#### ALGEBRAIC CYCLES AND TRIPLE K3 BURGERS

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ABSTRACT. We consider surfaces of geometric genus 3 with the property that their transcendental cohomology splits into 3 pieces, each piece coming from a K3 surface. For certain families of surfaces with this property, we can show there is a similar splitting on the level of Chow groups (and Chow motives).

#### 1. INTRODUCTION

This note is about a class of surfaces which we propose to call *triple* K3 *burgers*. These are complex smooth projective surfaces S of general type of geometric genus 3, with the property that there exist 3 K3 surfaces  $X_j$  such that the transcendental cohomology  $H_{tr}^2(S)$  splits

(1) 
$$H^2_{tr}(S) \cong H^2_{tr}(X_0) \oplus H^2_{tr}(X_1) \oplus H^2_{tr}(X_2)$$

(The precise definition of triple K3 burgers is more restrictive, cf. definition 3.1.)

The crystal ball of the Bloch–Beilinson–Murre conjectures [24], [25], [58], [35], [34] predicts that relation (1) also holds on the level of Chow groups (and provided the Hodge conjecture is true, the Chow motive of S should be of abelian type, in the sense of [49]). The main result of this note provides a verification of this prediction in certain cases:

**Theorem** (=theorem 5.1). Let *S* be a triple K3 burger. Assume that either (i)  $K_S^2 = 2$ , or

(ii)  $K_S^2 = 3$  and the canonical map of S is base point free.

Then there is an isomorphism (induced by a correspondence)

$$A^2_{hom}(S) \xrightarrow{\cong} A^2_{hom}(X_0) \oplus A^2_{hom}(X_1) \oplus A^2_{hom}(X_2)$$
,

where the  $X_i$  are the associated K3 surfaces.

(Here  $A_{hom}^2$ ) denotes the Chow group of 0-cycles of degree 0 with rational coefficients.) In each of the cases of theorem 5.1, these surfaces do exist (in case (i), they form a family of dimension at least 6; in case (ii) the moduli dimension is 4).

It is *not* a coincidence that the surfaces of theorem 5.1 lie on or close to the Noether line  $K^2 = 2p_g - 4$ . Indeed (as is known since the fundamental work of Horikawa [15], [16], [17], [18], [19]), the canonical model of a general type surface on or close to the Noether line admits a neat description as complete intersection in a certain weighted projective space. Thanks to such a description, surfaces as in theorem 5.1 fit in nicely behaved universal families. Then, one can

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apply the alchemy of Voisin's method of "spread" [54], [57], [58] to transmute the base metal of the homological relation (1) into the pure gold of a rational equivalence.

We also prove (subsection 6.1) that a triple K3 burger S as in theorem 5.1 admits a canonical 0-cycle  $\mathfrak{o}_S \in A^2(S)$ , such that there is a splitting

$$A^2(S) = \mathbb{Q}[\mathfrak{o}_S] \oplus A^2_{hom}(S)$$

The cycle  $\mathfrak{o}_S$  has the property that the intersection of certain divisors is proportional to  $\mathfrak{o}_S$  (proposition 6.8). Another characterization of  $\mathfrak{o}_S$  is as follows (proposition 6.4): for any positive integer k, the cycle  $k\mathfrak{o}_S$  is the unique degree k 0-cycle z for which the effective orbit  $O_z$  has dimension  $\geq k$ . These results are based on similar results for the canonical 0-cycle of a K3 surface [21], [3], [58], [56].

In a sense, the present note is a sequel to [30], which dealt with certain surfaces of geometric genus  $p_g = 2$ . The surfaces S of [30] are also studied in [14] and [37]; they have the property that their transcendental cohomology decomposes

$$H^{2}_{tr}(S) \cong H^{2}_{tr}(X_{0}) \oplus H^{2}_{tr}(X_{1})$$
,

where  $X_0, X_1$  are K3 surfaces. In [30], using arguments very similar to the present note, I proved there exists a similar splitting on the level of Chow groups.

Several open questions remain, which I hope someone will be able to answer (cf. section 7).

**Conventions.** In this article, the word variety will refer to a reduced irreducible scheme of finite type over  $\mathbb{C}$ . A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

By default, all Chow groups will be with rational coefficients: we will denote by  $A_j(X)$  the Chow group of *j*-dimensional cycles on X with Q-coefficients; for X smooth of dimension n the notations  $A_j(X)$  and  $A^{n-j}(X)$  are used interchangeably. When dealing with Chow groups with integral coefficients, we will make this clear by writing  $A_j(X)_{\mathbb{Z}}$ .

The notations  $A_{hom}^j(X)$ ,  $A_{AJ}^j(X)$  will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles. For a morphism  $f: X \to Y$ , we will write  $\Gamma_f \in$  $A_*(X \times Y)$  for the graph of f. The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [43], [35]) will be denoted  $\mathcal{M}_{rat}$ .

We use  $H^{j}(X)$  to indicate singular cohomology  $H^{j}(X, \mathbb{Q})$ , and  $H_{j}(X)$  to indicate Borel-Moore homology  $H_{j}^{BM}(X, \mathbb{Q})$ .

#### 2. PRELIMINARIES

#### 2.1. Relative Künneth projectors.

**Lemma 2.1.** Let  $S \to B$  be as in notation 3.13. There exist relative correspondences

$$\pi_0^{\mathcal{S}}, \ \pi_2^{\mathcal{S}}, \ \pi_4^{\mathcal{S}} \in A^2(\mathcal{S} \times_B \mathcal{S}),$$

with the property that for each  $b \in B$ , the restriction

 $\pi_i^{\mathcal{S}}|_b := \pi_i^{\mathcal{S}}|_{S_b \times S_b} \quad \in H^4(S_b \times S_b)$ 

is the ith Künneth component. Moreover,

$$(\pi_2^{\mathcal{S}}|_b)_* = \mathrm{id}: \quad A_{hom}^2(S_b) \rightarrow A_{hom}^2(S_b) .$$

*Proof.* This is well–known, and holds more generally for any family of surfaces with  $H^1(S_b) = 0$ . Let  $H \in A^1(S)$  be a relatively ample divisor, and let  $d := \deg(H^2|_{S_b})$ . One defines

$$\pi_0^{\mathcal{S}} := \frac{1}{d} (p_1)^* (H^2) ,$$
  

$$\pi_4^{\mathcal{S}} := \frac{1}{d} (p_2)^* (H^2) ,$$
  

$$\pi_2^{\mathcal{S}} := \Delta_{\mathcal{S}} - \pi_0^{\mathcal{S}} - \pi_4^{\mathcal{S}} \quad \in A^2 (\mathcal{S} \times_B \mathcal{S})$$

It is readily checked this does the job.

#### 2.2. Transcendental part of the motive.

**Theorem 2.2** (Kahn–Murre–Pedrini [26]). Let S be any smooth projective surface, and let  $h(S) \in \mathcal{M}_{rat}$  denote the Chow motive of S. There exists a self–dual Chow–Künneth decomposition  $\{\pi_i\}$  of S, with the property that there is a further splitting in orthogonal idempotents

$$\pi_2 = \pi_2^{alg} + \pi_2^{tr} \text{ in } A^2(S \times S) .$$

The action on cohomology is

$$(\pi_2^{alg})_*H^*(S) = N^1H^2(S), \ (\pi_2^{tr})_*H^*(S) = H^2_{tr}(S),$$

where the transcendental cohomology  $H^2_{tr}(S) \subset H^2(S)$  is defined as the orthogonal complement of  $N^1H^2(S)$  with respect to the intersection pairing. The action on Chow groups is

$$(\pi_2^{alg})_*A^*(S) = N^1H^2(S), \ (\pi_2^{tr})_*A^*(S) = A^2_{AJ}(S).$$

This gives rise to a well-defined Chow motive

$$h_2^{tr}(S) := (S, \pi_2^{tr}, 0) \subset h(S) \in \mathcal{M}_{\mathrm{rat}}$$

the so-called transcendental part of the motive of S.

*Proof.* Let  $\{\pi_i\}$  be a Chow–Künneth decomposition as in [26, Proposition 7.2.1]. The assertion then follows from [26, Proposition 7.2.3].

3. TRIPLE K3 BURGERS

### 3.1. Definition.

**Definition 3.1.** A surface S is called a triple K3 burger if the following conditions are satisfied: (0) S is minimal, of general type, with q = 0 and  $p_g = 3$ ;

(i) there exist involutions  $\sigma_j \colon S \to S$  (j = 0, 1, 2) that commute with one another, and such that the quotients

$$X_j := S / < \sigma_j > (j = 0, 1, 2)$$

are birational to a K3 surface  $X_j$ ; (ii) there is an isomorphism

$$((p_0)^*, (p_1)^*, (p_2)^*): H^2(\bar{X}_0, \mathcal{O}) \oplus H^2(\bar{X}_1, \mathcal{O}) \oplus H^2(\bar{X}_2, \mathcal{O}) \xrightarrow{\cong} H^2(S, \mathcal{O})$$

where  $p_i: S \to \overline{X}_i$  denotes the quotient morphism;

(iii) the involutions  $\sigma_i$  respect the canonical divisor:

$$(\sigma_j)^* K_S = K_S$$
 ,  $j = 0, 1, 2$ .

**Remark 3.2.** Let  $\Psi_i \in A^2(X_i \times S)$  (j = 0, 1, 2) be the correspondence defined by the diagram

$$\begin{array}{ccc} & S \\ \downarrow \\ X_j & \to & \bar{X}_j \end{array}$$

where  $X_j \to \bar{X}_j$  is a resolution of singularities and  $X_j$  is a K3 surface.

Since the  $\overline{X}_j$  have only quotient singularities and quotient singularities are rational, condition (ii) of definition 3.1 is equivalent to asking for an isomorphism

$$\left((\Psi_0)_*, (\Psi_1)_*, (\Psi_2)_*\right): \quad H^2(X_0, \mathcal{O}) \oplus H^2(X_1, \mathcal{O}) \oplus H^2(X_2, \mathcal{O}) \xrightarrow{\cong} H^2(S, \mathcal{O}) .$$

Also, since  $(\Psi_j)_*$  is a homomorphism of Hodge structures, condition (ii) is equivalent to an isomorphism

$$((\Psi_0)_*, (\Psi_1)_*, (\Psi_2)_*): \quad H^2_{tr}(X_0) \oplus H^2_{tr}(X_1) \oplus H^2_{tr}(X_2) \xrightarrow{\cong} H^2_{tr}(S).$$

(Here, by definition  $H_{tr}^2() \subset H^2()$  is the orthogonal complement of the Néron–Severi group under the cup product pairing.)

Also, since  $(p_j)^* H^2(\bar{X}_j)$  is contained in the  $\sigma_j$ -invariant part of  $H^2(S)$ , condition (ii) is equivalent to the condition

(2) 
$$H^2_{tr}(S) = H^2_{tr}(S)^{+--} \oplus H^2_{tr}(S)^{-+-} \oplus H^2_{tr}(S)^{-+-}$$

where  $H_{tr}^2(S)^{+--}$  denotes the part of  $H_{tr}^2(S)$  where  $\sigma_0$  acts as the identity and  $\sigma_1, \sigma_2$  act as minus the identity, and the other summands are defined similarly.

(This uses some Hodge theory. E.g., let us consider  $H^2_{tr}(S)^{++-}$ . This is a Hodge substructure of  $H^2_{tr}(S)$ , and so if it is non-trivial, it must have  $Gr^0_F$  of dimension  $\geq 1$ . But then, as it is contained in the image of  $H^2_{tr}(X_0)$ , it must have  $Gr^0_F$  of dimension = 1. This implies that

$$\Psi_0)_* H^2_{tr}(X_0) = H^2_{tr}(S)^{++-}$$

as both sides are Hodge substructures of  $H^2_{tr}(S)$  with dim  $Gr^0_F = 1$ . But for the same reason, we have

$$(\Psi_1)_* H^2_{tr}(X_1) = H^2_{tr}(S)^{++-},$$

and so

$$(\Psi_0)_* H^2_{tr}(X_0) = (\Psi_1)_* H^2_{tr}(X_1)$$
 in  $H^2_{tr}(S)$ 

But this is absurd, because it contradicts the surjectivity in condition (ii). We conclude that  $H_{tr}^2(S)^{++-}$  must be zero. Applying the same reasoning to the other eigenspaces, one arrives at the isomorphism (2).)

**Remark 3.3.** *Definition 3.1 is directly inspired by the definition of Todorov surfaces* [47], [28], [33], [41].

One could extend definition 3.1 to surfaces of any geometric genus: a surface S is called an m-tuple K3 burger if  $p_g(S) = m$  and there exist m involutions  $\sigma_1, \ldots, \sigma_m$  such that the quotients  $S/ < \sigma_i >$  are birational to K3 surfaces and their transcendental cohomology generates  $H_{tr}^2(S)$ 

as in condition (ii). For m = 1 (i.e., "simple K3 burgers"), one obtains certain Todorov surfaces. (NB: There is a slight difference with the definition of Todorov surfaces; in the definition of a Todorov surface one merely asks, instead of (iii), that the involution  $\sigma$  is composed with the bicanonical map).

Surfaces similar to the case m = 2 of definition 3.1 (i.e., "double K3 burgers") have been studied in [14], [37], [30].

**Remark 3.4.** A closely related construction (which also inspired the present note) appears in recent work of Garbagnati [14, Section 6.1]. Let S be the minimal model of the surface  $U_{10}$  of [14, Section 6.1]. Then S satisfies conditions (0), (i) and (ii) of definition 3.1 (and I am not sure about condition (iii)). Also, it follows from [14, Theorem 3.1] that  $K_S^2 = 9$ , and so S is not among the cases covered by theorem 5.1.

The fact that  $K_S^2 = 9$  means that S is quite far from the Noether line; hence there is (as far as I am aware) not a nice and simple, Horikawa–style description of the canonical model of S as a weighted complete intersection. Due to the lack of such a description, the method of "spread" does not seem to apply to S, and I do not know how to handle the Chow groups of S.

**Remark 3.5.** Condition (iii) in definition 3.1 is admittedly somewhat ad hoc. The reason I have added condition (iii) is that otherwise, I am not able to prove theorem 5.1.

(More precisely: condition (iii) ensures that the involutions  $\sigma_j$  come from involutions of the ambient space (which will be a weighted projective space); as such, the involutions exist family–wise, which will be crucial to the argument.)

**Remark 3.6.** Todorov surfaces have been classified: there are 11 irreducible families, each of dimension 12 [33]. Likewise, it is perhaps possible to classify triple K3 burgers. The next subsection provides a first step.

# 3.2. Structural results.

**Notation 3.7.** Let  $\mathbb{P}$  be some weighted projective space, with weighted homogeneous coordinates  $[x_0 : x_1 : \cdots : x_n]$ . We define involutions  $s_j \in Aut(\mathbb{P}), j = 0, \ldots, n$ , by

 $s_j[x_0:\ldots:x_n] = [x_0:\cdots:-x_j:\ldots:x_n].$ 

Similarly, for  $0 \le i < j \le n$  we define involutions  $s_{ij} \in Aut(\mathbb{P})$  by

$$s_{ij}[x_0:\ldots:x_n] = [x_0:\ldots:-x_i:x_{i+1}:\ldots:-x_j:x_{j+1}:\ldots:x_n].$$

Similarly, we define involutions  $s_{ijk}$  involving 3 minus signs.

**Proposition 3.8.** Let S be a triple K3 burger with  $K^2 = 2$ . Then S is isomorphic to a smooth degree 8 hypersurface in  $\mathbb{P}(1^3, 4)$  invariant under  $G = \langle \sigma_0, \sigma_1, \sigma_2 \rangle$ , where  $\{\sigma_0, \sigma_1, \sigma_2\}$  are one of the following:

(i)

$$\{\sigma_0, \sigma_1, \sigma_2\} = \{s_0, s_1, s_2\}$$
.

*(ii)* 

$$\{\sigma_0, \sigma_1, \sigma_2\} = \{s_0, s_1, s_{01}\}$$

(iii)

$$\{\sigma_0, \sigma_1, \sigma_2\} = \{s_{01}, s_{02}, s_0\}$$
.

(iv)

$$\{\sigma_0, \sigma_1, \sigma_2\} = \{s_{01}, s_{02}, s_{12}\}.$$

Conversely, any such surface S is a triple K3 burger with  $K^2 = 2$ , and the associated K3 surfaces are obtained as  $\bar{X}_j = S/\langle \sigma_j \rangle$ , where the  $\sigma_j$  are as in (i)–(iv).

*Proof.* Since S is minimal, of general type, with  $K^2 = 2$  and  $p_g = 3$ , we know that S is isomorphic to a smooth degree 8 hypersurface in  $\mathbb{P} := \mathbb{P}(1^3, 4)$  [17]. Since the involutions  $\sigma_j$  (j = 0, 1, 2) preserve the polarization  $K_S$ , they are induced by involutions of  $\mathbb{P}$ . Let  $[x_0 : x_1 : x_2 : x_3]$  be weighted homogeneous coordinates for  $\mathbb{P}$ . After a projective transformation, we may suppose the involutions are defined by adding a minus sign in front of one or two or three of the  $x_i$ , i.e. the  $\sigma_j$  are of the form  $s_i, s_{ij}, s_{012}$ , where  $i, j \in \{0, 1, 2\}$ .

Griffiths residue calculus (which also exists for weighted projective hypersurfaces, cf. [11], [2]) shows that  $H^{0,2}(S)$  is generated by the image under the residue map of the holomorphic forms with poles

(3) 
$$x_0\Omega/f$$
,  $x_1\Omega/f$ ,  $x_2\Omega/f$ .

Here, f is a defining equation for S and  $\Omega$  is the standard 3-form

$$\Omega := \sum_{i=0}^{2} (-1)^{i} x_{i} dx_{0} \wedge \ldots \hat{dx_{i}} \ldots dx_{3} - 4x_{3} dx_{0} \wedge dx_{1} \wedge dx_{2}$$

[11, 2.1.3], [2, Example 9.4].

The involution  $s_{012}$  acts as -1 on the form  $\Omega$ . Hence, the involution  $s_{012}$  acts either as (+1, +1, +1) or as (-1, -1, -1) on the three generators (3) (depending on whether  $s_{012}$  acts as +1 or as -1 on f). As such, the quotient  $S/< s_{012} > can not be a K3$  surface, and so  $s_{012}$  is not among the  $\sigma_j$ .

Suppose now the  $\sigma_j$  are all of type  $s_i$ . The involution  $s_i$  acts on  $\Omega$  as -1, and on f as  $\pm 1$ . Considering the action on generators (3), clearly the only possibility is (i).

Suppose next that exactly one of the  $\sigma_j$  is of type  $s_{ij}$  (and so the others are of type  $s_i$ ). Up to a coordinate change, we may suppose  $\sigma_2 = s_{01}$ . The involution  $s_{01}$  acts on  $\Omega$  as +1, and on f as  $\pm 1$ . Since the quotient  $S/\langle s_{01} \rangle$  is K3, the action on f has to be the identity, and so  $s_{01}$  acts on the generators (3) as (-1, -1, +1). Clearly, the only possibility for  $\{\sigma_0, \sigma_1\}$  is now  $\{s_0, s_1\}$ , and so we are in case (ii).

Next, let us suppose that exactly two of the  $\sigma_j$  are of type  $s_{ij}$ , say  $\sigma_0 = s_{01}$  and  $\sigma_1 = s_{02}$ . As per above, the case  $s_{ij}(f) = -f$  can be excluded. We conclude that  $\sigma_0$  acts on the generators (3) as (-1, -1, +1), and  $\sigma_1$  acts as (-1, +1, -1). The remaining involution  $\sigma_2 = s_i$  should act as (+1, -1, -1), and so  $\sigma_2 = \sigma_0$ , and we are in case (iii).

Finally, if all three  $\sigma_j$  are of type  $s_{ij}$ , they need to be different (for otherwise, there is a generator (3) not preserved by any of the  $\sigma_j$ ). Hence, we are in case (iv).

The converse is clear from the above argument. (Note that the involutions  $\sigma_j$  commute because they commute as automorphisms of  $\mathbb{P}$ .)

**Remark 3.9.** *Triple* K3 *burgers as in proposition* 3.8(*i*) *form a family of moduli dimension* 6. *Indeed, after a change of variables the equation defining* S *is of the form* 

$$(x_3)^2 = f(x_0, x_1, x_2) ,$$

*i.e.* S is a double cover of the plane branched along an octic f, where  $x_0, x_1, x_2$  occur only in even degrees. This family has 6 moduli.

(The degree 8 equation

$$(x_3)^2 = f(x_0, x_1, x_2)$$

(with  $x_0, x_1, x_2$  occurring in even degree) depends on 15 parameters, so smooth hypersurfaces of this type correspond to an open in  $\mathbb{P}^{14}$ . The group PGL(3) acts on these hypersurfaces, and so we get 14 - 8 = 6 moduli.)

One element in this family is the weighted Fermat hypersurface

$$x_0^8 + x_1^8 + x_2^8 + x_3^2 = 0 .$$

The surfaces of proposition 3.8(iii) and (iv) are the same family as that of (i); only the associated K3 surfaces are different, so there are different "burger structures" on elements of this family.

**Proposition 3.10.** Let *S* be a triple *K*3 burger with  $K^2 = 3$  and such that the canonical divisor is base–point free. Then S is isomorphic to a smooth degree 6 hypersurface in  $\mathbb{P}(1^3, 2)$  invariant under  $G = \langle \sigma_0, \sigma_1, \sigma_2 \rangle$ , where  $\{\sigma_0, \sigma_1, \sigma_2\}$  are one of the following: (i)

$$\{\sigma_0, \sigma_1, \sigma_2\} = \{s_0, s_1, s_2\}$$
.

*(ii)* 

$$\{\sigma_0, \sigma_1, \sigma_2\} = \{s_0, s_1, s_{01}\}$$

(iii)

$$\{\sigma_0, \sigma_1, \sigma_2\} = \{s_{01}, s_{02}, s_0\}.$$

(iv)

$$\{\sigma_0, \sigma_1, \sigma_2\} = \{s_{01}, s_{02}, s_{12}\}$$

Conversely, any such surface S is a triple K3 burger with  $K^2 = 3$ , and the associated K3 surfaces are obtained as  $\bar{X}_i = S/\langle \sigma_i \rangle$ , where the  $\sigma_i$  are as in (i)–(iv).

*Proof.* Since S is minimal, of general type, with  $K^2 = p_g = 3$  and base point free canonical divisor, we know that S is isomorphic to a degree 6 hypersurface in  $\mathbb{P}(1^3, 2)$  [23].

To classify the possible involutions, one proceeds exactly as in the proof of proposition 3.8.  $\Box$ 

**Remark 3.11.** Triple K3 burgers with  $K^2 = 3$  and  $K_S$  base–point free form a family of dimension 4. (Indeed, under the natural map

$$\mathbb{P}(1^3, 2) \to \mathbb{P}(2^4) ,$$

the hypersurfaces as in proposition 3.10 correspond to degree 6 hypersurfaces in  $\mathbb{P}(2^4)$ . But under the natural isomorphism

$$\mathbb{P}(2^4) \xrightarrow{\cong} \mathbb{P}(1^4) = \mathbb{P}^3$$
,

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the degree 6 hypersurfaces in  $\mathbb{P}(2^4)$  correspond to degree 3 hypersurfaces in  $\mathbb{P}^3$ , for which there are 4 moduli.)

We note that there is a subfamily given by triple covers of the plane, and this subfamily has moduli dimension 1.

(The degree 6 equation

$$(x_3)^3 = f(x_0, x_1, x_2)$$

with  $x_0, x_1, x_2$  occurring in even degree depends on 10 parameters. We get  $10-1-\dim PGL(3) = 1.$ )

One element in the family (which is also in the subfamily of triple planes) is given by the weighted Fermat hypersurface

$$x_0^6 + x_1^6 + x_2^6 + x_3^3 = 0$$

**Remark 3.12.** I have not been able to classify triple K3 burgers with  $K^2 = 3$  without the assumption that  $K_S$  be base point free. When  $K_S$  is not base–point free, it is known [23] there is exactly one base–point, and the canonical model of S is isomorphic to a bidegree (3,6) complete intersection in  $\mathbb{P}(1^3, 2, 3)$ . However, determining the possible involutions  $\sigma_j$  in this case seems to get messy.

Similarly, triple K3 burgers with  $K^2 = 4$  and  $K_S$  base point free are complete intersections in a weighted projective space [40]. I have not been able to classify them.

3.3. **Families.** This section establishes some notation. The two cases in notation 3.13 correspond to two cases of propositions 3.8 and 3.10.

Notation 3.13. Let

 $\mathcal{S} \rightarrow B$ 

denote one of the following families:

(i) (Case (i) of proposition 3.8) The family of all smooth hypersurfaces in  $\mathbb{P} := \mathbb{P}(1^3, 4)$  of type

$$f_b(x_0, x_1, x_2, x_3) = 0$$
,

where  $f_b$  is weighted homogeneous of degree 8, and  $x_0, x_1, x_2$  occur only in even degree. Let  $S_b$  denote the fibre of S over  $b \in B$ .

(ii) (Case (i) of proposition 3.10) The family of all smooth hypersurfaces in  $\mathbb{P} = \mathbb{P}(1^3, 2)$  of type

$$f_b(x_0, x_1, x_2, x_3) = 0$$
,

where  $f_b$  is weighted homogeneous of degree 6, and  $x_0, x_1, x_2$  occur only in even degree. Let  $S_b$  denote the fibre of S over  $b \in B$ .

**Remark 3.14.** Let  $S \to B$  be the family as in notation 3.13(i) (resp. (ii)). Then any fibre  $S_b$  is a triple K3 burger with  $K^2 = 2$  (resp.  $K^2 = 3$ ). This is immediate from proposition 3.8 (resp. proposition 3.10).

**Lemma 3.15.** Let  $S \to B$  be one of the two families of notation 3.13. The variety S is a smooth quasi-projective variety.

*Proof.* Let us treat case (i); the other case is similar. By construction, there are morphisms

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\pi} & \mathbb{P} \\ \downarrow \nu \\ B \end{array}$$

Let  $\bar{S} \to \bar{B}$  denote the universal family of all (not necessarily smooth) hypersurfaces in  $\mathbb{P}$  of type

$$f_b(x_0, x_1, x_2, x_3) = 0$$

where  $f_b$  is weighted homogeneous of degree 8 and  $x_0, x_1, x_2$  only occur in even degrees. Then  $\overline{B}$  is a projective space containing B as a Zariski open.

**Lemma 3.16.** For any  $x \in \mathbb{P}(1^3, 4)$ , there exists  $b \in \overline{B}$  such that  $x \notin S_b$ .

*Proof.* There is a  $(\mathbb{Z}/2\mathbb{Z})^3$  cover

 $\mathbb{P}(1^3, 4) \to \mathbb{P}(2^3, 4) \cong \mathbb{P}(1^3, 2) =: \mathbb{P}'.$ 

The surfaces in  $\bar{S} \to \bar{B}$  correspond to the complete linear system  $\mathbb{P}H^0(\mathbb{P}', \mathcal{O}_{\mathbb{P}'}(4))$  which is (ample hence) base point free.

Lemma 3.16 ensures that  $\overline{S}$  is a projective bundle over  $\mathbb{P}(1^3, 4)$ , in particular it is a projective quotient variety. Any surface  $S_b$  with  $b \in B$  avoids the singular point of  $\mathbb{P}(1^3, 4)$ , and so S is Zariski open inside a projective bundle over the non–singular locus of  $\mathbb{P}(1^3, 4)$ . It follows that S is smooth.

#### 4. TRIVIAL CHOW GROUPS

This intermediate section contains a result asserting the triviality of a certain Chow group. This result (proposition 4.1) will be the most essential ingredient in the proof of our main result (theorem 5.1 in the next section). The proof of proposition 4.1 occupies subsection 4.2, and uses a stratification argument borrowed from [29].

**Proposition 4.1.** Let  $S \to B$  be a family of surfaces as in notation 3.13. Let  $B^0 \subset B$  be a Zariski open, and let  $S^0 \to B^0$  be the family obtained by restriction. Then

$$A_{hom}^2(\mathcal{S}^0 \times_{B^0} \mathcal{S}^0) = 0$$

## 4.1. Weak and strong property.

**Definition 4.2** (Totaro [48]). For any (not necessarily smooth) quasi-projective variety X, let  $A_i(X, j)$  denote Bloch's higher Chow groups with rational coefficients (these groups are sometimes written  $A^{n-i}(X, j)_{\mathbb{Q}}$  or  $CH^{n-i}(X, j)_{\mathbb{Q}}$ , where  $n = \dim X$ ). As explained in [48, Section 4], the relation with algebraic K-theory ensures there are functorial cycle class maps

$$A_i(X,j) \rightarrow Gr^W_{-2i}H_{2i+j}(X)$$
,

compatible with long exact sequences (here  $W_*$  denotes Deligne's weight filtration on Borel-Moore homology [39]).

We say that X has the weak property if the cycle class maps induce isomorphisms

$$A_i(X) \xrightarrow{-} W_{-2i}H_{2i}(X)$$

for all i.

We say that X has the strong property if X has the weak property, and, in addition, the cycle class maps induce surjections

$$A_i(X,1) \twoheadrightarrow Gr^W_{-2i}H_{2i+1}(X)$$

for all i.

**Lemma 4.3** ([48]). Let X be a quasi-projective variety, and  $Y \subset X$  a closed subvariety with complement  $U = X \setminus Y$ . If X has the strong property and Y has the weak property, then U has the strong property.

Proof. This is [48, Lemma 6].

**Lemma 4.4.** Let X be a quasi-projective variety, and  $Y \subset X$  a closed subvariety with complement  $U = X \setminus Y$ . If Y and U have the strong property, then so does X.

*Proof.* This is the same argument as [48, Lemma 7], which is a slightly different statement. As in loc. cit., using the localization property of higher Chow groups [7], [31], one finds a commutative diagram with exact rows

A diagram chase reveals that under the assumptions of the lemma, the one but last vertical arrow is an isomorphism.

Continuing these long exact sequences to the left, there is a commutative diagram with exact rows

Chasing some more inside this diagram, one finds that the second vertical arrow is a surjection.  $\Box$ 

**Corollary 4.5.** Let X be a quasi-projective variety that admits a stratification such that each stratum is of the form  $\mathbb{A}^k \setminus L$ , where L is a finite union of linearly embedded affine subspaces. Then X has the strong property.

*Proof.* Affine space has the strong property (this is the homotopy invariance for higher Chow groups). The subvariety L has the weak property. Doing a diagram chase as in lemma 4.4 (or directly applying [48, Lemma 6]), it follows that the variety  $\mathbb{A}^k \setminus L$  has the strong property. The corollary now follows from lemma 4.4.

**Lemma 4.6.** Let X be a quasi-projective variety with the strong property. Let  $Y \to X$  be a projective bundle. Then Y has the strong property.

*Proof.* This follows from the projective bundle formula for higher Chow groups [6].  $\Box$ 

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# 4.2. **Proof of proposition 4.1.**

*Proof.* (i)  $(K^2 = 2)$  Let us use the shorthand

$$\mathbb{P} := \mathbb{P}(1^3, 4) ,$$
  

$$M := \mathbb{P} \times \mathbb{P} ,$$
  

$$N := \left\{ (f_b, p, p') \in \bar{B} \times \mathbb{P} \times \mathbb{P} \mid f_b(p) = f_b(p') = 0 \right\} \subset \bar{B} \times M$$

The goal is to prove that

(4)

$$A_{hom}^2(N) \stackrel{??}{=} 0$$
.

This implies proposition 4.1 for case (i), because (4) implies triviality of  $A^2$  of any open in N, and  $S \times_B S$  is an open in N.

Inside M, we have various "partial diagonals"

$$\begin{split} \Delta_{M} &= \Delta_{+++} := \left\{ (p,p') \in \mathbb{P} \times \mathbb{P} \mid p = p' \right\}, \\ \Delta_{+-+} &:= \left\{ (p,p') \in \mathbb{P} \times \mathbb{P} \mid [p_{0}:p_{1}:p_{2}:p_{3}] = [p'_{0}:-p'_{1}:p'_{2}:p'_{3}] \right\}, \\ \Delta_{-++} &:= \left\{ (p,p') \in \mathbb{P} \times \mathbb{P} \mid [p_{0}:p_{1}:p_{2}:p_{3}] = [-p'_{0}:p'_{1}:-p'_{2}:p'_{3}] \right\}, \\ \Delta_{++-} &:= \left\{ (p,p') \in \mathbb{P} \times \mathbb{P} \mid [p_{0}:p_{1}:p_{2}:p_{3}] = [p'_{0}:p'_{1}:-p'_{2}:p'_{3}] \right\}, \\ \Delta_{+--} &:= \left\{ (p,p') \in \mathbb{P} \times \mathbb{P} \mid [p_{0}:p_{1}:p_{2}:p_{3}] = [p'_{0}:-p'_{1}:-p'_{2}:p'_{3}] \right\}, \\ \Delta_{-+-} &:= \left\{ (p,p') \in \mathbb{P} \times \mathbb{P} \mid [p_{0}:p_{1}:p_{2}:p_{3}] = [-p'_{0}:-p'_{1}:-p'_{2}:p'_{3}] \right\}, \\ \Delta_{--+} &:= \left\{ (p,p') \in \mathbb{P} \times \mathbb{P} \mid [p_{0}:p_{1}:p_{2}:p_{3}] = [-p'_{0}:-p'_{1}:p'_{2}:p'_{3}] \right\}, \\ \Delta_{---} &:= \left\{ (p,p') \in \mathbb{P} \times \mathbb{P} \mid [p_{0}:p_{1}:p_{2}:p_{3}] = [-p'_{0}:p'_{1}:p'_{2}:p'_{3}] \right\}, \end{split}$$

(Here, we write  $p = [p_0 : p_1 : p_2 : p_3]$  and  $p' = [p'_0 : p'_1 : p'_2 : p'_3]$ . We observe that the various  $\Delta_{\pm \mp \pm}$  are just the graphs of the elements of the group  $(\mathbb{Z}/2\mathbb{Z})^3 = \langle \sigma_0, \sigma_1, \sigma_2 \rangle \subset \operatorname{Aut}(\mathbb{P})$ .)

Let us define the Zariski opens

$$M^{0} := M \setminus (\cup \Delta_{\pm \mp \pm}) ,$$
  
$$N^{0} := N \setminus \pi^{-1}(\cup \Delta_{\pm \mp \pm})$$

Corollary 4.5 implies that the union  $\cup \Delta_{\pm \mp \pm}$  has the strong property. Since  $M = \mathbb{P} \times \mathbb{P}$  has the strong property, so does  $M^0$  (lemma 4.3). The morphism from  $N^0$  to  $M^0$  has constant dimension (lemma 4.7), so it is a projective bundle and  $N^0$  also has the strong property (lemma 4.6).

### Lemma 4.7. Let

$$(p, p') \in M \setminus (\cup \Delta_{\pm \mp \pm})$$

Then (p, p') imposes 2 independent conditions on  $\overline{B}$ , i.e. there exists  $b \in \overline{B}$  such that  $S_b$  contains p but not p'.

Proof. Consider the map

 $r \times r$ :  $M = \mathbb{P} \times \mathbb{P} \to \mathbb{P}' \times \mathbb{P}'$ ,

where  $\mathbb{P}'$  is as before  $\mathbb{P}(2,2,2,4)$ . The condition  $(p,p') \notin (\cup \Delta_{\pm \mp \pm})$  implies that  $r(p) \neq r(p')$ . Since  $\mathbb{P}'$  is isomorphic to  $\mathbb{P}'' := \mathbb{P}(1,1,1,2)$  (and sections of  $\mathcal{O}_{\mathbb{P}'}(8)$  correspond under this isomorphism to sections of  $\mathcal{O}_{\mathbb{P}''}(4)$ ), lemma 4.8 below shows there exists  $S_b$  separating the points p and p'.

**Lemma 4.8.** Let  $\mathbb{P}''$  be the weighted projective space  $\mathbb{P}(1, 1, 1, 2)$ . Then the line bundle  $\mathcal{O}_{\mathbb{P}''}(4)$  is very ample.

*Proof.* The coherent sheaf  $\mathcal{O}_{\mathbb{P}''}(4)$  is locally free, because 4 is a multiple of the weights [11]. To see that this line bundle is very ample, we use the following numerical criterion:

**Proposition 4.9** (Deforme [10]). Let  $P = \mathbb{P}(q_0, q_1, \ldots, q_n)$  be a weighted projective space. Let *m* be the least common multiple of the  $q_j$ . Suppose every monomial

 $x_0^{b_0} x_1^{b_1} \cdots x_n^{b_n}$ 

of (weighted) degree km ( $k \in \mathbb{N}^*$ ) is divisible by a monomial of (weighted) degree m. Then  $\mathcal{O}_P(m)$  is very ample.

(This is the case E(x) = 0 of [10, Proposition 2.3(iii)].) Using proposition 4.9, lemma 4.8 is now easily established.

Let us now finish the proof of proposition 4.1 for case (i). Any point

 $(p, p') \in M^1 := (\cup \Delta_{\pm \mp \pm}) \subset M$ 

imposes exactly one condition on  $\overline{B}$ ; indeed p imposes one condition (lemma 3.16), and since r(p) = r(p') in  $\mathbb{P}' = \mathbb{P}(2, 2, 2, 4)$ , any  $S_b$  containing p also contains p'. This means that  $N^1$  has the structure of a projective bundle over  $M^1$ . We have seen above that  $M^1$  has the strong property. It follows from lemma 4.6 that

$$N^1 := \pi^{-1}(M^1) \quad \subset N$$

has the strong property. Lemma 4.4 now implies that N has the strong property, and so equality (4) is proven.

(ii)  $(K^2 = 3)$ . Similar to case (i), except that  $\mathbb{P}$  is now  $\mathbb{P}(1^3, 2)$  and the degree of the hypersurfaces is 6. Instead of lemma 4.8, we now use that  $\mathcal{O}_{\mathbb{P}^3}(3)$  is very ample.

### 5. MAIN

**Theorem 5.1.** Let S be a triple K3 burger, and let  $X_j (j = 0, 1, 2)$  be the associated K3 surfaces. Assume that either (i)  $K_S^2 = 2$ , or

(ii)  $\tilde{K}_S^2 = 3$  and the canonical map is base point free.

Then there is an isomorphism

$$(\Psi_0)_* + (\Psi_1)_* + (\Psi_2)_* : A^2_{hom}(X_0) \oplus A^2_{hom}(X_1) \oplus A^2_{hom}(X_2) \xrightarrow{\cong} A^2_{hom}(S)$$

Proof. First, a reduction step. Let us define eigenspaces

$$A^{2}(S)^{\pm \mp \pm} := \left\{ a \in A^{2}(S) \mid (\sigma_{0})^{*}(a) = \pm a , \ (\sigma_{1})^{*}(a) = \mp a , \ (\sigma_{2})^{*}(a) = \pm a \right\}.$$

We now make the following claim:

**Claim 5.2.** Let S be as in theorem 5.1. Any eigenspace with an odd number of minus signs is trivial, i.e.

$$A^{2}(S)^{---} = A^{2}(S)^{-++} = A^{2}(S)^{+-+} = A^{2}(S)^{++-} = 0.$$

Moreover,

$$A_{hom}^2(S)^{+++} = 0 \; .$$

Before proving the claim, let us verify that the claim suffices to prove the theorem: the claim implies there is a decomposition

(5) 
$$A_{hom}^2(S) = A_{hom}^2(S)^{+--} \oplus A_{hom}^2(S)^{-+-} \oplus A_{hom}^2(S)^{--+}$$

Also, since necessarily

$$(\Psi_0)_* A^2(S) \subset A^2(S)^{+\pm\pm},$$

the claim implies that

$$(\Psi_0)_* A^2_{hom}(S) \subset A^2(S)^{+--}$$

What's more, since

$$(\Psi_0)_*(\Psi_0)^* = 2 \operatorname{id}: A^2(S)^{+\pm\pm} \to A^2(S)^{+\pm\pm},$$

there is actually equality

$$(\Psi_0)_*(\Psi_0)^* A_{hom}^2(S) = A^2(S)^{+--}$$
.

(And similarly, for reasons of symmetry,

$$(\Psi_1)_*(\Psi_1)^* A_{hom}^2(S) = A^2(S)^{-+-},$$
  

$$(\Psi_2)_*(\Psi_2)^* A_{hom}^2(S) = A^2(S)^{--+}.$$

Therefore, the decomposition (5) is equivalent to the decomposition

$$A_{hom}^2(S) = (\Psi_0)_*(\Psi_0)^* A_{hom}^2(S) \oplus (\Psi_1)_*(\Psi_1)^* A_{hom}^2(S) \oplus (\Psi_2)_*(\Psi_2)^* A_{hom}^2(S) .$$

This proves the surjectivity statement of the theorem

Again using the claim, one deduces that the composition

$$\begin{array}{c} A_{hom}^{2}(X_{0}) \oplus A_{hom}^{2}(X_{1}) \oplus A_{hom}^{2}(X_{2}) \xrightarrow{(\Psi_{0})_{*} + (\Psi_{1})_{*} + (\Psi_{2})_{*}} A_{hom}^{2}(S) \\ \xrightarrow{((\Psi_{0})^{*}, (\Psi_{1})^{*}, (\Psi_{2})^{*})} A_{hom}^{2}(X_{0}) \oplus A_{hom}^{2}(X_{1}) \oplus A_{hom}^{2}(X_{2}) \end{array}$$

equals twice the identity. This proves the injectivity statement of the theorem.

#### ROBERT LATERVEER

It remains to prove the claim. First, let us treat case (ii) of propositions 3.8 and 3.10. In this case,  $\sigma_2 = \sigma_0 \circ \sigma_1$  (i.e.,  $G := \langle \sigma_0, \sigma_1, \sigma_2 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$ ), and so the first part of the claim is trivially true. The second part of the claim is also true for these cases: indeed, there is equality

$$A_{hom}^2(S)^{+++} = A_{hom}^2(S/G)$$

But the surface S/G is a degree 8 hypersurface in  $\mathbb{P}(1, 2, 2, 4)$  (resp. a degree 6 hypersurface in  $\mathbb{P}(1, 2^3)$ ), and so S/G is a surface with quotient singularities and ample anticanonical bundle. Such surfaces are rational [59, Theorem 2.3], and hence  $A_{hom}^2(S/G) = 0$ .

Next, let us consider the cases (i), (iii) and (iv) of propositions 3.8 and 3.10. In this case, the surfaces  $S_b$  are elements of the families of notation 3.13. The argument, in a nutshell, is now as follows: the correspondences  $\Psi_j$  exist as relative correspondences for the whole family of triple K3 burgers. Using the trivial Chow groups result (proposition 4.1), one can upgrade a vanishing in cohomology to a vanishing of Chow groups.

We now proceed to prove claim 5.2 for surfaces as in proposition 3.8(i), (iii) and (iv). (The cases of proposition 3.10(i), (iii) and (iv) are mostly the same, modulo some mutatis mutandis which we will indicate below).

Cases (i), (iii), (iv) of proposition 3.8: Let

$$\mathcal{S} \rightarrow B$$

denote the universal family of surfaces as in notation 3.13(i). Let  $\{\sigma_0, \sigma_1, \sigma_1\}$  be either  $\{s_0, s_1, s_2\}$  or  $\{s_{01}, s_{02}, s_0\}$ , and let

$$\mathcal{X}_j := \mathcal{S}/\sigma_j \quad (j = 0, 1, 2)$$

denote the universal families of associated K3 surfaces as in notation 3.13. For any  $b \in B$ , we will write  $S_b$  for the fibre of S over b, and  $X_{0b}$  (resp.  $X_{1b}$  resp.  $X_{2b}$ ) for the fibre of  $\mathcal{X}_0$  (resp.  $\mathcal{X}_1$  resp.  $\mathcal{X}_2$ ) over b. Likewise, we will write  $\sigma_{0b}$ ,  $\sigma_{1b}$ ,  $\sigma_{2b}$  for the restriction of  $\sigma_0$  (resp.  $\sigma_1$  resp.  $\sigma_2$ ) to  $S_b$ . For a relative correspondence  $\Gamma \in A^*(S \times_B S)$ , we will use the shorthand

$$\Gamma|_b := \Gamma|_{S_b \times S_b} \in A^*(S_b \times S_b)$$

for the restriction (i.e., the image of  $\Gamma$  under the Gysin homomorphism induced by the inclusion  $b \hookrightarrow B$ ).

By definition (cf. remark 3.2), we know that there is a fibrewise isomorphism

(6) 
$$H^{2}_{tr}(S_{b}) \cong H^{2}_{tr}(S_{b})^{+--} \oplus H^{2}_{tr}(S_{b})^{-+-} \oplus H^{2}_{tr}(S_{b})^{--+} \\ \cong H^{2}_{tr}(X_{0b}) \oplus H^{2}_{tr}(X_{1b}) \oplus H^{2}_{tr}(X_{2b}) \quad \forall b \in B .$$

That is, there are no eigenspaces with an odd number of minus signs:

(7) 
$$H_{tr}^{2}(S_{b})^{---} = H_{tr}^{2}(S_{b})^{-++} = H_{tr}^{2}(S_{b})^{+-+} = H_{tr}^{2}(S_{b})^{++-} = 0 \quad \forall b \in B.$$

Also, there is no eigenspace without minus signs:

(8) 
$$H^2_{tr}(S_b)^{+++} = 0 \quad \forall b \in B .$$

Let us define a relative correspondence

$$\Gamma^{---} := \frac{1}{8} (\Delta_{\mathcal{S}} - \Gamma_{\sigma_0}) \circ (\Delta_{\mathcal{S}} - \Gamma_{\sigma_1}) \circ (\Delta_{\mathcal{S}} - \Gamma_{\sigma_2}) \circ \pi_2^{\mathcal{S}} \in A^2(\mathcal{S} \times_B \mathcal{S}) .$$

(For details on the formalism of relative correspondences and their composition, cf. [35, Chapter 8] whose conventions are met with in our set–up.)

We observe that for any  $b \in B$ , the restriction

$$\Gamma^{---}|_b \in A^2(S_b \times S_b)$$

is a projector on  $H^2(S_b)^{---}$ .

In terms of correspondences, the vanishing  $H_{tr}^2(S_b)^{---} = 0$  in (7) is equivalent to the statement that

$$(\Gamma^{---}|_b) \circ \pi^{S_b}_{2,tr} = 0 \text{ in } H^4(S_b \times S_b) \quad \forall b \in B.$$

(Here,  $\pi_{2,tr}^{S_b}$  is a projector defining the transcendental part of the motive as in theorem 2.2.) This is in turn equivalent to the statement that for any  $b \in B$ , there exists a divisor  $D_b \subset S_b$ , and a cycle  $\gamma_b$  supported on  $D_b \times D_b \subset S_b \times S_b$ , such that

$$(\Gamma^{---}|_b) \circ \pi_2^{S_b} = \gamma_b \text{ in } H^4(S_b \times S_b)$$

Using a Baire category argument as in [54, Proposition 3.7] or [57, Lemma 1.4], these data can be "spread out" over the base B, i.e. one can find a divisor  $\mathcal{D} \subset S$  and a cycle  $\gamma$  supported on  $\mathcal{D} \times_B \mathcal{D} \subset S \times_B S$  such that

$$(\Gamma^{---} \circ \pi_2^{\mathcal{S}})|_b = \gamma|_b \text{ in } H^4(S_b \times S_b) \quad \forall b \in B.$$

In other words, the relative correspondence

$$\Gamma := \Gamma^{---} \circ \pi_2^{\mathcal{S}} - \gamma \quad \in A^2(\mathcal{S} \times_B \mathcal{S})$$

is fibrewise homologically trivial:

$$\Gamma|_b \in A^2_{hom}(S_b \times S_b) \quad \forall b \in B$$

The next step is to make  $\Gamma$  globally homologically trivial. Employing a Leray spectral sequence argument as in [54, Lemmas 3.11 and 3.12], this can be done by adding a cycle coming from the ambient space  $\mathbb{P}$ . More precisely, the argument of [54, Lemmas 3.11 and 3.12] proves the following: up to shrinking the base (i.e., after replacing *B* by a dense Zariski open  $B' \subset B$ , and writing B := B' for simplicity), there exists  $\delta \in A^2(\mathbb{P} \times \mathbb{P})$  such that

$$\Gamma + (\delta \times B)|_{\mathcal{S} \times_B \mathcal{S}} \quad \in A^2_{hom}(\mathcal{S} \times_B \mathcal{S})$$

In view of the fact that  $A_{hom}^2(\mathcal{S} \times_B \mathcal{S}) = 0$  (proposition 4.1), it follows that

$$\Gamma + (\delta \times B)|_{\mathcal{S} \times_B \mathcal{S}} = 0 \text{ in } A^2(\mathcal{S} \times_B \mathcal{S}) .$$

We know that for any  $b \in B$ , the restriction  $\delta|_b$  acts trivially on  $A^2_{hom}(S_b)$  (the action factors over  $A^*_{hom}(\mathbb{P}) = 0$ ). The above thus implies in particular that

$$(\Gamma|_b)_* = 0: \quad A^2_{hom}(S_b) \rightarrow A^2_{hom}(S_b) \quad \forall b \in B.$$

By definition of  $\Gamma$ , this means that

$$(\Gamma^{---}|_b - \gamma|_b)_* = 0: \quad A^2_{hom}(S_b) \rightarrow A^2_{hom}(S_b) \quad \forall b \in B.$$

Since for  $b \in B$  general, the restriction  $\gamma|_b$  will still be supported on (divisor)×(divisor), we know that

$$(\gamma|_b)_* = 0: \quad A^2_{hom}(S_b) \rightarrow A^2_{hom}(S_b) \text{ for general } b \in B$$
.

Thus, the above simplifies to

$$((\Gamma^{---} \circ \pi_2^{\mathcal{S}})|_b)_* = 0: \quad A^2_{hom}(S_b) \to A^2_{hom}(S_b) \text{ for general } b \in B$$

Using a Baire category argument as in [12, Lemma 3.1], this can be extended to *all* elements of the base *B*: we actually have

$$\left( \left( \Gamma^{---} \circ \pi_2^{\mathcal{S}} |_b \right)_* = 0 \colon A^2_{hom}(S_b) \to A^2_{hom}(S_b) \quad \forall b \in B , \right.$$

where B is now once more (as in the beginning of the proof) the parameter space parametrizing all triple K3 burgers as in notation 3.13.

By construction  $\Gamma^{---}|_b$  acts on  $A^2(S_b)^{---}$  as the identity, and

$$(\Gamma^{---} \circ \pi_2^{\mathcal{S}})|_b = \Gamma^{---}|_b \circ \pi_2^{S_b}$$

acts on  $A_{hom}^2(S_b)^{---}$  as the identity. The above thus implies the vanishing

$$A_{hom}^2(S_b)^{---} = 0 \quad \forall b \in B ,$$

which proves the first part of the claim. The other parts of the claim are proven similarly, by choosing a different correspondence: e.g., for the second vanishing statement one considers the relative correspondence

$$\Gamma^{-++} := \frac{1}{8} (\Delta_{\mathcal{S}} - \Gamma_{\sigma_0}) \circ (\Delta_{\mathcal{S}} + \Gamma_{\sigma_1}) \circ (\Delta_{\mathcal{S}} + \Gamma_{\sigma_2}) \circ \pi_2^{\mathcal{S}} \quad \in A^2(\mathcal{S} \times_B \mathcal{S}) .$$

<u>Cases (i), (iii), (iv) of proposition 3.10</u>: The claim is proven by the same argument as in case (i), applied to the family  $S \to B$  as specified in notation 3.13. The weighted projective space  $\mathbb{P}$  now has different weights, and the defining equation has a different degree. The trivial Chow groups statement (proposition 4.1) still holds for this family.

## 6. COROLLARIES

6.1. The canonical 0-cycle. In this subsection, we work with integral Chow groups  $A^i()_{\mathbb{Z}}$ , instead of Chow groups with rational coefficients. Let S be a triple K3 burger as in theorem 5.1. Thanks to Rojtman's theorem [42], theorem 5.1 implies that

$$A^2(S)^{+++}_{\mathbb{Z}} \cong \mathbb{Z}$$
.

**Definition 6.1.** Let S be a triple K3 burger as in theorem 5.1. The canonical 0-cycle  $\mathfrak{o}_S$  is defined as the unique degree 1 cycle such that

$$A^2(S)_{\mathbb{Z}}^{+++} = \mathbb{Z}[\mathfrak{o}_S]$$

(where  $A^2(S)_{\mathbb{Z}}^{+++}$  denotes as before the subspace where  $\sigma_j$  acts as the identity for j = 0, 1, 2). Equivalently,  $\mathfrak{o}_S$  is the unique degree 1 cycle z satisfying

$$(\Psi_j)^*(z) = \mathfrak{o}_{X_j} \text{ in } A^2(X_j)_{\mathbb{Z}} (j = 0, 1, 2),$$

where  $X_j$  are the associated K3 surfaces and the correspondences  $\Psi_j \in A^2(X_j \times S)_{\mathbb{Z}}$  are as above.

Equivalently,  $o_S$  is the unique degree 1 cycle *z* satisfying

$$(\Psi_j)_*(\Psi_j)^*(z) = 2z \text{ in } A^2(S)_{\mathbb{Z}} (j = 0, 1, 2)$$

The equivalences in definition 6.1 are valid because of the following lemma:

**Lemma 6.2.** Let S be a triple K3 burger as in theorem 5.1. Then

$$(\Psi_0)_*(\mathfrak{o}_{X_0}) = (\Psi_1)_*(\mathfrak{o}_{X_1}) = (\Psi_2)_*(\mathfrak{o}_{X_2}) \in A^2(S)_{\mathbb{Z}}.$$

*Proof.* The point is that there is a commutative diagram of surfaces



where all arrows are degree 2 morphisms, and  $A^2(W)_{\mathbb{Z}} = \mathbb{Z}$ . (In case (i) of theorem 5.1, the surface W is defined as the degree 8 hypersurface in  $\mathbb{P}(2^3, 4)$  defined by the equation  $f(t_0, t_1, t_2, x_3) = 0$ , where  $f(x_0^2, x_1^2, x_2^2, x_3) = 0$  is a defining equation for S. For cases (ii) and (iii), the construction is similar.)

Let us pick two divisors D, D' on W, and set

$$w := D \cdot D' \quad \in A^2(W) \; .$$

The pullbacks to the various  $\bar{X}_j$  are intersections of divisors, and so

$$(r_j)^*(w) = d \mathfrak{o}_{\bar{X}_j}$$
 in  $A^2(\bar{X}_j)$   $(j = 0, 1, 2)$ 

(Here,  $d = \deg(D \cdot D')$ , and we define  $\mathfrak{o}_{\bar{X}_j}$  to be  $(q_j)_*(\mathfrak{o}_{X_j})$ .) This implies that

$$(\Psi_j)_*(d \mathfrak{o}_{X_j}) = (p_j)^*(d \mathfrak{o}_{\bar{X}_j}) = (r_j \circ p_j)^*(w) \text{ in } A^2(S)_{\mathbb{Z}} \quad (j = 0, 1, 2) ,$$

and so

$$d(\Psi_0)_*(\mathfrak{o}_{X_0}) = d(\Psi_1)_*(\mathfrak{o}_{X_1}) = d(\Psi_2)_*(\mathfrak{o}_{X_2}) \in A^2(S)_{\mathbb{Z}}$$

Using Rojtman's theorem [42], this proves the lemma.

We now recall the definition of the "effective orbit under rational equivalence" of a 0-cycle:

**Definition 6.3** (Voisin [56]). Let S be any surface. Given a cycle  $z \in A^2(S)_{\mathbb{Z}}$  of degree  $k \ge 0$ , we define the "effective orbit"  $O_z$  as

$$O_z := \bigcup_{z' \in X^{(k)}, z' \sim_{\operatorname{rat}} z} \operatorname{supp}(z') \quad \subset X^{(k)} \,.$$

(Here, the union is taken over all k-tuples of points z' such that the 0-cycle associated to z' is rationally equivalent to the 0-cycle z in X.)

One defines

$$\dim O_z := \sup_{V \subset O_z} \dim V \; ,$$

where the supremum runs over all irreducible components  $V \subset O_z$  (we note that  $O_z$  is known to be a countable union of closed subvarieties, so this is well-defined).

Inspired by [56], one can give a nice characterization of the canonical 0-cycle  $o_S$ :

**Proposition 6.4.** Let S be a triple K3 burger as in theorem 5.1. Let k > 0 be an integer. Then  $k\mathfrak{o}_S$  is the unique degree k 0-cycle  $z \in A^2(S)_{\mathbb{Z}}$  satisfying dim  $O_z \ge k$ .

*Proof.* We actually prove a somewhat more general statement, which is based on Voisin's result [56, Theorem 1.4]. This result of Voisin's gives an alternative description of O'Grady's filtration  $S_d^k()$  on the Chow group of 0-cycles of a K3 surface, in terms of effective orbits. We recall that for any K3 surface X, O'Grady's filtration [36] is defined as

(9) 
$$S_d^k(X) := \left\{ z \in A^2(X)_{\mathbb{Z}} \mid z = z' + (k-d)\mathfrak{o}_X \right\},$$

where z' is effective of degree d and  $\mathfrak{o}_X$  is the canonical 0-cycle.

Voisin gives an interesting alternative description of the O'Grady filtration: for any  $k > d \ge 0$ , she proves [56, Theorem 1.4] that

(10) 
$$S_d^k(X) = \left\{ z \in A^2(X)_{\mathbb{Z}} \mid O_z \subset X^{(k)} \neq \emptyset \text{ and } \dim O_z \ge k - d \right\}.$$

Let us now consider a triple K3 burger S as in theorem 5.1. The canonical 0-cycle  $\mathfrak{o}_S$  exists, and so definition (9) makes sense for S.

Step 1 (Unicity): Let  $z \in A^2(S)_{\mathbb{Z}}$  of degree k, and let us assume that the orbit  $O_z \subset S^{(k)}$  is non-empty of dimension  $\geq k - d$ , for some  $k > d \geq 0$ . According to (the proof of) theorem 5.1, we can write z uniquely as

$$z = k \mathfrak{o}_S + z_0 + z_1 + z_2$$
 in  $A^2(S)_{\mathbb{Z}}$ ,

where  $z_0 \in A^2_{hom}(S)^{+--}_{\mathbb{Z}}$  and  $z_1, z_2$  are in  $A^2_{hom}(S)^{-+-}_{\mathbb{Z}}$  resp. in  $A^2_{hom}(S)^{-++}_{\mathbb{Z}}$ . The assumption on  $O_z$  implies that the cycles

$$(\Psi_j)^*(z) = k\mathfrak{o}_{X_j} + (\Psi_j)^*(z_j) \in A^2(X_j)_{\mathbb{Z}} \quad (j = 0, 1, 2)$$

also have orbits  $O_{z_i}$  of dimension  $\geq k - d$ . Therefore, Voisin's result (10) implies that

$$(\Psi_j)^*(z) \in S_d^k(X_j) \ (j=0,1,2),$$

i.e. one can write

$$(\Psi_j)^*(z) = k \mathfrak{o}_{X_j} + (\Psi_j)^*(z_j) = z'_j + (k-d)\mathfrak{o}_{X_j} \quad \text{in } A^2(X_j)_{\mathbb{Z}} \quad (j=0,1,2) ,$$

where  $z'_i$  is effective of degree d. It follows that

$$(\Psi_j)^*(z_j) = z'_j - d \mathfrak{o}_{X_j} \text{ in } A^2(X_j)_{\mathbb{Z}} (j = 0, 1, 2).$$

Using the proof of theorem 5.1, we find that

$$\begin{aligned} 2z &= 2k \,\mathfrak{o}_S + 2z_0 + 2z_1 + 2z_2 \\ &= 2k \,\mathfrak{o}_S + (\Psi_0)_* (\Psi_0)^* (z_0) + (\Psi_1)_* (\Psi_1)^* (z_1) + (\Psi_2)_* (\Psi_2)^* (z_2) \\ &= 2k \,\mathfrak{o}_S (\Psi_0)_* (z'_0 - d \,\mathfrak{o}_{X_0}) + (\Psi_1)_* (z'_1 - d \,\mathfrak{o}_{X_1}) + (\Psi_2)_* (z'_2 - d \,\mathfrak{o}_{X_2}) \\ &= 2(k - 3d) \mathfrak{o}_S + b_0 + b_1 + b_2 \quad \text{ in } A^2(S)_{\mathbb{Z}} ,\end{aligned}$$

where  $b_0 + b_1 + b_2$  is effective of degree 6d. That is, we have

$$2z \in S^{2k}_{6d}(S)$$
.

In particular, taking d = 0 we obtain the following implication: if z is a degree k cycle with orbit  $O_z$  of dimension  $\geq k$ , then

$$2z = 2k \mathfrak{o}_S \text{ in } A^2(S)_{\mathbb{Z}}$$
.

As  $A_{hom}^2(S)_{\mathbb{Z}}$  is torsion free, it follows that

$$z = k \mathfrak{o}_S$$
 in  $A^2(S)_{\mathbb{Z}}$ .

Step 2 (Existence): We now prove that the cycle  $z = k \mathfrak{o}_S$  has orbit of dimension  $\geq k$ . This is the easier direction. Take  $j \in \{0, 1, 2\}$ , and let  $\overline{C} \subset \overline{X}_j$  be any rational curve. Using lemma 6.2, one finds that the curve  $C := (p_j)^{-1}(\overline{C}) \subset S$  is a constant cycle curve, and any point  $p \in C$  is such that  $(\Psi_j)^*(p) = \mathfrak{o}_{X_j}$  and so p represents  $\mathfrak{o}_S$ . This proves the statement for k = 1. For k > 1, one notes that  $C^{(k)} \subset S^{(k)}$  is contained in the orbit of  $k \mathfrak{o}_S$ .

Let Z be any smooth projective variety (say of dimension n), and let  $z \in A^n_{hom}(Z)$  be a degree 0 0-cycle. It is known that z is *smash-nilpotent*, meaning that

$$z^{\times(N)} := \underbrace{z \times \cdots \times z}_{(N \text{ times})} = 0 \text{ in } A^{Nn}(Z^n)$$

for N >> 0 [51], [52]. In the special case of the varieties under consideration in this note, one can give a precise estimate for the smash–nilpotence index N:

**Proposition 6.5.** Let S be a triple K3 burger as in theorem 5.1. Let  $z \in A^2_{hom}(S)$  be a 0-cycle of the form

$$z = z' - d\mathfrak{o}_S \quad \in A^2_{hom}(S)$$

where z' is an effective cycle of degree d. Then

$$z^{\times(3d+1)} := \underbrace{z \times \cdots \times z}_{((3d+1) \text{ times})} = 0 \text{ in } A^{6d+2}(S^{3d+1}) .$$

*Proof.* The assumption means that z is in the subgroup  $S_d^0(S)$  of the O'Grady filtration mentioned in the proof of proposition 6.4 above. This implies that

$$(\Psi_j)^*(z) \in S^0_d(X_j), \quad j = 0, 1, 2.$$

For any positive integer r, theorem 6.11 gives an isomorphism of Chow motives

$$t(S^r) \cong \bigoplus_{r_0+r_1+r_2=r} t((X_0)^{r_0}) \otimes t((X_1)^{r_1}) \otimes t((X_2)^{r_2}) \quad \text{in } \mathcal{M}_{\text{rat}}$$

(induced by the  $\Psi_i$ ), and so there is an isomorphism of Chow groups

$$\sum_{\substack{r_0+r_1+r_2=r\\ A^{2r}(t(S)^{\otimes r}) \xrightarrow{\cong} \\ r_0+r_1+r_2=r}} \left( ((\Psi_1)^{r_1})^*, ((\Psi_2)^{r_2})^* \right) :$$

In particular, this implies that there is an injection

(11) 
$$\sum_{\substack{r_0+r_1+r_2=r\\ A^{2r}(t(S)^{\otimes r}) \hookrightarrow \bigoplus_{r_0+r_1+r_2=r} A^{2r_0}((X_0)^{r_0}) \otimes A^{2r_1}((X_1)^{r_1}) \otimes A^{2r_2}((X_2)^{r_2})} A^{2r_2}(X_2)^{r_2} \dots$$

Consider now the element  $z^{\times r}$  for  $r \ge 3d + 1$ . Since  $z \in A^2_{hom}(S) = A^2(t(S))$ , we have  $z^{\times r} \in A^{2r}(t(S)^{\otimes r})$ .

The image of  $z^{\times r}$  in the right-hand side of the injection (11) is a sum of 0-cycles on the various products  $(X_0)^{r_0} \times (X_1)^{r_1} \times (X_2)^{r_2}$ . In each summand, one of the integers  $r_0, r_1, r_2$  must be  $\geq d + 1$ . The proposition now follows from the following lemma:

**Lemma 6.6** (O'Grady [36]). Let X be a K3 surface, and let  $z \in S_d^0(X)$ . Then  $z^{\times (d+1)} = 0$  in  $A^{2d+2}(X^{d+1})$ .

*Proof.* This is established in [36, (5.0.1)]. The reason is that z can be represented by a degree 0 0-cycle w on a curve  $C \subset X$  of geometric genus d. This proves the lemma, for it is known since [52] that  $w^{\times (d+1)} = 0$  in  $A^{d+1}(C^{d+1})$ .

#### 6.2. The canonical 0-cycle, bis.

**Definition 6.7.** Let S be a triple K3 burger, and let  $X_j$  (j = 0, 1, 2) be the associated K3 surfaces. By definition, the subgroup of K3-type divisors  $A_{K3}^1(S)_{\mathbb{Z}} \subset A^1(S)_{\mathbb{Z}}$  is defined as

$$A_{K3}^{1}(S)_{\mathbb{Z}} := \left( (\Psi_{0})_{*}A^{1}(X_{0})_{\mathbb{Z}} \cap (\Psi_{1})_{*}A^{1}(X_{1})_{\mathbb{Z}} \right) + \left( (\Psi_{0})_{*}A^{1}(X_{0})_{\mathbb{Z}} \cap (\Psi_{2})_{*}A^{1}(X_{2})_{\mathbb{Z}} \right) + \left( (\Psi_{1})_{*}A^{1}(X_{1})_{\mathbb{Z}} \cap (\Psi_{2})_{*}A^{1}(X_{2})_{\mathbb{Z}} \right).$$

That is,

$$A_{K3}^{1}(S)_{\mathbb{Z}} = A^{1}(S)_{\mathbb{Z}}^{+++} \oplus A^{1}(S)_{\mathbb{Z}}^{++-} \oplus A^{1}(S)_{\mathbb{Z}}^{+-+} \oplus A^{1}(S)_{\mathbb{Z}}^{-++}$$

**Proposition 6.8.** Let S be a triple K3 burger as in theorem 5.1. Let  $D, D' \in A^1_{K3}(S)_{\mathbb{Z}}$ . Then

$$D \cdot D' = \deg(D \cdot D') \mathfrak{o}_S$$
 in  $A^2(S)_{\mathbb{Z}}$ .

*Proof.* Since  $A_{hom}^2(S)_{\mathbb{Z}}$  is torsion free [42], it suffices to prove the statement for Chow groups with  $\mathbb{Q}$ -coefficients. We have seen that

$$A^{1}_{K3}(S) = A^{1}(S)^{+++} \oplus A^{1}(S)^{++-} \oplus A^{1}(S)^{+-+} \oplus A^{1}(S)^{-++}$$

Assuming that D and D' are in the same summand of this decomposition, we have

$$D \cdot D' \in A^2(S)^{+++} = \mathbb{Q}[\mathfrak{o}_S],$$

and we are done.

Next, let us assume D is in the first summand and D' is in another summand (say the second). Then

$$D \cdot D' \in A^2(S)^{++-}$$

But  $A^2(S)^{++-} = 0$  (proof of theorem 5.1), and so  $D \cdot D' = 0$ .

Finally, let us assume D and D' are in two different summands and neither is in the first summand (say  $D \in A^1(S)^{+-+}$  and  $D' \in A^1(S)^{-++}$ ). Then

$$D \cdot D' \in A^2(S)^{--+}$$
.

We have seen (proof of theorem 5.1) that  $A^2(S)^{--+}$  is mapped isomorphically (under  $(\Psi_2)^*$ ) to  $A^2_{hom}(X_2)$ , and so to prove that  $D \cdot D' = 0$ , it suffices to prove that

$$(\Psi_2)^*(D \cdot D') \stackrel{??}{=} 0 \text{ in } A^2_{hom}(X_2)$$

To this end, recall that (by construction)  $(\Psi_2)^* = (q_2)^*(p_2)_*$  (where  $p_2: S \to \overline{X}_2$  is projection to the K3 surface with double points, and  $q_2: X_2 \to \overline{X}_2$  is a resolution of singularities). Hence,

$$(\Psi_2)^*(D \cdot D') = (q_2)^*(p_2)_*(D \cdot D')$$
  
=  $(q_2)^*(\bar{F} \cdot (p_2)_*(D'))$   
=  $(q_2)^*(\bar{F}) \cdot (q_2)^*(p_2)_*(D')$   
=  $0$  in  $A_{hom}^2(X_2)$ .

Here,  $\overline{F} \in A^1(\overline{X}_2)$  is a divisor such that  $D = (p_2)^*(\overline{F})$ . The last line follows from the celebrated Beauville–Voisin result that

$$(A^1(X_2) \cdot A^1(X_2)) \cap A^2_{hom}(X_2) = 0$$

for any K3 surface  $X_2$  [3].

**Remark 6.9.** The behaviour displayed in proposition 6.8 is remarkable, because the dimension of  $A_{K3}^1(S)$  tends to be large. For example, let S be a triple K3 burger with  $K^2 = 2$ . Then  $A^1(S)^{+++}$  coincides with  $A^1(T)$ , where

$$T := S / < \sigma_0, \sigma_1, \sigma_2 > .$$

The surface T can be identified with a degree 4 hypersurface in  $\mathbb{P}(1^3, 2)$ . Hence, T is isomorphic to the double cover of  $\mathbb{P}^2$  branched along a quartic curve. In case the quartic curve is smooth, one has dim  $A^1(T) = \dim H^2(T) = 8$  [46], and so

$$\dim A^{1}_{K3}(S) \ge \dim A^{1}(S)^{+++} = 8.$$

### 6.3. Bloch conjecture.

**Corollary 6.10.** Let S be a triple K3 burger as in theorem 5.1, and let  $\sigma_0, \sigma_1, \sigma_2$  be the three covering involutions. Let  $f \in Aut(S)$  be a finite-order automorphism that commutes with the  $\sigma_j$ , and such that

$$f^* = \mathrm{id}: \quad H^{2,0}(S) \to H^{2,0}(S)$$

Then also

$$f^* = \operatorname{id}: A^2(S) \to A^2(S)$$
.

*Proof.* Since f commutes with the  $\sigma_j$ , f induces finite-order automorphisms  $f_j \in Aut(X_j)$ , j = 0, 1, 2 that are symplectic. Huybrechts has proven [20] that one has

$$(f_j)^* = \mathrm{id}: A^2(X_j) \to A^2(X_j) \ (j = 0, 1, 2).$$

Theorem 5.1, combined with the commutative diagram

$$\begin{array}{cccc} A_{hom}^2(S) & \xrightarrow{f^*} & A_{hom}^2(S) \\ \uparrow (\Psi_j)^* & \uparrow (\Psi_j)^* & (j = 0, 1, 2) \\ A_{hom}^2(X_j) & \xrightarrow{(f_j)^*} & A_{hom}^2(X_j) \end{array}$$

implies that

$$f^* = \mathrm{id}: A^2_{hom}(S) \to A^2_{hom}(S)$$

Since the 1-dimensional subspace  $A^2(S)^{+++}$  is fixed by f, this proves the corollary.

# 6.4. Finite-dimensionality.

**Corollary 6.11.** Let S be a triple burger as in theorem 5.1, and let  $X_j$  be the associated K3 surfaces. The morphism of Chow motives

$$(\Psi_0, \Psi_1, \Psi_2)$$
:  $t(X_0) \oplus t(X_1) \oplus t(X_2) \rightarrow t(S)$  in  $\mathcal{M}_{rat}$ 

is an isomorphism. (Here, t() denotes the transcendental part of the motive, as in theorem 2.2.)

*Proof.* We may suppose S and the  $X_j$  are defined over some subfield  $k \subset \mathbb{C}$  which is finitely generated over  $\mathbb{Q}$ . To prove the isomorphism of motives, it suffices to prove there is an isomorphism

$$((\Psi_0)_*, (\Psi_1)_*, (\Psi_2)_*): \quad A^2_{hom}((X_0)_K) \oplus A^2_{hom}((X_1)_K) \oplus A^2_{hom}((X_2)_K) \xrightarrow{\cong} A^2_{hom}(S_K)$$

for all function fields K = k(Z) of varieties Z defined over k [22, Lemma 1.1]. This is equivalent to proving claim 5.2 for the surface  $S_K$ . Since  $\mathbb{C}$  is a universal domain, one can choose an embedding  $K \subset \mathbb{C}$ . As is well-known (cf. [5, Appendix to Lecture 1]), this induces an injection

$$A^2(S_K) \hookrightarrow A^2(S_{\mathbb{C}}),$$

and so claim 5.2 for  $S_K$  follows from claim 5.2 for  $S_{\mathbb{C}}$ .

**Corollary 6.12.** Let S be as in theorem 5.1, and assume

$$\dim H^2_{tr}(S) \le 7 \; .$$

Then S has finite-dimensional motive (in the sense of Kimura [27]). What's more, S has motive of abelian type (in the sense of [49]).

*Proof.* Let  $X_0, X_1, X_2$  be the associated K3 surfaces. Recall (proposition 3.2) that there is an isomorphism

$$H^{2}_{tr}(S) \cong H^{2}_{tr}(X_{0}) \oplus H^{2}_{tr}(X_{1}) \oplus H^{2}_{tr}(X_{2})$$
.

The  $X_j$  being K3 surfaces, the dimension of  $H^2_{tr}(X_j)$  is at least 2, and so the assumption on  $H^2_{tr}(S)$  implies that

$$\dim H^2_{tr}(X_j) \le 3 \ (j=0,1,2)$$

It follows from [38] that the  $X_j$  have finite-dimensional motive. In view of corollary 6.11, the motive t(S) is isomorphic to  $t(X_0) \oplus t(X_1) \oplus t(X_2)$ , and so this implies the corollary.

To see that S has motive of abelian type, one remarks that the K3 surfaces  $X_j$  either have a Shioda–Inose structure, or are rationally dominated by a Kummer surface [45], [32]. This implies that their motive is actually a submotive of the motive of an abelian surface.

**Remark 6.13.** In fairness, I hasten to add that I am not sure whether surfaces S as in corollary 6.12 exist. Indeed, one might naively expect that inside the families

$$\mathcal{X}_j \rightarrow B \quad (j=0,1,2)$$

of K3 surfaces associated to the family  $S \to B$  (cf. notation 3.13),  $\rho$ -maximal surfaces lie analytically dense (and so  $\rho$ -maximal triple K3 burgers would also be analytically dense). But to prove this, one would need to know a Torelli result for this type of K3 surfaces.

For this reason, corollary 6.12 is only a conditional result.

# 7. OPEN QUESTIONS

**Question 7.1.** Can one prove Torelli type theorems for families of triple K3 burgers as in theorem 5.1 ? As noted in remark 6.13, this would have interesting consequences for the distribution of Picard numbers, and for the existence of certain finite–dimensional motives.

**Question 7.2.** Let *S* be a triple *K*3 burger as in theorem 5.1. I wonder whether a stronger version of proposition 6.8 might be true: is it the case that (as for K3 surfaces)

$$A^1(S)_{\mathbb{Z}} \cdot A^1(S)_{\mathbb{Z}} = \mathbb{Z}[\mathfrak{o}_S] \subset A^2(S)_{\mathbb{Z}}$$
??

On a related note, does S have a multiplicative Chow–Künneth decomposition, in the sense of [44]?

**Question 7.3.** Let S be a triple K3 burger as in theorem 5.1. Is it the case that (as for K3 surfaces) the second Chern class  $c_2(T_S) \in A^2(S)$  lies in the subgroup  $\mathbb{Q}[\mathfrak{o}_S]$ ?

**Question 7.4.** Let X be a K3 surface, and let F be a simple rigid vector bundle on X. Voisin has proven [56, Theorem 1.9] that  $c_2(F) \in A^2(X)$  lies in the subgroup  $\mathbb{Q}[\mathfrak{o}_X]$ . Can one prove a similar statement for triple K3 burgers ?

(Presumably, Voisin's argument for K3 surfaces can be adapted to triple K3 burgers? At least the "dimension of orbit" part goes through unchanged (proposition 6.4). However, Voisin's argument also involves Riemann–Roch calculations, which rely on having trivial canonical bundle. I have not pursued this.)

**Question 7.5.** Let  $\pi: S \to B$  be a family of surfaces (i.e., a smooth projective morphism with 2–dimensional fibres). According to Deligne [8], there is a decomposition isomorphism

$$R\pi_*\mathbb{Q}\cong\bigoplus_i R^i\pi_*\mathbb{Q}[-i]$$

in the derived category of sheaves of  $\mathbb{Q}$ -vector spaces on B. If the fibres of  $\pi$  are K3 surfaces, then according to Voisin [53], one can choose an isomorphism that becomes multiplicative after shrinking the base B. Can one do the same for a family of triple K3 burgers ?

(*This is closely related to the existence of a multiplicative Chow–Künneth decomposition, cf.* [50, Section 4].)

**Question 7.6.** What are the generic and maximal Picard numbers for the families of triple K3 burgers of theorem 5.1 ?

**Question 7.7.** Constructing quadruple K3 burgers (i.e., surfaces satisfying the m = 4 analogon of definition 3.1) seems a daunting task.

(For example: if we suppose S is a canonical surface of general type with  $p_g = 4$  and  $K^2 = 5$ , then we know [15] that S is isomorphic to a quintic in  $\mathbb{P}^3$  with rational double points. Consider the involutions

$\sigma_0[x_0:x_1:x_2:x_3]$	$= [-x_0: x_1: x_2: x_3],$
$\sigma_1[x_0:x_1:x_2:x_3]$	$= [x_0: -x_1: x_2: x_3],$
$\sigma_2[x_0:x_1:x_2:x_3]$	$= [x_0: x_1: -x_2: x_3] ,$
$\sigma_3[x_0:x_1:x_2:x_3]$	$= [x_0: x_1: x_2: -x_3],$

If S is a hypersurface invariant under  $\sigma_j$  (i.e., the defining equation of S contains only even powers of  $x_j$ ), the quotient  $S / < \sigma_j >$  is a K3 surface with double points. However, clearly there is no quintic hypersurface invariant under all 4 involutions  $\sigma_j$  !)

The following is a weaker question: can one at least find general type surfaces S with  $p_g(S) = 4$  such that the transcendental cohomology of S splits in 4 pieces of K3 type ? And what about  $p_g > 4$  ?

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