

ON THE CHOW GROUPS OF CERTAIN EPW SEXTICS

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ABSTRACT. This note is about the Hilbert square $X = S^{[2]}$, where S is a general K3 surface of degree 10, and the anti-symplectic birational involution ι of X constructed by O'Grady. The main result is that the action of ι on certain pieces of the Chow groups of X is as expected by Bloch's conjecture. Since X is birational to a double EPW sextic X' , this has consequences for the Chow ring of the EPW sextic $Y \subset \mathbb{P}^5$ associated to X' .

1. INTRODUCTION

For a smooth projective variety X over \mathbb{C} , let $A^i(X) := CH^i(X)_{\mathbb{Q}}$ denote the Chow groups (i.e. the groups of codimension i algebraic cycles on X with \mathbb{Q} -coefficients, modulo rational equivalence). Let $A_{hom}^i(X)$ and $A_{AJ}^i(X) \subset A^i(X)$ denote the subgroup of homologically trivial (resp. Abel–Jacobi trivial) cycles. It seems fair to say that Chow groups of codimension $i > 1$ cycles are still poorly understood. To cite but one example, there is Bloch's famous conjecture (which famously is still open for surfaces of general type with geometric genus 0):

Conjecture 1.1 (Bloch [5]). *Let S be a smooth projective surface. Let $\Gamma \in A^2(S \times S)$ be a correspondence such that*

$$\Gamma^* = 0: \quad H^2(X, \mathcal{O}_X) \rightarrow H^2(X, \mathcal{O}_X).$$

Then

$$\Gamma^* = 0: \quad A_{AJ}^2(X) \rightarrow A_{AJ}^2(X).$$

For varieties of higher dimension, versions of conjecture 1.1 can be stated for 0-cycles and for codimension 2-cycles:

Conjecture 1.2. *Let X be a smooth projective variety of dimension n . Let $\Gamma \in A^n(X \times X)$ be a correspondence such that*

$$\Gamma^* = 0: \quad H^n(X, \mathcal{O}_X) \rightarrow H^n(X, \mathcal{O}_X).$$

Then

$$\Gamma^* = 0: \quad F^n A^n(X) \rightarrow A^n(X).$$

Here $F^* A^*(\)$ denotes the conjectural Bloch–Beilinson filtration, expected to exist for all smooth projective varieties [16], [17], [26], [27].

2010 *Mathematics Subject Classification.* Primary 14C15, 14C25, 14C30.

Key words and phrases. Algebraic cycles, Chow groups, motives, hyperkähler varieties, anti-symplectic involution, K3 surfaces, (double) EPW sextics, Beauville's splitting principle, multiplicative Chow–Künneth decomposition, spread of algebraic cycles.

Conjecture 1.3. *Let X be a smooth projective variety of dimension n . Let $\Gamma \in A^n(X \times X)$ be a correspondence such that*

$$\Gamma^* = 0: \quad H^2(X, \mathcal{O}_X) \rightarrow H^2(X, \mathcal{O}_X).$$

Then

$$\Gamma^* = 0: \quad A_{AJ}^2(X) \rightarrow A_{AJ}^2(X).$$

Let us now restrict attention to hyperkähler varieties X (by which we mean: projective irreducible holomorphic symplectic manifolds [2], [1]). For the purposes of this introduction, we will optimistically assume the Chow ring of X has a splitting

$$A^i(X) = \bigoplus_j A_{(j)}^i(X)$$

such that $A_{(*)}^*(X)$ is a bigraded ring, and the piece $A_{(j)}^i(X)$ is isomorphic to the graded $\mathrm{Gr}_F^j A^i(X)$ for the conjectural Bloch–Beilinson filtration mentioned above. (This is expected to be the case for all hyperkähler varieties [4].)

Here is what conjectures 1.2 and 1.3 predict for the action of an anti-symplectic involution (i.e., an involution acting as -1 on the symplectic form) on the Chow groups of X :

Conjecture 1.4. *Let X be a hyperkähler variety of dimension 4. Let ι be an anti-symplectic involution of X . Then*

$$\begin{aligned} \iota^* &= -\mathrm{id}: \quad A_{(2)}^i(X) \rightarrow A_{(2)}^i(X) \quad \text{for } i = 2, 4; \\ \iota^* &= \mathrm{id}: \quad A_{(4)}^4(X) \rightarrow A_{(4)}^4(X). \end{aligned}$$

(The statement for $A_{(4)}^4(X) = F^4 A^4(X)$ is conjecture 1.2 applied to the graph of ι . The statement for $A_{(2)}^2(X) = A_{AJ}^2(X)$ is conjecture 1.3 applied to the graph of ι . The statement for $A_{(2)}^4(X)$ then follows from the expected “hard Lefschetz” isomorphism $A_{(2)}^2(X) \xrightarrow{\cong} A_{(2)}^4(X)$.)

The main result of this note establishes a weak form of conjecture 1.4 for a 19-dimensional family of hyperkähler fourfolds:

Theorem (=theorem 3.1). *Let X be the Hilbert scheme $S^{[2]}$, where S is a very general K3 surface of degree $d = 10$. Let $\iota \in \mathrm{Bir}(X)$ be the anti-symplectic involution constructed by O’Grady [29]. Then*

$$\begin{aligned} \iota^* &= \mathrm{id}: \quad A_{(0)}^4(X) \rightarrow A_{(0)}^4(X); \\ \iota^* &= -\mathrm{id}: \quad A_{(2)}^4(X) \rightarrow A_{(2)}^4(X); \\ (\Pi_2^X)_* \iota^* &= -\mathrm{id}: \quad A_{(2)}^2(X) \rightarrow A_{(2)}^2(X); \\ (\Pi_4^X)_* \iota^* &= \mathrm{id}: \quad A_{(4)}^4(X) \rightarrow A_{(4)}^4(X). \end{aligned}$$

The birational involution ι of [29] is briefly explained in proposition 2.20 below. The notation $A_{(*)}^*(X)$ in theorem 3.1 refers to the bigraded ring structure constructed unconditionally for all Hilbert squares of K3 surfaces by Shen–Vial, using their version of the Fourier transform on the Chow ring [35] (cf. also section 2.2 below). The Π_j^X refer to the Chow–Künneth decomposition of [35]; by construction, these have the property that $(\Pi_j^X)_* A^i(X) = A_{(2i-j)}^i(X)$.

It is known that a variety X as in theorem 3.1 has a birational model X' which is a hyperkähler variety; X' is a so-called *double EPW sextic* [30], [29], [32]. The variety X' has a generically $2 : 1$ morphism to a slightly singular sextic hypersurface $Y \subset \mathbb{P}^5$, called an *EPW sextic* [10], [30]. Theorem 3.1 has interesting consequences for the Chow ring of this EPW sextic:

Corollary (=corollary 4.2). *Let X be as in theorem 3.1, and let $Y \subset \mathbb{P}^5$ be the associated EPW sextic. For any $r \in \mathbb{N}$, let*

$$E^*(Y^r) \subset A^*(Y^r)$$

be the subring generated by (pullbacks of) $A^1(Y)$ and $A^2(Y)$. The cycle class map

$$E^k(Y^r) \rightarrow \mathrm{Gr}_{2k}^W H^{2k}(Y^r)$$

is injective for $k \geq 4r - 1$.

(Here, the Chow ring $A^*(Y^r)$ is taken to mean the operational Chow cohomology of Fulton–MacPherson [13].)

In particular, taking $r = 1$, we find that the subspaces

$$\begin{aligned} \mathrm{Im}\left(A^2(Y) \otimes A^1(Y) \rightarrow A^3(Y)\right), \\ \mathrm{Im}\left(A^2(Y) \otimes A^2(Y) \rightarrow A^4(Y)\right) \end{aligned}$$

are of dimension 1 (corollary 4.6). This is analogous to known results for 0–cycles on $K3$ surfaces [3] and on certain Calabi–Yau varieties [41], [11] (cf. remark 4.7 below).

Theorem 3.1 is proven using the technique of “spread” of algebraic cycles in a family, as developed by Voisin in her seminal work on the Bloch/Hodge equivalence for complete intersections [42], [43], [44], [45].

In a final section (section 5), some questions related to theorem 3.1 are stated, which we hope may spurn further research.

Conventions. *In this article, the word variety will refer to a reduced irreducible scheme of finite type over \mathbb{C} . A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.*

All Chow groups will be with rational coefficients: *we will denote by $A_j(X)$ the Chow group of j –dimensional cycles on X with \mathbb{Q} –coefficients; for X smooth of dimension n the notations $A_j(X)$ and $A^{n-j}(X)$ are used interchangeably.*

The notations $A_{\mathrm{hom}}^j(X)$, $A_{\mathrm{AJ}}^j(X)$ will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles. For a morphism $f: X \rightarrow Y$, we will write $\Gamma_f \in A_(X \times Y)$ for the graph of f . The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [34], [27]) will be denoted $\mathcal{M}_{\mathrm{rat}}$.*

If $\tau: Y \rightarrow X$ is an inclusion of smooth varieties and $b \in A^j(X)$, we will often write

$$b|_Y \in A^j(Y)$$

to indicate the class $\tau^(b)$.*

We use $H^j(X)$ to indicate singular cohomology $H^j(X, \mathbb{Q})$.

We write $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ to denote the group of automorphisms, resp. of birational automorphisms, of X .

Given an involution $\iota \in \text{Bir}(X)$, we will write $A^j(X)^\iota$ (and $H^j(X)^\iota$) for the subgroup of $A^j(X)$ (resp. $H^j(X)$) invariant under ι .

2. PRELIMINARIES

2.1. MCK decomposition.

Definition 2.1 (Murre [26]). *Let X be a smooth projective variety of dimension n . We say that X has a CK decomposition if there exists a decomposition of the diagonal*

$$\Delta_X = \Pi_0^X + \Pi_1^X + \cdots + \Pi_{2n}^X \text{ in } A^n(X \times X),$$

such that the Π_i^X are mutually orthogonal idempotents and $(\Pi_i^X)_* H^*(X) = H^i(X)$.

(NB: “CK decomposition” is shorthand for “Chow–Künneth decomposition”.)

Remark 2.2. *The existence of a CK decomposition for any smooth projective variety is part of Murre’s conjectures [26], [16].*

Definition 2.3 (Shen–Vial [35]). *Let X be a smooth projective variety of dimension n . Let $\Delta_{sm}^X \in A^{2n}(X \times X \times X)$ be the class of the small diagonal*

$$\Delta_{sm}^X := \{(x, x, x) \mid x \in X\} \subset X \times X \times X.$$

An MCK decomposition is a CK decomposition $\{\Pi_i^X\}$ of X that is multiplicative, i.e. it satisfies

$$\Pi_k^X \circ \Delta_{sm}^X \circ (\Pi_i^X \times \Pi_j^X) = 0 \text{ in } A^{2n}(X \times X \times X) \text{ for all } i + j \neq k.$$

(NB: “MCK decomposition” is shorthand for “multiplicative Chow–Künneth decomposition”.)

Remark 2.4. *The small diagonal (seen as a correspondence from $X \times X$ to X) induces the multiplication morphism*

$$\Delta_{sm}^X: h(X) \otimes h(X) \rightarrow h(X) \text{ in } \mathcal{M}_{\text{rat}}.$$

Suppose X has a CK decomposition

$$h(X) = \bigoplus_{i=0}^{2n} h^i(X) \text{ in } \mathcal{M}_{\text{rat}}.$$

By definition, this decomposition is multiplicative if for any i, j the composition

$$h^i(X) \otimes h^j(X) \rightarrow h(X) \otimes h(X) \xrightarrow{\Delta_{sm}^X} h(X) \text{ in } \mathcal{M}_{\text{rat}}$$

factors through $h^{i+j}(X)$. It follows that if X has an MCK decomposition, then setting

$$A_{(j)}^i(X) := (\Pi_{2i-j}^X)_* A^i(X),$$

one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends $A_{(j)}^i(X) \otimes A_{(j')}^{i'}(X)$ to $A_{(j+j')}^{i+i'}(X)$.

The property of having an MCK decomposition is severely restrictive, and is closely related to Beauville’s “weak splitting property” [4]. For more ample discussion, and examples of varieties with an MCK decomposition, we refer to [35, Section 8] and also [40], [36], [12], [23].

Lemma 2.5 (Vial [40]). *Let X, X' be birational hyperkähler varieties. Then X has an MCK decomposition if and only if X' has one.*

Proof. This is noted in [40, Introduction]; the idea (as indicated in loc. cit.) is that Rieß’s result [33] implies that X and X' have isomorphic Chow motives and the isomorphism is compatible with the multiplicative structure. □

2.2. MCK for $K3^{[2]}$.

Theorem 2.6 (Shen–Vial [35]). *Let S be a $K3$ surface, and $X = S^{[2]}$. There exists an MCK decomposition $\{\Pi_j^X\}$ for X . In particular, setting*

$$A_{(j)}^i(X) := (\Pi_{2i-j}^X)_* A^i(X)$$

defines a bigraded ring structure $A_{()}^*(X)$ on $A^*(X)$. Moreover, $A_{(*)}^*(X)$ coincides with the bigrading on $A^*(X)$ defined by the Fourier transform.*

Proof. The existence of $\{\Pi_j^X\}$ is a special case of [35, Theorem 13.4]. The “moreover” part is [35, Theorem 15.8]. □

Remark 2.7. *The first statement of theorem 2.6 actually holds for $X = S^{[r]}$ for any $r \in \mathbb{N}$ [40].*

Any $K3$ surface S has an MCK decomposition [35, Example 8.17]. Since this property is stable under products [35, Theorem 8.6], S^2 also has an MCK decomposition. The following lemma records a basic compatibility between the bigradings on $A^*(S^{[2]})$ and on $A^*(S^2)$:

Lemma 2.8. *Let S be a $K3$ surface, and $X = S^{[2]}$. Let $\Psi \in A^4(X \times S^2)$ be the correspondence coming from the diagram*

$$\begin{array}{ccc} S^{[2]} & \leftarrow & \widetilde{S^2} \\ h \downarrow & & \downarrow \\ S^{(2)} & \xleftarrow{g} & S^2 \end{array}$$

(the arrow labelled h is the Hilbert–Chow morphism; the right vertical arrow is the blow–up of the diagonal). Then

$$\begin{aligned} (\Psi)_* R(X) &\subset R(S^2), \\ ({}^t\Psi)_* R(S^2) &\subset R(X), \end{aligned}$$

where $R = A_{(4)}^4$ or $A_{(2)}^4$ or $A_{(2)}^2$ or $A_{(0)}^2 \cap A_{hom}^2$.

Proof. We prove the statement for ${}^t\Psi$ and $R = A_{(2)}^2$ or $A_{(0)}^2 \cap A_{hom}^2$, which are the only cases we’ll be using (the other statements can be proven similarly). By construction of the MCK decomposition for X , there is a relation

$$(1) \quad \Pi_k^X = \frac{1}{2} {}^t\Psi \circ \Pi_k^{S^2} \circ \Psi + \text{Rest} \quad \text{in } A^4(X \times X), \quad (k = 0, 2, 4, 6, 8),$$

where $\{\Pi_k^{S^2}\}$ is a product MCK decomposition for S^2 , and “Rest” is a term coming from $\Delta_S \subset S \times S$ which does not act on $A^4(X)$ and on $A_{AJ}^2(X)$. Since $\frac{1}{2} {}^t\Psi \circ \Psi$ is the identity on $A_{hom}^2(X) = A_{AJ}^2(X)$, we can write

$$({}^t\Psi)_*(\Pi_k^{S^2})_* = ({}^t\Psi \circ \Pi_k^{S^2})_* = \left(\frac{1}{2} {}^t\Psi \circ \Psi \circ {}^t\Psi \circ \Pi_k^{S^2}\right)_*: A_{hom}^2(S^2) \rightarrow A_{hom}^2(X).$$

In view of sublemma 2.9 below, this implies

$$(2) \quad ({}^t\Psi)_*(\Pi_k^{S^2})_* = \left(\frac{1}{2} {}^t\Psi \circ \Pi_k^{S^2} \circ \Psi \circ {}^t\Psi\right)_*: A_{hom}^2(S^2) \rightarrow A_{hom}^2(X).$$

But then, plugging in relation (1), we find

$$({}^t\Psi)_*(\Pi_k^{S^2})_* A_{hom}^2(S^2) \subset (\Pi_k^X)_* A_{hom}^2(X).$$

Taking $k = 2$, this proves

$$({}^t\Psi)_* A_{(2)}^2(S^2) \subset A_{(2)}^2(X).$$

Taking $k = 4$, this proves

$$({}^t\Psi)_* \left(A_{(0)}^2(S^2) \cap A_{hom}^2(S^2) \right) \subset A_{(0)}^2(X) \cap A_{hom}^2(X).$$

Sublemma 2.9. *There is commutativity*

$$\Psi \circ {}^t\Psi \circ \Pi_k^{S^2} = \Pi_k^{S^2} \circ \Psi \circ {}^t\Psi \text{ in } A^4(S^4).$$

To prove the sublemma, we remark that $h_* h^* = 2 \text{id}: A^i(S^{(2)}) \rightarrow A^i(S^{(2)})$, and so

$$(\Psi \circ {}^t\Psi)_* = 2 g^* g_* = 2(\Delta_{S^2} + \Gamma_\tau)_*: A^i(S^2) \rightarrow A^i(S^2),$$

where τ denotes the involution switching the two factors. But $\{\Pi_k^{S^2}\}$, being a product decomposition, is symmetric and hence

$$\Gamma_\tau \circ \Pi_k^{S^2} \circ \Gamma_\tau = (\tau \times \tau)^* \Pi_k^{S^2} = \Pi_k^{S^2} \text{ in } A^4(S^4).$$

This implies commutativity

$$\Gamma_\tau \circ \Pi_k^{S^2} = \Pi_k^{S^2} \circ \Gamma_\tau \text{ in } A^4(S^4),$$

which proves the sublemma. \square

Remark 2.10. *Lemma 2.8 is probably true for any (i, j) (i.e., Ψ should be “of pure grade 0” in the language of [36, Definition 1.1]). I have not been able to prove this.*

2.3. Relative MCK for S^2 .

Notation 2.11. *Let $\mathcal{S} \rightarrow B$ be a family (i.e., a smooth projective morphism). For $r \in \mathbb{N}$, we write $\mathcal{S}^{r/B}$ for the relative r -fold fibre product*

$$\mathcal{S}^{r/B} := \mathcal{S} \times_B \mathcal{S} \times_B \cdots \times_B \mathcal{S}$$

(r copies of \mathcal{S}).

Proposition 2.12. *Let $\mathcal{S} \rightarrow B$ be a family of K3 surfaces. There exist relative correspondences*

$$\Pi_j^{\mathcal{S}^{2/B}} \in A^4(\mathcal{S}^{4/B}) \quad (j = 0, 2, 4, 6, 8),$$

such that for each $b \in B$, the restriction

$$\Pi_j^{(S_b)^2} := \Pi_j^{\mathcal{S}^{2/B}}|_{(S_b)^4} \in A^4((S_b)^4)$$

defines a self-dual MCK decomposition for $(S_b)^2$.

Proof. On any K3 surface S_b , there is the distinguished 0-cycle \mathfrak{o}_{S_b} such that $c_2(S_b) = 24\mathfrak{o}_{S_b}$ [3]. Let $p_i: \mathcal{S} \times_B \mathcal{S} \rightarrow \mathcal{S}$, $i = 1, 2$, denote the projections to the two factors. Let $T_{\mathcal{S}/B}$ denote the relative tangent bundle. The assignment

$$\begin{aligned} \Pi_0^{\mathcal{S}} &:= (p_1)^* \left(\frac{1}{24} c_2(T_{\mathcal{S}/B}) \right) \in A^2(\mathcal{S} \times_B \mathcal{S}), \\ \Pi_4^{\mathcal{S}} &:= (p_2)^* \left(\frac{1}{24} c_2(T_{\mathcal{S}/B}) \right) \in A^2(\mathcal{S} \times_B \mathcal{S}), \\ \Pi_2^{\mathcal{S}} &:= \Delta_{\mathcal{S}} - \Pi_0^{\mathcal{S}} - \Pi_4^{\mathcal{S}} \end{aligned}$$

defines (by restriction) an MCK decomposition for each fibre:

$$\Pi_j^{S_b} := \Pi_j^{\mathcal{S}}|_{S_b \times S_b} \in A^2(S_b \times S_b) \quad (j = 0, 2, 4)$$

is an MCK decomposition for any $b \in B$ [35, Example 8.17].

Next, we consider the fourfold relative fibre product $\mathcal{S}^{4/B}$. Let

$$p_{ij}: \mathcal{S}^{4/B} \rightarrow \mathcal{S}^{2/B} \quad (1 \leq i < j \leq 4)$$

denote projection to the i -th and j -th factor. We set

$$\Pi_j^{\mathcal{S}^{2/B}} := \sum_{k+\ell=j} (p_{13})^*(\Pi_k^{\mathcal{S}}) \cdot (p_{24})^*(\Pi_\ell^{\mathcal{S}}) \in A^4(\mathcal{S}^{4/B}), \quad (j = 0, 2, 4, 6, 8).$$

By construction, the restriction to each fibre induces an MCK decomposition (the ‘‘product MCK decomposition’’)

$$\Pi_j^{(S_b)^2} := \Pi_j^{\mathcal{S}^{2/B}}|_{(S_b)^4} = \sum_{k+\ell=j} \Pi_k^{S_b} \times \Pi_\ell^{S_b} \in A^4((S_b)^4), \quad (j = 0, 2, 4, 6, 8).$$

□

Proposition 2.13. *Let $\mathcal{S} \rightarrow B$ be a family of K3 surfaces. There exist relative correspondences*

$$\Theta_1, \Theta_2 \in A^4((\mathcal{S} \times_B \mathcal{S}) \times_B \mathcal{S}), \quad \Xi_1, \Xi_2 \in A^2(\mathcal{S} \times_B (\mathcal{S} \times_B \mathcal{S}))$$

such that for each $b \in B$, the composition

$$A^2(S_b \times S_b) \xrightarrow{((\Theta_1|_{(S_b)^3})^*, (\Theta_2|_{(S_b)^3})^*)} A^2(S_b) \oplus A^2(S_b) \xrightarrow{(\Xi_1|_{(S_b)^3})^* + (\Xi_2|_{(S_b)^3})^*} A^2(S_b \times S_b)$$

acts as a projector on $A_{(2)}^2(S_b \times S_b)$.

Proof. As before, let

$$p_{ij}: \mathcal{S}^{4/B} \rightarrow \mathcal{S}^{2/B} \quad (1 \leq i < j \leq 4)$$

denote projection to the i -th and j -th factor, and let

$$p_i: \mathcal{S}^{2/B} \rightarrow \mathcal{S} \quad (i = 1, 2)$$

denote projection to the i -th factor.

By construction of $\Pi_2^{\mathcal{S}^{2/B}}$, for each $b \in B$ we have equality

$$(3) \quad \begin{aligned} (\Pi_2^{\mathcal{S}^{2/B}})|_{(S_b)^4} &= \frac{1}{24^2} \left({}^t\Gamma_{p_1} \circ \Pi_2^{\mathcal{S}} \circ \Gamma_{p_1} \circ ((p_{13})^*(\Delta_{\mathcal{S}}) \cdot (p_2)^*c_2(T_{\mathcal{S}/B}) \cdot (p_4)^*c_2(T_{\mathcal{S}/B})) \right. \\ &\quad \left. + {}^t\Gamma_{p_2} \circ \Pi_2^{\mathcal{S}} \circ \Gamma_{p_2} \circ ((p_{24})^*(\Delta_{\mathcal{S}}) \cdot (p_1)^*c_2(T_{\mathcal{S}/B}) \cdot (p_3)^*c_2(T_{\mathcal{S}/B})) \right)|_{(S_b)^4} \\ &\quad \text{in } A^4((S_b)^4). \end{aligned}$$

Indeed, using Lieberman's lemma [13, 16.1.1], we find that

$$\begin{aligned} ({}^t\Gamma_{p_1} \circ \Pi_2^{\mathcal{S}} \circ \Gamma_{p_1})|_{(S_b)^4} &= (({}^t\Gamma_{p_{13}})_*(\Pi_2^{\mathcal{S}}))|_{(S_b)^4} = ((p_{13})^*(\Pi_2^{\mathcal{S}}))|_{(S_b)^4}, \\ ({}^t\Gamma_{p_2} \circ \Pi_2^{\mathcal{S}} \circ \Gamma_{p_2})|_{(S_b)^4} &= (({}^t\Gamma_{p_{24}})_*(\Pi_2^{\mathcal{S}}))|_{(S_b)^4} = ((p_{24})^*(\Pi_2^{\mathcal{S}}))|_{(S_b)^4}, \end{aligned}$$

and so both sides of (3) are equal to

$$\Pi_2^{S_b} \times \Pi_0^{S_b} + \Pi_0^{S_b} \times \Pi_2^{S_b} \in A^2((S_b)^4).$$

It follows that if we define

$$\begin{aligned} \Theta_1 &:= \frac{1}{24^2} \Gamma_{p_1} \circ ((p_{13})^*(\Delta_{\mathcal{S}}) \cdot (p_2)^*c_2(T_{\mathcal{S}/B}) \cdot (p_4)^*c_2(T_{\mathcal{S}/B})) \in A^4((\mathcal{S} \times_B \mathcal{S}) \times_B \mathcal{S}), \\ \Theta_2 &:= \frac{1}{24^2} \Gamma_{p_2} \circ ((p_{24})^*(\Delta_{\mathcal{S}}) \cdot (p_1)^*c_2(T_{\mathcal{S}/B}) \cdot (p_3)^*c_2(T_{\mathcal{S}/B})) \in A^4((\mathcal{S} \times_B \mathcal{S}) \times_B \mathcal{S}), \\ \Xi_1 &:= {}^t\Gamma_{p_1} \circ \Pi_2^{\mathcal{S}} \in A^2(\mathcal{S} \times_B (\mathcal{S} \times_B \mathcal{S})), \\ \Xi_2 &:= {}^t\Gamma_{p_2} \circ \Pi_2^{\mathcal{S}} \in A^2(\mathcal{S} \times_B (\mathcal{S} \times_B \mathcal{S})), \end{aligned}$$

then we have

$$\left((\Xi_1 \circ \Theta_1 + \Xi_2 \circ \Theta_2)|_{(S_b)^4} \right)_* = (\Pi_2^{(S_b)^2})_*: A^2(S_b \times S_b) \rightarrow A_{(2)}^2(S_b \times S_b) \quad \forall b \in B.$$

This proves the proposition. \square

2.4. Relative MCK for $\mathcal{S}^{[2]}$.

Proposition 2.14. *Let $\mathcal{S} \rightarrow B$ be a family of K3 surfaces (i.e. each fibre S_b is a K3 surface), and let $\mathcal{X} \rightarrow B$ be the family of associated Hilbert schemes (i.e., a fibre X_b is $(S_b)^{[2]}$). There exist relative correspondences*

$$\Pi_j^{\mathcal{X}} \in A^4(\mathcal{X} \times_B \mathcal{X}) \quad (j = 0, 2, 4, 6, 8),$$

such that for each $b \in B$, the restrictions

$$\Pi_j^{X_b} := \Pi_j^{\mathcal{X}}|_{X_b \times X_b} \in A^4(X_b \times X_b) \quad (j = 0, 2, 4, 6, 8)$$

define an MCK decomposition for X_b .

Proof. The construction of an MCK decomposition for X_b given in [35, Theorem 13.4] can be done in a relative setting. That is, let $\{\Pi_j^{\mathcal{S}}\}$ be a relative MCK decomposition for \mathcal{S} as in proposition 2.12, and let $\{\Pi_j^{\mathcal{S}^{2/B}}\}$ be the induced relative MCK decomposition for $\mathcal{S}^{2/B}$ as in proposition 2.12. Let

$$\mathcal{Z} \rightarrow B$$

be the family obtained by blowing-up $\mathcal{S} \times_B \mathcal{S}$ along the relative diagonal $\Delta_{\mathcal{S}}$. As in the proof of [35, Propositions 13.2 and 13.3]¹, one can use $\{\Pi_j^{\mathcal{S}^{2/B}}\}$ and $\{\Pi_j^{\mathcal{S}}\}$ to define relative correspondences

$$\Pi_j^{\mathcal{Z}} \in A^4(\mathcal{Z} \times_B \mathcal{Z}) \quad (j = 0, 2, 4, 6, 8),$$

which restrict to an MCK decomposition of each fibre Z_b . Let

$$p: \mathcal{Z} \rightarrow \mathcal{X}$$

denote the morphism of B -schemes induced by the action of the symmetric group \mathfrak{S}_2 , and let $\Gamma_p \in A^4(\mathcal{Z} \times_B \mathcal{X})$ be the graph of p . We define

$$\Pi_j^{\mathcal{X}} := \frac{1}{2} \Gamma_p \circ \Pi_j^{\mathcal{Z}} \circ {}^t \Gamma_p \in A^4(\mathcal{X} \times_B \mathcal{X}) \quad (j = 0, 2, 4, 6, 8).$$

The restrictions $\Pi_j^{X_b} := \Pi_j^{\mathcal{X}}|_{X_b \times X_b}$ define an MCK decomposition for each fibre by [35, Theorem 13.4]. \square

2.5. Multiplicative structure of Chow ring of $K3$ ^[2].

Theorem 2.15 (Shen–Vial [35]). *Let S be a $K3$ surface, and $X = S^{[2]}$.*

(i) *Intersection product induces a surjection*

$$A_{(2)}^2(X) \otimes A_{(2)}^2(X) \twoheadrightarrow A_{(4)}^4(X).$$

(ii) *There is a distinguished class $l \in A_{(0)}^2(X)$ such that intersection induces an isomorphism*

$$\cdot l: A_{(2)}^2(X) \xrightarrow{\cong} A_{(2)}^4(X).$$

Proof. This is [35, Theorem 3]. \square

2.6. Refined CK decomposition.

Theorem 2.16 (Vial [38]). *Let X be a smooth projective variety of dimension $n \leq 5$. Assume the Lefschetz standard conjecture $B(X)$ holds (in particular, the Künneth components $\pi_i \in H^{2n}(X \times X)$ are algebraic). Then there is a splitting into mutually orthogonal idempotents*

$$\pi_i = \sum_j \pi_{i,j} \in H^{2n}(X \times X),$$

such that

$$(\pi_{i,j})_* H^*(X) = gr_{\tilde{N}}^j H^i(X),$$

where wtN^* is the niveau filtration of [38].

¹The statement and proof of [35, Proposition 13.2] should be slightly modified, as noted in [36, Remark 2.8].

In particular,

$$\begin{aligned} (\pi_{2,1})_* H^j(X) &= H^2(X) \cap F^1, \\ (\pi_{2,0})_* H^j(X) &= H_{tr}^2(X). \end{aligned}$$

(Here F^* denotes the Hodge filtration, and $H_{tr}^2(X)$ is the orthogonal complement to $H^2(X) \cap F^1$ under the pairing

$$\begin{aligned} H^2(X) \otimes H^2(X) &\rightarrow \mathbb{Q}, \\ a \otimes b &\mapsto a \cup h^{n-2} \cup b. \end{aligned}$$

The projector $\pi_{2,1}$ is supported on $C \times D$, where $C \subset X$ is a curve and $D \subset X$ is a divisor.

Proof. This is [38, Theorem 1]. □

2.7. Mukai models.

Theorem 2.17 (Mukai [24]). *Let S be a general K3 surface of degree 10 (i.e. genus $g(S) = 6$). Let $G = G(2, 5)$ denote the Grassmannian of lines in \mathbb{P}^4 . Then S is isomorphic to the zero locus of a section of $\mathcal{O}_G(1)^{\oplus 3} \oplus \mathcal{O}_G(2)$.*

This result can be exploited as follows:

Proposition 2.18 (Voisin [42]). *Let $\mathcal{S} \rightarrow B$ be the universal family of degree 10 K3 surfaces (i.e., B is a Zariski open in a product of projective spaces parametrizing sections of $\mathcal{O}_G(1)^{\oplus 3} \oplus \mathcal{O}_G(2)$ that are smooth). We have*

$$A_{hom}^2(\mathcal{S} \times_B \mathcal{S}) = 0.$$

Proof. The family $\mathcal{S} \rightarrow B$ is the family of smooth complete intersections $S_b \subset G$ defined by the very ample line bundles $\mathcal{O}_G(1)$ (3 copies) and $\mathcal{O}_G(2)$. The Grassmannian G has trivial Chow groups. The result is thus a special case of [42, Proposition 3.13] (NB: as explained in [42, Section 3.3], the hypothesis that [42, Conjecture 1.6] holds is satisfied in codimension 2, and so the result is unconditional in codimension 2). □

2.8. EPW sextics.

Definition 2.19 ([10]). *Let $A \subset \wedge^3 \mathbb{C}^6$ be a subspace which is Lagrangian with respect to the symplectic form on $\wedge^3 \mathbb{C}^6$ given by the wedge product. The EPW sextic associated to A is*

$$Y_A := \left\{ [v] \in \mathbb{P}(\mathbb{C}^6) \mid \dim(A \cap (v \wedge \wedge^2 \mathbb{C}^6)) \geq 1 \right\} \subset \mathbb{P}(\mathbb{C}^6).$$

An EPW sextic is an Y_A for some $A \subset \wedge^3 \mathbb{C}^6$ Lagrangian.

Proposition 2.20 (O’Grady [29]). *Let $X = S^{[2]}$ where S is a very general degree 10 K3 surface. There exists a non-trivial birational involution*

$$\iota: X \dashrightarrow X.$$

There exists a ι -invariant divisor $D \subset X$ (of Beauville–Bogomolov square 2), such that the action of ι on the Néron–Severi group $NS(X)$ is given by reflection in the span of D .

Proof. This is [29, Section 4.3] (cf. also [15, Section 3.1]). The idea of the construction of ι is as follows. Using Mukai’s work (theorem 2.17), the $K3$ surface S can be realized as a quadratic section $S = V_5 \cap Q$ of the del Pezzo threefold $V_5 := G \cap \mathbb{P}^6$. Hence, a general unordered pair of points on S gives a general unordered pair of points (x, y) on V_5 . One checks (by a dimension count) that there is a unique conic $q = q_{x,y} \subset V_5$ passing through the pair of points (x, y) . Since $S = V_5 \cap Q$ is a quadratic section, the conic q meets S in x, y plus 2 other points x', y' . The involution is defined by this residual intersection, i.e.

$$\iota(x, y) := (x', y') \in X .$$

□

Theorem 2.21 (O’Grady [30]). *Let X and $\iota \in \text{Bir}(X)$ be as in proposition 2.20. There exists a hyperkähler fourfold X' birational to X , and a generically $2 : 1$ morphism $p: X' \rightarrow Y$ to an EPW sextic Y .*

Moreover, let $\iota' \in \text{Bir}(X')$ be the birational involution induced by ι . Then $Y \subset \mathbb{P}^5$ is the closure of the quotient U'/ι' , where $U' \subset X'$ is the open on which the involution ι' is defined.

Proof. This is contained in [32, Theorem 4.15]. The idea is that there is a generically $2 : 1$ rational map $X \dashrightarrow Y_A$ to an EPW sextic with $A \in \Delta$ in the notation of loc. cit. For S very general, the subspace A will be generic in Δ and thus $Y_A[3]$ will consist of a single point v_0 . Let $X_A \rightarrow Y_A$ be the singular double cover of the EPW sextic as in loc. cit. According to [32, Theorem 4.15], X_A has one singular point p_0 (lying over $v_0 \in Y_A$), and there exists a small resolution $s: X_A^\epsilon \rightarrow X_A$ with exceptional locus $E := s^{-1}(p_0)$ isomorphic to \mathbb{P}^2 , and such that X_A^ϵ is isomorphic to the Hilbert square of a certain $K3$ surface (the $K3$ surface denoted $S_A(v_0)$ in loc. cit.). We define $X' := X_A^\epsilon$ and $Y := Y_A$.

The singular variety X_A has an involution $\iota_A \in \text{Aut}(X_A)$ (coinciding with $\iota \in \text{Bir}(X)$ on an open) such that $Y = X_A/\iota_A$. Since $X_A^\epsilon \rightarrow X_A$ is birational, this proves the “moreover” statement. □

3. MAIN RESULT

Theorem 3.1. *Let X be the Hilbert scheme $S^{[2]}$, where S is a very general $K3$ surface of degree $d = 10$. Let $\iota \in \text{Bir}(X)$ be the anti-symplectic involution of proposition 2.20. Then*

$$\begin{aligned} \iota^* &= \text{id}: A_{(0)}^4(X) \rightarrow A_{(0)}^4(X); \\ \iota^* &= -\text{id}: A_{(2)}^4(X) \rightarrow A_{(2)}^4(X); \\ (\Pi_2^X)_* \iota^* &= -\text{id}: A_{(2)}^2(X) \rightarrow A_{(2)}^2(X); \\ (\Pi_4^X)_* \iota^* &= \text{id}: A_{(4)}^4(X) \rightarrow A_{(4)}^4(X). \end{aligned}$$

Proof. We first prove the statement for $A_{(2)}^2(X)$. The statements for $A_{(2)}^4(X)$ and for $A_{(4)}^4(X)$ will be deduced from the statement for $A_{(2)}^2(X)$ using theorem 2.15.

We consider the universal family

$$S \rightarrow B$$

of all smooth degree 10 $K3$ surfaces S_b . Here the base B is a Zariski–open in a product of projective spaces

$$B \subset \bar{B} := \mathbb{P}H^0(G, \mathcal{O}(1))^{\times 3} \times \mathbb{P}H^0(G, \mathcal{O}(3)) ,$$

corresponding to theorem 2.17.

We will write

$$\mathcal{X} \rightarrow B$$

for the universal family of Hilbert squares of degree 10 $K3$ surfaces, and X_b for a fibre of $\mathcal{X} \rightarrow B$ over $b \in B$. This family is obtained from the family $\mathcal{S} \times_B \mathcal{S}$ (whose fibres are products $S_b \times S_b$) by a “hat” of morphisms over B

$$(4) \quad \begin{array}{ccc} & \widetilde{\mathcal{S} \times_B \mathcal{S}} & \\ \swarrow & & \searrow \\ \mathcal{X} & & \mathcal{S} \times_B \mathcal{S} \end{array}$$

where $\widetilde{\mathcal{S} \times_B \mathcal{S}}$ is the blow–up of $\mathcal{S} \times_B \mathcal{S}$ with centre the relative diagonal, and the southwest arrow is the quotient morphism for the natural action of the symmetric group on 2 elements. This diagram (4) gives rise to relative correspondences

$$\Psi \in A^4(\mathcal{X} \times_B \mathcal{S} \times_B \mathcal{S}) , \quad {}^t\Psi \in A^4(\mathcal{S} \times_B \mathcal{S} \times_B \mathcal{X}) .$$

(For details on relative correspondences, cf. [27], and also [9], [8], [28].) Restricting to a fibre over $b \in B$, diagram (4) induces the familiar diagram

$$\begin{array}{ccc} & \widetilde{S_b \times S_b} & \\ \swarrow & & \searrow \\ X_b = (S_b)^{[2]} & & S_b \times S_b \end{array}$$

(where $\widetilde{S_b \times S_b}$ is the blow–up of $S_b \times S_b$ along the diagonal), and the (absolute) correspondences

$$\Psi_b \in A^4(X_b \times S_b \times S_b) , \quad {}^t\Psi_b \in A^4(S_b \times S_b \times X_b) .$$

Since the construction of the birational involution $\iota_b \in \text{Bir}(X_b)$ of proposition 2.20 is geometric in nature, it naturally extends to the relative setting. More precisely, let

$$\mathcal{V} \rightarrow B' \rightarrow B$$

denote the family of smooth codimension 3 linear sections of the Grassmannian $G = G(2, 5)$ of lines in \mathbb{P}^4 (so B' is an open in $(\mathbb{P}H^0(G, \mathcal{O}_G(1)))^{\times 3}$, and each fibre V_b of the family $\mathcal{V} \rightarrow B$ is the del Pezzo threefold usually denoted V_5). Let $\mathcal{F} \rightarrow B$ denote the family of Fano varieties of conics contained in V_b (so the family $\mathcal{F} \rightarrow B$ is isotrivial with fibre $F(V_5)$ according to the previous parenthesis). Associating to a general unordered pair of 2 points on S_b the unique conic in V_b containing this pair of points defines a rational map of B –schemes

$$\mathcal{X} \dashrightarrow \mathcal{F} .$$

Taking the residual intersection of the conic with the surface S_b , we get a birational involution of B –schemes

$$\iota: \mathcal{X} \rightarrow \mathcal{X} ,$$

such that restriction to a fibre gives the birational involution $\iota_b: X_b \rightarrow X_b$ of proposition 2.20.

Let $\Gamma_\iota \in A^4(\mathcal{X} \times_B \mathcal{X})$ denote the closure of the graph of the birational map ι . The fact that ι_b acts as -1 on $H^{2,0}(X_b)$ for all $b \in B$, combined with the fact that $H_{tr}^2(X_b) \subset H^2(X_b)$ is the smallest Hodge substructure containing $H^{2,0}$, implies that

$$({}^t\Gamma_{\iota_b} + \Delta_{X_b}) \circ (\pi_{2,tr}^{X_b}) = 0 \quad \text{in } H^8(X_b \times X_b), \quad \forall b \in B.$$

In view of the refined Chow–Künneth decomposition (theorem 2.16), this implies that

$$(5) \quad ({}^t\Gamma_{\iota_b} + \Delta_{X_b}) \circ (\pi_2^{X_b}) = \gamma_b \quad \text{in } H^8(X_b \times X_b), \quad \forall b \in B,$$

where γ_b is some cycle supported on $Y_b \times Y_b$, for $Y_b \subset X_b$ a divisor.

Let $\{\Pi_j^{\mathcal{X}}\}$ be a relative MCK decomposition as in proposition 2.14. The relation (5) implies the following: the relative correspondence

$$\Gamma_0 := ({}^t\Gamma_\iota + \Delta_{\mathcal{X}}) \circ \Pi_2^{\mathcal{X}} \in A^4(\mathcal{X} \times_B \mathcal{X})$$

has the property that for each $b \in B$, there exists a divisor $Y_b \subset X_b$ and a cycle γ_b supported on $Y_b \times Y_b$ such that

$$(\Gamma_0)|_{X_b \times X_b} = \gamma_b \quad \text{in } H^8(X_b \times X_b).$$

At this point, we recall Voisin’s “spread–out” result:

Proposition 3.2 (Voisin [42]). *Let $\mathcal{X} \rightarrow B$ be a smooth projective morphism of relative dimension n . Let $\Gamma \in A^n(\mathcal{X} \times_B \mathcal{X})$ be a cycle such that for all $b \in B$, there exists a closed algebraic subset $Y_b \subset X_b$ of codimension c , and a cycle $\gamma_b \in A_n(Y_b \times Y_b)$ such that*

$$\Gamma|_{X_b \times X_b} = \gamma_b \quad \text{in } H^{2n}(X_b \times X_b).$$

Then there exists a closed algebraic subset $\mathcal{Y} \subset \mathcal{X}$ of codimension c , and a cycle $\gamma \in A_(\mathcal{Y} \times_B \mathcal{Y})$ such that*

$$\Gamma|_{X_b \times X_b} = \gamma|_{X_b \times X_b} \quad \text{in } H^{2n}(X_b \times X_b) \quad \forall b \in B.$$

Proof. This is a Hilbert schemes argument [42, Proposition 3.7]. □

Applying proposition 3.2 to Γ_0 , it follows there exists a divisor $\mathcal{Y} \subset \mathcal{X}$ and a cycle $\gamma \in A_*(\mathcal{Y} \times_B \mathcal{Y})$ such that

$$(\Gamma_0 - \gamma)|_{X_b \times X_b} = 0 \quad \text{in } H^8(X_b \times X_b), \quad \forall b \in B.$$

That is, the relative correspondence

$$\Gamma_1 := \Gamma_0 - \gamma \in A^4(\mathcal{X} \times_B \mathcal{X})$$

has the property of being homologically trivial on every fibre:

$$(\Gamma_1)|_{X_b \times X_b} = 0 \quad \text{in } H^8(X_b \times X_b), \quad \forall b \in B.$$

At this point, it is convenient to consider the family $\mathcal{S} \times_B \mathcal{S}$ (of products of surfaces $S_b \times S_b$), rather than the family \mathcal{X} (of Hilbert schemes $(S_b)^{[2]}$). That is, we consider the relative correspondence

$$\Gamma_2 := \Psi \circ \Gamma_1 \circ {}^t\Psi \in A^4(\mathcal{S}^{4/B}),$$

where

$$\mathcal{S}^{4/B} := \mathcal{S} \times_B \mathcal{S} \times_B \mathcal{S} \times_B \mathcal{S}.$$

Since

$$(\Gamma_2)|_{(S_b)^4} = (\Psi_b) \circ ((\Gamma_1)|_{X_b \times X_b}) \circ {}^t\Psi_b \quad \text{in } A^4((S_b)^4)$$

(restriction and composition commute), the relative correspondence Γ_2 has the property of being homologically trivial on every fibre:

$$(6) \quad (\Gamma_2)|_{(S_b)^4} = 0 \quad \text{in } H^8((S_b)^4), \quad \forall b \in B.$$

Let us now define four relative correspondences

$$\Gamma_3^{k,\ell} := \Theta_k \circ \Gamma_2 \circ \Xi_\ell \in A^2(\mathcal{S}^{2/B}), \quad k, \ell \in \{1, 2\},$$

where Ξ_ℓ, Θ_k are as in proposition 2.13.

It follows from (6) there is fibrewise homological vanishing

$$(\Gamma_3^{k,\ell})|_{S_b \times S_b} = 0 \quad \text{in } H^4(S_b \times S_b) \quad \forall b \in B \quad (k, \ell \in \{1, 2\}).$$

Applying the Leray spectral sequence argument of [42, Lemmas 3.11 and 3.12], one finds that there exist

$$\delta_{k,\ell} \in \text{Im}(A^4(G \times G \times B) \rightarrow A^2(\mathcal{S} \times_B \mathcal{S})) \quad (k, \ell \in \{1, 2\})$$

such that (after replacing B by a smaller Zariski open subset) there is global homological vanishing

$$\Gamma_3^{k,\ell} + \delta_{k,\ell} \in A_{hom}^2(\mathcal{S} \times_B \mathcal{S}) \quad (k, \ell \in \{1, 2\}).$$

But then, in view of proposition 2.18, we have that

$$\Gamma_3^{k,\ell} + \delta_{k,\ell} = 0 \in A^2(\mathcal{S} \times_B \mathcal{S}) \quad (k, \ell \in \{1, 2\}).$$

Composing on both sides, this implies there are also rational equivalences

$$(7) \quad \Xi_k \circ \Gamma_3^{k,\ell} \circ \Theta_\ell + \delta'_{k,\ell} = 0 \in A^4(\mathcal{S}^{4/B}) \quad (k, \ell \in \{1, 2\}),$$

where we define $\delta'_{k,\ell} := \Xi_k \circ \delta_{k,\ell} \circ \Theta_\ell$.

We note that the action of the restricted correspondences $\delta_{k,\ell}|_{S_b \times S_b}$ on $A^i(S_b)$ factors over $A^{i+4}(G)$. Since the Grassmannian G has trivial Chow groups, this implies that

$$(\delta_{k,\ell}|_{S_b \times S_b})_* = 0: \quad A_{hom}^i(S_b) \rightarrow A_{hom}^i(S_b) \quad \forall b \in B \quad (k, \ell \in \{1, 2\}).$$

As $\delta'_{k,\ell}$ is composed with $\delta_{k,\ell}$, the same property holds for $\delta'_{k,\ell}$:

$$(\delta'_{k,\ell}|_{(S_b)^4})_* = 0: \quad A_{hom}^i(S_b \times S_b) \rightarrow A_{hom}^i(S_b \times S_b) \quad \forall b \in B \quad \forall i \quad (k, \ell \in \{1, 2\}).$$

Plugging this in the restriction of equality (7) to a fibre, we see that

$$\begin{aligned} ((\Xi_k \circ \Gamma_3^{k,\ell} \circ \Theta_\ell)|_{(S_b)^4})_* = 0: \quad A_{hom}^i(S_b \times S_b) &\rightarrow A_{hom}^i(S_b \times S_b), \\ &\text{for all } i \text{ and for all } b \in B \quad (k, \ell \in \{1, 2\}). \end{aligned}$$

In view of the definition of the $\Gamma_3^{k,\ell}$, this implies that

$$\begin{aligned} & \left((\Xi_1 \circ \Theta_1 + \Xi_2 \circ \Theta_2) \circ \Gamma_2 \circ (\Xi_1 \circ \Theta_1 + \Xi_2 \circ \Theta_2) |_{(S_b)^4} \right)_* \\ &= \sum_{k,\ell \in \{1,2\}} \left((\Xi_k \circ \Theta_k \circ \Gamma_2 \circ \Xi_\ell \circ \Theta_\ell) |_{(S_b)^4} \right)_* \\ &= 0: \quad A_{hom}^i(S_b \times S_b) \rightarrow A_{hom}^i(S_b \times S_b), \quad \text{for all } i \text{ and for all } b \in B. \end{aligned}$$

But

$$\left((\Xi_1 \circ \Theta_1 + \Xi_2 \circ \Theta_2) |_{(S_b)^4} \right)_* = (\Pi_2^{(S_b)^2})_*: \quad A^2(S_b \times S_b) \rightarrow A^2(S_b \times S_b)$$

(proposition 2.13), and $A_{(2)}^2 \subset A_{hom}^2$, and so this simplifies to

$$(8) \quad (\Pi_2^{(S_b)^2})_*(\Gamma_2 |_{(S_b)^4})_* = 0: \quad A_{(2)}^2(S_b \times S_b) \rightarrow A_{(2)}^2(S_b \times S_b) \text{ for } b \in B \text{ general}.$$

To finish the proof of the $A_{(2)}^2(X)$ part of theorem 3.1, it remains to connect the action of (the restriction of) Γ_2 and the action of (the restriction of) the relative correspondence Γ_0 that we started out with. We make this connection in the next two lemmas:

Lemma 3.3. *Notation as above. There is equality*

$$(\Gamma_2 |_{(S_b)^4})_* = ((\Psi \circ \Gamma_0 \circ {}^t\Psi) |_{(S_b)^4})_*: \quad A_{hom}^2(S_b \times S_b) \rightarrow A_{hom}^2(S_b \times S_b) \text{ for } b \in B \text{ general}.$$

Proof. Unravelling the various definitions we made, we find

$$\begin{aligned} \Gamma_2 &= \Psi \circ \Gamma_1 \circ {}^t\Psi \circ \Pi_2^{\mathcal{S}^{2/B}} \\ &= \Psi \circ (\Gamma_0 - \gamma) \circ {}^t\Psi \circ \Pi_2^{\mathcal{S}^{2/B}} \\ &= \Psi \circ \Gamma_0 \circ {}^t\Psi \circ \Pi_2^{\mathcal{S}^{2/B}} - \gamma' \quad \text{in } A^4(\mathcal{S}^{4/B}), \end{aligned}$$

where $\gamma' := \Psi \circ \gamma \circ {}^t\Psi \circ \Pi_2^{\mathcal{S}^{2/B}}$ is a completely decomposed cycle. The restriction of γ' to a general fibre $(S_b)^4$ will be a completely decomposed cycle, and as such will not act on $A_{hom}^*(S_b \times S_b)$. This proves the lemma. \square

Lemma 3.4. *Notation as above. There is equality*

$$({}^t\Psi_b)_*(\Psi_b)_* = 2 \text{ id}: \quad A_{hom}^2(X_b) \rightarrow A_{hom}^2(X_b) \quad \forall b \in B.$$

Proof. This is noted in the proof of lemma 2.8. \square

Obviously, lemmas 3.3 and 3.4 suffice to prove the $A_{(2)}^2(X)$ part of theorem 3.1: indeed, the combination of lemma 3.3 with (8) implies that

$$(\Pi_2^{(S_b)^2})_*((\Psi \circ \Gamma_0 \circ {}^t\Psi) |_{(S_b)^4})_* = 0: \quad A_{(2)}^2(S_b \times S_b) \rightarrow A^2(S_b \times S_b) \text{ for } b \in B \text{ general}.$$

Composing with $({}^t\Psi_b)_*$ on the left and with $(\Psi_b)_*$ on the right, we find that

$$({}^t\Psi_b)_*(\Pi_2^{(S_b)^2})_*((\Psi \circ \Gamma_0 \circ {}^t\Psi) |_{(S_b)^4})_*(\Psi_b)_* = 0: \quad A_{(2)}^2(X_b) \rightarrow A^2(X_b) \text{ for } b \in B \text{ general}.$$

Applying relation (2) and lemma 3.4, it follows that

$$(\Pi_2^{X_b})_*(\Gamma_0)_* = 0: \quad A_{(2)}^2(X_b) \rightarrow A^2(X_b) \text{ for } b \in B \text{ general}.$$

By virtue of the definition of Γ_0 (and the fact that $A_{(2)}^2(X_b) = (\Pi_2^{X_b})_* A_{hom}^2(X_b)$), it follows that

$$(\Pi_2^{X_b})_*(\Gamma_{\iota_b} + \Delta_{X_b})_* = 0: \quad A_{(2)}^2(X_b) \rightarrow A_{(2)}^2(X_b) \quad \text{for } b \in B \text{ general,}$$

as asserted by theorem 3.1.

We have now proven the $A_{(2)}^2(X)$ part of theorem 3.1. The statement for $A_{(4)}^4(X)$ follows easily from this. Indeed, Shen–Vial have proven the multiplication map

$$A_{(2)}^2(X) \otimes A_{(2)}^2(X) \rightarrow A_{(4)}^4(X)$$

is surjective (theorem 2.15). Given $b \in A_{(4)}^4(X)$, we can thus write

$$b = a_1 \cdot a_2 \quad \text{in } A^4(X),$$

where $a_1, a_2 \in A_{(2)}^2(X)$. The statement we have just proven for $A_{(2)}^2$ implies that

$$\iota^*(a_j) = -a_j + r_j \quad \text{in } A^2(X) \quad (j = 1, 2),$$

where $r_j \in A_{(0)}^2(X) \cap A_{hom}^2(X)$. It follows that

$$\iota^*(b) = \iota^*(a_1 \cdot a_2) = \iota^*(a_1) \cdot \iota^*(a_2) = (-a_1 + r_1) \cdot (-a_2 + r_2) = a_1 \cdot a_2 - r_1 \cdot a_2 - r_2 \cdot a_1 \quad \text{in } A^4(X).$$

(NB: note that ι is not a morphism, and so the second equality is *not* trivial. The second equality happens to be true since $a_1, a_2 \in A_{AJ}^2(X)$; this is [35, Proposition B.6].) But then, since $-r_1 \cdot a_2 - r_2 \cdot a_1 \in A_{(2)}^4(X)$, we have

$$(\Pi_4^X)_* \iota^*(b) = a_1 \cdot a_2 = b \quad \text{in } A_{(4)}^4(X)$$

as requested.

It remains to prove theorem 3.1 for $A_{(0)}^4(X)$ and $A_{(2)}^4(X)$. As we have seen (theorem 2.15), Shen–Vial have shown there exists a class $l \in A_{(0)}^2(X)$ inducing an isomorphism

$$(9) \quad \cdot l: \quad A_{(2)}^2(X) \xrightarrow{\cong} A_{(2)}^4(X).$$

We need to understand the action of ι on the class $l \in A^2(X)$. To this end, we will prove the following:

Proposition 3.5. *Let X and ι be as in theorem 3.1. Let $l \in A_{(0)}^2(X)$ be the class as in theorem 2.15. Then*

$$\iota^*(l) = \pm l \quad \text{in } A^2(X).$$

Proposition 3.5 suffices to complete the proof of theorem 3.1. Indeed, one has

$$A_{(0)}^4(X) = \mathbb{Q}[l^2]$$

[35, Theorem 4.6]. The action of ι on l^2 satisfies

$$\iota^*(l^2) = \iota^*(l) \cdot \iota^*(l) + \iota^*((l - \iota_* \iota^*(l)) \cdot (l - \iota_* \iota^*(l))) = l^2 \quad \text{in } A^4(X).$$

(Here, for the first equality we have used [35, Lemma B.4], and the second equality follows from proposition 3.5.) This proves the $A_{(0)}^4$ part of theorem 3.1.

It remains to prove the $A_{(2)}^4$ part of theorem 3.1. For this, let us suppose for a moment that proposition 3.5 is true with a minus sign, i.e.

$$\iota^*(l) = -l \text{ in } A^2(X) .$$

Using the isomorphism (9), [35, Proposition B.6], and the fact that (as proven above) $\iota^* = -\text{id}$ on $A_{(2)}^2(X)$, this would imply

$$(10) \quad \iota^* = \text{id}: A_{(2)}^4(X) \rightarrow A^4(X) .$$

The statement for $A_{(4)}^4(X)$ we have just proven is that for any $b \in A_{(4)}^4(X)$ we have

$$\iota^*(b) = b + r \text{ in } A^4(X)$$

where $r \in A_{(2)}^4(X)$. Since $\iota^*\iota^*(b) = b$, this implies that $\iota^*(r) = -r$, and so (using equality (10)) $r = 0$. That is, we find that

$$\iota^* = \text{id}: A^4(X) \rightarrow A^4(X) .$$

Applying [35, Lemma 3.1] to ${}^t\bar{\Gamma}_\iota - \Delta_X$, this would imply that

$${}^t\bar{\Gamma}_\iota - \Delta_X = \gamma \text{ in } A^4(X \times X) ,$$

where γ is a cycle supported on $X \times D$ for $D \subset X$ a divisor. In particular, this would imply

$$\iota^* = \text{id}: H^{2,0}(X) \rightarrow H^{2,0}(X) ,$$

which is absurd since we know that ι is non-symplectic. The minus sign in proposition 3.5 can thus be excluded; assuming proposition 3.5 is true, we must have $\iota^*(l) = l$.

Now let $c \in A_{(2)}^4(X)$. Using the isomorphism (9), we can find $a \in A_{(2)}^2(X)$ such that

$$c = l \cdot a \text{ in } A^4(X) .$$

We know (from the $A_{(2)}^2(X)$ statement proven above) that $\iota^*(a) = -a + r$, where $r \in A_{(0)}^2(X) \cap A_{hom}^2(X)$. But then

$$\iota^*(c) = \iota^*(l \cdot a) = \iota^*(l) \cdot \iota^*(a) = l \cdot (-a + r) = -l \cdot a = -c \text{ in } A^4(X) .$$

Here, the second equality holds thanks to [35, Proposition B.6], and the third equality comes from proposition 3.5 and the statement for $A_{(2)}^2$. The fourth equality uses $A_{(0)}^2(X) \cdot (A_{(0)}^2(X) \cap A_{hom}^2(X)) = 0$, which is a consequence of $A_{(0)}^4(X) \cap A_{hom}^4(X) = 0$. This proves theorem 3.1, assuming proposition 3.5.

We now proceed with the proof of proposition 3.5. The first step is to prove the corresponding statement in homology:

Lemma 3.6. *Let S be any K3 surface and let $X = S^{[2]}$. Let $l \in A^2(X)$ be the class of theorem 2.15, and let $\iota \in \text{Bir}(X)$ be any birational involution. Then we have*

$$\iota^*(l) = \pm l \text{ in } H^4(X) .$$

Proof. Shen and Vial have constructed a distinguished cycle $L \in A^2(X \times X)$ (whose cohomology class is the Beauville–Bogomolov class denoted \mathfrak{B} in loc. cit.), and an eigenspace decomposition

$$(11) \quad A^2(X) = \Lambda_{25}^2 \oplus \Lambda_2^2 \oplus \Lambda_0^2 ,$$

where

$$\Lambda_\lambda^i := \{a \in A^i(X) \mid (L^2)_*(a) = \lambda a\},$$

and

$$\Lambda_{25}^2 = \mathbb{Q}[l]$$

(This is [35, Theorem 14.5, Propositions 14.6 and 14.8], combined with [35, Theorem 2.2]).

We now observe the following commutativity relation in cohomology:

Lemma 3.7. *Set-up as in lemma 3.6. Then*

$$(L^2)_* \iota^* = \iota^* (L^2)_*: H^i(X) \rightarrow H^i(X).$$

Proof. Let $L \in A^2(X \times X)$ be the Shen–Vial cycle as above. As proven in [35, Proposition 1.3(i)], the Shen–Vial cycle satisfies a quadratic relation

$$(12) \quad L^2 = 2\Delta_X - \frac{2}{25}(l_1 + l_2)L - \frac{1}{23 \cdot 25}(2l_1^2 - 23l_1l_2 + 2l_2^2) \quad \text{in } H^8(X \times X),$$

where $l := (i_\Delta)^*(L)$ (and $i_\Delta: X \rightarrow X \times X$ is the diagonal embedding) and $l_i := (p_i)^*(l)$ (and p_i are the obvious projections).

Let us define a modified cycle

$$L' := \Gamma_\iota \circ L \circ \Gamma_\iota \in A^2(X \times X).$$

Using Lieberman’s lemma ([13, 16.1.1] or [39, Lemma 3.3]), plus the fact that ${}^t\Gamma_\iota = \Gamma_\iota$, we see that

$$L' = (\iota \times \iota)^*(L) \quad \text{in } A^2(X \times X).$$

Define also $l' := (i_\Delta)^*(L') \in A^2(X)$ and $l'_i := (p_i)^*(l') \in A^2(X \times X)$, $i = 1, 2$. Since the diagram

$$\begin{array}{ccccc} X \times X & \xrightarrow{p_i} & X & \xrightarrow{i_\Delta} & X \times X \\ \downarrow \iota \times \iota & & \downarrow \iota & & \downarrow \iota \times \iota \\ X \times X & \xrightarrow{p_i} & X & \xrightarrow{i_\Delta} & X \times X \end{array}$$

commutes, we have the relations

$$(13) \quad l'_i = (\iota \times \iota)^*(l_i) \quad \text{in } A^2(X \times X), \quad i = 1, 2.$$

Let us apply $(\iota \times \iota)^*$ to the quadratic relation (12). The result is a relation

$$(14) \quad (\iota \times \iota)^*(L^2) = 2\Delta_X - \frac{2}{25}(\iota \times \iota)^*(l_1 + l_2)L' - \frac{1}{23 \cdot 25}(\iota \times \iota)^*(2l_1^2 - 23l_1l_2 + 2l_2^2) \\ \text{in } H^8(X \times X).$$

But

$$(15) \quad (\iota \times \iota)^*(L^2) = ((\iota \times \iota)^*L)^2 = (L')^2 \quad \text{in } A^4(X \times X).$$

Plugging this in equality (14), and also using the relations (13), we find that the cycle L' satisfies a quadratic relation

$$(16) \quad (L')^2 = 2\Delta_X - \frac{2}{25}(l'_1 + l'_2)L' - \frac{1}{23 \cdot 25}(2(l'_1)^2 - 23l'_1l'_2 + 2(l'_2)^2) \quad \text{in } H^8(X \times X).$$

But then, applying the unicity result [35, Proposition 1.3 (v)], we find there is equality

$$L' = \pm L \quad \text{in } H^4(X \times X) .$$

In particular, there is equality

$$(L')^2 = L^2 \quad \text{in } H^8(X \times X) .$$

In view of equality (15), this means

$$\Gamma_\iota \circ (L^2) \circ \Gamma_\iota = L^2 \quad \text{in } H^8(X \times X) ,$$

and so (by composing with Γ_ι)

$$\Gamma_\iota \circ (L^2) = (L^2) \circ \Gamma_\iota \quad \text{in } H^8(X \times X) .$$

This proves lemma 3.7. □

The eigenspace decomposition (11) induces an eigenspace decomposition modulo homological equivalence:

$$\text{Im}(A^2(X) \rightarrow H^4(X)) = \Lambda_{25}^2 + \frac{\Lambda_2^2}{A_{(0)}^2(X) \cap A_{hom}^2(X)}$$

(this is the algebraic part of the eigenspace decomposition of $H^4(X)$ given in [35, Proposition 1.3(iii)]).

Lemma (3.7) implies ι preserves this eigenspace decomposition modulo homological equivalence. In particular, $\iota^* \Lambda_{25}^2 \subset \Lambda_{25}^2$ (modulo homologically trivial cycles), and so

$$\iota^*(l) = dl \quad \text{in } H^4(X) ,$$

for some $d \in \mathbb{Q}$. Since ι is an involution, we must have $d = \pm 1$. This proves lemma 3.6. □

The next step (in proving proposition 3.5) is to upgrade to rational equivalence. Here, we use again the method of “spread” developed in [42], [43]. As above, let $\mathcal{S} \rightarrow B$ resp. $\mathcal{X} \rightarrow B$ denote the family of all smooth degree 10 $K3$ surfaces $S_b \subset \mathbb{P}^3$, resp. of all Hilbert schemes $X_b = (S_b)^{[2]}$. We note that there exists a relative cycle

$$\mathcal{L} \in A^2(\mathcal{X})$$

such that restriction

$$(17) \quad \mathcal{L}|_{X_b} = l_b \in A^2(X_b) \quad \forall b \in B$$

is the distinguished class (denoted l in theorem 2.15) for the fibre X_b . Indeed, one can define \mathcal{L} as

$$\mathcal{L} := \frac{5}{6} c_2(T_{\mathcal{X}/B}) \in A^2(\mathcal{X}) ,$$

where $T_{\mathcal{X}/B}$ is the relative tangent bundle of the smooth morphism $\mathcal{X} \rightarrow B$. Since for any $b \in B$ there is a relation

$$l_b = \frac{5}{6} c_2(X_b) \quad \text{in } A^2(X_b)$$

[35, Equation (93)], this implies (17).

The relative cycle

$$\Gamma_0 := \mathcal{L} \pm \iota^*(\mathcal{L}) \in A^2(\mathcal{X})$$

is such that the restriction to each fibre is homologically trivial:

$$(\Gamma_0)|_{X_b} = 0 \text{ in } H^4(X_b) .$$

(Here, “ \pm ” is taken to mean $+$ (resp. $-$) if lemma 3.6 is true with a $-$ (resp. a $+$.) Thus, the relative cycle

$$\Gamma_1 := \Psi_*(\Gamma_0) \in A^2(\mathcal{S} \times_B \mathcal{S})$$

also is homologically trivial on each fibre. (Here, Ψ is the relative correspondence from \mathcal{X} to $\mathcal{S} \times_B \mathcal{S}$ as in the proof of theorem 3.1.)

Applying [42, Lemma 3.12], up to shrinking B we can make Γ_1 globally homologically trivial. That is, there exists

$$\psi \in \text{Im}(A^2(B \times \mathbb{P}^3 \times \mathbb{P}^3) \rightarrow A^2(\mathcal{S} \times_B \mathcal{S}))$$

such that (after replacing B by a non-empty open subset $B' \subset B$)

$$\Gamma_2 := \Gamma_1 + \psi \in A^2(\mathcal{S} \times_{B'} \mathcal{S})$$

is actually in $A_{hom}^2(\mathcal{S} \times_{B'} \mathcal{S})$.

But $A_{hom}^2(\mathcal{S} \times_{B'} \mathcal{S}) = 0$ (proposition 2.18), and so

$$\Gamma_2 = 0 \text{ in } A^2(\mathcal{S} \times_{B'} \mathcal{S}) .$$

Restricting to a fibre, we find

$$(\Gamma_1)|_{S_b \times S_b} + \psi|_{S_b \times S_b} = 0 \text{ in } A^2(S_b \times S_b) \quad \forall b \in B' .$$

As Γ_1 is fibrewise homologically trivial, the same goes for ψ :

$$(18) \quad \psi|_{S_b \times S_b} = 0 \text{ in } H^4(S_b \times S_b) \quad \forall b \in B' .$$

But $A^2(\mathbb{P}^3 \times \mathbb{P}^3) = \bigoplus_i A^i(\mathbb{P}^3) \otimes A^{2-i}(\mathbb{P}^3)$ and so

$$\psi|_{S_b \times S_b} = \lambda_0[S_b] \times H_b^2 + \lambda_1 H_b \times H_b + \lambda_2 H_b^2 \times [S_b] \text{ in } A^2(S_b \times S_b) ,$$

where $\lambda_i \in \mathbb{Q}$ and $H_b \in A^1(S_b)$ is an ample class on S_b . It follows from the vanishing (18) that the λ_i must be 0, and so $\psi|_{S_b \times S_b}$ is rationally trivial, and hence also

$$(\Gamma_1)|_{S_b \times S_b} = 0 \text{ in } A^2(S_b \times S_b) .$$

Composing with ${}^t\Psi_b$, it follows that also

$$({}^t\Psi_b)_*((\Gamma_1)|_{S_b \times S_b}) = ({}^t\Psi_b)_*(\Psi_b)_*((\Gamma_0)|_{X_b}) = 0 \text{ in } A^2(X_b) \quad \forall b \in B' .$$

On the other hand, as we have seen above $(\Gamma_0)|_{X_b} \in A_{hom}^2(X_b)$ and $({}^t\Psi_b)_*(\Psi_b)_*$ is the identity on $A_{hom}^2(X_b)$. It follows that

$$(\Gamma_0)|_{X_b} = (l_b \pm (l_b)^*(l_b))|_{X_b} = 0 \text{ in } A^2(X_b) \quad \forall b \in B' .$$

This proves proposition 3.5 for general $b \in B$. To extend to *all* $b \in B$, one can invoke [45, Lemma 3.2]. Proposition 3.5 and hence theorem 3.1 are now proven. \square

Remark 3.8. *Can one prove the commutativity of lemma 3.7 also modulo rational equivalence, i.e. can one prove*

$$(19) \quad (L^2)_* \iota^* \stackrel{??}{=} \iota^* (L^2)_* : A^i(X) \rightarrow A^i(X) ?$$

This would imply that ι respects the eigenspace decomposition Λ_λ^i of [35] (and in particular, that ι respects the bigraded ring structure $A_{()}^*(X)$).*

The proof of lemma 3.7 given above does not extend to rational equivalence, for the following reason: The quadratic relation (12) still holds modulo rational equivalence [35, Theorem 14.5], and so L' satisfies the quadratic relation (16) modulo rational equivalence. However, the unicity result ([35, Proposition 1.3(v)]), that allowed us to conclude from this that $L = \pm L'$, is only known modulo homological equivalence.

(This unicity result modulo rational equivalence is conjecturally true, and would follow from the Bloch–Beilinson conjectures [35, Proposition 3.4].)

4. SOME CONSEQUENCES

4.1. A birational statement. Theorem 3.1 can be extended to other birational models:

Corollary 4.1. *Let X and ι be as in theorem 3.1. Let X' be a hyperkähler variety birational to X , and let $\iota' \in \text{Bir}(X')$ be the birational involution induced by ι . Then*

$$\begin{aligned} (\iota')^* &= \text{id} : A_{(0)}^4(X') \rightarrow A_{(0)}^4(X') ; \\ (\iota')^* &= -\text{id} : A_{(2)}^4(X') \rightarrow A_{(2)}^4(X') ; \\ (\Pi_2^{X'})_* (\iota')^* &= -\text{id} : A_{(2)}^2(X') \rightarrow A_{(2)}^2(X') ; \\ (\Pi_4^{X'})_* (\iota')^* &= \text{id} : A_{(4)}^4(X') \rightarrow A_{(4)}^4(X') . \end{aligned}$$

Proof. First, let us recall (lemma 2.5) that X' has an MCK decomposition (induced by an MCK decomposition for X , and the birational map $\phi : X \dashrightarrow X'$ induces isomorphisms

$$\phi_* : A_{(j)}^i(X) \xrightarrow{\cong} A_{(j)}^i(X') .$$

To deduce corollary 4.1 from theorem 3.1, it only remains to establish commutativity of the diagram

$$(20) \quad \begin{array}{ccc} A_{(j)}^i(X) & \xrightarrow{\phi_*} & A_{(j)}^i(X') \\ \downarrow \iota^* & & \downarrow (\iota')^* \\ A^i(X) & \xrightarrow{\phi_*} & A^i(X') \end{array}$$

for the relevant (i, j) . Let $U \subset X$, $U' \subset X'$ be opens such that ι is everywhere defined on U and ϕ induces an isomorphism between U and U' . Any 0-cycle $a \in A^4(X)$ is represented by a cycle α with support contained in U . Then $\phi_*(a)$ is represented by the cycle with isomorphic support in U' . This proves commutativity of the square (20) for $i = 4$. The Bloch–Srinivas argument [7] (or, more precisely, [35, Lemma 3.1]) applied to the correspondence

$$\bar{\Gamma}_\phi \circ \bar{\Gamma}_\iota - \bar{\Gamma}_{\iota'} \circ \bar{\Gamma}_\phi \in A^4(X \times X)$$

(where $\bar{\Gamma}$ indicates closure of the cycle $\Gamma \subset X \times X$) then shows commutativity for $A_{AJ}^2(X) = A_{hom}^2(X)$, and so (since $A_{(2)}^2 \subset A_{hom}^2$) the square (20) commutes for $(i, j) = (2, 2)$. \square

4.2. EPW sextics. Let X and ι be as in theorem 3.1. As we have seen, there is a birational modification X' of X that is a hyperkähler fourfold, and such that there is a generically $2 : 1$ morphism from X' to an EPW sextic Y ([30], cf. theorem 2.21 above). The following result is about the Chow ring A^* (in the sense of operational Chow cohomology [13]) of the EPW sextic Y . We note that for any variety M , there exists a ‘‘cycle class’’ map

$$A^i(M) \rightarrow \text{Gr}_{2i}^W H^{2i}(M)$$

which is functorial, and agrees with the usual cycle class map for smooth M [37].

Corollary 4.2. *Let X be as in theorem 3.1, and let $Y \subset \mathbb{P}^5$ be the associated EPW sextic. For any $r \in \mathbb{N}$, let*

$$E^*(Y^r) \subset A^*(Y^r)$$

be the subring generated by (pullbacks of) $A^1(Y)$ and $A^2(Y)$. The cycle class map

$$E^k(Y^r) \rightarrow \text{Gr}_{2k}^W H^{2k}(Y^r)$$

is injective for $k \geq 4r - 1$.

Proof. Let X' and ι' be as in theorem 2.21. The point is that X' , and hence also $(X')^r$, has an MCK decomposition [35]. Let $p: X' \rightarrow Y$ denote the morphism of theorem 2.21.

Lemma 4.3. *We have*

$$p^* A^2(Y) \subset A_{(0)}^2(X').$$

Proof. As explained in the proof of theorem 2.21, the morphism p decomposes as

$$p: X' := X_A^c \xrightarrow{s} X_A \xrightarrow{q} Y,$$

where s is a small resolution of the singular variety X_A , and q is a double cover with covering involution ι_A (and ι_A agrees with ι' on the open where ι' is defined). Because of the equality $q \circ \iota_A = q: X_A \rightarrow Y$, one has an inclusion

$$q^* A^i(Y) \subset A^i(X_A)^{\iota_A} \quad \forall i.$$

Because of the equality $\iota_A \circ s = s \circ \iota': X' \rightarrow X_A$, one has an inclusion

$$s^*(A^i(X_A)^{\iota_A}) \subset A^i(X')^{\iota'} \quad \forall i.$$

Combining these two inclusions and taking $i = 2$, we find in particular that

$$p^* A^2(Y) \subset A^2(X')^{\iota'}.$$

Given $b \in A^2(Y)$, let us write

$$p^*(b) = c_0 + c_2 \in A_{(0)}^2(X') \oplus A_{(2)}^2(X').$$

Applying ι' , we find

$$(21) \quad (\iota')^* p^*(b) = p^*(b) = c_0 + c_2 \in A_{(0)}^2(X') \oplus A_{(2)}^2(X').$$

On the other hand, corollary 4.1 implies that

$$(\iota')^*(c_2) = -c_2 + r \quad \text{in } A^2(X'),$$

where $r \in A_{(0)}^2(X')$. It follows that

$$(22) \quad (\iota')^*p^*(b) = (\iota')^*(c_0) + (\iota')^*(c_2) = (\iota')^*(c_0) + r - c_2 \in A_{(0)}^2(X') \oplus A_{(2)}^2(X')$$

(where we have used lemma 4.4 below to obtain that $(\iota')^*(c_0) \in A_{(0)}^2(X')$). Comparing expressions (21) and (22), we find

$$(\iota')^*(c_0) + r = c_0 \quad \text{in } A_{(0)}^2(X'), \quad -c_2 = c_2 \quad \text{in } A_{(2)}^2(X'),$$

proving lemma 4.3.

Lemma 4.4. *Set-up as above. Let $b \in A^2(Y)$, and write*

$$p^*(b) = c_0 + c_2 \in A_{(0)}^2(X') \oplus A_{(2)}^2(X').$$

Then

$$(\iota')^*(c_0) \in A_{(0)}^2(X').$$

Proof. Suppose

$$(\iota')^*(c_0) = d_0 + d_2 \quad \text{in } A^2(X'),$$

with $d_0 \in A_{(0)}^2(X')$ and $d_2 \in A_{(2)}^2(X')$.

Let $\gamma \in A^4(X \times X')$ be the correspondence of [33] (cf. also lemma 2.5) inducing an isomorphism of bigraded rings

$$(23) \quad \gamma_*: A_{(*)}^*(X) \xrightarrow{\cong} A_{(*)}^*(X').$$

Let $l \in A_{(0)}^2(X)$ be the distinguished class of theorem 2.15, and define

$$l' := \gamma_*(l) \in A_{(0)}^2(X').$$

The 0-cycle $c_0 \cdot l'$ is in $A_{(0)}^4(X')$, and so corollary 4.1 implies that

$$(24) \quad (\iota')^*(c_0 \cdot l') = c_0 \cdot l' \quad \text{in } A_{(0)}^4(X').$$

On the other hand, we have

$$(25) \quad (\iota')^*(c_0 \cdot l') = (\iota')^*(c_0) \cdot (\iota')^*(l') = (d_0 + d_2) \cdot l' = d_0 \cdot l' + d_2 \cdot l' \quad \text{in } A^4(X').$$

(Here in the first equality, we have used sublemma 4.5 below, and in the second equality we have used proposition 3.5, which we have seen must be true with a + sign.)

Since $d_0 \cdot l' \in A_{(0)}^4(X')$ and $d_2 \cdot l' \in A_{(2)}^4(X')$, comparing expressions (25) and (24), we see that we must have

$$d_0 \cdot l' = c_0 \cdot l' \quad \text{in } A_{(0)}^4(X'), \quad d_2 \cdot l' = 0 \quad \text{in } A_{(2)}^4(X').$$

In view of the isomorphism (23) and the injectivity part of theorem 2.15, this implies that $d_2 = 0$, thus proving lemma 4.4. \square

Above, we have used the following sublemma:

Sublemma 4.5. *Let $c_0 \in A_{(0)}^2(X')$ be as above. For any $e \in A^2(X')$, there is equality*

$$(\iota')^*(c_0 \cdot e) = (\iota')^*(c_0) \cdot (\iota')^*(e).$$

Proof. As we have seen above, the morphism p is a composition of morphisms

$$p: X' \xrightarrow{s} X_A \rightarrow Y,$$

where X_A is the ‘‘singular double EPW sextic’’ on which there exists an involution $\iota_A \in \text{Aut}(X_A)$ extending ι . We thus have

$$p^*A^2(Y) \subset s^*A^2(X_A)$$

(where we recall that $A^*(\cdot)$ of the singular varieties Y and X_A means the operational Chow cohomology of [13]). Let $b_A := q^*(b) \in A^2(X_A)$, so that $p^*(b) = s^*(b_A)$ in $A^2(X')$. We claim that there is equality

$$(26) \quad (\iota')^*(s^*(b_A) \cdot e) = (\iota')^*s^*(b_A) \cdot (\iota')^*(e) \quad \forall e \in A^2(X').$$

Since there is also equality

$$(\iota')^*(c_2 \cdot e) = (\iota')^*(c_2) \cdot (\iota')^*(e) \quad \forall e \in A^2(X')$$

(because $c_2 \in A_{(2)}^2(X') \subset A_{AJ}^2(X')$, this follows from [35, Proposition B.6]), and the definitions imply that $c_0 = s^*(b_A) - c_2$, the claim suffices to prove the sublemma.

To prove the claim, we exploit the fact that there is a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{\iota'} & X' \\ \downarrow s & & \downarrow s \\ X_A & \xrightarrow{\iota_A} & X_A \end{array}$$

Since $s_*: A_0(X') \rightarrow A_0(X_A)$ is an isomorphism, it suffices to prove equality (26) holds after pushing forward under s . The pushforward of the left–hand side of (26) is

$$\begin{aligned} s_*((\iota')^*(s^*(b_A) \cdot e)) &= ((\iota_A)^{-1})_*s_*(s^*(b_A) \cdot e) \\ &= (\iota_A)^*s_*(s^*(b_A) \cdot e) \\ &= (\iota_A)^*(b_A \cap s_*(e)) \\ &= (\iota_A)^*(b_A) \cap (\iota_A)_*s_*(e) \quad \text{in } A_0(X_A). \end{aligned}$$

(Here the notation $\alpha \cap \beta$ means the action of an operational Chow cohomology class $\alpha \in A^*(X_A)$ on $\beta \in A_*(X_A)$. The third equality is an application of the projection formula as given in [13, Definition 17.3].)

The pushforward of the right–hand side of (26) is

$$\begin{aligned} s_*((\iota')^*s^*(b_A) \cdot (\iota')^*(e)) &= s_*(s^*(\iota_A)^*(b_A) \cdot (\iota')^*(e)) \\ &= (\iota_A)^*(b_A) \cap s_*(\iota')^*(e) \\ &= (\iota_A)^*(b_A) \cap ((\iota_A)^{-1})_*s_*(e) \\ &= (\iota_A)^*(b_A) \cap (\iota_A)_*s_*(e) \quad \text{in } A_0(X_A). \end{aligned}$$

This proves the claim, and hence sublemma 4.5. □

□

Lemma 4.3, combined with the obvious fact that $A^1(X') = A_{(0)}^1(X')$, implies that

$$(p^r)^* E^*(Y^r) \subset A_{(0)}^*((X')^r).$$

Since there is a commutative diagram

$$\begin{array}{ccc} A_{(0)}^k((X')^r) & \rightarrow & H^{2k}((X')^r) \\ \uparrow (p^r)^* & & \uparrow (p^r)^* \\ E^k(Y^r) & \rightarrow & \mathbf{Gr}_{2k}^W H^{2k}(Y^r), \end{array}$$

and the cycle class map

$$A_{(0)}^k((X')^r) \rightarrow H^{2k}((X')^r)$$

is known to be injective for $k \geq 4r - 1$ ([40, Introduction]; this follows for instance from [38, Section 4.3]), this establishes corollary 4.2. □

We single out a particular case of corollary 4.2:

Corollary 4.6. *Let $Y \subset \mathbb{P}^5$ be an EPW sextic as in corollary 4.2. The subspaces*

$$\begin{aligned} & \text{Im}\left(A^2(Y) \otimes A^1(Y) \rightarrow A^3(Y)\right), \\ & \text{Im}\left(A^2(Y) \otimes A^2(Y) \rightarrow A^4(Y)\right) \end{aligned}$$

are of dimension 1.

Proof. This follows from corollary 4.2, combined with the fact that

$$N^3(Y) := \text{Im}\left(A^3(Y) \rightarrow \mathbf{Gr}_6^W H^6(Y)\right)$$

is of dimension 1. To see this, since the pairing

$$NS(X)^\iota \otimes N^3(X)^\iota \rightarrow N^4(X)^\iota \cong \mathbb{Q}$$

is non-degenerate, it suffices to prove

$$(27) \quad \dim NS(Y) = \dim NS(X)^\iota = 1.$$

But $\dim \mathbf{Gr}_2^W H^2(Y) = 1$ (weak Lefschetz for the hypersurface $Y \subset \mathbb{P}^5$), and so

$$\dim H^2(X)^\iota = 1,$$

proving (27).

(Alternatively, (27) can also be proven directly: ι acts on $NS(X)$ as reflection in the span of D (proposition 2.20), and so $NS(X)^\iota = \mathbb{Q}[D]$ is of dimension 1.) □

Remark 4.7. *It is instructive to compare corollary 4.6 with known results concerning the Chow ring of K3 surfaces and of Calabi–Yau varieties. For any K3 surface S , it is known that*

$$\dim \text{Im}\left(A^1(S) \otimes A^1(S) \rightarrow A^2(S)\right) = 1$$

[3]. For a generic Calabi–Yau complete intersection X of dimension n , it is known that

$$\dim \operatorname{Im} \left(A^i(X) \otimes A^{n-i}(X) \rightarrow A^n(X) \right) = 1, \quad \forall 0 < i < n$$

[41], [11].

The new part of corollary 4.6, with respect to these results, is the part about

$$\operatorname{Im} \left(A^2(X) \otimes A^1(X) \rightarrow A^3(X) \right).$$

This part is conjecturally true for all EPW sextics [22], but presumably not true for general Calabi–Yau varieties (or even general Calabi–Yau complete intersections). This is related to the question of determining which varieties satisfy Beauville’s weak splitting property [4].

5. SOME QUESTIONS

In this final section, we record some questions suggested by the results of this note.

Question 5.1. Let X and ι be as in theorem 3.1. Can one prove that ι preserves the bigrading $A_{(*)}^*(X)$ of the Chow ring (so that the Π_j^X in theorem 3.1 can be omitted)? It is a matter of regret that I have not been able to prove this (cf. remark 3.8).

Question 5.2. Let X and ι be as in theorem 3.1. What can one say about the action of ι on $A^3(X)$? This seems more difficult than theorem 3.1. Indeed, the action of ι on A_{hom}^2 and on A^4 is determined by “behaviour up to codimension 1 phenomena”. The action of ι on $A_{(2)}^3$, on the other hand, should be determined by the action of ι on $H^{3,1}(X)$, which is not as neat as the action on $H^{2,0}(X)$ and $H^{4,0}(X)$. I am not even sure what the conjectural statement should be.

Question 5.3. Let X' be a double EPW sextic as in theorem 2.21, and let $E \cong \mathbb{P}^2$ denote the exceptional locus of the small resolution $X' \rightarrow X_A$. Since $E \subset X'$ is a constant cycle subvariety, the ideas developed in [47] suggest that E should lie in $A_{(0)}^2(X')$. Can this actually be proven?

Question 5.4. Let X be a Hilbert square $X = S^{[2]}$, where S is a very general K3 surface of degree

$$d = 2(4n^2 + 8n + 5), \quad n \in \mathbb{N}.$$

Generalizing O’Grady’s result (theorem 2.21, which corresponds to $n = 0$), Iliev–Madonna show that X is birational to a double EPW sextic [15, Theorem 1.1], and so there exists an anti-symplectic birational involution ι on X . It follows from Shen–Vial’s work (theorem 2.6) that X has an MCK decomposition. Is it possible to prove theorem 3.1 for X and ι ?

The case $n = 1$ (i.e. $d = 34$ and $g(S) = 18$) can perhaps be done similarly to theorem 3.1: in this case, the general K3 surface S is given as zero locus of some vector bundle V on the orthogonal Grassmannian $\mathbb{O}\mathbb{G}(3, 9)$ [25]. The only problem is that I’m not sure whether the vector bundle V is sufficiently ample for Voisin’s method of spread to work in this case.

The case $n > 1$ seems considerably more difficult, due to the absence of Mukai models for S .

Question 5.5. Double EPW sextics form a 20–dimensional family of hyperkähler fourfolds of $K3^{[2]}$ type (and the above–mentioned work [15] constructs a countably infinite number of codimension 1 subfamilies, elements of which are birational to Hilbert squares of K3 surfaces). More

ambitiously than the above question 5.4, can one somehow prove theorem 3.1 for all double EPW sextics ?

One obvious problem is that first (in order for the statement of theorem 3.1 even to make sense), one would need to construct an MCK decomposition for a general double EPW sextic.

Acknowledgements. *Thanks to all participants of the Strasbourg 2014/2015 “groupe de travail” based on the monograph [45] for a very pleasant atmosphere. Much thanks to Jiji and Baba for hospitably receiving me in Kokuryo, where this note was written ☺.*

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