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RATIONAL PARALLELISM ON COMPLEX MANIFOLDS

INDRANIL BISWAS AND SORIN DUMITRESCU

ABSTRACT. A Theorem of Wang in [Wa] implies that any holomorphic parallelism on a compact complex manifold M is flat with respect to some complex Lie algebra structure whose dimension coincides with that of M . We study here rational parallelisms on complex manifolds. We exhibit rational parallelisms on compact complex manifolds which are not flat with respect to any complex Lie algebra structure.

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1. INTRODUCTION

A well-known result of Wang [Wa] classifies compact complex manifolds M with holomorphically trivial holomorphic tangent bundle TM .

Recall that a holomorphic trivialization of TM is defined by a holomorphic one-form $\omega \in \Omega^1(M, V)$ with values in a complex vector space V with $\dim V = \dim_{\mathbb{C}} M$ such that $\omega_m : T_m M \rightarrow V$ is a linear isomorphism at every $m \in M$. Hence ω -constant vector fields define a parallelism on M and trivialize TM . A compact complex manifold with trivial tangent bundle is called (holomorphically) *parallelizable*.

In this context Wang proved that there exists a complex Lie algebra structure \mathfrak{L} on V such that ω realizes a Lie algebra isomorphism between TM and \mathfrak{L} (see Theorem 2.2 here). Notice that this is equivalent to the assertion that ω , seen as Cartan geometry modeled on \mathfrak{L} , is *flat* ([Sh], Chapter 3 and Chapter 5), meaning that ω satisfies the Maurer-Cartan equation with respect to the Lie algebra structure \mathfrak{L} . In view of Darboux-Cartan Theorem, [Sh, Chapter 3, p. 116], this implies that M inherits a (L, L) -structure in the sense of Ehresmann–Thurston [Eh] with L being a complex connected Lie group with Lie algebra \mathfrak{L} . The compactness assumption on M ensures that the (L, L) -structure is complete and M is biholomorphic to

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a quotient of L by a lattice in it. This result was extended by Winkelmann to certain open complex manifolds [Wi1, Wi2].

We study here rational parallelisms on compact complex manifolds given by holomorphic one-forms $\omega \in \Omega^1(M, V)$ such that $\omega_m : T_m M \rightarrow V$ is a linear isomorphism for m lying in an open dense subset $U = M \setminus S$ (the subset S is an analytic divisor, see Proposition 3.3). In this case, ω -constant vector fields are actually meromorphic on M with poles lying in S and they holomorphically trivialize TU . This definition corresponds to the particular case of a *branched Cartan geometry*, whose model is a Lie group, as introduced and studied by the authors in [BD]. We prove here (see Theorem 3.4) that on compact complex manifolds M which are either in the Fujiki class \mathcal{C} (e.g. M is the meromorphic image of a Kähler manifold [Fu]), or of complex dimension two or of algebraic dimension zero (e.g. all meromorphic functions on M are constant), all such rational parallelisms are flat with respect to some complex Lie algebra structure on V (meaning ω is a Lie algebra isomorphism). For such a situation we deduce that the fundamental group of M is infinite.

Flat rational parallelisms were studied in particular in [BC].

The main result proved here (Theorem 3.5) exhibits examples of rational parallelisms on compact complex manifolds which are non-flat with respect to any complex Lie algebra structure.

2. HOLOMORPHIC PARALLELISMS

In this section we recall the proof of Wang's classification Theorem of (holomorphically) parallelizable manifolds. The idea of this proof will be useful later on in the study of flatness for rational parallelisms.

Let us first recall the following:

Definition 2.1. A holomorphic trivialization (parallelization) of the holomorphic tangent bundle TM of a compact complex manifold M of (complex) dimension m is a holomorphic one-form $\omega \in \Omega^1(M, V)$ with values in a complex vector space V of dimension m such that $\omega_m : T_m M \rightarrow V$ is a linear isomorphism at every $m \in M$.

Theorem 2.2 (Wang, [Wa]). *Let M be a compact complex manifold of dimension m and*

$$\omega : TM \rightarrow V$$

a holomorphic trivialization of its holomorphic tangent bundle. Then the universal cover of M is biholomorphic to a complex Lie group L , and the pull-back of ω on L coincides with the Maurer-Cartan form of L . Consequently, $d\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{L}} = 0$, where \mathfrak{L} is the Lie algebra of L . The manifold M is biholomorphic to a quotient of L by a lattice in it.

Moreover, M is Kähler if and only if L is abelian. In this case M is a compact complex torus.

Proof. Consider a basis (e_1, \dots, e_m) of the complex vector space V , and let X_1, X_2, \dots, X_m be global holomorphic ω -constant vector fields on M such that $\omega(X_i) = e_i$ for all i . Consequently, the vector fields X_1, \dots, X_m span TM .

Notice that for all $1 \leq i, j \leq m$, we have

$$[X_i, X_j] = f_1^{ij} X_1 + f_2^{ij} X_2 + \dots + f_m^{ij} X_m$$

with f_k^{ij} being holomorphic functions on M . Since M is compact, these (holomorphic) functions are constant and, consequently, X_1, X_2, \dots, X_m generate a m -dimensional complex Lie algebra \mathfrak{L} . When V is endowed with the Lie algebra structure of \mathfrak{L} , the form ω produces a Lie algebra isomorphism.

By Lie's theorem, there exists a unique connected simply connected complex Lie group L corresponding to \mathfrak{L} . The holomorphic parallelization of M by ω -constant holomorphic vector fields is locally isomorphic to the parallelization given by the left translation-invariant vector fields on the Lie group L .

Since M is compact, all vector fields X_i are complete and they define a holomorphic locally free transitive action of L on M (with discrete kernel). Hence M is biholomorphic to a quotient of L by a cocompact discrete subgroup Γ in L .

The Lie-Cartan formula

$$d\omega(X_i, X_j) = X_i \cdot \omega(X_j) - X_j \cdot \omega(X_i) - \omega([X_i, X_j]) = -\omega([X_i, X_j]) = -[\omega(X_i), \omega(X_j)]$$

shows that ω satisfies the Maurer-Cartan equation of the Lie group L , which can also be expressed more formally as $d\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{L}} = 0$.

Assume now that M is Kähler. Then, any holomorphic form on M is closed. The Maurer-Cartan formula shows that the one-forms composing the isomorphism ω are all closed if and only if L is abelian and thus M is a compact complex torus, which is the quotient of a complex vector space by a lattice. \square

With the terminology of Cartan geometries [Sh], ω defines a flat Cartan geometry with respect to the Lie algebra structure \mathfrak{L} on V . Indeed, the vanishing of Cartan's curvature is equivalent to the fact that ω satisfies the Maurer-Cartan equation $d\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{L}} = 0$.

3. RATIONAL PARALLELISMS

We study here rational parallelisms defined by a *branched Cartan geometry* modeled on a Lie group in the sense of [BD]. Let us begin with the following:

Definition 3.1. A branched holomorphic co-parallelism on the holomorphic tangent bundle TM of a compact complex manifold M of (complex) dimension m is a holomorphic one-form $\omega \in \Omega^1(M, V)$ with values in a complex vector space V of dimension m such that $\omega_m : T_m M \rightarrow V$ is a linear isomorphism on an open dense subset U in M (which is necessarily the complement of a divisor in M , see Proposition 3.3).

The branched holomorphic co-parallelism is *flat* in the sense of branched Cartan geometries [BD], with respect to some Lie algebra structure \mathfrak{L} on V , if and only if ω satisfies the Maurer-Cartan equation $d\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{L}} = 0$.

Notice that a basis (e_1, \dots, e_m) of V uniquely defines a family (X_1, \dots, X_m) of meromorphic vector fields on M such that $\omega(X_i) = e_i$. This family of meromorphic vector fields

X_i holomorphically span TM at the generic point in M : they form a *rational parallelism* of TM .

Moreover, the branched holomorphic co-parallelism is flat with respect to the Lie algebra structure \mathfrak{L} , if and only if $[X_i, X_j] = [e_i, e_j]_{\mathfrak{L}}$, for all i, j .

Remark 3.2. The pull-back of a (flat) holomorphic parallelism through a holomorphic map whose differential is invertible at the generic point gives rise to a (flat) branched holomorphic co-parallelism. Consequently, a blow-up (or a ramified cover) of a parallelizable manifold is endowed with a flat branched holomorphic co-parallelism.

The subset $M \setminus U$ where ω fails to be an isomorphism is called the *branching locus* of the co-parallelism.

Let us first prove:

Proposition 3.3. *The branching locus of a branched co-parallelism ω is either empty, or it is an effective divisor in M representing the canonical class. Consequently, if the canonical class of M is trivial (or more generally, does not have an effective representative), any branched holomorphic co-parallelism on TM has empty branching locus and therefore M is a parallelizable manifold.*

Proof. Choose a basis (e_1, \dots, e_m) of V over \mathbb{C} and consider the corresponding components ω_i of ω in this basis:

$$\omega = (\omega_1, \dots, \omega_m) \in \Omega^1(M, \mathbb{C}^m).$$

The branching locus of the co-parallelism ω coincides with the vanishing set of the m -form $\omega_1 \wedge \dots \wedge \omega_m$, considered as a holomorphic section of the canonical bundle K_M . Consequently, the branching locus of ω is either empty or it coincides with an effective divisor representing the canonical class of M .

If the canonical class of M does not admit an effective representative, then the co-parallelism ω has empty branching locus. It follows that M is parallelizable manifold, and a quotient of a complex Lie group L by a lattice in it (see Theorem 2.2). \square

Let us now prove:

Theorem 3.4. *Let M be a compact complex manifold endowed with a branched holomorphic co-parallelism $\omega \in \Omega^1(M, V)$.*

(i) *If M is either in the Fujiki class \mathcal{C} or a complex surface, then ω is flat with respect to the abelian Lie algebra structure on V ;*

(ii) *If M is of algebraic dimension zero (so all meromorphic functions on M are constants), then ω is flat with respect to some complex Lie algebra structure \mathfrak{L} on V ;*

(iii) *If ω is flat with respect to some complex Lie algebra structure \mathfrak{L} on V , then the fundamental group of M is infinite.*

Proof. To prove (i), let us first deal with the case where M is in Fujiki class \mathcal{C} (meaning that M is the meromorphic image of a Kähler manifold [Fu]). Fix a basis (e_1, \dots, e_m) of V over

\mathbb{C} and consider the corresponding components ω_i of ω in this basis:

$$\omega = (\omega_1, \dots, \omega_m) \in \Omega^1(M, \mathbb{C}^m).$$

The manifold M being in class \mathcal{C} , by a result of Varouchas, [Va], it must be bimeromorphic to a Kähler manifold. Consequently, as for Kähler manifolds, all holomorphic one-forms on M must be closed. This implies $d\omega_i = 0$ and consequently ω is flat with respect to the abelian Lie algebra \mathbb{C}^m .

Moreover, since M admits non-trivial closed holomorphic one-forms ω_i , the abelianization of the fundamental group of M is infinite. This gives the proof of statement (iii) when \mathfrak{L} is abelian.

The same proof works for any compact complex surface, since any holomorphic one-form on a compact complex surface is closed (see, for example, [Br], p. 644).

Proof of (ii): Let us assume that M is of algebraic dimension zero and is endowed with a branched holomorphic co-parallelism $\omega \in \Omega^1(M, V)$. Fix a basis (e_1, \dots, e_m) of V over \mathbb{C} and consider the corresponding meromorphic vector fields X_i on M defined by $\omega(X_i) = e_i$. They form a rational parallelism on M .

Notice that for all $1 \leq i, j \leq m$, we have

$$[X_i, X_j] = f_1^{ij} X_1 + f_2^{ij} X_2 + \dots + f_m^{ij} X_m$$

with f_k^{ij} being meromorphic functions on M . Since M is of algebraic dimension zero, the functions f_k^{ij} are all constants and consequently X_1, X_2, \dots, X_m generate a m -dimensional complex Lie algebra \mathfrak{L} . When V is endowed with the Lie algebra structure of \mathfrak{L} , the form ω produces a Lie algebra isomorphism. Hence the rational parallelism is flat with respect to the structure of the Lie algebra \mathfrak{L} .

The Darboux-Cartan Theorem ([Sh], Chapter 3, p. 116) implies that the open dense subset $U \subset M$ where ω is an isomorphism, inherits a (L, L) -structure in the sense of Ehresmann-Thurston [Eh], where L is a complex connected Lie group with Lie algebra \mathfrak{L} .

Proof of (iii): To prove by contradiction, assume that the fundamental group of M is finite. Then, up to replacing M by its universal cover endowed with the pullback of ω , we shall assume that M is simply connected. Since M is simply connected we get a holomorphic developing map

$$d : M \longrightarrow L$$

which is a submersion on an open dense subset (outside the branching locus described in Proposition 3.3) [BD]. The manifold M being compact, the image $d(M)$ must be closed (and open) in L , hence $L = d(M)$ is compact. But compact complex Lie groups are abelian. This implies \mathfrak{L} is abelian, and we conclude as before in (i) that the components ω_i of ω are closed holomorphic one-forms. The complex manifold M being simply connected, for each i there exists a holomorphic function h_i on M such that $\omega_i = dh_i$. But holomorphic functions on compact manifolds are constant and, consequently, $\omega_i = dh_i = 0$, for all i : a contradiction. \square

In contrast to the unbranched case (Theorem 2.2) we exhibit the following non-flat examples, which are inspired by a construction of non-closed holomorphic one-forms in [Br] (p. 648).

Theorem 3.5. *There exists a branched holomorphic co-parallelism $\omega \in \Omega^1(P_E, V)$ on some compact (non-Kähler) principal elliptic bundle P_E , over the product of two Riemann surfaces S_1 and S_2 of genus $g \geq 2$, such that ω is non-flat with respect to any complex Lie algebra structure \mathcal{L} on V .*

Proof. We shall first construct a holomorphic two-form Ω on the product $S_1 \times S_2$ of two Riemann surfaces such that the periods of Ω belong to a lattice Λ in \mathbb{C} .

Consider the standard elliptic curve $E = \mathbb{C}/\Lambda$, with $\Lambda = \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}$. Let z be the coordinate on \mathbb{C} and dz the associated standard (translation invariant) one-form on E . Notice that the periods of dz form the lattice $\Lambda = \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}$.

Choose two Riemann surfaces S_1 and S_2 (of genus $g \geq 2$) admitting holomorphic ramified covers $f_1 : S_1 \rightarrow E$ and $f_2 : S_2 \rightarrow E$. Let us denote by Ω the holomorphic two-form on the complex surface $S_1 \times S_2$ defined by the pull-back:

$$\pi_1^*(f_1^*dz) \wedge \pi_2^*(f_2^*dz),$$

where π_1 and π_2 are the projections of $S_1 \times S_2$ on the first and the second factor respectively. The periods of Ω belong to the lattice Λ in \mathbb{C} .

Fix an open cover $\{U_i\}$ of $S_1 \times S_2$ such that all U_i and all connected components of $U_i \cap U_j$ are contractible. Then on each U_i there exists a holomorphic one-form ω_i such that $\Omega_i = d\omega_i$, where Ω_i denotes the restriction of Ω to U_i . On any intersections $U_i \cap U_j$, we have $\omega_i - \omega_j = dF_{ij}$, where F_{ij} is a holomorphic function defined on $U_i \cap U_j$.

On triple intersections $U_i \cap U_j \cap U_k$, the functions $F_{ij} + F_{jk} + F_{ki}$ form a locally constant two-cocycle which represents the class of Ω in $H^2(S_1 \times S_2, \mathbb{C})$. Hence we can choose the forms ω_i and the associated functions F_{ij} in such a way that $F_{ij} + F_{jk} + F_{ki}$ belongs to the lattice Λ of periods for every triple i, j, k .

Consider then every holomorphic function F_{ij} as taking values in the translations group $E = \mathbb{C}/\Lambda$. In this way we construct an associate one-cocycle with values in E . Let us form the corresponding holomorphic principal elliptic bundle

$$\pi : P_E \rightarrow S_1 \times S_2$$

with E as the structure group.

On each local trivialization $U_i \times E$ of P_E , consider the local one-form $p_1^*(\omega_i) + p_2^*dz$, where p_1 and p_2 are the projections of $U_i \times E$ on the first and the second factor respectively. By construction, these local one-forms glue to a global holomorphic one-form θ on the principal elliptic bundle P_E . Moreover we have $d\theta = \pi^*\Omega$. Since Ω does not vanish, θ is a non-closed holomorphic one-form on P_E , in particular the complex manifold P_E is non-Kähler. In fact θ is a holomorphic connection form on the principal elliptic bundle P_E whose curvature is $\pi^*\Omega$ [At].

Consider a section ω_1 of the canonical bundle K_{S_1} of S_1 such that $\omega_1 = g_1 \cdot f_1^*(dz)$, where g_1 is a non-constant meromorphic function on S_1 . Also consider a holomorphic section ω_2 of the canonical bundle K_{S_2} of S_2 such that $\omega_2 = g_2 \cdot f_2^*(dz)$ with g_2 being a non-constant meromorphic function on S_2 .

Let

$$\omega = (\theta, \pi^* \pi_1^* \omega_1, \pi^* \pi_2^* \omega_2) \in \Omega^1(P_E, \mathbb{C}^3)$$

be the associated branched holomorphic co-parallelism on P_E . Then

$$d\theta = \pi^* \Omega = (g \circ \pi) \cdot \pi^*(\pi_1^*(\omega_1) \wedge \pi_2^*(\omega_2)),$$

with $g = (g_1^{-1} \circ \pi_1) \cdot (g_2^{-1} \circ \pi_2)$ being a non-constant meromorphic function on $S_1 \times S_2$. Since $g \circ \pi$ is a non-constant (meromorphic) function on P_E , this implies that the branched holomorphic parallelism ω is non-flat for any Lie algebra structure on the vector space \mathbb{C}^3 . Moreover, ω is not locally homogeneous on any nonempty open subset in P_E . \square

We have seen in Theorem 3.4 (iii) that compact complex manifolds admitting flat branched holomorphic co-parallelisms have infinite fundamental group. We do not know if there exists compact complex simply connected manifolds bearing branched holomorphic co-parallelism (necessarily non-flat with respect to any Lie algebra structure, as that in Theorem 3.5).

Remark 3.6. The manifold P_E in Theorem 3.5 is a ramified cover of the parallelizable manifold H/Γ , where H is the complex Heisenberg group of upper triangular unipotent (3×3) matrices with complex entries and Γ is the lattice of matrices with Gaussian integers as entries. This quotient H/Γ is biholomorphic to a principal elliptic bundle with fiber $E = \mathbb{C}/\Lambda$ (recall that $\Lambda = \mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}$) over the two-dimensional compact complex torus $E \times E$. The map

$$f = (f_1 \circ \pi_1, f_2 \circ \pi_2) : S_1 \times S_2 \longrightarrow E \times E$$

in the proof of Theorem 3.5 is a ramified cover. The bundle P_E is the pull-back of the elliptic bundle $H/\Gamma \longrightarrow E \times E$ through f . Consequently, P_E is a ramified cover of H/Γ and inherits a branched holomorphic co-parallelism which is flat with respect to the Lie algebra of H (the three dimensional complex Heisenberg algebra) (see Remark 3.2).

We do not know examples of compact complex manifolds admitting branched holomorphic co-parallelisms and which are not ramified covers of parallelizable manifolds. In particular, we do not know if all compact complex manifold admitting branched holomorphic co-parallelisms also admit flat branched holomorphic co-parallelisms (with respect to some Lie algebra structure).

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