



Surjectivity of cycle maps for singular varieties

Robert Laterveer

► To cite this version:

Robert Laterveer. Surjectivity of cycle maps for singular varieties. *Geometriae Dedicata*, 2015, 179 (1), pp.265-278. [⟨10.1007/s10711-015-0080-x⟩](https://doi.org/10.1007/s10711-015-0080-x). [⟨hal-02306863⟩](https://hal.science/hal-02306863)

HAL Id: hal-02306863

<https://hal.science/hal-02306863v1>

Submitted on 23 Oct 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Surjectivity of cycle maps for singular varieties

Robert Laterveer

Abstract A theorem of Jannsen asserts that if a smooth projective variety has injective cycle class maps, it has surjective cycle class maps. The object of this note is to present a version of Jannsen’s theorem for singular quasi-projective varieties.

Keywords Algebraic cycles · Chow groups · Pure motives · Singular varieties

Mathematics Subject Classification (2010) 14C15 · 14C25 · 14C30

1 Introduction

Let X be a smooth complex projective variety. The cycle class maps

$$cl^i : A^i X_{\mathbb{Q}} \rightarrow H^{2i}(X, \mathbb{Q})$$

from Chow groups to singular cohomology have given rise to some of the most profound and fascinating conjectures in algebraic geometry: the Hodge conjecture (concerning the image of cl^i), and the Bloch–Beilinson conjectures (concerning the structure of the kernel of cl^i).

Since Mumford’s work [19], it is well-known that if the Chow groups $A^i X_{\mathbb{Q}}$ are “small” (in the sense of being supported on some subvariety), then also the singular cohomology groups are small (in the sense that they are supported on some subvariety). The following result can be seen as an extreme instance of this general principle:

Theorem 1 (Jannsen [12]) *Suppose X is a smooth projective variety, such that*

$$cl^i : A^i X_{\mathbb{Q}} \rightarrow H^{2i}(X, \mathbb{Q})$$

is injective for all i . Then there is an isomorphism

$$\bigoplus_{i \geq 0} cl^i : \bigoplus_{i \geq 0} A^i X_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{j \geq 0} H^j(X, \mathbb{Q}).$$

(In particular, $H^{p,q}(X, \mathbb{C}) = 0$ for all $p \neq q$.)

This result can be proven using the Bloch–Srinivas method of decomposing the diagonal, and the formalism of correspondences [4], [26].

In this note, we look at the following question:

CNRS - IRMA, Université de Strasbourg
 7 rue René Descartes
 67084 Strasbourg cedex
 France
 E-mail: laterv@math.unistra.fr

Question 1 If X is only quasi-projective, and/or singular, in what sense is theorem 1 still true ?

This question has been treated by Lewis [18, Corollary 0.3], assuming a generalized version of the generalized Hodge conjecture holds. In this note, by contrast, we wanted to see how far we could get unconditionally. Our main result gives a version of theorem 1, provided the singular locus of X is not too large:

Theorem 2 *Let X be a quasi-projective variety of dimension n , and suppose there exists a compactification of X with singular locus of dimension $\leq \frac{n+1}{3}$. Suppose all cycle class maps are injective. Then the cl_i induce an isomorphism*

$$\bigoplus_{i \geq 0} : \bigoplus_{i \geq 0} A_i X_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{\ell \geq 0} W_{-\ell} H_{\ell}(X, \mathbb{Q}) .$$

That is,

$$W_{-\ell} H_{\ell} X = \begin{cases} 0 & \text{if } \ell \text{ is odd;} \\ \text{Im } cl_i & \text{if } \ell = 2i. \end{cases}$$

To prove this result, we adapt Jannsen's original method (i.e. the decomposition of the diagonal, plus the formalism of correspondences) to the singular and quasi-projective case. The decomposition of the diagonal goes through unchanged, except that in the quasi-projective case the boundary of a compactification appears in the decomposition (this is lemma 1). As to correspondences: suppose X is projective (not necessarily smooth) of dimension n . Then a correspondence, i.e. a cycle $C \in A_n(X \times X)_{\mathbb{Q}}$, induces an action

$$C_* : H^i(X, \mathbb{Q}) \rightarrow H_{2n-i}(X, \mathbb{Q})$$

in a natural way (using the cap-product). Since Δ_* is just the canonical map (capping with the class of X), and since we can control the image of this canonical map in favourable cases (lemma 4), we can conclude by looking at the action of the components occurring in the decomposition of Δ .

We raise several questions in the course of this note. Indeed, in several respects even the smooth quasi-projective case is far from being as clear-cut and well-understood as the smooth projective case; a relation with ‘Voisin's standard conjecture’ [25] naturally appears (remark 3).

Convention *In this note, the word variety refers to a quasi-projective algebraic variety over \mathbb{C} .*

2 The Bloch–Srinivas argument

Definition 1 We will use A_i to denote the Chow group of i -dimensional cycles, and A^i to denote the Fulton–MacPherson operational Chow cohomology [7]. By construction, A^* acts on Chow groups A_* ; in particular, for any projective X there are natural maps

$$\begin{aligned} A^i X &\rightarrow \text{Hom}(A_i X, \mathbb{Z}) , \\ b &\mapsto \deg(b \cap -) . \end{aligned}$$

We recall [21] there are functorial cycle class maps

$$cl^i : A^i X \rightarrow \text{Gr}_{2i}^W H^{2i}(X, \mathbb{Q}) ,$$

where W is Deligne's weight filtration.

Definition 2 We will use the notation A_c^i for ‘compactly supported operational Chow cohomology’. This is defined as follows: for any quasi-projective X , let $X \subset \bar{X}$ be a compactification with boundary D . Define $A_c^i X$ by the exact sequence

$$0 \rightarrow A_c^i X \rightarrow A^i \bar{X} \rightarrow A^i D .$$

This is independent of choice of \bar{X} [8] (actually, $A_c^i X$ is what is denoted $R^0 A^i X$ in [8]). Moreover A_c^i is a contravariant functor for arbitrary morphisms [8]. There are natural maps

$$A_c^i X_{\mathbb{Q}} \rightarrow \text{Hom}(A_i X_{\mathbb{Q}}, \mathbb{Q}) ,$$

defined by the above exact sequence. There are also functorial cycle class maps

$$cl^i: A_c^i X \rightarrow \mathrm{Gr}_{2i}^W H_c^{2i}(X, \mathbb{Q}),$$

again defined by the above exact sequence.

Definition 3 Let X be a quasi-projective variety. Following Voisin [25], [26], we say that X has trivial Chow groups if the cycle class maps

$$cl_i: A_i X_{\mathbb{Q}} \rightarrow H_{2i}(X, \mathbb{Q})$$

are injective for all i .

Definition 4 Let X be a quasi-projective variety. We say that

$$\mathrm{Niveau}(A_i X_{\mathbb{Q}}) \leq r$$

if there exists a closed $(i+r)$ -dimensional subvariety $Y \subset X$ such that $A_i(X \setminus Y)_{\mathbb{Q}} = 0$.

The key to the whole argument is the following decomposition lemma. This is the Bloch–Srinivas argument [4]; in his book, Bloch attributes this argument to Colliot–Thélène [1, appendix to lecture 1].

Lemma 1 Let \bar{X} be a projective variety of dimension n , and $X \subset \bar{X}$ the complement of a closed subvariety D . Suppose

$$\mathrm{Niveau}(A_i X_{\mathbb{Q}}) \leq r \quad \text{for all } i.$$

Then there is a decomposition of the diagonal

$$\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_n + \Gamma \in A_n(\bar{X} \times \bar{X})_{\mathbb{Q}},$$

where Δ_j is supported on $V_j \times W_j$, and $V_j \subset \bar{X}$ is of dimension $j+r$, $W_j \subset \bar{X}$ is of dimension $n-j$, and Γ is supported on $\bar{X} \times D$.

Proof This is an application of the Bloch–Srinivas method [4]. We use the following two well-known lemmas:

Lemma 2 Let X and Z be quasi-projective varieties, and suppose Z is irreducible of dimension n . Then for any i

$$A_i(X_{k(Z)}) \cong \varinjlim A_{i+n}(X \times U),$$

where the limit is taken over opens $U \subset Z$.

Proof This is usually stated for smooth projective varieties [1, appendix to Lecture 1]. If one is brave, one goes checking in Quillen’s work to see that the proof given in loc. cit. for the smooth case still goes on for singular varieties. Alternatively, take a resolution of singularities and reduce to the smooth case using the “descent” exact sequences, and the fact that \varinjlim is an exact functor.

Lemma 3 Let X be a quasi-projective variety defined over a field k , and let $k \subset K$ be a field extension. Then

$$A_i(X_k)_{\mathbb{Q}} \rightarrow A_i(X_K)_{\mathbb{Q}}$$

is injective.

Proof This is usually stated for smooth varieties [1, appendix to Lecture 1], but the same argument works in general: use lemma 2 to reduce to the case of a finite extension. For a finite extension, take a resolution of singularities; for smooth varieties, the existence of the norm implies the extension map is a split injection; by descent, the same is true for singular varieties.

Now we proceed with the proof of lemma 1. We can reduce to some subfield $k \subset \mathbb{C}$ which is finitely generated over its prime subfield. Consider the restriction

$$\Delta \in A_n(\bar{X} \times \bar{X})_{\mathbb{Q}} \rightarrow A_n(X \times \bar{X})_{\mathbb{Q}} \rightarrow A_0(X_{k(\bar{X})})_{\mathbb{Q}}.$$

The last group is supported in dimension r , so we get a rational equivalence

$$\Delta = \Delta_0 + \Delta^1 + \Gamma^1 \in A_n(\bar{X} \times \bar{X})_{\mathbb{Q}},$$

where Δ_0 is supported on $V_0 \times \bar{X}$, where V_0 has dimension r , and Δ^1 is supported on $\bar{X} \times W_1$ for some divisor W_1 , and Γ^1 is supported on $D \times \bar{X}$.

Applying the same process to Δ^1 and continuing inductively, we end up with a decomposition

$$\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_n + \Gamma' \in A_n(\bar{X} \times \bar{X})_{\mathbb{Q}},$$

where the Δ_j are as desired, but Γ' is supported on $D \times \bar{X}$. Taking the transpose and renumbering, we end up with a decomposition as desired.

Remark 1 In case X is smooth projective, lemma 1 was proven in [15], inspired by [4] and [20].

3 The smooth projective case

The following is well-known.

Proposition 1 (Jannsen [12], Kimura, Vial) *Let X be a smooth projective variety of dimension n . The following are equivalent:*

- (i) *The groups $A_i^{alg} X_{\mathbb{Q}}$ are 0 for $i < \frac{n}{2}$;*
- (ii) *X has trivial Chow groups;*
- (iii) *$\text{Niveau}(A_i X_{\mathbb{Q}}) \leq 0$ for all i ;*
- (iv) *The cycle class maps induce a ring isomorphism*

$$A_* X_{\mathbb{Q}} \xrightarrow{\cong} H_*(X, \mathbb{Q});$$

- (v) *For any variety Z , and for any i , the product map induces an isomorphism*

$$\bigoplus_{l+m=i} A_l X_{\mathbb{Q}} \otimes A_m Z_{\mathbb{Q}} \xrightarrow{\cong} A_i(X \times Z)_{\mathbb{Q}};$$

- (vi) *For any variety Z , and for any i, j , the product map induces an isomorphism*

$$\bigoplus_{l+m=i} A^l X_{\mathbb{Q}} \otimes A^m(Z, j)_{\mathbb{Q}} \xrightarrow{\cong} A^i(X \times Z, j)_{\mathbb{Q}}$$

of higher Chow groups [2].

Proof We will recall the proof, as a warm-up for what follows. To see that (i) \Rightarrow (ii), we work inductively. First, the hypothesis $A_0^{alg} X_{\mathbb{Q}} = A_0^{hom} X_{\mathbb{Q}} = 0$ implies a decomposition of the diagonal

$$\Delta = X \times x + \Gamma^1 \in A^n(X \times X)_{\mathbb{Q}},$$

with Γ^1 supported on $D \times X$, for some divisor D . Considering the action of Δ on $\text{Griff}^{n-1} X_{\mathbb{Q}}$, we find

$$\text{Griff}^{n-1} X_{\mathbb{Q}} = 0.$$

(Indeed, the action factors over $\text{Griff}^{n-1}(\tilde{D})_{\mathbb{Q}} = 0$, for some desingularisation \tilde{D} .) Taken together with the hypothesis $A_{alg}^{n-1} X_{\mathbb{Q}} = 0$, we find that $A_1^{hom} X_{\mathbb{Q}} = 0$, and we continue likewise.

After $\frac{n}{2}$ steps, we end up with a decomposition

$$\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_{\lfloor \frac{n}{2} \rfloor} + \Gamma \in A^n(X \times X)_{\mathbb{Q}},$$

where Δ_j comes from $A^j X_{\mathbb{Q}} \otimes A^{n-j} X_{\mathbb{Q}}$, and Γ is supported on $V \times X$ with $\dim V \leq \frac{n}{2}$. We can apply this decomposition to $A_i^{hom} X_{\mathbb{Q}}$ to check that (ii) holds. (Indeed, for $i \geq \frac{n}{2}$, the component Γ does not act on $A_i^{hom} X_{\mathbb{Q}}$ for dimension reasons; neither do the Δ_j act.)

To see that (ii) \Rightarrow (iii), remark that the cohomology groups $H^{2i}(X, \mathbb{Q})$ are finite-dimensional \mathbb{Q} -vector spaces.

To get the implication (iii) \Rightarrow (iv), let the decomposition of the diagonal act on the kernel and cokernel to see that both vanish.

Now, let's prove the implication (iv) \Rightarrow (v). Let $S \subset Z$ denote the singular locus, and let $\tilde{Z} \rightarrow Z$ denote a resolution of singularities with exceptional divisor E . There is a commutative diagram with exact rows

$$\begin{array}{ccccccc} \rightarrow & A_i(X \times E)_{\mathbb{Q}} & \rightarrow & A_i(X \times \tilde{Z})_{\mathbb{Q}} \oplus A_i(X \times S)_{\mathbb{Q}} & \rightarrow & A_i(X \times Z)_{\mathbb{Q}} & \rightarrow 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ \rightarrow & \bigoplus_{l+m=i} A_l X_{\mathbb{Q}} \otimes A_m E_{\mathbb{Q}} & \rightarrow & \bigoplus_{l+m=i} A_l X_{\mathbb{Q}} \otimes (A_m(\tilde{Z}) \oplus A_m S)_{\mathbb{Q}} & \rightarrow & \bigoplus_{l+m=i} A_l X_{\mathbb{Q}} \otimes A_m Z_{\mathbb{Q}} & \rightarrow 0 \end{array}$$

By noetherian induction, we are thus reduced to the case where Z is smooth. Writing out a similar diagram for a compactification, we reduce to the case where Z is smooth and projective.

Let's suppose now Z is smooth projective, say of dimension d . We first prove surjectivity of the product map: Take c an element of $A_i(X \times Z)_{\mathbb{Q}}$. We may suppose everything (X , Z , c and the subvarieties supporting the $A_i X_{\mathbb{Q}}$) is defined over a field $k \subset \mathbb{C}$ finitely generated over its prime subfield. Consider what happens to c under the restriction

$$c \in A_i(X \times Z)_{\mathbb{Q}} \rightarrow A_{i-d}(X_{k(Z)})_{\mathbb{Q}} = \varinjlim A_i(X \times U)_{\mathbb{Q}},$$

where the limit is taken over opens $U \subset Z$, and the equality is established in lemma 2. Since $k(Z) \subset \mathbb{C}$, lemma 3 implies that

$$A_{i-d}(X_{k(Z)})_{\mathbb{Q}} \rightarrow A_{i-d}(X_{\mathbb{C}})_{\mathbb{Q}}$$

is injective, so that

$$\text{Niveau}\left(A_{i-d}(X_{k(Z)})_{\mathbb{Q}}\right) \leq 0.$$

It follows that the cycle c can be written

$$c = b \times Z + c' \in A_i(X \times Z)_{\mathbb{Q}},$$

where $b \in A_{i-n} X_{\mathbb{Q}}$ and c' supported on $X \times Z'$, for $Z' \subset Z$ some divisor. By induction, the statement is true for $X \times Z'$, and so we find that c is a sum of product cycles as desired.

Next, we prove the product map is injective. So let

$$p: \bigoplus_{l+m=i} A_l X_{\mathbb{Q}} \otimes A_m Z_{\mathbb{Q}} \rightarrow A_i(X \times Z)_{\mathbb{Q}}$$

denote the product map, and let a be an element in $\text{Ker } p$. We write

$$a = \sum_{l+m=i} a_{l,m} \in \bigoplus_{l+m=i} A_l X_{\mathbb{Q}} \otimes A_m Z_{\mathbb{Q}},$$

and we let L be the maximum l for which $a_{l,m} \neq 0$. Hypothesis (iv) implies that $A_L X_{\mathbb{Q}}$ is finite-dimensional, and that there is a perfect pairing

$$A_L X_{\mathbb{Q}} \times A_{n-L} X_{\mathbb{Q}} \rightarrow \mathbb{Q}.$$

Let b_1, \dots, b_r be a basis of $A_L X_{\mathbb{Q}}$, and let $b_1^{\vee}, \dots, b_r^{\vee}$ denote the dual basis of $A_{n-L} X_{\mathbb{Q}}$. We write

$$a_{L,m} = \sum_j c_j b_j \otimes d_j \in A_L X_{\mathbb{Q}} \otimes A_m Z_{\mathbb{Q}}.$$

Since by hypothesis, $p(a) = 0$, we have

$$p(a_{L,m}) \cdot (b_j^{\vee} \times Z) = p(a) \cdot (b_j^{\vee} \times Z) = 0 \in A_m(X \times Z)_{\mathbb{Q}}.$$

But $p(a_{L,m}) \cdot (b_j^\vee \times Z) = c_j(\text{point}) \times d_j$ projects to $c_j d_j \in A_m(Z)_\mathbb{Q}$ under projection to the second factor, so we find

$$c_j d_j = 0 \in A_m(Z)_\mathbb{Q} \quad \forall j.$$

But this means $a_{L,m} = 0$; contradiction.

To get that (v) implies (i): taking $Z = X$, one obtains a complete decomposition of the diagonal of X ; having the diagonal act on $A_i^{\text{alg}} X_\mathbb{Q}$, one obtains the required vanishing.

It remains to establish an equivalence with (vi): using a commutative diagram extending the above diagram to the left (this exists, thanks to the notorious moving lemma for higher Chow groups [3], [17]), one is again reduced to the case Z smooth projective. Now we use a result of Kimura [14] and Vial [23, Theorem 5], which states that (ii) is equivalent to the fact that the Chow motive of X is a sum of twisted Lefschetz motives. Hence the motive of $X \times Z$ is a sum of twists of the motive of Z ; as higher Chow groups only depend on the Chow motive, this implies (vi).

Finally, (vi) \Rightarrow (v) is obvious, and we are done.

Remark 2 Jannsen proved that properties (ii), (iii) and (iv) in proposition 1 are equivalent [12].

The fact that it suffices to consider algebraically trivial cycles (i.e. point (i) in proposition 1) is a particular instance of a more general phenomenon, discovered by Vial: if a morphism of Chow motives $f: N \rightarrow M$, with N finite-dimensional, induces a surjection $A_*^{\text{alg}}(N)_\mathbb{Q} \rightarrow A_*^{\text{alg}}(M)_\mathbb{Q}$, then also $A_*^{\text{hom}}(N)_\mathbb{Q} \rightarrow A_*^{\text{hom}}(M)_\mathbb{Q}$ is surjective [24, Theorem 7]. The above manifestation is just the case where N is a Lefschetz motive.

Property (v) is studied in depth in [22], where it is called the ‘‘Chow–K nneth property’’. Notably, [22, Theorem 4.1] generalizes the result of Kimura and Vial evoked in the above proof.

4 The smooth quasi-projective case

In case X is a smooth quasi-projective variety, the situation is not as well-understood as in the smooth projective case; the equivalences of proposition 1 become difficult open problems. For instance, we raise the following questions:

Question 2 Does the implication (i) \Rightarrow (ii) of proposition 1 still hold for X smooth quasi-projective ?

Question 3 Does the implication (iii) \Rightarrow (ii) of proposition 1 still hold for X smooth quasi-projective ?

Remark 3 For both questions, the answer is positive provided the ‘‘Voisin standard conjecture’’ [25, Conjecture 0.6], [26] is true. Moreover, a positive answer to either question would imply the following result: if X is smooth quasi-projective with trivial Chow groups, then any open $U \subset X$ has trivial Chow groups. As shown in [25], this result (i.e. that ‘‘having trivial Chow groups’’ transfers from a variety to its open subsets) would be a consequence of the truth of the Voisin standard conjecture.

Here is what we can prove unconditionally:

Proposition 2 *Let X be a smooth quasi-projective variety with trivial Chow groups. Then the cycle class maps induce isomorphisms*

$$\bigoplus_i A_i X_\mathbb{Q} \xrightarrow{\cong} \bigoplus_\ell W_{-\ell} H_\ell(X, \mathbb{Q}).$$

That is,

$$W_{-\ell} H_\ell(X, \mathbb{Q}) = \begin{cases} 0 & \text{if } \ell \text{ is odd;} \\ \text{Im } cl_i & \text{if } \ell = 2i. \end{cases}$$

Since ‘‘having trivial Chow groups’’ obviously implies that the niveau of all Chow groups is ≤ 0 , proposition 2 follows from the following more general proposition:

Proposition 3 *Let X be a smooth quasi-projective variety of dimension n . Suppose*

$$\text{Niveau}(A_i X_{\mathbb{Q}}) \leq 0 \text{ for all } i.$$

Then

$$W_{-\ell} H_{\ell}(X, \mathbb{Q}) = \begin{cases} 0 & \text{if } \ell \text{ is odd;} \\ \text{Im } cl_i & \text{if } \ell = 2i. \end{cases}$$

Moreover,

$$cl^i : A_c^i X_{\mathbb{Q}} \rightarrow Gr_{2i}^W H_c^{2i}(X, \mathbb{Q})$$

is injective.

Proof Let $\tau : X \subset \bar{X}$ be a smooth compactification, with boundary D . From lemma 1, we obtain a decomposition of the diagonal of \bar{X}

$$\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_n + \Gamma \in A_n(\bar{X} \times \bar{X})_{\mathbb{Q}},$$

where Δ_j is supported on $V_j \times W_j$, and V_j (resp. W_j) is of dimension j (resp. $n - j$), and Γ is supported on $\bar{X} \times D$. Let $a \in Gr_F^k W_{-\ell} H_{\ell}(X, \mathbb{C})$, with $\ell \neq -2k$. Let $\bar{a} \in Gr_F^k H_{\ell}(\bar{X}, \mathbb{C})$ be an element restricting to a .

The action of Δ_j on \bar{a} is 0 for dimension reasons. (Indeed, let \widetilde{V}_j and \widetilde{W}_j denote resolutions of singularities. Then the action of Δ_j factors over

$$H^{k+n, n-\ell-k}(\widetilde{V}_j, \mathbb{C})$$

and

$$H^{k+n-j, n-\ell-k-j}(\widetilde{W}_j, \mathbb{C}),$$

and one of these groups is 0 for dimension reasons.) It follows that

$$a = \tau^* \bar{a} = \tau^* ((\Delta_0)_*(\bar{a}) + \cdots + (\Delta_n)_*(\bar{a})) = 0 \in Gr_F^k W_{-\ell} H_{\ell}(X, \mathbb{C}),$$

so $Gr_F^k W_{-\ell} H_{\ell}(X, \mathbb{C})$ for all $\ell \neq -2k$. In particular, for ℓ odd we find that

$$W_{-\ell} H_{\ell}(X, \mathbb{C}) = \bigoplus_j Gr_F^j W_{-\ell} H_{\ell}(X, \mathbb{C}) = 0.$$

Next, let $a \in W_{-2i} H_{2i}(X, \mathbb{Q}) \cap F^{-i}$. Using the polarisation on $H_{2i}(\bar{X}, \mathbb{Q})$ (cf. [26,]), one finds there exists a Hodge class $\bar{a} \in H_{2i}(\bar{X}, \mathbb{Q})$ which restricts to a (i.e. $\tau^* \bar{a} = a$).

For $j \neq n - i$, the action of Δ_j on \bar{a} is 0 for dimension reasons. (This is similar to the prior parenthesis: the action of Δ_j factors over

$$H^{n-i, n-i}(\widetilde{V}_j, \mathbb{Q})$$

and

$$H^{n-i-j, n-i-j}(\widetilde{W}_j, \mathbb{Q}),$$

and one of these groups is 0 for dimension reasons.) For $j = n - i$, we have that

$$(\Delta_{n-i})_* \bar{a} \subset \text{Im}(H_{2i}(W_{n-i}) \rightarrow H_{2i} \bar{X}) \subset \text{Im } cl_i.$$

It follows that

$$a = \tau^* \bar{a} = \tau^* ((\Delta_{n-i})_* \bar{a} + \Gamma_* \bar{a}) = \tau^* ((\Delta_{n-i})_* \bar{a}) \subset \text{Im } cl_i.$$

It remains to prove the statement for cl^i . Taking the transpose of all elements involved, we may suppose we have a decomposition

$$\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_n + \Gamma \in A_n(\bar{X} \times \bar{X})_{\mathbb{Q}},$$

where Δ_j are as before, but Γ is now supported on $D \times \bar{X}$.

Let $a \in A_c^i X_{\mathbb{Q}}$, and let \bar{a} be the image of a in $A^i \bar{X}_{\mathbb{Q}}$. The restriction of \bar{a} to D is 0 (i.e., if $\psi : D \rightarrow \bar{X}$ denotes the inclusion, we have $\psi^*(\bar{a}) = 0 \in A^i(D, \mathbb{Q})$), hence $\Gamma_* \bar{a} = 0$. Just as above, the correspondence Δ_j does not act on \bar{a} except for $j = i$, hence

$$\bar{a} = (\Delta_i)_* \bar{a}.$$

Suppose now $\bar{a} \in A_{\text{hom}}^i \bar{X}_{\mathbb{Q}}$. The action of Δ_i on $A_{\text{hom}}^i \bar{X}_{\mathbb{Q}}$ factors over $A_{\text{hom}}^0(\widetilde{W}_i)_{\mathbb{Q}}$, which is 0; it follows that $\bar{a} = 0$, whence $a = 0$.

5 The singular case

In this section, we consider quasi-projective (possibly singular) varieties X . We prove our main result as promised in the introduction; this is a version of Jannsen's theorem for varieties whose singular locus is not too large:

Theorem 3 *Let X be a quasi-projective variety of dimension n , and suppose there is a compactification of X with singular locus of dimension $\leq \frac{n+1}{3}$. Suppose X has trivial Chow groups. Then cycle class maps induce an isomorphism*

$$\bigoplus_i A_i X_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{\ell} W_{-\ell} H_{\ell}(X, \mathbb{Q}) .$$

That is,

$$W_{-\ell} H_{\ell}(X, \mathbb{Q}) = \begin{cases} 0 & \text{if } \ell \text{ is odd;} \\ \text{Im } cl_i & \text{if } \ell = 2i. \end{cases}$$

This follows from the following more precise version:

Theorem 4 *Let X be a quasi-projective variety of dimension n , and suppose a compactification of X has singular locus of dimension $\leq s$. Suppose*

$$\text{Niveau}(A_i X_{\mathbb{Q}}) \leq 0 \text{ for all } i .$$

Then

$$W_{-\ell} H_{\ell}(X, \mathbb{Q}) = \begin{cases} 0 & \text{if } \ell \text{ is odd;} \\ \text{Im } cl_i & \text{if } \ell = 2i, \end{cases}$$

provided $\ell \in [0, n-s] \cup [2s, 2n]$.

Moreover,

$$cl^i : A_c^i X_{\mathbb{Q}} \rightarrow Gr_{2i}^W H_c^{2i}(X, \mathbb{Q})$$

is injective in the range $i > s$.

Proof Let $\tau : X \rightarrow \bar{X}$ denote the given compactification, with boundary $D = \bar{X} \setminus X$. Applying lemma 1, we find a decomposition of the diagonal of \bar{X} of the form

$$\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_n \in A_n(\bar{X} \times \bar{X})_{\mathbb{Q}} ,$$

where Δ_j is supported on $V_j \times W_j$, and $V_j \subset \bar{X}$ has dimension j and W_j has dimension $n-j$. We can view Δ (and the Δ_j) as a correspondence

$$\Delta_* : H^i(\bar{X}, \mathbb{Q}) \rightarrow W_{i-2n} H_{2n-i}(\bar{X}, \mathbb{Q}) ,$$

where for $\bar{a} \in H^i(\bar{X}, \mathbb{Q})$, we define

$$\Delta_*(\bar{a}) := (\pi_2)_* ((\pi_1^* \bar{a}) \cap \Delta) \in W_{i-2n} H_{2n-i}(\bar{X}, \mathbb{Q})$$

(here π_1 resp. π_2 denotes projection on the first resp. second factor). It is easily checked that

$$\Delta_*(\bar{a}) = \bar{a} \cap [\bar{X}] \in H_{2n-i}(\bar{X}, \mathbb{Q}) .$$

(Indeed, let $f : \tilde{X} \rightarrow \bar{X}$ be a resolution of singularities, with projections $\tilde{\pi}_1, \tilde{\pi}_2$ from $\tilde{X} \times \tilde{X}$ to the two factors. Let $\tilde{\Delta}$ denote the diagonal of \tilde{X} , so that $\Delta = (f \times f)_* \tilde{\Delta}$. Then

$$\begin{aligned} \Delta_*(\bar{a}) &= (\pi_2)_* (\pi_1^* \bar{a} \cap \Delta) \\ &= (\pi_2)_* (f \times f)_* ((f \times f)^* \pi_1^* \bar{a} \cap \tilde{\Delta}) \\ &= f_* (\tilde{\pi}_2)_* (\tilde{\pi}_1^* f^* \bar{a} \cap \tilde{\Delta}) \\ &= f_* (\tilde{\Delta}_* (f^* \bar{a})) \\ &= f_* (f^* \bar{a} \cap [\tilde{X}]) = \bar{a} \cap [\bar{X}] . \end{aligned}$$

Case 1: $\ell \leq n - s$. Let $a \in \mathrm{Gr}_F^p W_{-\ell} H_\ell(X, \mathbb{C})$, and let $\bar{a} \in \mathrm{Gr}_F^p W_{-\ell} H_\ell(\bar{X}, \mathbb{C})$ be an element restricting to a . According to lemma 4 below, we can find $b \in \mathrm{Gr}_{2n-\ell}^W H^{2n-\ell}(\bar{X}, \mathbb{C})$ such that

$$\bar{a} = b \cap [\bar{X}] \in W_{-\ell} H_\ell(\bar{X}, \mathbb{C}) .$$

The ‘‘Poincaré duality’’ map from H^{2i} to H_{2n-i} , being a map of Hodge structures, is strictly compatible with the Hodge filtration, so we may suppose $b \in \mathrm{Gr}_F^{p+n}$. Note that we have

$$a = \tau^* \bar{a} = \tau^* \left((\Delta_0)_* b + \cdots + (\Delta_n)_* b \right) \in \mathrm{Gr}_F^p W_{-\ell} H_\ell(X, \mathbb{C}) .$$

On the other hand, for dimension reasons we have

$$(\Delta_j)_* \mathrm{Gr}_F^{p+n} \mathrm{Gr} H^{2n-\ell}(\bar{X}, \mathbb{C}) = 0 \quad \text{unless } 2n - \ell = 2j = 2(p+n) .$$

It follows that $a = 0$ if $\ell \neq -2p$; in particular $W_{-\ell} H_\ell(X, \mathbb{Q}) = 0$ for ℓ odd.

Next, let $\ell = 2i$ and consider $a \in W_{-2i} H_{2i}(X, \mathbb{Q})$. Using lemma 4, we can find again $b \in H^{2n-2i}(\bar{X}, \mathbb{Q})$ such that

$$a = \tau^* (b \cap [\bar{X}]) = \tau^* ((\Delta_0)_* b + \cdots + (\Delta_n)_* b) \in H_{2i}(X, \mathbb{Q}) .$$

But for reasons of dimension,

$$(\Delta_j)_* b = 0 \quad \text{for } j \neq n - i ,$$

and clearly

$$(\Delta_{n-i})_* b \in \mathrm{Im} \, cl_i .$$

Case 2: $\ell \geq 2s$. Let $S \subset X$ denote the singular locus, and $U = X \setminus S$ the non-singular locus. Then obviously

$$\mathrm{Niveau}(A_i(U)_{\mathbb{Q}}) \leq 0 \quad \text{for all } i .$$

This implies (by proposition 3 above) that

$$cl_i : A_i U_{\mathbb{Q}} \rightarrow W_{-2i} H_{2i}(U, \mathbb{Q})$$

is surjective for all i , and $W_{-\ell} H_\ell(U, \mathbb{Q}) = 0$ for ℓ odd. But the map

$$W_{-\ell} H_\ell(X, \mathbb{Q}) \rightarrow W_{-\ell} H_\ell(U, \mathbb{Q})$$

is an isomorphism for $\ell > 2s$, so

$$W_{-\ell} H_\ell(X, \mathbb{Q}) = 0 \quad \text{if } \ell > 2s \text{ is odd} .$$

Restriction induces a surjection

$$W_{-2i} H_{2i}(X, \mathbb{Q}) \rightarrow W_{-2i} H_{2i}(U, \mathbb{Q})$$

for reasons of weight; this fits into a commutative diagram with exact rows

$$\begin{array}{ccccccc} A_i S_{\mathbb{Q}} & \rightarrow & A_i X_{\mathbb{Q}} & \rightarrow & A_i U_{\mathbb{Q}} & \rightarrow & 0 \\ \downarrow cl_i & & \downarrow cl_i & & \downarrow cl_i & & \\ W_{-2i} H_{2i}(S, \mathbb{Q}) & \rightarrow & W_{-2i} H_{2i}(X, \mathbb{Q}) & \rightarrow & W_{-2i} H_{2i}(U, \mathbb{Q}) & \rightarrow & 0 . \end{array}$$

The right vertical arrow is surjective, as we just noted, and the left vertical arrow is an isomorphism for $i \geq s$.

The ‘‘Moreover’’ part follows from the commutative diagram

$$\begin{array}{ccc} A_c^i U_{\mathbb{Q}} & \rightarrow & A_c^i X \\ \downarrow & & \downarrow \\ \mathrm{Gr}_{2i}^W H_c^{2i}(U, \mathbb{Q}) & \rightarrow & \mathrm{Gr}_{2i}^W H_c^{2i}(X, \mathbb{Q}) \end{array}$$

The horizontal maps are isomorphisms since $i > s$; the left vertical arrow is injective by proposition 3.

Lemma 4 *Let X be a projective variety of dimension n , and with singular locus of dimension $\leq s$. Then the natural map*

$$\mathrm{Gr}_j^W H^j(X, \mathbb{Q}) \rightarrow W_{j-2n} H_{2n-j}(X, \mathbb{Q})$$

is injective for $j \leq n - s$, and surjective for $j \geq n + s$.

Proof Let $IH^j X$ denote middle-perversity intersection homology with rational coefficients. It follows from work of Durfee [5] that

$$IH^j X = \begin{cases} \mathrm{Gr}_j^W H^j(X, \mathbb{Q}), & j \geq n + s; \\ W_{j-2n} H_{2n-j}(X, \mathbb{Q}), & j \leq n - s. \end{cases}$$

It is well-known [9], [10] that the ‘‘Poincaré duality’’ map factors as

$$\mathrm{Gr}_j^W H^j(X, \mathbb{Q}) \rightarrow IH^j X \rightarrow W_{j-2n} H_{2n-j}(X, \mathbb{Q}).$$

Moreover, it is known by work of Weber [27] (cf. also [11]) that the first arrow is injective, and the second arrow surjective.

Remark 4 It seems likely theorem 4 is true without any condition on the singular locus. This is proven by Lewis [18, Corollary (0.2)], under the assumption of (a generalized version of) the generalized Hodge conjecture.

Remark 5 Linear varieties (in the sense of [21]) form a subclass of the class of varieties with trivial Chow groups. For a projective (possibly singular) linear variety X , Totaro has shown [21] that

$$\begin{aligned} cl_i : A_i X_{\mathbb{Q}} &\rightarrow W_{-2i} H_{2i}(X, \mathbb{Q}), \\ cl^i : A^i X_{\mathbb{Q}} &\rightarrow \mathrm{Gr}_{2i}^W H^{2i}(X, \mathbb{Q}) \end{aligned}$$

are isomorphisms for all i . (The first isomorphism is [21, Theorem 3]; the second isomorphism is obtained by combining the first isomorphism with [21, Theorem 2].)

The argument in the proof of theorem 4 suggests the following question:

Question 4 Let X be any quasi-projective variety with $\mathrm{Niveau}(A_i X_{\mathbb{Q}}) \leq 0$ for all i . Is it true that the natural map

$$\mathrm{Im}(A^* X_{\mathbb{Q}} \rightarrow A_* X_{\mathbb{Q}}) \rightarrow \mathrm{Im}(H^*(X, \mathbb{Q}) \rightarrow H_*(X, \mathbb{Q}))$$

is an isomorphism ?

A partial answer is given by the following result: for X projective, the right-hand side is generated by algebraic cycles.

Proposition 4 *Let X be a projective variety of dimension n , and suppose $\mathrm{Niveau}(A_i X_{\mathbb{Q}}) \leq 0 \forall i$. Then*

$$\mathrm{Im}(H^*(X, \mathbb{Q}) \rightarrow H_*(X, \mathbb{Q}))$$

is generated by algebraic cycles. That is,

$$\mathrm{Im}(H^{\ell}(X, \mathbb{Q}) \rightarrow H_{2n-\ell}(X, \mathbb{Q})) = \begin{cases} 0 & \text{if } \ell \text{ is odd;} \\ \subset \mathrm{Im} \, cl_{n-i} & \text{if } \ell = 2i. \end{cases}$$

Proof First, let’s suppose ℓ is odd. Then the vanishing of $\mathrm{Im}(H^{\ell}(X, \mathbb{Q}) \rightarrow H_{2n-\ell}(X, \mathbb{Q}))$ follows from the following lemma:

Lemma 5 *Set-up as in the proposition. Then*

$$\mathrm{Gr}_F^{p-n}(\mathrm{Im}(H^{\ell}(X, \mathbb{C}) \rightarrow H_{2n-\ell}(X, \mathbb{C}))) = 0 \text{ for } \ell \neq 2p.$$

Proof (of the lemma) By strict compatibility of the Hodge filtration, the group in the statement of the lemma is the same as

$$\mathrm{Im}(\mathrm{Gr}_F^p H^\ell(X, \mathbb{C}) \rightarrow \mathrm{Gr}_F^{p-n} H_{2n-\ell}(X, \mathbb{C})) .$$

This is the same as

$$\Delta_* \mathrm{Gr}_F^p H^\ell(X, \mathbb{C}) ,$$

where correspondences act as defined in the proof of proposition 4. Now we apply the decomposition of Δ given by lemma 1. The action of the component Δ_j factors as follows:

$$\begin{array}{ccc} & \dots & \\ & \uparrow & \downarrow \\ \mathrm{Gr}_F^p H^\ell(\widetilde{V}_j, \mathbb{C}) & & \mathrm{Gr}_F^{p-n} H_{2n-\ell}(\widetilde{W}_j, \mathbb{C}) \\ & \uparrow & \downarrow \\ \mathrm{Gr}_F^p H^\ell(X, \mathbb{C}) & \xrightarrow{(\Delta_j)_*} & \mathrm{Gr}_F^{p-n} H_{2n-\ell}(X, \mathbb{C}) \end{array}$$

Since $\dim \widetilde{V}_j = j$, the upper left group vanishes for $p > j$; likewise, since $\dim \widetilde{W}_j = n - j$, the upper right group vanishes for $p < j$. It follows that the only non-trivial action is for $p = j$. But the group $\mathrm{Gr}_F^j H^\ell(\widetilde{V}_j, \mathbb{C})$ vanishes unless $\ell = 2j$.

It remains to treat the case $\ell = 2i$. Let $a \in H^{2i}(X, \mathbb{Q})$. From the proof of the above lemma, we find that

$$a \cap [X] = \Delta_* a = (\Delta_i)_* a \in H_{2n-2i}(X, \mathbb{Q})$$

(indeed, for $j \neq i$, the action

$$(\Delta_j)_* H^{2i}(X, \mathbb{C}) = 0 \in H_{2n-2i}(X, \mathbb{C}) ,$$

since it is 0 on each Gr_F^p . But

$$\begin{array}{ccc} H^{2i}(X, \mathbb{Q}) & \xrightarrow{(\Delta_j)_*} & H_{2n-2i}(X, \mathbb{Q}) \\ \downarrow & & \downarrow \\ H^{2i}(X, \mathbb{C}) & \xrightarrow{(\Delta_j)_*} & H_{2n-2i}(X, \mathbb{C}) \end{array}$$

commutes, so that also $(\Delta_j)_* H^{2i}(X, \mathbb{Q}) = 0$ for $j \neq i$.)

But

$$(\Delta_i)_* H^{2i} X \subset \mathrm{Im}(H_{2n-2i}(W_i, \mathbb{Q})) \rightarrow H_{2n-2i}(X, \mathbb{Q}) \subset \mathrm{Im} \, cl_{n-i} .$$

Remark 6 A result analogous to Jannsen's theorem is proven by Esnault–Levine [6]. They prove that if X is smooth projective such that all cycle class maps into Deligne cohomology are injective, these cycle class maps are surjective (this is reproven, and rendered more precise, in [23, Theorem 4].) Lewis extends this to singular and quasi-projective varieties, again assuming (a generalized version of) the generalized Hodge conjecture [18, Corollary (0.3)]. It would be interesting to try whether the approach of the present note can be applied to this problem; I haven't looked into this.

Remark 7 The argument of the proof of theorem 4 can also be used to obtain a new version of Mumford's theorem for singular varieties, plus a verification of the Hodge conjecture for certain singular varieties [16].

Acknowledgements This note was stimulated by the Strasbourg “groupe de travail” based on the monograph [26]. It is a pleasure to thank the participants of this groupe de travail for a very pleasant atmosphere and stimulating interactions.

References

1. S. Bloch, Lectures on algebraic cycles, Duke Univ. Press 1980,
2. S. Bloch, Algebraic cycles and higher K-theory, *Advances in Math.* vol. 61 (1986), 267—304,
3. S. Bloch, The moving lemma for higher Chow groups, *J. Alg. Geom.* 3 (1994), 537—568,
4. S. Bloch and V. Srinivas, Remarks on correspondences and algebraic cycles, *American Journal of Mathematics* Vol. 105, No 5 (1983), 1235—1253,
5. A. Durfee, Intersection homology Betti numbers, *Proc. Amer. Math. Soc.* 123 (1995), 989—993,
6. H. Esnault and M. Levine, Surjectivity of cycle maps, in: *Journées de Géométrie algébrique d'Orsay*, *Astérisque* 218 (1993),
7. W. Fulton, Intersection theory, Springer-Verlag Berlin Heidelberg New York 1984,
8. H. Gillet and C. Soulé, Descent, motives and K-theory, *J. Reine Angew. Math.* 478 (1996), 127—176,
9. M. Goresky and R. MacPherson, Intersection homology theory, *Topology* 19 (1980), 135—162,
10. M. Goresky and R. MacPherson, Intersection homology II, *Inv. Math.* 71 (1983), 77—129,
11. M. Hanamura and M. Saito, Weight filtrations on the cohomology of algebraic varieties, arxiv 0605603v2,
12. U. Jannsen, Mixed motives and algebraic K-theory, *Springer Lecture Notes in Mathematics* 1400 (1990),
13. U. Jannsen, Motivic sheaves and filtrations on Chow groups, in: *Motives (Jannsen et alii, eds.)*, *Proceedings of Symposia in Pure Mathematics* Vol. 55 (1994), Part 1,
14. S. Kimura, Surjectivity of the cycle map for Chow motives, in: *Motives and algebraic cycles*, *Fields Institute Comm.* vol. 56, Amer. Math. Soc., Providence 2009,
15. R. Laterveer, Algebraic varieties with small Chow groups, *J. Math. Kyoto Univ.* Vol. 38 No 4 (1998), 673—694,
16. R. Laterveer, Correspondences and singular varieties, to appear in *Monatshefte für Mathematik*,
17. M. Levine, Techniques of localization in the theory of algebraic cycles, *J. Alg. Geom.* 10 (2001), 299—363,
18. J. Lewis, A generalization of Mumford's theorem, II, *Illinois Journal of Mathematics*, Vol. 39 No 2 (1995), 288—304,
19. D. Mumford, Rational equivalence of 0-cycles on surfaces, *J. Math. Kyoto Univ.* Vol. 9 No 2 (1969), 195—204,
20. K. Paranjape, Cohomological and cycle-theoretic connectivity, *Annals of Mathematics* 140 (1994), 641—660,
21. B. Totaro, Chow groups, Chow cohomology, and linear varieties, *Forum of Mathematics, Sigma* (2014), vol. 1, e1,
22. B. Totaro, The motive of a classifying space, preprint,
23. C. Vial, Projectors on the intermediate algebraic Jacobians, *New York J. Math.* 19 (2013), 793—822,
24. C. Vial, Remarks on motives of abelian type, arXiv:1112.1080v2,
25. C. Voisin, The generalized Hodge and Bloch conjectures are equivalent for general complete intersections, *Annales scientifiques de l'ENS* 46, fascicule 3 (2013), 449—475,
26. C. Voisin, *Chow Rings, Decomposition of the Diagonal, and the Topology of Families*, Princeton University Press, Princeton and Oxford, 2014,
27. A. Weber, Pure homology of algebraic varieties, *Topology* 43 (2004), 635—644.