamma) = \text{Slk}^{DX}(\gamma) - R^X(\gamma),$$

where only the term  $\text{Slk}^{\zeta_X}(\gamma)$  is always an integer.

The Ruelle invariant may be extended to any  $X$ -invariant measure using long arcs of orbits, but we do not need this here (see [GaG97]).



**3.b. Asymptotic genus for right-handed vector fields.** Assume that  $X$  is now a right-handed vector field on an integral homology sphere  $M$ , meaning that all  $X$ -invariant positive measures have positive linking number [Ghy09]. Then Ghys proved that every periodic orbit bounds a Birkhoff section. It is also known that such a section is genus-minimizing (this is a folklore result among 3-dimensional topologists, a possible reference is [Thu86] although the statement is older). Therefore the genus of a periodic orbit  $\gamma$  is given by  $1 - (\text{Slk}^{\zeta^X}(\gamma) - 1)/2$ .

*Proof of Corollary B.* Arnold and Vogel proved that if  $(\gamma_n)$  is a sequence a periodic orbits that tend to an invariant volume  $\mu$  in the weak sense, then writing  $t_n$  for the period of  $\gamma_n$ , the sequence  $\frac{1}{t_n^2} \text{Slk}^{DX}(\gamma_n)$  tends to the helicity  $\text{Hel}(X, \mu)$  [Arn73, Vog02]. Similarly, Gambaudo and Ghys proved that the sequence  $\frac{1}{t_n} R^X(\gamma_n)$  tends to the Ruelle invariant  $R^X(\mu)$  [GaG97].

Since one terms grows quadratically and the other one linearly on  $t_n$ , in the right-hand side of Equation (1), the term  $R^X(\gamma_n)$  is negligible, and the asymptotic is dictated by  $\text{Slk}^{DX}(\gamma_n)$ . In particular we have

$$\frac{1}{t_n^2} |\chi_{\min}(\gamma_n)| = \frac{1}{t_n^2} |\text{Slk}^{\zeta^X}(\gamma_n)| \rightarrow |\text{Hel}(X, \mu)|.$$

Then

$$\lim_{t_n \rightarrow \infty} \frac{1}{t_n^2} g(\gamma_n) = \lim_{t_n \rightarrow \infty} \frac{1}{t_n^2} \left( 1 - \frac{\text{Slk}^{\zeta^X}(\gamma_n) - 1}{2} \right) = \lim_{t_n \rightarrow \infty} \frac{|\text{Slk}^{\zeta^X}(\gamma_n)|}{2t_n^2} = -\frac{1}{2} |\text{Hel}(X, \mu)|.$$

□

In other words, the genus is an asymptotic invariant of order 2 for right-handed volume-preserving vector fields, and its asymptotic is half the helicity.

Remark that Baader proved that the slice genus (for arbitrary vector fields, not only right-handed) is an asymptotic invariant of order 2, and that it is also equal to half the helicity [Baa11]. So for right-handed vector fields, the long periodic orbits tend to have genus and slice genus of the same order.

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