Self-similar measures and the Rajchman property

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Abstract

For classical Bernoulli convolutions, the convergence to zero at infinity of the Fourier transform was characterized by successive works of Erdös [4] and Salem [15]. We provide a quasi-complete extension of these results for general self-similar measures on the real line.

1 Introduction

Rajchman measures. In the present article we consider the extension of some well-known results concerning Bernoulli convolutions to a more general context of self-similar measures. For a Borel probability measure μ on $\mathbb{R}$, define its Fourier transform as :

$$\hat{\mu}(t) = \int_{\mathbb{R}} e^{2i\pi tx} \ d\mu(x), \ t \in \mathbb{R}.$$

We shall say that μ is Rajchman, whenever $\hat{\mu}(t) \to 0$, as $t \to +\infty$. When μ is a Borel probability measure on the torus $\mathbb{T} = \mathbb{R} \backslash \mathbb{Z}$, we introduce its Fourier coefficients, defined as :

$$\hat{\mu}(n) = \int_{\mathbb{T}} e^{2i\pi nx} \ d\mu(x), \ n \in \mathbb{Z}.$$

In this study, starting from a Borel probability measure μ on $\mathbb{R}$, Borel probability measures on $\mathbb{T}$ will naturally appear, quantifying the non-Rajchman character of μ.

For a Borel probability measure μ on $\mathbb{R}$, the Rajchman property holds for example if μ has a density with respect to Lebesgue measure $\mathcal{L}_{\mathbb{R}}$, by the Riemann-Lebesgue lemma. It can be verified without density and for instance there exist Cantor sets of zero Lebesgue measure and even of zero-Hausdorff dimension which support a Rajchman measure; cf Menshov [12], Bluhm [2]. Questions on the Rajchman property of a measure naturally arise in Harmonic Analysis, for example when studying sets of multiplicity for trigonometric series, cf Lyons [11] or Zygmund [23]. We shall say a word on this topic at the end of the article. Mention the classical counter-example of the uniform measure μ on the standard triadic Cantor set, which is a singular continuous measure, not Rajchman (because $\hat{\mu}(3n) = \hat{\mu}(n)$, $n \in \mathbb{Z}$). As in this last example, the obstructions for a measure to be Rajchman are often of arithmetical nature. The present work goes in this direction.

Naively, as it concerns $t \to +\infty$, the Rajchman character of a measure μ on $\mathbb{R}$ is an information of local regularity. As is well-known, it says for example that μ has no atom; if ever the convergence to zero is fast enough, then μ has a density; etc. Stricto sensu, the Rajchman character can be reformulated as an equidistribution property modulo 1. Since $\hat{\mu}(t) \to 0$ is equivalent to $\hat{\mu}(mt) \to 0$ for any integer $m \neq 0$, if X is a real random variable with law μ, then μ is Rajchman if and only if the law of $tX \mod 1$ converges, as $t \to +\infty$, to Lebesgue measure $\mathcal{L}_{\mathbb{T}}$ on $\mathbb{T}$.

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Self-similar measures. Let us now recall standard notions on self-similar measures on the real line $\mathbb{R}$, with a probabilistic point of view. We write $\mathcal{L}(X)$ for the law of a real random variable X. Let $N \geq 0$ and real affine maps $\varphi_k(x) = r_k x + b_k$, with $r_k > 0$, for $0 \leq k \leq N$, and at least one $r_k < 1$. We call (C) the condition that the $(\varphi_k)_{0 \leq k \leq N}$ are all strict contractions, in other words :

$$(C): \qquad 0 < r_k < 1, \text{ for all } 0 \leq k \leq N.$$

Introduce the vectors $r = (r_k)_{0 \leq k \leq N}$ and $b = (b_k)_{0 \leq k \leq N}$. Notice for what follows that for $n \geq 0$, a composition $\varphi_{k_{n-1}} \circ \cdots \circ \varphi_{k_0}$ has the form :

$$\varphi_{k_{n-1}} \circ \cdots \circ \varphi_{k_0}(x) = r_{k_{n-1}} \cdots r_{k_0} x + \sum_{l=0}^{n-1} b_{k_l} r_{k_{n-1}} \cdots r_{k_{l+1}}.$$

Consider the convex set $\mathcal{C}_N = \{p = (p_0, \cdots, p_N) \mid p_i \geq 0,\ \sum_i p_i = 1\}$ and $\mathcal{C}_N^o$ as the subset where $p_j > 0$, for all $0 \leq j \leq N$. Define :

$$\mathcal{D}_N(r) = \left\{ p \in \mathcal{C}_N \mid \sum_{0 \leq j \leq N} p_j \log r_j < 0 \right\}.$$

This is a non-empty open subset of $\mathcal{C}_N$, for the relative topology. Notice that $\mathcal{D}_N(r) = \mathcal{C}_N$, when condition (C) holds. Fixing a probability vector $p \in \mathcal{D}_N(r)$, we now compose the contractions at random, independently, according to p. Precisely, let X_0 be any real random variable and $(\varepsilon_n)_{n \geq 0}$ be independent and identically distributed random variables (*i.i.d.*), independent from X_0, and with law p, in other words $\mathbb{P}(\varepsilon_0 = k) = p_k$, $0 \leq k \leq N$. We consider the Markov chain $(X_n)_{n \geq 0}$ on $\mathbb{R}$ defined by $X_n = \varphi_{\varepsilon_{n-1}} \circ \cdots \circ \varphi_{\varepsilon_0}(X_0)$, $n \geq 0$.

The condition $p \in \mathcal{D}_N(r)$ is a hypothesis of contraction on average, rewritten as $\mathbb{E}(\log r_{\varepsilon_0}) < 0$. Classically, it implies that $(X_n)_{n \geq 0}$ has a unique invariant (or stationary) measure, written as ν. This for example follows from the fact that $\mathcal{L}(X_n) = \mathcal{L}(Y_n)$, where :

$$Y_n := \varphi_{\varepsilon_0} \circ \cdots \circ \varphi_{\varepsilon_{n-1}}(X_0) = r_{\varepsilon_0} \cdots r_{\varepsilon_{n-1}} X_0 + \sum_{l=0}^{n-1} b_{\varepsilon_l} r_{\varepsilon_0} \cdots r_{\varepsilon_{l-1}}.$$

As a standard fact, (Y_n) is more stable than (X_n). Using the Law of Large Numbers, we obtain $n^{-1} \log(r_{\varepsilon_0} \cdots r_{\varepsilon_{n-1}}) \to \mathbb{E}(\log r_{\varepsilon_0}) < 0$, a.-s., as $n \to +\infty$, so Y_n converges a.-s., as $n \to +\infty$, to :

$$X := \sum_{l \geq 0} b_{\varepsilon_l} r_{\varepsilon_0} \cdots r_{\varepsilon_{l-1}}.$$

Setting $\nu = \mathcal{L}(X)$, we obtain that $\mathcal{L}(X_n)$ weakly converges to ν. By construction, $\mathcal{L}(X_{n+1}) = \sum_{0 \leq j \leq N} p_j \mathcal{L}(X_n) \circ \varphi_j^{-1}$, hence, taking the limit as $n \to +\infty$, the measure ν verifies :

$$\nu = \sum_{0 \leq j \leq N} p_j \nu \circ \varphi_j^{-1}. \tag{1}$$

The previous convergence implies that the solution of this "stable fixed point equation" is unique among Borel probability measures. Also, ν has to be of pure type, i.e. either purely atomic or absolutely continuous with respect to Lebesgue measure $\mathcal{L}_{\mathbb{R}}$ or else singular continuous, since each term in its Radon-Nikodym decomposition with respect to $\mathcal{L}_{\mathbb{R}}$ verifies (1). A few remarks :

i) The measure ν is purely atomic if and only if the φ_j with $p_j > 0$ have a common fixed point c, in which case ν is the Dirac mass at c. Indeed (considering the necessity), suppose that ν has an atom. Let $a > 0$ be the maximal mass of an atom and E the finite set of points having mass a. Fixing any $c \in E$, the relation $\nu(\{c\}) = \sum_j p_j \nu(\{\varphi_j^{-1}(c)\})$ furnishes $\varphi_j^{-1}(c) \in E$, whenever $p_j > 0$. Hence $\varphi_j^{-n}(c) \in E$, $n \geq 0$. If $\varphi_j \neq id$, then $\varphi_j^{-1}(c) = c$, the set $\{\varphi_j^{-n}(c),\ n \geq 0\}$ being infinite otherwise. If $\varphi_j = id$, it fixes all points.

ii) The equation for a potential density f of ν with respect to $\mathcal{L}_{\mathbb{R}}$, coming from (1) is :

$$f = \sum_{0\le j\le N} p_j r_j^{-1} f\circ\varphi_j^{-1}.$$

This is essentially an "unstable fixed point equation", difficult to solve directly, equivalently reformulated into the fact that $((r_{\varepsilon_{n-1}}^{-1}\cdots r_{\varepsilon_0}^{-1})f\circ\varphi_{\varepsilon_{n-1}}^{-1}\cdots\circ\varphi_{\varepsilon_0}^{-1}(x))_{n\ge 0}$ is a non-negative martingale (for its natural filtration), for Lebesgue a.-e. $x\in\mathbb{R}$.

iii) Let $f(x)=ax+b$ be an affine map, with $a\neq 0$. With the same $p\in\mathcal{C}_N$, consider the conjugate system $(\psi_j)_{0\le j\le N}$, with $\psi_j(x)=f\circ\varphi_j\circ f^{-1}(x)=r_jx+b(1-r_j)+ab_j$. It has an invariant measure $w=\mathcal{L}(aX+b)$ verifying the relation $\hat{w}(t)=\hat{\nu}(at)e^{2i\pi tb}$, $t\in\mathbb{R}$. In particular ν is Rajchman if and only if w is Rajchman.

iv) When supposing condition (C), some self-similar set F can be introduced, where $F\subset\mathbb{R}$ is the unique non-empty compact set verifying the self-similarity relation :

$$F=\cup_{0\le k\le N}\varphi_k(F).$$

See for example Huchinson [7] for general properties of such sets. Introducing $\mathbb{N}=\{0,1,\cdots\}$ and the compact $S=\{0,\cdots,N\}^{\mathbb{N}}$, condition (C) implies that F is a continuous (and even hölderian) image of S, in other words we have the following description :

$$F = \left\{\sum_{l\ge 0} b_{x_l} r_{x_0}\cdots r_{x_{l-1}},\ (x_0,x_1,\cdots)\in S\right\}.$$

Whereas in the general case a self-similar invariant measure can have $\mathbb{R}$ as topological support, under condition (C) the compact self-similar set F exists and supports any self-similar measure.

Background and content of the article. Back to the general case, we assume in the sequel that the φ_j with $p_j>0$ do not have a common fixed point; in particular $N\ge 1$. A difficult problem is to characterize the absolute continuity of ν in terms of the parameters r, b and p. An example with a long and well-known history is that of Bernoulli convolutions, corresponding to $N=1$, the affine contractions $\varphi_0(x)=\lambda x-1$, $\varphi_1(x)=\lambda x+1$, $0<\lambda<1$, and the probability vector $p=(1/2,1/2)$. Notice that when the r_i are equal (to some real in $(0,1)$), the situation is a little simplified, as ν is an infinite convolution (this is not true in general). Although we discuss below some works in this context, we will not present here the vast subject of Bernoulli convolutions, addressing the reader to detailed surveys, Peres-Schlag-Solomyak [14] or Solomyak [19].

For general self-similar measures, an important aspect of the problem, that we shall not enter, and an active line of research, concerns the Hausdorff dimension of the measure ν, cf Hochman [6]. In a large generality, cf for example Falconer [5] and more recently Jaroszewska and Rams [8], there is an "entropy/Lyapunov exponent" upper-bound :

$$\mathrm{Dim}_{\mathcal{H}}(\nu)\le\min\{1,s(p,r)\},\text{ where } s(p,r):=\frac{-\sum_{i=0}^N p_i\log p_i}{-\sum_{i=0}^N p_i\log r_i}.$$

The quantity $s(p,r)$ is called the singularity dimension of the measure and can be >1. The equality $\mathrm{Dim}_{\mathcal{H}}(\nu)=1$ does not mean that ν is absolutely continuous, but the inequality $s(p,r)<1$ implies that ν is singular. The interesting domain of parameters for the question of the absolute continuity of the invariant measure therefore corresponds to $s(p,r)\ge 1$.

We focus in this work on another fundamental tool, the Fourier transform $\hat{\nu}$. If ν is not Rajchman, the Riemann-Lebesgue lemma implies that ν is singular. This property was used by Erdös [4] in the context of Bernoulli convolutions. Erdös proved that if $1/2<\lambda<1$ is such that $1/\lambda$ is a Pisot number, then ν is not Rajchman. The reciprocal statement was next shown by Salem [15]. As a result, for Bernoulli convolutions the Rajchman property always holds, except for a very particular set of parameters. For general self-similar measures, supposing condition (C), the non-Rajchman character was recently shown to hold for only a very small set of parameters, by Solomyak [20] : if $p \in \mathcal{C}_N^o$, $N \geq 1$, and the $(\varphi_k)_{0\leq k\leq N}$ do not have a common fixed point, then outside a set of r of zero-Hausdorff dimension, $\hat{\nu}$ even has a power decay at infinity.

The aim of the present article is to study for general self-similar measures the exceptional set of parameters where the Rajchman property is not true. We essentially show that r and b have to be very specific, as for Bernoulli convolutions. We shall first prove a general extension of the result of Salem [15], drastically reducing the set of parameters where the Rajchman property does not hold. Focusing then on this particular case, we provide a general characterization of the Rajchman character. Next, restricting to condition (C), we prove a partial extension of the theorem of Erdös [4]. We next give some complements, first rather surprising numerical simulations involving the Plastic number, then an application to sets of uniqueness for trigonometric series.

2 Statement of the results

Let us place in the general situation considered in the introduction. Without surprise, Pisot numbers come out of the analysis. Let us introduce a few definitions concerning Algebraic Number Theory; cf for example Samuel [17] for more details.

Definition 2.1
A Pisot number is a real algebraic integer $\theta > 1$, with conjugates (i.e. the other roots of its minimal unitary polynomial) of modulus strictly less than 1. We fix such a $\theta > 1$ and denote as :

$$Q = X^{s+1} + a_s X^s + \cdots + a_0 \in \mathbb{Z}[X],$$

its minimal polynomial, of degree $s+1$, with $s \geq 0$. If $s = 0$, then θ is an integer ≥ 2. The images of $\mu \in \mathbb{Q}[\theta]$ by the $s+1$ $\mathbb{Q}$-homomorphisms $\mathbb{Q}[\theta] \to \mathbb{C}$ are called conjugates of μ and are in general denoted by $\mu^{(0)} = \mu, \mu^{(1)}, \cdots, \mu^{(s)}$. Let us also introduce :

i) For $\alpha \in \mathbb{Q}[\theta]$, the trace $Tr(\alpha)$ is the trace of the linear multiplication operator $x \longmapsto \alpha x$, considered from $\mathbb{Q}[\theta]$ to itself. As a general fact, $Tr(\alpha) \in \mathbb{Q}$.

ii) Let $\mathbb{Z}[\theta] = \mathbb{Z}\theta^0 + \cdots + \mathbb{Z}\theta^s$, the subring generated by θ of the ring of algebraic integers of $\mathbb{Q}[\theta]$. We write $\mathcal{D}(\theta)$ for its $\mathbb{Z}$-dual, as a $\mathbb{Z}$-lattice :

$$\mathcal{D}(\theta) = \{\alpha \in \mathbb{Q}[\theta],\ Tr(\theta^n \alpha) \in \mathbb{Z},\ \text{for } 0 \leq n \leq s\}$$

It can be shown that $\mathcal{D}(\theta) = (1/Q'(\theta))\mathbb{Z}[\theta]$.

iii) Classically, $Tr(\theta^n \alpha) \in \mathbb{Z}$, for all $n \geq 0$, if this holds for $0 \leq n \leq s$. Let us define :

$$\mathcal{T}(\theta) = \{\alpha \in \mathbb{Q}[\theta],\ Tr(\theta^n \alpha) \in \mathbb{Z},\ \text{for large } n \geq 0\} = \cup_{n\geq 0}\theta^{-n}\mathcal{D}(\theta) = \frac{1}{Q'(\theta)}\mathbb{Z}[\theta, 1/\theta],$$

where $\mathbb{Z}[\theta, 1/\theta]$ is the subring of $\mathbb{Q}[\theta]$ generated by θ and $1/\theta$.

Let us now introduce special families of affine maps, that will somehow play the role of canonical models for the analysis of the Rajchman property.

Definition 2.2
Let $N \geq 1$. A family of real affine maps $\varphi_k(x) = r_k x + b_k$, with $r_k > 0$, for $0 \leq k \leq N$, and at least one $r_k < 1$ and no common fixed point, is in reduced Pisot form, if there exist a Pisot number $1/\lambda > 1$, relatively prime integers $(n_k)_{0\leq k\leq N}$ and $\mu_k \in \mathcal{T}(1/\lambda)$, $0 \leq k \leq N$, such that :

$$\varphi_j(x) = \lambda^{n_j} x + \mu_j,\ \text{for all } 0 \leq j \leq N,$$

with moreover some Bezout relation $1 = \sum_{0\leq j\leq N, n_j\neq 0} l_j n_j$ verifying the following condition :

$$0 = \sum_{0\le j\le N, n_j\ne 0} \left(\frac{\mu_j}{1-\lambda^{n_j}}\right) \lambda^{\sum_{0\le k<j, n_k\ne 0} l_k n_k}(1-\lambda^{l_j n_j}).$$

We call such an identity a Bezout centering relation.

Remark. — If a family $(\varphi_j)_{0\le j\le N}$ is in reduced Pisot form, then the $(\lambda, (n_j), (\mu_j))$ are uniquely determined. Indeed, if $(\lambda', (n'_j), (\mu'_j))$ also convenes, it suffices to show that $\lambda = \lambda'$. Taking some collection of integers (a_j) realizing a Bezout relation for the (n_j), we have :

$$\lambda = \lambda^{\sum_j a_j n_j} = \lambda'^{\sum_j a_j n'_j} = \lambda'^p,$$

for some $p \ge 1$. Idem, $\lambda' = \lambda^q$, for some $q \ge 1$. Hence $pq = 1$, giving $p = q = 1$ and $\lambda = \lambda'$. Notice that if some Bezout relation satisfies a centering condition, it may not be the case for another one.

As a first result, extending [15], the analysis of the non-Rajchman character of the invariant measure necessitates to consider families in reduced Pisot form.

Theorem 2.3
Let $N \ge 1$, $p \in \mathcal{C}_N^o$ and affine maps $\varphi_k(x) = r_k x + b_k$, $r_k > 0$, for $0 \le k \le N$, with no common fixed point and $\sum_{0\le j\le N} p_j \log r_j < 0$. The invariant measure ν is not Rajchman if and only if there exists $f(x) = ax + b$, $a \ne 0$, such that the conjugate system $(f \circ \varphi_j \circ f^{-1})_{0\le j\le N}$ is in reduced Pisot form, for some Pisot number $1/\lambda > 1$, with invariant measure w verifying $\hat{w}(\lambda^{-k}) \not\to_{k+\infty} 0$.

As above and in [4], sequences of the form $(\alpha\lambda^{-k})_{k\ge 0}$, $\alpha \ne 0$, will play a central role. In a second step, we provide a general analysis of families in reduced Pisot form, specifying when we require the existence of a Bezout centering relation.

Fixing a Pisot number $1/\lambda > 1$, an important preliminary remark is that when $\mu \in \mathcal{T}(1/\lambda)$ and $k \ge 0$ is large enough, we have :

$$\lambda^{-k}\mu + \sum_{1\le j\le s} \alpha_j^k \mu^{(j)} = Tr(\lambda^{-k}\mu) \in \mathbb{Z},$$

where the $(\alpha_j)_{0\le j\le s}$ are the other conjugates of $1/\lambda = \alpha_0$ and the $(\mu^{(j)})_{1\le j\le s}$ that of $\mu = \mu^{(0)}$, in the field $\mathbb{Q}[\lambda]$. Since $|\alpha_j| < 1$, for $1 \le j \le s$, and (S_l) is a.-s. transient with a non-zero linear speed to $-\infty$, as $l \to -\infty$, this ensures that for any $k \in \mathbb{Z}$, the random variable $\sum_{l\in\mathbb{Z}} \mu_{\varepsilon_l} \lambda^{k+S_l} \mod 1$ is a well-defined $\mathbb{T}$-valued random variable.

Theorem 2.4
Let $N \ge 1$ and $\varphi_k(x) = \lambda^{n_k} x + \mu_k$, for $0 \le k \le N$, with $1/\lambda > 1$ a Pisot number of degree $s+1$, relatively prime integers $(n_k)_{0\le k\le N}$ and $\mu_k \in \mathcal{T}(1/\lambda)$, for $0 \le k \le N$.

Let $p \in \mathcal{C}_N^o$ be such that $\sum_{0\le j\le N} p_j n_j > 0$ and i.i.d. random variables $(\varepsilon_n)_{n\in\mathbb{Z}}$, with $\mathbb{P}(\varepsilon_0 = k) = p_k$, $0 \le k \le N$. Set $S_l = n_{\varepsilon_0} + \cdots + n_{\varepsilon_{l-1}}$, $l > 0$, $S_0 = 0$ and $S_l = -n_{\varepsilon_l} - \cdots - n_{\varepsilon_{-1}}$, $l < 0$. With this notation, the real random variable $X = \sum_{l\ge 0} \mu_{\varepsilon_l} \lambda^{S_l}$ has law ν.

i) Introduce the $\mathbb{T}$-valued random variables $Z_k = \sum_{l\in\mathbb{Z}} \mu_{\varepsilon_l} \lambda^{k+S_l}$, $k \in \mathbb{Z}$. Then $\lambda^{-n} X \mod 1$ converges to a probability measure m on $\mathbb{T}$, verifying, for all $f \in C(\mathbb{T}, \mathbb{R})$ and all $k \in \mathbb{Z}$:

$$\int_{\mathbb{T}} f(x)\ dm(x) = \frac{1}{\mathbb{E}(n_{\varepsilon_0})} \sum_{0\le r<n^*} \mathbb{E}\left[f\left(Z_{k+r}\right) 1_{S_{-u}<-r, u\ge 1}\right],$$

where $n^* = \max_{0\le k\le N} n_k$. More generally, $(\lambda^{-n}X, \lambda^{-n-1}X, \cdots, \lambda^{-n-s}X) \mod \mathbb{Z}^{s+1}$ converges in law, as $n \to +\infty$, to a probability measure $\mathcal{M}$ on $\mathbb{T}^{s+1}$, with marginals m, verifying :

$$\int_{\mathbb{T}^{s+1}} f(x)\ d\mathcal{M}(x) = \frac{1}{\mathbb{E}(n_{\varepsilon_0})} \sum_{0\le r<n^*} \mathbb{E}\left[f\left(Z_{k+r}, Z_{k+r-1}, \cdots, Z_{k+r-s}\right) 1_{S_{-u}<-r, u\ge 1}\right],$$

for all $f \in C(\mathbb{T}^{s+1}, \mathbb{R})$ and all $k \in \mathbb{Z}$.

ii) Let $\mathcal{P} = \{\alpha \in \mathbb{Q}[\lambda],\ \alpha \neq 0$ and $\alpha\mu_j \in \mathcal{T}(1/\lambda)$, for all $0 \leq j \leq N\}$. For $\alpha \in \mathcal{P}$, let m_α and $\mathcal{M}_\alpha$ be the measures corresponding to m and $\mathcal{M}$, when replacing the (μ_j) by the $(\alpha\mu_j)$.

- *a) If ν is Rajchman, then $\mathcal{M}_\alpha = \mathcal{L}_{\mathbb{T}^{s+1}}$, for all $\alpha \in \mathcal{P}$.*
- *b) If ν is continuous and $\mathcal{M}_{|\mathbb{T}^{s+1}\setminus\{0\}} \ll \mathcal{L}_{\mathbb{T}^{s+1}}$, for example if ν is Rajchman, then $\nu \ll \mathcal{L}_{\mathbb{R}}$.*
- *c) If some Bezout centering relation holds and $m_\alpha = \mathcal{L}_{\mathbb{T}}$, $\alpha \in \mathcal{P} \cap [1, 1/\lambda]$, then ν is Rajchman.*

A general analysis of m and $\mathcal{M}$ has to be done. A corollary of Theorem 2.4 is for example that whenever $\mathcal{M} \ll \mathcal{L}_{\mathbb{T}^{s+1}}$, then in fact $\mathcal{M} = \mathcal{L}_{\mathbb{T}^{s+1}}$. We finally consider families in reduced Pisot form, without the condition of existence of a Bezout centering relation, under condition (C).

Theorem 2.5
Let $N \geq 1$ and $\varphi_k(x) = \lambda^{n_k}x + \mu_k$, for $0 \leq k \leq N$, with $1/\lambda > 1$ a Pisot number, relatively prime integers $(n_k)_{0\leq k\leq N}$, with $n_k \geq 1$ and $\mu_k \in \mathcal{T}(1/\lambda)$, for $0 \leq k \leq N$.

i) For any $p \in \mathcal{C}_N^o$, the invariant measure ν is Rajchman if and only if it has a density, bounded and with compact support, with respect to Lebesgue measure $\mathcal{L}_{\mathbb{R}}$.

ii) There exists $0 \neq a \in \mathbb{Z}$ such that for any $k \neq 0$, for all $p \in \mathcal{C}_N^o$ outside a finite set (depending on k), $\hat{m}(ak) \neq 0$, where m is the measure of Theorem 2.4, i). Thus, ν is not Rajchman, for any $p \in \mathcal{C}_N^o$ outside a finite set. Moreover, for all $p \in \mathcal{C}_N^o$ outside a countable set, $\hat{m}(ak) \neq 0$, $k \in \mathbb{Z}$.

Remark. — Part *ii*) of Theorem 2.5 relies on an indirect argument, based on the analysis of the regularity of $\hat{m}(n)$, for some fixed $n \in \mathbb{Z}$, as a function of $p \in \mathcal{C}_N^o$. We give very concrete examples in the last section, with $N = 1$ and $1/\lambda$ the Plastic number, where ν is not Rajchman for all $p \in \mathcal{C}_1$. On the existence of singular measures in the inhomogeneous case, we were previously essentially aware of the non-explicit examples, using algebraic curves, of Neunhäuserer [13].

Remark. — Still on Theorem 2.5 *ii*), under condition (C), observe that when the Pisot number $1/\lambda$ is an integer ≥ 2, then the involved finite set can be non-empty. For instance, if $N \geq 1$ and $\varphi_k(x) = (x + k)/(N + 1)$, for $0 \leq k \leq N$, with $p = (1/(N + 1), \cdots, 1/(N + 1))$, then ν is Lebesgue measure on $[0, 1]$. It would be important to find examples (if there are some) when the Pisot number $1/\lambda$ is irrational and more generally to determine, in the context of Theorem 2.5, the exceptional parameters where ν is Rajchman.

Remark. — Some number theoretic question appears in Theorem 2.4 *ii*). Fix a Pisot number $\theta > 1$ and a non-zero family $(\mu_j)_{0\leq j\leq N} \in \mathcal{T}(\theta)^{N+1}$. Since $\theta^M\mu_j \in \mathcal{D}(\theta)$, for all $0 \leq j \leq N$, as soon as $M \geq 0$ is large enough, assume for example that $(\mu_j)_{0\leq j\leq N} \in \mathcal{D}(\theta)^{N+1}$.

- For $\mu \in \mathbb{Q}[\theta]$, define $den(\mu)$ as the lcm of the denominators of the vector of irreducible fractions $(Tr(\theta^l\mu))_{0\leq l\leq s}$. Then $den(\mu)\mu \in \mathcal{D}(\theta)$. Taking any $0 \neq \alpha \in \mathbb{Q}[\theta]$, introduce $q = \mathrm{lcm}\{den(\alpha\mu_j),\ 0 \leq j \leq N\}$. Then $n\alpha\mu_j \in \mathcal{D}(\theta)$, for all $0 \leq j \leq N$, if and only if $n \in q\mathbb{Z}$.
- Another approach is to take some other family $0 \neq (\mu'_j)_{0\leq j\leq N} \in \mathcal{D}(\theta)^{N+1}$. There exists $\alpha \neq 0$ (necessarily in $\mathbb{Q}[\theta]$) such that $\mu'_j = \alpha\mu_j$, for all $0 \leq j \leq N$, if and only if $\mu_i\mu'_j - \mu'_i\mu_j = 0$, for all $0 \leq i \neq j \leq N$. Since $\mathcal{D}(\theta) = (1/Q'(\theta))\mathbb{Z}[\theta]$, where $Q(x) = X^{s+1} + a_sX^s + \cdots + a_0 \in \mathbb{Z}[X]$ is the minimal polynomial of θ, consider the equality $yz' - y'z = 0$, for elements y, y', z, z' in $\mathbb{Z}[\theta]$. Let $y = \sum_{0\leq u\leq s} n_u\theta^u$ and $z = \sum_{0\leq v\leq s} m_v\theta^v$, with similar expressions with $'$ for y' and z'. Introduce the integer-valued $(s + 1) \times (s + 1)$-companion matrix M of Q :

$$M = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & 1 \\ -a_0 & \cdots & -a_{s-1} & -a_s \end{pmatrix}.$$

  If y has coordinates $(n_0, \cdots, n_s)$ in the basis $(\theta^0, \cdots, \theta^s)$ of $\mathbb{Q}[\theta]$, then in the same basis θy has coordinates $(n_0, \cdots, n_s)M$. From this it is not difficult to infer that the conditions on the $(n_u), (m_u), (n'_u), (m'_u)$ for the equality $yz' - y'z = 0$ can be reformulated as :

$$(n_0,\cdots,n_s)\sum_{0\le u\le s} m'_u M^u - (n'_0,\cdots,n'_s)\sum_{0\le u\le s} m_u M^u = 0.$$

In the same fashion, observe that for any $\mu \in \mathbb{Q}[\theta]$ and when setting $V = (Tr(\theta^0\mu),\cdots,Tr(\theta^s\mu))$, then $\mu \in \mathcal{T}(\theta)$ if and only if there exists $n \ge 0$ such that VM^n has integral entries.

3 Proof of Theorem 2.3

For the sequel, introduce *i.i.d.* random variables $(\varepsilon_n)_{n\ge 0}$ with law p, to which $\mathbb{P}$ and $\mathbb{E}$ refer. Recall that ν is the law of the random variable $\sum_{l\ge 0} b_{\varepsilon_l} r_{\varepsilon_0}\cdots r_{\varepsilon_{l-1}}$. Without loss of generality, we also assume that $0 < r_0 \le r_1 \le \cdots \le r_N$, with necessarily $r_0 < 1$.

Step 1. We prove that if $\log r_i/\log r_j \notin \mathbb{Q}$, for some $0 \le i \ne j \le N$, then ν is Rajchman. This was shown by Li and Sahlsten [10], under condition (C). We simplify their proof. In [10], some logarithmic decay at infinity of $\hat{\nu}$ is also established, under an additional Diophantine condition.

For $n \ge 1$, consider the random walk $S_n = -\log r_{\varepsilon_0} - \cdots - \log r_{\varepsilon_{n-1}}$, with $S_0 = 0$. For a real $s \ge 0$, introduce the finite stopping time $\tau_s = \min\{n \ge 0,\ S_n > s\}$ and write $\mathcal{T}_s$ for the corresponding sub-σ-algebra of the underlying σ-algebra. Taking $\alpha > 0$ and $s \ge 0$:

$$\begin{aligned}\hat{\nu}(\alpha e^s) &= \mathbb{E}\left(e^{2\pi i\alpha e^s\sum_{l\ge 0} b_{\varepsilon_l} e^{-S_l}}\right)\\ &= \mathbb{E}\left(e^{2\pi i\alpha e^s\sum_{0\le l<\tau_s} b_{\varepsilon_l} e^{-S_l}} e^{2\pi i\alpha e^{-S_{\tau_s}+s}\sum_{l\ge\tau_s} b_{\varepsilon_l} e^{-S_l+S_{\tau_s}}}\right).\end{aligned}$$

In the expectation, the first exponential term is $\mathcal{T}_s$-measurable. Also, the conditional expectation of the second exponential term with respect to $\mathcal{T}_s$ is just $\hat{\nu}(\alpha e^{-S_{\tau_s}+s})$, as a consequence of the strong Markov property. It follows that :

$$\hat{\nu}(\alpha e^s) = \mathbb{E}\left(\hat{\nu}(\alpha e^{-S_{\tau_s}+s}) e^{2\pi i\alpha e^s\sum_{0\le l<\tau_s} b_{\varepsilon_l} e^{-S_l}}\right).$$

This gives $|\hat{\nu}(\alpha e^s)| \le \mathbb{E}\left(|\hat{\nu}(\alpha e^{-S_{\tau_s}+s})|\right)$, so by the Cauchy-Schwarz inequality and a safe Fubini theorem consecutively :

$$\begin{aligned}|\hat{\nu}(\alpha e^s)|^2 \le \mathbb{E}\left(|\hat{\nu}(\alpha e^{-S_{\tau_s}+s})|^2\right) &= \mathbb{E}\left(\int_{\mathbb{R}^2} e^{2\pi i\alpha e^{-S_{\tau_s}+s}(x-y)}\, d\nu(x)d\nu(y)\right)\\ &= \int_{\mathbb{R}^2}\mathbb{E}\left(e^{2\pi i\alpha e^{-S_{\tau_s}+s}(x-y)}\right)\, d\nu(x)d\nu(y).\end{aligned}$$

Let $Y := -\log r_{\varepsilon_0}$. As the law of Y is non-lattice (since some $\log r_i/\log r_j \notin \mathbb{Q}$ and $p_k > 0$ for all $0 \le k \le N$) and with $0 < \mathbb{E}(Y) < \infty$, it is a well-known consequence of the Blackwell theorem on the law of the overshoot that (see for instance Woodroofe [22], chap. 2, thm 2.3), that :

$$\mathbb{E}(g(S_{\tau_s} - s)) \to \frac{1}{\mathbb{E}(S_{\tau_0})}\int_0^{+\infty} g(x)\mathbb{P}(S_{\tau_0} > x)\, dx, \text{ as } s \to +\infty,$$

for any Riemann-integrable g on $\mathbb{R}_+$. Here, all $S_{\tau_s} - s$, $s \ge 0$, (in particular S_{τ_0}) have support in some $[0, A]$. Thus, also, $\mathbb{P}(S_{\tau_0} > x) = 0$ for large $x > 0$. By dominated convergence, for any $\alpha > 0$:

$$\limsup_{t\to+\infty} |\hat{\nu}(t)|^2 \le \frac{1}{\mathbb{E}(S_{\tau_0})}\int_{\mathbb{R}^2}\left|\int_0^{+\infty} e^{2\pi i\alpha e^{-u}(x-y)}\mathbb{P}(S_{\tau_0} > u)du\right| d\nu(x)d\nu(y).$$

The inside term (in the modulus) is uniformly bounded with respect to $(x, y) \in \mathbb{R}^2$. We shall use dominated convergence once more, this time with $\alpha \to +\infty$. It is sufficient to show that for

$\nu^{\otimes 2}$-almost every (x,y), the inside term goes to zero. Since ν is non-atomic, $\nu^{\otimes 2}$-almost-surely, $x \neq y$. If for example $x > y$:

$$\int_0^{+\infty} e^{2\pi i\alpha e^{-u}(x-y)}\mathbb{P}(S_{\tau_0} > u)du = \int_0^{x-y} e^{2\pi i\alpha t}\mathbb{P}(S_{\tau_0} > \log((x-y)/t)\ \frac{dt}{t},$$

making the change of variable $t = e^{-u}(x-y)$. The last integral now converges to 0, as $\alpha \to +\infty$, by the Riemann-Lebesgue lemma. Hence, $\lim_{t\to+\infty}\hat{\nu}(t) = 0$. This ends the proof of this step.

Step 2. Assuming ν not Rajchman, from *Step 1*, $\log r_i/\log r_j \in \mathbb{Q}$, for all (i,j). Hence $r_j = r_0^{p_j/q_j}$, with integers $p_j \in \mathbb{Z}$, $q_j \geq 1$, for $1 \leq j \leq N$. Let :

$$n_0 = \prod_{1\leq l\leq N} q_l \geq 1 \text{ and } n_j = p_j \prod_{1\leq l\leq N, l\neq j} q_l \in \mathbb{Z},\ 1 \leq j \leq N.$$

Recall that $0 < r_0 < 1$. Setting $\lambda = r_0^{1/n_0} \in (0,1)$, one has $r_j = \lambda^{n_j}$, $0 \leq j \leq N$. Up to taking some positive integral power of λ, one can assume that $\gcd(n_0,\cdots,n_N) = 1$. Recall in passing that the set of Pisot numbers is stable under positive integral powers. The condition $\mathbb{E}(\log r_{\varepsilon_0}) < 0$ rewrites into $\mathbb{E}(n_{\varepsilon_0}) > 0$ and we have $n_N \leq \cdots \leq n_0$, with $n_0 \geq 1$.

Using now some sub-harmonicity, one can reinforce the assumption that $\hat{\nu}(t)$ is not converging to 0, as $t \to +\infty$.

Lemma 3.1
There exists $1 \leq \alpha \leq 1/\lambda$ *and* $c > 0$ *such that* $\hat{\nu}(\alpha\lambda^{-k}) = c_k e^{2i\pi\theta_k}$*, written in polar form, verifies* $c_k \to c$*, as* $k \to +\infty$*.*

Proof of the lemma :
Let us write this time $S_n = n_{\varepsilon_0} + \cdots + n_{\varepsilon_{n-1}}$, for $n \geq 1$, with $S_0 = 0$. Since $\mathbb{E}(n_{\varepsilon_0}) > 0$, (S_n) is transient to $+\infty$. Introduce the random ladder epochs $0 = \sigma_0 < \sigma_1 < \cdots$, where inductively σ_{k+1} is the first time $n \geq 0$ with $S_n > S_{\sigma_k}$. Let $S'_k = S_{\sigma_k}$. The $(S'_k - S'_{k-1})_{k\geq 1}$ are *i.i.d.* random variables with law $\mathcal{L}(S_{\tau_0})$ and support in $\{1,\cdots,n_0\}$. Since $\gcd(n_0,\cdots,n_N) = 1$, the support of the law of S_{τ_0} generates $\mathbb{Z}$ as an additive group (cf for example Woodroofe [22], thm 2.3, second part). For an integer $u \geq 1$ large enough, we can fix integers $r \geq 1$ and $s \geq 1$ such that the support of the law of S'_r contains u and that of S'_s contains $u+1$, both supports being included in some $\{1,\cdots,M\}$, with therefore $1 \leq u \leq u+1 \leq M$. Proceeding as in *Step 1*, for any $t \in \mathbb{R}$:

$$\hat{\nu}(t) = \mathbb{E}\left(e^{2\pi it\sum_{l\geq 0} b_{\varepsilon_l}\lambda^{S_l}}\right) = \mathbb{E}\left(\hat{\nu}(t\lambda^{S'_r})e^{2\pi it\sum_{0\leq l<\sigma_r} b_{\varepsilon_l}\lambda^{S_l}}\right).$$

Doing the same thing with S'_s and taking modulus gives :

$$|\hat{\nu}(t)| \leq \mathbb{E}\left(|\hat{\nu}(t\lambda^{S'_r})|\right) \text{ and } |\hat{\nu}(t)| \leq \mathbb{E}\left(|\hat{\nu}(t\lambda^{S'_s})|\right). \tag{2}$$

In particular, $|\hat{\nu}(t)| \leq \max_{1\leq l\leq M}|\hat{\nu}(\lambda^l t)|$. We now set :

$$V_\alpha(k) := \max_{k\leq l<k+M}|\hat{\nu}(\alpha\lambda^l)|,\ k \in \mathbb{Z},\ \alpha > 0.$$

The previous remarks imply that $V_\alpha(k) \leq V_\alpha(k+1)$, $k \in \mathbb{Z}$, $\alpha > 0$.

Since ν is not Rajchman, $|\hat{\nu}(t_l)| \geq c' > 0$, along some sequence $t_l \to +\infty$. Write $t_l = \alpha_l\lambda^{-k_l}$, with $1 \leq \alpha_l \leq 1/\lambda$ and $k_l \to +\infty$. Up to taking a subsequence, $\alpha_l \to \alpha \in [1, 1/\lambda]$. Fixing $k \in \mathbb{Z}$:

$$c' \leq V_{\alpha_l}(-k_l) \leq V_{\alpha_l}(-k),$$

as soon as l is large enough. By continuity, letting $l \to +\infty$, we get $c' \leq V_\alpha(-k)$, $k \in \mathbb{Z}$. As $k \longmapsto V_\alpha(-k)$ is non-increasing, $V_\alpha(-k) \to c \geq c'$, as $k \to +\infty$. We now show that necessarily $|\hat{\nu}(\alpha\lambda^{-k})| \to c$, as $k \to +\infty$.

If this were not true, there would exist $\varepsilon > 0$ and $(m_k) \to +\infty$, with $|\hat{\nu}(\alpha\lambda^{-m_k})| \leq c-\varepsilon$. Using $V_\alpha(-k) \to c$ and $|\hat{\nu}(\alpha\lambda^{-m_k})| \leq c - \varepsilon$, as $k \to +\infty$, consider (2) with r and $t = \alpha\lambda^{-m_k-u}$ and next with s and $t = \alpha\lambda^{-m_k-u-1}$. Since u is in the support of the law of S'_r and $u+1$ is in the support of the law of S'_s, we obtain the existence of some $c_1 < c$ such that for k large enough :

$$\max\{|\hat{\nu}(\alpha\lambda^{-m_k-u})|, |\hat{\nu}(\alpha\lambda^{-m_k-u-1})|\} \leq c_1 < c.$$

Again via (2), with successively r and $t = \alpha\lambda^{-m_k-2u}$, next r and $t = \alpha\lambda^{-m_k-2u-1}$ and finally s and $t = \alpha\lambda^{-m_k-2u-2}$, still using that u is in the support of the law of S'_r and $u+1$ in the support of the law of S'_s, we get some $c_2 < c$ such that for k large enough :

$$\max\{|\hat{\nu}(\alpha\lambda^{-m_k-2u})|,\ |\hat{\nu}(\alpha\lambda^{-m_k-2u-1})|,\ |\hat{\nu}(\alpha\lambda^{-m_k-2u-2})|\} \leq c_2 < c.$$

Etc, for some $c_{M-1} < c$ and k large enough :

$$\max\{|\hat{\nu}(\alpha\lambda^{-m_k-(M-1)u})|,\ \cdots,\ |\hat{\nu}(\alpha\lambda^{-m_k-(M-1)u-(M-1)})|\} \leq c_{M-1} < c.$$

This contradicts the fact that $V_\alpha(-k) \to c$, as $k \to \infty$. We conclude that $|\hat{\nu}(\alpha\lambda^{-k})| \to c$, as $k \to \infty$, and this ends the proof of the lemma.

$\square$

Step 3. We complete the proof of Theorem 2.3. In this part, introduce the notation $\|x\| = \operatorname{dist}(x, \mathbb{Z})$, for $x \in \mathbb{R}$. Let us consider any $1 \leq \alpha \leq 1/\lambda$, with $\hat{\nu}(\alpha\lambda^{-k}) = c_k e^{2i\pi\theta_k}$, verifying $c_k \to c > 0$, as $k \to +\infty$. We start from the relation :

$$\hat{\nu}(\alpha\lambda^{-k}) = \sum_{0\leq j\leq N} p_j e^{2i\pi\alpha\lambda^{-k}b_j}\hat{\nu}(\alpha\lambda^{-k+n_j}),$$

obtained by conditioning with respect to the value of ε_0. This furnishes for $k \geq 0$:

$$c_k = \sum_{0\leq j\leq N} p_j e^{2i\pi(\alpha\lambda^{-k}b_j+\theta_{k-n_j}-\theta_k)}c_{k-n_j}.$$

We rewrite this as :

$$\sum_{0\leq j\leq N} p_j \left[e^{2i\pi(\alpha\lambda^{-k}b_j+\theta_{k-n_j}-\theta_k)} - 1\right] c_{k-n_j} = c_k - \sum_{0\leq j\leq N} p_j c_{k-n_j} = \sum_{0\leq j\leq N} p_j(c_k - c_{k-n_j}).$$

Let $K > 0$ be such that $c_{k-n_j} \geq c/2 > 0$, for $k \geq K$ and all $0 \leq j \leq N$. For $L > n^*$, where $n^* = \max_{0\leq j\leq N}|n_j|$, we sum the previous equality on $K \leq k \leq K+L$:

$$\sum_{0\leq j\leq N} p_j \sum_{k=K}^{K+L} c_{k-n_j}\left[e^{2i\pi(\alpha\lambda^{-k}b_j+\theta_{k-n_j}-\theta_k)} - 1\right] = \sum_{0\leq j\leq N} p_j \left(\sum_{k=K}^{K+L} c_k - \sum_{k=K-n_j}^{K+L-n_j} c_k\right).$$

Observe that the right-hand side involves a telescopic sum and is bounded by $2n^*$ (using that $|c_k| \leq 1$), uniformly in K and L. In the left hand-hand side, we take the real part and use that $1 - \cos(2\pi x) = 2(\sin\pi x)^2$, which, as is well-known, has the same order as $\|x\|^2$. We obtain, for some constant C, that for K and L large enough :

$$\frac{c}{2}\sum_{0\leq j\leq N} p_j \sum_{k=K}^{K+L} \|\alpha\lambda^{-k}b_j + \theta_{k-n_j} - \theta_k\|^2 \leq C.$$

Introducing the constants $p_* = \min_{0\leq j\leq N} p_j > 0$ and $C' = 2C/(cp_*)$, we get that for all $0 \leq j \leq N$ and K, L large enough :

$$\sum_{k=K}^{K+L} \|\alpha\lambda^{-k} b_j + \theta_{k-n_j} - \theta_k\|^2 \leq C'. \tag{3}$$

In the sequel, we distinguish two cases : there is a non-zero translation (case 1) or not (case 2).

- *Case 1.* For any non-zero-translation $\varphi_j(x) = x + b_j$, we have $n_j = 0$ and $b_j \neq 0$. Then (3) gives that for K, L large enough :

$$\sum_{k=K}^{K+L} \|\alpha\lambda^{-k} b_j\|^2 \leq C'.$$

This implies that $(\|\alpha b_j \lambda^{-k}\|)_{k\geq 0} \in l^2(\mathbb{N})$. By a classical theorem of Pisot, cf Cassels [3], chap. 8, Theorems I and II, we obtain that $1/\lambda$ is a Pisot number and $b_j = (1/\alpha)\mu_j$, with $\mu_j \in \mathcal{T}(1/\lambda)$. Consider now the non-translations $\varphi_j(x) = \lambda^{n_j} x + b_j$, $n_j \neq 0$. By (3), for any $r \geq 0$ and K, L large enough (depending on r) :

$$\sum_{k=K}^{K+L} \|\alpha\lambda^{-k+rn_j} b_j + \theta_{k-(r+1)n_j} - \theta_{k-rn_j}\|^2 \leq C'.$$

Fixing $l_j \geq 1$ and summing over $0 \leq r \leq l_j - 1$, making use of the triangular inequality and of $(x_1 + \cdots + x_n)^2 \leq n(x_1^2 + \cdots + x_n^2)$, we obtain, for K, L large enough (depending on l_j) :

$$\sum_{k=K}^{K+L} \left\| \alpha\lambda^{-k} b_j \left(\frac{1 - \lambda^{l_j n_j}}{1 - \lambda^{n_j}} \right) + \theta_{k-l_j n_j} - \theta_k \right\|^2 \leq l_j C'. \tag{4}$$

Changing k into $k + l_j n_j$, we obtain, for K, L large enough (depending on l_j) :

$$\sum_{k=K}^{K+L} \left\| \alpha\lambda^{-k} b_j \left(\frac{1 - \lambda^{-l_j n_j}}{1 - \lambda^{n_j}} \right) + \theta_{k+l_j n_j} - \theta_k \right\|^2 \leq l_j C'. \tag{5}$$

Let $1 = \sum_{0\leq j\leq N} l_j n_j$ be a Bezout relation and $J \subset \{0, \cdots, N\}$ be the subset where $l_j n_j \neq 0$, equipped with its natural order. Using successively for $j \in J$ either (4) or (5), according to the sign of l_j, we obtain with :

$$b = \sum_{j\in J} b_j \lambda^{\sum_{k\in J, k<j} l_k n_k} \left(\frac{1 - \lambda^{l_j n_j}}{1 - \lambda^{n_j}} \right), \tag{6}$$

the following relation, for a new constant C' and all K, L large enough :

$$\sum_{k=K}^{K+L} \|\alpha\lambda^{-k} b + \theta_{k-1} - \theta_k\|^2 \leq C'.$$

Now, for any $n_j \neq 0$, whatever the sign of n_j is, we arrive at, for some constant C' and all K, L large enough :

$$\sum_{k=K}^{K+L} \|\alpha\lambda^{-k} b \left(\frac{1 - \lambda^{n_j}}{1 - \lambda} \right) + \theta_{k-n_j} - \theta_k\|^2 \leq C'.$$

Set $b' = b/(1 - \lambda)$. Hence, for any $0 \leq j \leq N$ with $n_j \neq 0$, for some new constant C' and all K, L large enough, using (3) :

$$\sum_{k=K}^{K+L} \|\alpha\lambda^{-k}(b_j - b'(1 - \lambda^{n_j}))\|^2 \leq C'.$$

Let $0 \leq j \leq N$, with $n_j \neq 0$. If $b_j \neq b'(1-\lambda^{n_j})$, then we deduce again that $1/\lambda$ is a Pisot number and $b_j = b'(1-\lambda^{n_j}) + (1/\alpha)\mu_j$, with $\mu_j \in \mathcal{T}(1/\lambda)$. The other case is $b_j = b'(1-\lambda^{n_j})$. In any case, we obtain that for all $0 \leq j \leq N$:

$$\varphi_j(x) = b' + \lambda^{n_j}(x - b') + (1/\alpha)\mu_j, \tag{7}$$

for some $\mu_j \in \mathcal{T}(1/\lambda)$. Finally, remark that (7) says that the $(\varphi_j)_{0 \leq j \leq N}$ are conjugated with the $(\psi_j)_{0 \leq j \leq N}$, where $\psi_j(x) = \lambda^{n_j} x + \mu_j$; precisely $\varphi_j = f \circ \psi_j \circ f^{-1}$, with $f(x) = x/\alpha + b'$.

- *Case 2.* Any φ_j with $n_j = 0$ is the identity. The conclusion is the same, because there now necessarily exists some $0 \leq j \leq N$ with $n_j \neq 0$ and $b_j \neq b'(1-\lambda^{n_j})$, otherwise b' is a common fixed point for all $(\varphi_j)_{0 \leq j \leq N}$.

Let us finally check the Bezout centering relation. From relation (6), injecting the value of each $b_j = b(1-\lambda^{n_j})/(1-\lambda) + (1/\alpha)\mu_j$, we get a telescopic sum and it is immediate that the $(\mu_j)_{0 \leq j \leq N}$ are centered. Reciprocally, if starting from a centered family $(b_j)_{0 \leq j \leq N}$, one can choose the Bezout relation giving the centering. In this case, $b = 0$, $b' = 0$, so $b_j = (1/\alpha)\mu_j$, for all $0 \leq j \leq N$.

This ends the proof of the theorem and shows a part of the third item of *ii*) in Theorem 2.4.

□

4 Proof of Theorem 2.4

Let $N \geq 1$ and affine maps $\varphi_k(x) = \lambda^{n_k} x + \mu_k$, for $0 \leq k \leq N$, with $1/\lambda > 1$ a Pisot number, relatively prime integers $(n_k)_{0 \leq k \leq N}$ and $\mu_k \in \mathcal{T}(1/\lambda)$, for $0 \leq k \leq N$. Let $p \in \mathcal{C}_N$, with $p_j > 0$ for all $0 \leq j \leq N$, and denote by $(\varepsilon_n)_{n \in \mathbb{Z}}$ *i.i.d.* random variables with law p, to which the probability $\mathbb{P}$ and the expectation $\mathbb{E}$ refer. We suppose that $\mathbb{E}(n_{\varepsilon_0}) > 0$. Without loss of generality, we assume that $n_N \leq \cdots \leq n_0$. Thus $n_0 \geq 1$. For general background on Markov chains, cf Spitzer [21].

Recall the cocycle notations introduced in the statement of the theorem and denote by θ the formal shift such that $\theta\varepsilon_l = \varepsilon_{l+1}$, $l \in \mathbb{Z}$. We have for all k and l in $\mathbb{Z}$:

$$S_{k+l} = S_k + \theta^k S_l.$$

Recall that ν is the law of $X = \sum_{l \geq 0} \mu_{\varepsilon_l} \lambda^{S_l}$. We write $Q \in \mathbb{Z}[X]$ for the minimal polynomial of $1/\lambda$, of degree $s+1$, with roots $\alpha_0 = 1/\lambda$, $\alpha_1, \cdots, \alpha_s$, where $|\alpha_k| < 1$, for $1 \leq k \leq s$. Recall that the case $s = 0$ corresponds to $1/\lambda$ an integer ≥ 2 (using then usual conventions regarding sums or products). As explained before the statement of Theorem 2.4, for any $k \in \mathbb{Z}$, the random variable $\sum_{l \in \mathbb{Z}} \mu_{\varepsilon_l} \lambda^{k+S_l} \mod 1$ is a well-defined $\mathbb{T}$-valued random variable.

Step 1. In order to prove the convergence in law of $(\lambda^{-n}X, \lambda^{-n-1}X, \cdots, \lambda^{-n-s}X) \mod \mathbb{Z}^{s+1}$, as $n \to +\infty$, it is enough to prove, for any $(m_0, \cdots, m_s) \in \mathbb{Z}^{s+1}$, the convergence of :

$$\mathbb{E}\left(e^{2i\pi \sum_{0 \leq u \leq s} m_u \lambda^{-n-u} X}\right) = \mathbb{E}\left(e^{2i\pi \sum_{l \geq 0} (\alpha\mu_{\varepsilon_l}) \lambda^{-n+S_l}}\right),$$

with $\alpha = \sum_{0 \leq u \leq s} m_u \lambda^{-u}$. Notice that $\alpha\mu_j \in \mathcal{T}(1/\lambda)$, for $0 \leq j \leq N$. We make the proof when $\alpha = 1$, the one for α being obtained by changing (μ_j) into $(\alpha\mu_j)$.

Next, since $\sum_{l<0} \mu_{\varepsilon_l} \lambda^{-n+S_l} \mod 1$ converges a.-s. to 0 in $\mathbb{T}$, as $n \to +\infty$, it is enough to consider expectations with $\sum_{l \in \mathbb{Z}} \mu_{\varepsilon_l} \lambda^{-n+S_l} \mod 1$. Let $k \in \mathbb{Z}$ be a fixed integer. Looking at $(S_l)_{l \in \mathbb{Z}}$ and the first $q \in \mathbb{Z}$ such that $S_q \geq n$, we have :

$$\begin{aligned}
\mathbb{E}\left(e^{2i\pi\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_l}\lambda^{k-n+S_l}}\right) &= \sum_{0\le r<n_0}\sum_{q\in\mathbb{Z}}\mathbb{E}\left(e^{2i\pi\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_l}\lambda^{(k-n+S_q)+(S_l-S_q)}}1_{S_{q-u}<n,u\ge1,S_q=n+r}\right)\\
&= \sum_{0\le r<n_0}\sum_{q\in\mathbb{Z}}\mathbb{E}\left(e^{2i\pi\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_l}\lambda^{k+r+\theta^q S_{l-q}}}1_{\theta^q S_{-u}<-r,u\ge1,\theta^q S_{-q}=-n-r}\right)\\
&= \sum_{0\le r<n_0}\sum_{q\in\mathbb{Z}}\mathbb{E}\left(e^{2i\pi\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_{l-q}}\lambda^{k+r+S_{l-q}}}1_{S_{-u}<-r,u\ge1,S_{-q}=-n-r}\right)\\
&= \sum_{0\le r<n_0}\sum_{q\in\mathbb{Z}}\mathbb{E}\left(e^{2i\pi\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}1_{S_{-u}<-r,u\ge1,S_{-q}=-n-r}\right).
\end{aligned}$$

For each $0\le r<n_0$, we can move the sum $\sum_{q\in\mathbb{Z}}$ inside the expectation, using the theorem of Fubini, if we for example show the finiteness of :

$$\sum_{q\in\mathbb{Z}}\mathbb{E}\left(1_{S_{-q}=-n-r}\right) = \mathbb{E}\left(\sum_{q\ge0}1_{S_{-q}=-n-r}\right)+\mathbb{E}\left(\sum_{q\ge1}1_{S_q=-n-r}\right).$$

This is true, since, as soon as n is larger than some constant (because of the missing term for $q=0$ in the second sum), this equals $G^-(0,-n-r)+G^+(0,-n-r)<+\infty$, where $G^-(x,y)$ and $G^+(x,y)$ are the Green functions, finite for every integers x and y, respectively associated to the *i.i.d.* transient random walks $(S_{-q})_{q\ge0}$ and $(S_q)_{q\ge0}$. Let σ_k^+, for $k\in\mathbb{Z}$, be the first time ≥0 when $(S_q)_{q\ge0}$ touches k. Then $G^+(x,y)=\mathbb{P}_0(\sigma_{y-x}^+<\infty)G^+(0,0)$. With some symmetric quantities, we have $G^-(x,y)=\mathbb{P}_0(\sigma_{y-x}^-<\infty)G^-(0,0)$.

We therefore obtain :

$$\mathbb{E}\left(e^{2i\pi\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_l}\lambda^{k-n+S_l}}\right)=\sum_{0\le r<n_0}\mathbb{E}\left(e^{2i\pi\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}1_{S_{-u}<-r,u\ge1}\left(\sum_{q\in\mathbb{Z}}1_{S_{-q}=-n-r}\right)\right).$$

Let us now fix $0\le r<n_0$ and consider the corresponding term of the right-hand side. First of all, for $n>0$ larger than some constant :

$$\mathbb{E}\left(\sum_{q<0}1_{S_{-q}=-n-r}\right)=\mathbb{P}_0(\sigma_{-n-r}^+<\infty)G^+(0,0)\to0, \tag{8}$$

as $n\to+\infty$, since $(S_q)_{q\ge0}$ is transient to the right. We thus only need to consider :

$$T(-n):=\mathbb{E}\left(e^{2i\pi\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}1_{S_{-u}<-r,u\ge1}N(-n-r)\right),$$

where we set $N(-k-r)=\sum_{q\ge0}1_{S_{-q}=-n-r}$. Consider an integer M_0, that we will let tend to $+\infty$ at the end. The difference of $T(-n)$ with the following expression :

$$\mathbb{E}\left(e^{2i\pi\sum_{l\ge-M_0}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}1_{S_{-u}<-r,1\le u\le M_0}N(-n-r)\right)$$

is bounded by $A+B$, where, first :

$$
\begin{aligned}
A &= \mathbb{E}\left[\left|e^{2i\pi\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_l}\lambda^{k+r+S_l}} - e^{2i\pi\sum_{l\geq -M_0}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}\right| N(-n-r)\right] \\
&= \mathbb{E}\left[\left|1 - e^{2i\pi\sum_{l<-M_0}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}\right| N(-n-r)\right] \\
&\leq \left(\mathbb{E}\left[\left|1 - e^{2i\pi\sum_{l<-M_0}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}\right|^2\right]\right)^{1/2} \left(\mathbb{E}(N(-n-r)^2)\right)^{1/2} \\
&\leq \left(\mathbb{E}\left[\left|1 - e^{2i\pi\sum_{l<-M_0}\mu_{\varepsilon_l}\lambda^{k+r+S_l}}\right|^2\right]\right)^{1/2} \left(\mathbb{E}(N(0)^2)\right)^{1/2},
\end{aligned}
$$

because $N(-n-r)$ is stochastically dominated by $N(0)$. Notice that $N(0)$ is square integrable, as it has exponential tail. The first term on the right-hand side also goes to 0, as $M_0 \to +\infty$, by dominated convergence. The other term B is :

$$
\begin{aligned}
B &= \mathbb{E}\left(1_{S_{-u}<-r, 1\leq u\leq M_0, \exists v>M_0, S_{-v}\geq -r} N(-n-r)\right) \\
&\leq \mathbb{P}(\exists v > M_0, S_{-v} \geq -r)^{1/2} \left(\mathbb{E}(N(-n-r)^2)\right)^{1/2} \\
&\leq \mathbb{P}(\exists v > M_0, S_{-v} \geq -r)^{1/2} \left(\mathbb{E}(N(0)^2)\right)^{1/2},
\end{aligned}
$$

as before. The first term on the right-hand side goes to 0, as $M_0 \to +\infty$, since (S_{-v}) is transient to $-\infty$, as $v \to +\infty$. As a result :

$$
T(-n) = \mathbb{E}\left(e^{2i\pi\sum_{l\geq -M_0}\mu_{\varepsilon_l}\lambda^{k+r+S_l}} 1_{S_{-u}<-r, 1\leq u\leq M_0} N(-n-r)\right) + o_{M_0}(1),
$$

where $o_{M_0}(1)$ goes to 0, as $M_0 \to +\infty$, uniformly in n. Now, when $n > 0$ is large enough, $N(-k-r) = \sum_{q\geq 0} 1_{S_{-q}=-n-r} = \sum_{q\geq M_0} 1_{S_{-q}=-n-r}$, for all ω. Taking inside the expectation the conditional expectation with respect to the σ-algebra generated by the $(\varepsilon_l)_{l\geq -M_0}$, we obtain :

$$
T(-n) = \mathbb{E}\left(e^{2i\pi\sum_{l\geq -M_0}\mu_{\varepsilon_l}\lambda^{k+r+S_l}} 1_{S_{-u}<-r, 1\leq u\leq M_0} G^-(S_{-M_0}, -n-r)\right) + o_{M_0}(1).
$$

Now, things are simpler because $G^-(S_{-M_0}, -n-r)$ is bounded by the constant $G^-(0,0)$. Hence, for some new $o_{M_0}(1)$, with the same properties :

$$
T(-n) = \mathbb{E}\left(e^{2i\pi\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_l}\lambda^{k+r+S_l}} 1_{S_{-u}<-r, u\geq 1} G^-(S_{-M_0}, -n-r)\right) + o_{M_0}(1).
$$

Since $G^-(S_{-M_0}, -n-r) \to 1/\mathbb{E}(n_{\varepsilon_0})$, as $n \to \infty$, by renewal theory (since the (n_j) are relatively prime and $p_j > 0$, for all $0 \leq j \leq N$; cf Woodroofe [22], chap. 2, thm 2.1), staying bounded by $G^-(0,0)$, we get by dominated convergence and next $M_0 \to +\infty$:

$$
\lim_{n\to+\infty} T(-n) = \frac{1}{\mathbb{E}(n_{\varepsilon_0})} \mathbb{E}\left(e^{2i\pi\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_l}\lambda^{k+r+S_l}} 1_{S_{-u}<-r, u\geq 1}\right).
$$

From the initial expression, the limit, if existing, had to be independent on the parameter k. So this gives the announced convergence and invariance, hence proving item $i)$ in Theorem 2.4.

Step 2. In the proof of Theorem 2.4, we now consider $ii)$. Recall that $\alpha \neq 0$ is in $\mathcal{P}$ if $\alpha\mu_j \in \mathcal{T}(1/\lambda)$, for $0 \leq j \leq N$. Suppose that ν is Rajchman and let $0 \neq \alpha \in \mathcal{P}$. Fix any $0 \neq (m_0, \cdots, m_s) \in \mathbb{Z}^{s+1}$ and set $\beta = \sum_{0\leq u\leq s} m_u \lambda^{-u}$. We have $\beta \neq 0$, since $(\lambda^{-u})_{0\leq u\leq s}$ is a basis of $\mathbb{Q}[\lambda]$ over $\mathbb{Q}$. Now :

$$
\sum_{0\leq u\leq s} m_u(\lambda^{-n-u}\alpha X) = \alpha\beta\lambda^{-n} X.
$$

Since $\alpha\beta \neq 0$ and ν is Rajchman, $\mathbb{E}(e^{2i\pi\alpha\beta\lambda^{-n}X}) \to 0$, as $n \to +\infty$. As a result, the Fourier coefficient of $\mathcal{M}_\alpha$ corresponding to $(m_0, \cdots, m_s)$ is zero. Hence $\mathcal{M}_\alpha = \mathcal{L}_{\mathbb{T}^{s+1}}$. Also, $m_\alpha = \mathcal{L}_{\mathbb{T}}$.

Considering the third item, suppose that a Bezout centering relation is satisfied and that for all $\alpha \in \mathcal{P} \cap [1, 1/\lambda]$, we have $m_\alpha = \mathcal{L}_{\mathbb{T}}$. If the $(\varphi_j)_{0\le j\le N}$ have a common fixed point c, then $X = c$ a.-s. and the law of $\lambda^{-k}c \mod 1$ is a Dirac mass on $\mathbb{T}$, which cannot converge to $\mathcal{L}_{\mathbb{T}}$, as $k \to +\infty$. This contradicts the hypothesis when $\alpha = 1$. We are thus in the context of Theorem 2.3. If ν is not Rajchman, it was shown in *Step 2* of this theorem (Lemma 3.1) that there exists $1 \le \alpha \le 1/\lambda$ such that $\hat{\nu}(\alpha\lambda^{-k}) \not\to 0$. It was then detailed at the end of *Step 3* of the proof of the same theorem that $\alpha \in \mathcal{P}$. This is a contradiction, so ν is Rajchman.

Focusing on the second item, suppose that ν is continuous and $\mathcal{M}_{|\mathbb{T}^{s+1}\backslash\{0\}} \ll \mathcal{L}_{\mathbb{T}^{s+1}}$, with density h in restriction to $\mathbb{T}^{s+1}\backslash\{0\}$. This holds in particular if ν is Rajchman. Recall that $Z_k = \sum_{l\in\mathbb{Z}} \mu_{\varepsilon_l}\lambda^{k+S_l} \mod 1$. For any $f \in C(\mathbb{T}^{s+1}, \mathbb{R})$ and $k \in \mathbb{Z}$:

$$\frac{1}{\mathbb{E}(n_{\varepsilon_0})} \sum_{0\le r<n^*} \mathbb{E}\left[f(Z_{-k+r}, Z_{-k+r-1}, \cdots, Z_{-k+r-s})1_{S_{-v}<-r, v\ge 1}\right] = \int_{\mathbb{T}^{s+1}} f(x)\ d\mathcal{M}(x).$$

Recall that $n^* = \max_{0\le j\le N} n_j$ and fix $k \ge n^*$ so that $Tr(\lambda^{-l}\mu_j) \in \mathbb{Z}$, $0 \le j \le N$, $l \ge k - n^*$.

For $0 \le j \le N$, denote by $(\mu_j^{(t)})_{0\le t\le s}$ the conjugates of $\mu_j = \mu_j^{(0)}$ in the field $\mathbb{Q}[\lambda]$. Taking any $0 \le u \le s$ and $l < 0$, we have :

$$\mu_{\varepsilon_l}\lambda^{-u-k+r+S_l} = Tr(\mu_{\varepsilon_l}\lambda^{-u-k+r+S_l}) - \sum_{1\le t\le s} \mu_{\varepsilon_l}^{(t)}\alpha_t^{u+k-r-S_l}.$$

The role of the indicator function is now fundamental. On the event $\{S_{-v} < -r, v \ge 1\}$, we have $Tr(\mu_{\varepsilon_l}\lambda^{-u-k+r+S_l}) \in \mathbb{Z}$, by our choice of k. As a result, introducing the real random variables :

$$Y_u^{(r)} = \lambda^{-u}\sum_{l\ge 0} \mu_{\varepsilon_l}\lambda^{-k+r+S_l} - \sum_{1\le t\le s} \alpha_t^{u+k-r} \sum_{l<0} \mu_{\varepsilon_l}^{(t)}\alpha_t^{-S_l},$$

we obtain that for any $f \in C(\mathbb{T}^{s+1}, \mathbb{R})$:

$$\frac{1}{\mathbb{E}(n_{\varepsilon_0})} \sum_{0\le r<n^*} \mathbb{E}\left[f(Y_0^{(r)}, \cdots, Y_s^{(r)})1_{S_{-v}<-r, v\ge 1}\right] = \int_{\mathbb{T}^{s+1}} f(x)\ d\mathcal{M}(x).$$

Let $B = \{S_{-v} < 0, v \ge 1\}$, which verifies $\mathbb{P}(B) > 0$. In the previous formula, taking some $f \ge 0$ in $C(\mathbb{T}^{s+1}, \mathbb{R})$, we obtain, conserving only the term for $r = 0$:

$$\frac{1}{\mathbb{E}(n_{\varepsilon_0})}\mathbb{E}\left[f(Y_0^{(0)}, \cdots, Y_s^{(0)})1_B\right] \le \int_{\mathbb{T}^{s+1}} f(x)\ d\mathcal{M}(x).$$

If moreover $f(0) = 0$, then :

$$\frac{1}{\mathbb{E}(n_{\varepsilon_0})}\mathbb{E}\left[f(Y_0^{(0)}1_B, \cdots, Y_s^{(0)}1_B)\right] \le \int_{\mathbb{T}^{s+1}} f(x)h(x)\ dx.$$

Hence, in restriction to $\mathbb{T}^{s+1}\backslash\{0\}$, the law of $(Y_0^{(0)}1_B, \cdots, Y_s^{(0)}1_B) \mod \mathbb{Z}^{s+1}$ has a density with respect to $\mathcal{L}_{\mathbb{T}^{s+1}}$, bounded by $\mathbb{E}(n_{\varepsilon_0})h(x)$, $x \in \mathbb{T}^{s+1}\backslash\{0\}$.

Hence, in restriction to $\mathbb{R}^{s+1}\backslash\mathbb{Z}^{s+1}$, the law on $\mathbb{R}^{s+1}$ of $(Y_0^{(0)}1_B, \cdots, Y_s^{(0)}1_B)$ has a density with respect to $\mathcal{L}_{\mathbb{R}^{s+1}}$, bounded by $\mathbb{E}(n_{\varepsilon_0})h(x \mod \mathbb{Z}^{s+1})$, $x \in \mathbb{R}^{s+1}\backslash\mathbb{Z}^{s+1}$.

Let now $X_0 = \sum_{l\ge 0} \mu_{\varepsilon_l}\lambda^{-k+S_l}$ and for $1 \le j \le s$, $X_j = -\sum_{l<0} \mu_{\varepsilon_l}^{(j)}\alpha_j^{k-S_l}$. Notice first that :

$$Y_0^{(0)} = X_0 + \sum_{1\le j\le s} X_j = \lambda^{-k}X + \sum_{1\le j\le s} X_j.$$

As $\nu = \mathcal{L}(X)$ is continuous, it has no atom. Since X is independent from $\sum_{1\le j\le s} X_j$, the law of $Y_0^{(0)}$ is continuous. Hence, restricted to $\mathbb{R}^{s+1}\backslash\{0\}$, the law on $\mathbb{R}^{s+1}$ of $(Y_0^{(0)}1_B, \cdots, Y_s^{(0)}1_B)$ has a density bounded by $\mathbb{E}(n_{\varepsilon_0})h(x \mod \mathbb{Z}^{s+1})$, $x \in \mathbb{R}^{s+1}\backslash\mathbb{Z}^{s+1}$, with respect to $\mathcal{L}_{\mathbb{R}^{s+1}}$.

Introducing the Vandermonde matrix :

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda^{-1} & \alpha_1 & \cdots & \alpha_s \\ \vdots & \vdots & \vdots & \vdots \\ \lambda^{-s} & \alpha_1^s & \cdots & \alpha_s^s \end{pmatrix}, \text{ by definition, } Y^{(0)} := \begin{pmatrix} Y_0^{(0)} \\ \vdots \\ Y_s^{(0)} \end{pmatrix} = V \begin{pmatrix} X_0 \\ \vdots \\ X_s \end{pmatrix}.$$

The matrix V is invertible (the roots of the minimal polynomial Q of $1/\lambda$ are simple), so, by Cramer's rule, $X_0 = \sum_{0\leq i\leq s} Y_i^{(0)}\gamma_i$, with $\gamma_i = \det(V^{(i)})/\det(V)$, where $V^{(i)}$ is obtained from V by replacing the first column by e_i, denoting by $(e_i)_{0\leq i\leq s}$ the canonical basis of $\mathbb{R}^{s+1}$.

Notice that each γ_i is real ($1/\lambda$ is a real root of Q and regrouping the other roots in conjugate pairs, when conjugating γ_i, one gets permutations in the numerator and the denominator, the same ones, so $\bar{\gamma_i} = \gamma_i$). Evidently, $\gamma = (\gamma_i)_{0\leq i\leq s} \neq 0$.

Since $X_0 = \lambda^{-k}X$, it is enough to show that X_0 has a density with respect to $\mathcal{L}_{\mathbb{R}}$. Recall that $\mathcal{L}(X_0)$ is continuous. Let now A be a Borel set of $\mathbb{R}$, such that $0 \notin A$. Since X_0 and the set $B = \{S_{-v} < 0, v \geq 1\}$ are independent, we can write :

$$\mathbb{P}(X_0 \in A) = \frac{\mathbb{P}(X_0 1_B \in A)}{\mathbb{P}(B)} = \frac{1}{\mathbb{P}(B)}\mathbb{P}\left(\sum_{0\leq i\leq s} Y_i^{(0)} 1_B \gamma_i \in A\right) = \frac{\mathbb{P}(\langle Y^{(0)}1_B, \gamma\rangle \in A)}{\mathbb{P}(B)}.$$

Using the Euclidean norm on $\mathbb{R}^{s+1}$, let $M = (f_0 = \gamma/\|\gamma\|, f_1, \cdots, f_s)$ be the matrix of an orthonormal basis of $\mathbb{R}^{s+1}$, written in the canonical basis. Then, in restriction to $\mathbb{R}^{s+1}\backslash\{0\}$, the law of $Z := M^{-1}Y^{(0)}1_B$ on $\mathbb{R}^{s+1}$ has a density g with respect to $\mathcal{L}_{\mathbb{R}}$. Then :

$$\begin{aligned}\mathbb{P}(X_0 \in A) &= \frac{\mathbb{P}(\langle Z, {}^tM\gamma\rangle \in A)}{\mathbb{P}(B)} = \frac{\mathbb{P}(\langle Z, e_0\rangle \in A/\|\gamma\|)}{\mathbb{P}(B)} \\ &= \frac{1}{\mathbb{P}(B)}\int_{\mathbb{R}} 1_A(x_0)\left(\int_{\mathbb{R}^s} g(x_0/\|\gamma\|, x_1, , \cdots, x_s)\ dx_1\cdots dx_s\right)\frac{dx_0}{\|\gamma\|}.\end{aligned}$$

We conclude that the law of X_0 has a density with respect to $\mathcal{L}_{\mathbb{R}}$. In the special case when $\mathcal{M} = \mathcal{L}_{\mathbb{T}^{s+1}}$, for example when ν is Rajchman, then $h = 1$ and g is bounded. If moreover condition (C) holds, then $\mathbb{P}(B) = 1$. In the previous proof 1_B can be removed and both $Y^{(0)}$ and Z have a law with a density with respect to $\mathcal{L}_{\mathbb{R}^{s+1}}$. Since in this case $Y^{(0)}$ and Z are obviously bounded, we deduce that the density g of Z is then bounded and with compact support. The previous formula for the density of X_0 easily furnishes that the latter is bounded and with compact support. This ends the proof of Theorem 2.4 and of Theorem 2.5 *i*).

□

5 Proof of Theorem 2.5

Assume now that condition (C) holds. Precisely, let $N \geq 1$ and affine contractions $\varphi_k(x) = \lambda^{n_k}x + \mu_k$, for $0 \leq k \leq N$, with $1/\lambda > 1$ a Pisot number, relatively prime integers $(n_k)_{0\leq k\leq N}$, with $n_k \geq 1$ and $\mu_k \in \mathcal{T}(1/\lambda)$, for $0 \leq k \leq N$. We shall suppose that $n_0 \geq \cdots \geq n_N \geq 1$.

Step 1. Considering $p \in \mathcal{C}_N$, with $p_j > 0$, for $0 \leq j \leq N$, denote by $(\varepsilon_n)_{n\in\mathbb{Z}}$ a sequence of *i.i.d.* random variables with law p. We fix an integer $n \neq 0$, whose exact value will be precised at the end. We focus on the Fourier coefficient $\hat{m}(n)$ of the measure m appearing in Theorem 2.4 *i*). Removing the normalizing constant $\mathbb{E}(n_{\varepsilon_0})$ and observing that it has a simplified expression under condition (C), we introduce the following quantity (a constant multiple of it) :

$$\Delta_p = \Delta_p(k) = \sum_{0\leq r<n_0} \mathbb{E}\left(e^{2i\pi n\sum_{l\in\mathbb{Z}}\mu_{\varepsilon_l}\lambda^{k+r+S_l}} 1_{n_{\varepsilon_{-1}}>r}\right),$$

which is independent of $k \in \mathbb{Z}$ and where we mark the dependence in $p \in \mathcal{C}_N^o$. We now focus on the regularity of $p \longmapsto \Delta_p$ on $\mathcal{C}_N^o$. For any $k \in \mathbb{Z}$, observe first that $\Delta_p(k)$ is well-defined, with the same formula as above, on the whole set $\mathcal{C}_N$, even on the boundary. Fixing $k \in \mathbb{Z}$, the map $p \longmapsto \Delta_p(k)$ is continuous, as this function is the uniform limit on $\mathcal{C}_N$, as $L \to +\infty$, of the continuous maps :

$$p \longmapsto \sum_{0 \le r < n_0} \mathbb{E}\left(e^{2i\pi n \sum_{-L \le l \le L} \mu_{\varepsilon_l} \lambda^{k+r+S_l}} 1_{n_{\varepsilon_{-1}} > r}\right).$$

It thus follows that $p \longmapsto \Delta_p(k) = \Delta_p$ is well-defined on $\mathcal{C}_N$, is continuous and independent on k. We shall now prove using standard methods that it is in fact real-analytic on $\mathcal{C}_N$, in a classical sense (precised below). Let us take $k = 0$ and fix $0 \le r < n_0$. Using independence, write :

$$\mathbb{E}\left(e^{2i\pi n \sum_{l \in \mathbb{Z}} \mu_{\varepsilon_l} \lambda^{r+S_l}} 1_{n_{\varepsilon_{-1}} > r}\right) = \mathbb{E}\left(e^{2i\pi n \sum_{l \ge 0} \mu_{\varepsilon_l} \lambda^{r+S_l}}\right) \mathbb{E}\left(e^{2i\pi n \sum_{l \le -1} \mu_{\varepsilon_l} \lambda^{r+S_l}} 1_{n_{\varepsilon_{-1}} > r}\right).$$

Let us call $F(p)$ and $G(p)$ respectively the terms appearing in the right-hand side. We shall show that both functions are real-analytic functions (in the below sense) of p. This property will be inheritated by $p \longmapsto \Delta_p$. We treat the case of $p \longmapsto F(p)$, the case of $G(p)$ needing only to rewrite first the $\mu_{\varepsilon_l} \lambda^{r+S_l}$, appearing in the definition of $G(p)$ and as soon as $l < 0$ is large enough (depending only the $(\mu_j)_{0 \le j \le N}$, since $n_k \ge 1$, for all k), as $-\sum_{1 \le j \le s} \alpha_j^{-r-S_l} \mu_{\varepsilon_l}^{(j)}$, quantity equal to $\mu_{\varepsilon_l} \lambda^{r+S_l}$ in $\mathbb{T}$, where the $(\mu_k^{(j)})_{1 \le j \le s}$ are the conjugates of μ_k in the field $\mathbb{Q}[1/\lambda]$.

Fix now $p \in \mathcal{C}_N$. Let $\mathbb{N} = \{0, 1, \cdots\}$ and the symbolic space $S = \{0, \cdots, N\}^{\mathbb{N}}$, equipped with the left shift σ. For $x = (x_0, x_1, \cdots) \in S$, we define :

$$g(x) = e^{2i\pi n \left(\sum_{l \ge 0} \mu_{x_l} \lambda^{r+n_{x_0}+\cdots+n_{x_{l-1}}}\right)}.$$

Introducing the product measure $\mu_p = (\sum_{0 \le j \le N} p_j \delta_j)^{\otimes \mathbb{N}}$ on S, we can write :

$$F(p) = \int_S g \; d\mu_p.$$

Denote by $C(S)$ the space of continuous functions $f : S \to \mathbb{C}$ and introduce the operator $P_p : C(S) \to C(S)$ defined by :

$$P_p(f)(x) = \sum_{0 \le j \le N} p_j f((j, x)), \; x \in S,$$

where $(j, x) \in S$ is the word obtained by the left concatenation of the symbol j to x. The operator P_p is Markovian, i.e. $f \ge 0 \Rightarrow P_p(f) \ge 0$ and verifies $P_p \mathbf{1} = \mathbf{1}$, where $\mathbf{1}(x) = 1$, $x \in S$. The measure μ_p has the invariance property $\int_S P_p(f) \; d\mu_p = \int_S f \; d\mu_p$, $f \in C(S)$. For $f \in C(S)$ and $k \ge 0$, introduce the variation :

$$\mathrm{Var}_k(f) = \sup\{|f(x) - f(y)|, \; (x, y) \in S^2, x_i = y_i, \; 0 \le i < k\}.$$

For any $0 < \theta < 1$, let $|f|_\theta = \sup\{\theta^{-k} \mathrm{Var}_k(f), \; k \ge 0\}$, as well as $\|f\|_\theta = |f|_\theta + \|f\|_\infty$. We denote by $\mathcal{F}_\theta$ the complex Banach space of fonctions f on S such that $\|f\|_\theta < \infty$. Any $\mathcal{F}_\theta$ is preserved by P_p. Observe now that $g \in \mathcal{F}_\theta$ for $\lambda \le \theta < 1$. We fix $\theta = \lambda$.

As a classical fact from Spectral Theory, cf for example Baladi [1], the operator $P_p : \mathcal{F}_\lambda \to \mathcal{F}_\lambda$ satisfies a Perron-Frobenius theorem. Let us show this elementarily. For $f \in \mathcal{F}_\lambda$, we have :

$$P_p^n f(x) = \sum_{0 \le j_1, \cdots, j_n \le N} p_{j_1} \cdots p_{j_n} f((j_1, \cdots, j_n, x)).$$

This furnishes $\mathrm{Var}_k(P_p^n f - \mathbf{1} \int_S f \; d\mu_p) = \mathrm{Var}_k(P_p^n f) \le \mathrm{Var}_{k+n}(f)$, therefore :

$$\left| P_p^n(f) - \mathbf{1} \int_S f \; d\mu_p \right|_\lambda \le \lambda^n |f|_\lambda.$$

In a similar way, we can write :

$$\begin{aligned}(P_p^n f - \mathbf{1}\int_S f\ d\mu_p)(x) &= P_p^n(f)(x) - \mathbf{1}(x)\int_S P_p^n(f)\ d\mu_p \\ &= \sum_{0\le j_1,\cdots,j_n\le N} p_{j_1}\cdots p_{j_n}\int_S (f((j_1,\cdots,j_n,x)) - f((j_1,\cdots,j_n,y)))\ d\mu_p(y).\end{aligned}$$

Consequently, $\|P_p^n f - \mathbf{1}\int_S f\ d\mu_p\|_\infty \le \mathrm{Var}_n(f) \le \lambda^n |f|_\lambda$. Putting things together, finally :

$$\|P_p^n(f - \mathbf{1}\int_S f\ d\mu_p)\|_\lambda \le 2\lambda^n \|f\|_\lambda.$$

This shows that the eigenvalue 1 is simple and that the rest of the spectrum of P_p is contained in the closed disk of radius $\lambda < 1$, independent on $p \in \mathcal{C}_N$. By standard functional holomorphic calculus, cf Kato [9], fixing some circle Γ centered at 1 and with radius $0 < r < 1-\lambda$, the following operator, involving the resolvent, is a continuous (Riesz) projector on $\mathrm{Vect}(\mathbf{1})$:

$$\Pi_p = \int_\Gamma (zI - P_p)^{-1} dz.$$

Notice that the formula is valid for any $p \in \mathcal{C}_N$. Classically, $\Pi_p(\mathcal{F}_\lambda)$ and $(I - \Pi_p)(\mathcal{F}_\lambda)$ are closed P_p-invariant subspaces with $\mathcal{F}_\lambda = \Pi_p(\mathcal{F}_\lambda) \oplus (I - \Pi_p)(\mathcal{F}_\lambda)$. In restriction to $(I - \Pi_p)(\mathcal{F}_\lambda)$, the spectral radius of P_p is less than λ. In particular $\int_S f\ d\mu_p = 0$, for $f \in (I - \Pi_p)(\mathcal{F}_\lambda)$. We therefore deduce that for any $f \in \mathcal{F}_\lambda$:

$$\Pi_p(f) = \left(\int_S f\ d\mu_p\right)\mathbf{1}.$$

Applying this to the function g of interest to us, we obtain that :

$$F(p)\mathbf{1} = \int_\Gamma (zI - P_p)^{-1}(g) dz.$$

Recall now that $N \ge 1$. Let $\eta' = (\eta_0, \cdots, \eta_{N-1})$ and $\eta = (\eta_0, \cdots, \eta_{N-1}, -(\eta_0 + \cdots + \eta_{N-1}))$. The condition on η' for $p + \eta \in \mathcal{C}_N$ is written as $\eta' \in D_N(p)$. Explicitly the condition is :

$$-p_i \le \eta_i \le 1 - p_i,\ 0 \le i \le N-1, \text{ and } p_N - 1 \le \eta_0 + \cdots + \eta_{N-1} \le p_N.$$

For the sequel, let $B_N(0, R)$ be the open Euclidean ball in $\mathbb{R}^N$ centered at 0, of radius R.

Definition 5.1
A function $h : \mathcal{C}_N \to \mathbb{C}$ admits a development in series around a point $p \in \mathcal{C}_N$, if there exists $\varepsilon > 0$ such that for $\eta' = (\eta_0, \cdots, \eta_{N-1}) \in D_N(p) \cap B_N(0, \varepsilon)$ and writing $\eta = (\eta_0, \cdots, \eta_{N-1}, -(\eta_0 + \cdots + \eta_{N-1}))$, then $h(p + \eta)$ is given by an absolutely converging series :

$$h(p + \eta) = \sum_{l_0 \ge 0, \cdots, l_{N-1} \ge 0} A_{l_0, \cdots, l_{N-1}} \eta_0^{l_0} \cdots \eta_{N-1}^{l_{N-1}}.$$

A function is real-analytic in $\mathcal{C}_N$ if it admits a development in series around every $p \in \mathcal{C}_N$.

For such a function, when non-constant, its zeroes are in finite number in $\mathcal{C}_N$, by the standard argument that the set of points where there is a null development in series is open and closed for the relative topology and thus equal to $\mathcal{C}_N$ by connexity if non-empty. In case of infinitely many zeros, any accumulation point (which exists, as $\mathcal{C}_N$ is compact) is such a point.

We now check that $p \longmapsto F(p)$ is real-analytic in the previous sense. As already indicated, this property will be inheritated by $p \longmapsto \Delta_p$. In this direction, notice that :

$$P_{p+\eta} = P_p + \sum_{0\le j\le N-1} \eta_j Q_j,$$

where $Q_j(f)(x) = f(j,x) - f(N,x)$. For $z \in \Gamma$ and η' small enough :

$$\begin{aligned}(zI - P_{p+\eta})^{-1} &= \left(I - (zI-P_p)^{-1} \sum_{0\le j\le N-1} \eta_j Q_j\right)^{-1} (zI-P_p)^{-1} \\ &= \sum_{n\ge 0} \sum_{0\le j_1,\cdots,j_n\le N-1} \eta_{j_1}\cdots\eta_{j_n} (zI-P_p)^{-1}Q_{j_1}\cdots(zI-P_p)^{-1}Q_{j_n}(zI-P_p)^{-1}.\end{aligned}$$

This is clearly absolutely convergent in the Banach operator algebra, for small enough η', uniformly in $z \in \Gamma$. We rewrite it as :

$$(zI - P_{p+\eta})^{-1} = \sum_{l_0\ge 0,\cdots,l_{N-1}\ge 0} B_{l_0,\cdots,l_{N-1}}(z)\eta_0^{l_0}\cdots\eta_{N-1}^{l_{N-1}},$$

converging for the operator norm, uniformly in $z \in \Gamma$. This leads to :

$$F(p+\eta)\mathbf{1} = \int_\Gamma (zI - P_{p+\eta})^{-1}(g)\ dz = \sum_{l_0\ge 0,\cdots,l_{N-1}\ge 0} \eta_0^{l_0}\cdots\eta_{N-1}^{l_{N-1}} \int_\Gamma B_{l_0,\cdots,l_{N-1}}(z)(g)\ dz.$$

Applying this equality at some particular $x \in S$, we obtain the desired development in series around p. This completes this step.

Step 2. If ever $\Delta_p = 0$ for infinitely many $p \in \mathcal{C}_N$, then by *Step 1*, $p \longmapsto \Delta_p$ has to be constant and equal to zero on $\mathcal{C}_N$. We shall show that if $n \neq 0$ has been appropriately chosen at the beginning it is not possible. We start with a lemma. We write as $x \equiv y$ equality of x and y in $\mathbb{T}$.

Lemma 5.2
Let $d \ge 1$ and $\mu \in \mathcal{T}(1/\lambda)$. The series $\sum_{l\in\mathbb{Z}} \mu\lambda^{ld}$, well-defined as an element of $\mathbb{T}$, equals a rational number modulo 1.

Proof of the lemma :
Let $l_0 \ge 1$ be such that $Tr(\lambda^{-l}\mu) \in \mathbb{Z}$, for $l > l_0$. Denote by $(\mu^{(j)})_{0\le j\le s}$ the conjugates of μ, with $\mu^{(0)} = \mu$, and $\alpha_1, \cdots, \alpha_s$ that of $\alpha_0 = 1/\lambda$. Then, we have the following equalities on the torus :

$$\begin{aligned}\sum_{l\in\mathbb{Z}} \mu\lambda^{ld} &\equiv \frac{\mu\lambda^{-l_0d}}{1-\lambda^d} + \sum_{l>l_0} \mu\lambda^{-ld} \equiv \frac{\mu\lambda^{-l_0d}}{1-\lambda^d} - \sum_{1\le i\le s} \mu^{(i)} \sum_{l>l_0} \alpha_i^{ld} \equiv \frac{\mu\lambda^{-l_0d}}{1-\lambda^d} - \sum_{1\le i\le s} \mu^{(i)} \frac{\alpha_i^{(l_0+1)d}}{1-\alpha_i^d} \\ &\equiv -\left(\frac{\mu\lambda^{-(l_0+1)d}}{1-\lambda^{-d}} + \sum_{1\le i\le s} \mu^{(i)} \frac{\alpha_i^{(l_0+1)d}}{1-\alpha_i^d}\right) = -Tr\left(\frac{\mu\lambda^{-(l_0+1)d}}{1-\lambda^{-d}}\right) \in \mathbb{Q}.\end{aligned}$$

□

We conclude the argument. Fixing $0 \le j \le N$ and $p^j = (0,\cdots,0,1,0,\cdots,0)$, where the 1 is at place j, we have for $k \in \mathbb{Z}$, recalling that $1 \le n_j \le n_0$:

$$\Delta_{p^j} = \Delta_{p^j}(k) = \sum_{0\le r<n_0} e^{2i\pi n \sum_{l\in\mathbb{Z}} \mu_j \lambda^{k+r+ln_j}} 1_{n_j>r} = \sum_{0\le r<n_j} e^{2i\pi n \sum_{l\in\mathbb{Z}} \mu_j \lambda^{k+r+ln_j}}.$$

Notice that the invariance with respect to k is now obvious, as we sum over r on a full period of length n_j. Now, taking $k = 0$, we have :

$$\Delta_{p^j} = \sum_{0\leq r<n_j} e^{2i\pi n (A_{j,r}/B_{j,r})},$$

for rational numbers $A_{j,r}/B_{j,r}$, making use of the previous lemma, since $\lambda^r \mu_j \in \mathcal{T}(1/\lambda)$, for any r. If for example n is a multiple of $B_{j,r}$ for any $0 \leq r < n_j$, we get $\Delta_{p^j} = n_j \geq 1$, which gives what was desired. This ends the proof of the theorem.

$\square$

Remark. — In the general case, without condition (C), the method seems to reach some limit. When trying to analyze the regularity of $p \longmapsto F(p)$ on $\mathcal{D}_N((\lambda^{n_k})_{0\leq k\leq N})$, continuity seems rather clear, but the real-analytic character, if ever true, certainly requires more work. Still setting $S = \{0, \cdots, N\}^{\mathbb{N}}$ and $\mu_p = (\sum_{0\leq j\leq N} p_j \delta_j)^{\otimes \mathbb{N}}$ on S, we again have :

$$F(p) = \int_S g \; d\mu_p,$$

with $g(x) = e^{2i\pi n\left(\sum_{l\geq 0} \mu_{x_l} \lambda^{r+n_{x_0}+\cdots+n_{x_{l-1}}}\right)}$, for $x = (x_0, x_1, \cdots) \in S$. However, this function is not continuous on S and in fact only defined μ_p-almost-everywhere.

6 Complements

6.1 A numerical example

Considering an example as simple as possible which is not homogeneous, take $N = 1$ and the two contractions $\varphi_0(x) = \lambda x$, $\varphi_1(x) = \lambda^2 x + 1$, where $1/\lambda > 1$ is a Pisot number, with probability vector $p = (p_0, p_1)$. Then $n_0 = 1$, $n_1 = 2$ and ν is the law of $\sum_{l\geq 0} \varepsilon_l \lambda^{n_{\varepsilon_0}+\cdots+n_{\varepsilon_{l-1}}}$, with $(\varepsilon_n)_{n\geq 0}$ *i.i.d.*, with common law Ber(p_1), i.e. $\mathbb{P}(\varepsilon_0 = 1) = p_1$ and $\mathbb{P}(\varepsilon_0 = 0) = 1 - p_1$. We shall take $0 \leq p_1 \leq 1$ as parameter for simulations. Notice that $\mathbb{E}(n_{\varepsilon_0}) = p_0 + 2p_1 = 1 + p_1$,

Taking $n = 1$, $k \in \mathbb{Z}$ and $r \in \{0, 1\}$, let us define :

$$F_p(k) = \mathbb{E}\left(e^{2i\pi\lambda^k \sum_{l\geq 0} \varepsilon_l \lambda^{n_{\varepsilon_0}+\cdots+n_{\varepsilon_{l-1}}}}\right), \; G_p(k, r) = \mathbb{E}\left(e^{2i\pi \sum_{l\geq 0} \varepsilon_l \lambda^{k-(n_{\varepsilon_0}+\cdots+n_{\varepsilon_l})}} 1_{n_{\varepsilon_0}>r}\right),$$

leading to $\Delta_p = F_p(k)G_p(k, 0) + F_p(k+1)G_p(k+1, 1)$, for all $k \in \mathbb{Z}$. Writing m_p in place of m for the measure on $\mathbb{T}$ in Theorem 2.4 *i*), when $0 < p_1 < 1$, we get $\hat{m}_p(1) = \Delta_p/(1 + p_1)$. Let us first discuss the choice of probability vector $p = (1 - p_1, p_1)$ and Pisot number $1/\lambda$.

A degenerated example (the invariant measure being automatically singular) is for instance given by $\lambda = (3 - \sqrt{5})/2 < 1/2$. Nevertheless, it is interesting to notice that $\lambda^{-n} \equiv -\lambda^n$, $n \geq 0$. Taking $p_1 = 1/2$, one can check that $\Delta_p = |F_p(1)|^2 + |F_p(2)|^2/2$. Necessarily $\Delta_p > 0$. Indeed, $k \longmapsto F_p(k)$ verifying a linear recurrence of order two, the equality $\Delta_p = 0$ would give $F_p(k) = 0$ for all k, but $F_p(k) \to 1$, as $k \to +\infty$. Notice that $(3 - \sqrt{5})/2$ is the largest λ with this property (it has to be a root of some $X^2 - aX + 1$, for some integer $a \geq 0$). Mention that in general Δ_p is not real; cf the pictures below.

To study an interesting example, we take into account the similarity dimension $s(p, r)$, rewritten here as $s(p, \lambda)$:

$$s(p, \lambda) = \frac{(1 - p_1)\ln(1 - p_1) + p_1 \ln p_1}{(1 - p_1)\ln\lambda + p_1 \ln(\lambda^2)}.$$

The condition $s(p, \lambda) \geq 1$ is equivalent to $(1 - p_1)\ln(1 - p_1) + p_1 \ln p_1 - (1 + p_1)\ln\lambda \leq 0$. As a function of p_1, the left-hand side has a minimum value $-\ln(\lambda + \lambda^2)$, attained at $p_1 = \lambda/(1 + \lambda)$. As a first attempt, taking for $1/\lambda$ the golden mean $(\sqrt{5}+1)/2 = 1,618...$ appears in fact not to be a good idea, as in this case $\lambda + \lambda^2 = 1$, giving $s(p, \lambda) \leq 1$.

We instead take for $1/\lambda$ the Plastic number, the smallest Pisot number (cf Siegel [18]). It is defined as the unique real root of $X^3 - X - 1$. Approximately, $1/\lambda = 1.324718....$ For this λ :

$$s(p,\lambda) > 1 \Longleftrightarrow 0,203... < p_0 < 0,907....$$

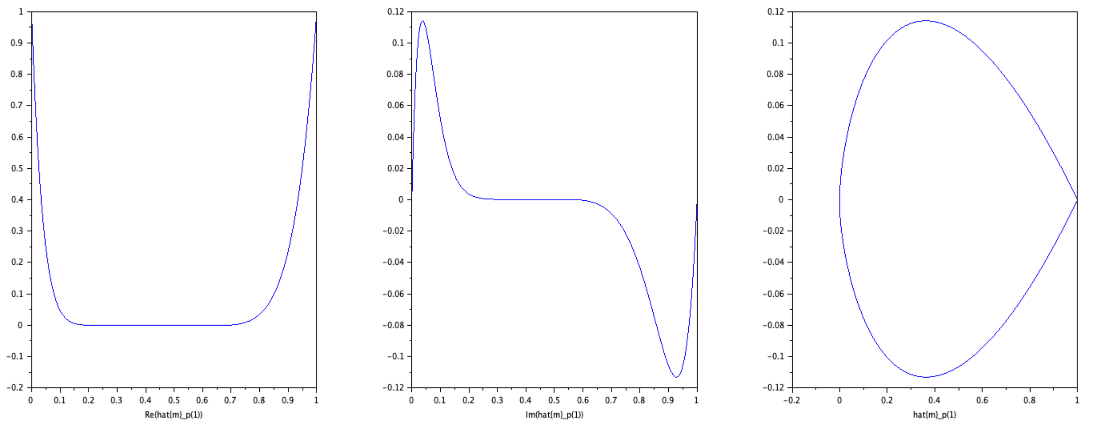
The other roots of $X^3 - X - 1 = 0$ are conjugate numbers $\rho e^{\pm i\theta}$. From the relations $1/\lambda + 2\rho\cos\theta = 0$ and $(1/\lambda)\rho^2 = 1$, we deduce $\rho = \sqrt{\lambda}$ and $\cos\theta = -1/(2\lambda^{3/2})$, thus $\theta = \pm 2.43...$ rad. For computations, the relations $\lambda^{-n} + \rho^n e^{in\theta} + \rho^n e^{-in\theta} \in \mathbb{Z}$, $n \geq 0$, furnish $\lambda^{-n} \equiv -2(\sqrt{\lambda})^n \cos(n\theta)$.

Let us finally compute the extreme values of $p_1 \longmapsto \hat{m}_p(1)$, abusively written as $\hat{m}_{(1,0)}(1)$ and $\hat{m}_{(0,1)}(1)$, since m_p has only been defined for $0 < p_1 < 1$. We first observe that $\hat{m}_{(1,0)}(1) = \Delta_{(1,0)} = F_{(1,0)}(0)G_{(1,0)}(0,0) = 1$. At the other extremity :

$$\begin{aligned}
\Delta_{(0,1)} &= F_{(0,1)}(0)G_{(0,1)}(0,0) + F_{(0,1)}(1)G_{(0,1)}(1,1) \\
&= e^{2i\pi \sum_{l\geq 0}\lambda^{2l}} e^{2i\pi \sum_{l\geq 0}\lambda^{-2(l+1)}} + e^{2i\pi\lambda \sum_{l\geq 0}\lambda^{2l}} e^{2i\pi \sum_{l\geq 0}\lambda^{1-2(l+1)}} \\
&= e^{2i\pi\left(\frac{1}{1-\lambda^2} - 2\sum_{l\geq 0}(\sqrt{\lambda})^{2l}\cos(2l\theta)\right)} + e^{2i\pi\left(\frac{\lambda}{1-\lambda^2} - 2\sum_{l\geq 0}(\sqrt{\lambda})^{2l+1}\cos((2l+1)\theta)\right)} \\
&= e^{2i\pi\left(\frac{1}{1-\lambda^2} - 2\mathrm{Re}\left(\frac{\lambda e^{2i\theta}}{1-\lambda e^{2i\theta}}\right)\right)} + e^{2i\pi\left(\frac{\lambda}{1-\lambda^2} - 2\mathrm{Re}\left(\frac{\sqrt{\lambda}e^{i\theta}}{1-\lambda e^{2i\theta}}\right)\right)}.
\end{aligned}$$

A not difficult computation, shortened by the observation that $(1 - \lambda e^{2i\theta})(1 - \lambda e^{-2i\theta}) = 1/\lambda$, shows that the arguments in the exponential terms (after the $2i\pi$) are respectively equal to 3 and 0, leading to $\Delta_{(0,1)} = 2$ and therefore $\hat{m}_{(0,1)}(1) = 1$.

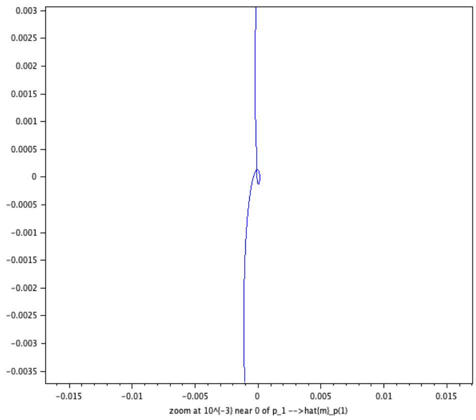
Recalling that $p = (1 - p_1, p_1)$, below are respectively drawn the real-analytic maps $p_1 \longmapsto \mathrm{Re}(\hat{m}_p(1))$, $p_1 \longmapsto \mathrm{Im}(\hat{m}_p(1))$ and the parametric curve $p_1 \longmapsto \hat{m}_p(1)$, $0 \leq p_1 \leq 1$.

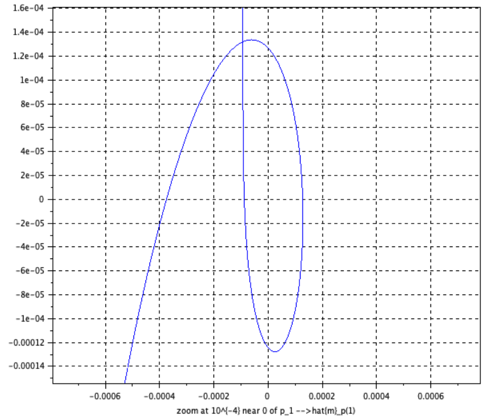


The first two pictures indicate that $p_1 \longmapsto \hat{m}_p(1)$ spends a rather long time near 0, with $\mathrm{Re}(\hat{m}_p(1))$ and $\mathrm{Im}(\hat{m}_p(1))$ both around 10^{-4}. Let us precise here that one can exploit the product form (given by the exponential) inside the expectation appearing in $F_p(k)$ and $G_p(k,r)$ and make a deterministic numerical computation of $\hat{m}_p(1)$, with nearly an arbitrary precision, based on a dynamical programming (using a binomial tree). For example, one can obtain the rather remarquable value :

$$\hat{m}_{(1/2,1/2)}(1) = 0,0001186... + i0,0000327...,$$

where all digits are exact. In this case, $s((1/2,1/2),\lambda) = 1,64... > 1$. The above pictures were drawn with 1000 points, each one determined with a sufficient precision. This allows to safely zoom on the neighbourhood of 0 of $p_1 \longmapsto \hat{m}_p(1)$, the interesting region. We obtain the following surprising pictures, the one on the right-hand side containing around 500 points :





There are certainly profound reasons behind these pictures, that would in particular clarify the condition of non-nullity of the Fourier coefficient $\hat{m}_p(1)$ and more generally of $\hat{m}_p(n)$, $n \in \mathbb{Z}$. Further investigations are necessary.

From the previous numerical analysis, we conclude that the curve $p_1 \longmapsto \hat{m}_p(1)$ is rather convincingly not touching 0. It may certainly be possible to build a rigorous numerical proof of this fact, but this is not the purpose of the present paper. Being confident in this, we can state :

Numerical Theorem 6.1
Let $N = 1$ and the two contractions $\varphi_0(x) = \lambda x$ and $\varphi_1(x) = 1 + \lambda^2 x$, where $1/\lambda > 1$ is the Plastic number. Then for any probability vector $p \in \mathcal{C}_1$, the invariant measure ν is not Rajchman.

Remark. — A similar study developed with $1/\lambda$ the supergolden ratio, i.e. the fourth Pisot number (the real root of $X^3 - X^2 - 1$) leads to essentially the same pictures. Further numerical investigations with the family $\varphi_0(x) = \lambda x$, $\varphi_1(x) = \lambda^2 x$ and $\varphi_2(x) = \lambda^2 x + 1$, for $1/\lambda$ the Plastic number, reveal rather clearly the existence of parameters $p = (p_0, p_1, p_2) \in \mathcal{C}_2^o$ for which $\hat{m}_p(1) = 0$.

6.2 Applications to sets of uniqueness for trigonometric series

Let $N \geq 1$ and for $0 \leq k \leq N$ affine contractions $\varphi_k(x) = r_k x + b_k$, with reals (r_k) and (b_k), with $0 < r_k < 1$ for all k (i.e. condition (C) holds). As a general fact, Theorem 2.3 has some consequences in terms of sets of multiplicity for trigonometric series, cf for example Salem [16] or Zygmund [23] for details. As in the introduction, let $F \subset \mathbb{R}$ be the unique non-empty compact set, verifying the self-similarity relation $F = \cup_{0 \leq k \leq N} \varphi_k(F)$. With $\mathbb{N} = \{0, 1, \cdots\}$ and $S = \{0, \cdots, N\}^{\mathbb{N}}$, recall that :

$$F = \left\{ \sum_{l \geq 0} b_{x_l} r_{x_0} \cdots r_{x_{l-1}}, \ (x_0, x_1, \cdots) \in S \right\}.$$

Let us place on the torus $\mathbb{T}$ and consider trigonometric series. Recall that a subset E of $\mathbb{T}$ is a set of uniqueness (U-set), if whenever a trigonometric series $\sum_{n \geq 0} (a_n \cos(2\pi x) + b_n \sin(2\pi x))$, with complex numbers (a_n) and (b_n), converges to 0 for all $x \notin E$, then $a_n = b_n = 0$ for all $n \geq 0$. Otherwise E is said of multiplicity (M-set).

Theorem 6.1
Let $N \geq 1$ and for $0 \leq k \leq N$ affine contractions $\varphi_k(x) = r_k x + b_k$, where $0 < r_k < 1$, with no common fixed point. Suppose that the system $(\varphi_k)_{0 \leq k \leq N}$ is not conjugated to a family in reduced Pisot form. Then $F \mod 1 \subset \mathbb{T}$ is a M-set.

Proof of the theorem :
Taking any $p \in \mathcal{C}_N$ with $p_j > 0$, for all $0 \le j \le N$, gives a Rajchman invariant probability measure ν supported by $F \subset \mathbb{R}$. Hence $F \mod (1) \subset \mathbb{T}$ supports the probability measure $\tilde{\nu}$, image of ν under the projection $x \longmapsto x \mod 1$, from $\mathbb{R}$ to $\mathbb{T}$. The measure $\tilde{\nu}$ is thus a Rajchman measure on $\mathbb{T}$, so, cf Salem [16] (chap. V), $F \mod 1$ is a M-set.

□

The analysis in the other direction is in general more delicate. For the following statement, fixing $0 < \lambda < 1$ and integers $n_k \ge 1$, for $0 \le k \le N$, notice that for any $(x_0, x_1, \cdots) \in S$ we have $\sum_{l \ge 0} \lambda^{n_{x_0} + \cdots + n_{x_{l-1}}} (1 - \lambda^{n_{x_l}}) = 1$.

Theorem 6.2
Let $N \ge 1$ and suppose that the (φ_k) are affine contractions of the form $\varphi_k(x) = \lambda^{n_k} x + b_k$, with $b_k = b a_k + c(1 - \lambda^{n_k})$, for some $0 < \lambda < 1$ with $1/\lambda$ a Pisot number $> N + 2$, relatively prime positive integers $n_k \ge 1$, $0 \le a_k \in \mathbb{Q}[\lambda]$ and real numbers $b \ge 0$ and c. Then the non-empty compact self-similar set $F = \cup_{0 \le k \le N} \varphi_k(F) \subset \mathbb{R}$ can be written as $F = bG + c$, where G is the compact set :

$$G = \left\{ \sum_{l \ge 0} a_{x_l} \lambda^{n_{x_0} + \cdots + n_{x_{l-1}}},\ (x_0, x_1, \cdots) \in S \right\}.$$

Assume that $bG \subset [0, 1)$, so that bG and F can be seen as subsets of $\mathbb{T}$. Then F is U-set.

Proof of the theorem :
Up to replacing b and the (a_k) respectively by br and (a_k/r), for some $r > 1$ in $\mathbb{Q}$, we may assume that $0 \le a_k < 1/(1 - \lambda)$, for all $0 \le k \le N$. Then :

$$G \subset H := \left\{ \sum_{l \ge 0} \eta_l \lambda^l,\ \eta_l \in \{0, a_0, \cdots, a_N\},\ l \ge 0 \right\} \subset [0, 1).$$

Since $1/\lambda > N + 2$ is a Pisot number and all $a_0, \cdots, a_N$ are in $\mathbb{Q}[\lambda]$, it follows from the Salem-Zygmund theorem, cf Salem [16], chap. VII, paragraph 3, on perfect homogeneous sets, that H is a perfect U-set. Mention that in this theorem, one also assumes that $\max_{0 \le k \le N} a_k = 1/(1 - \lambda)$ and that successive $a_u < a_v$ in $[0, 1)$ verify $a_v - a_u \ge \lambda$. These conditions serve to give a geometrical description of the perfect homogeneous set H in terms of dissection, without overlaps. They are in fact not used in the proof, where only the above description of H is important (one can indeed start reading Salem [16], chap. VII, paragraph 3, directly from line 9 of the proof).

As a subset of a U-set, G is also a U-set. This is also the case of bG, by hypothesis a subset of $[0, 1)$, using Zygmund, Vol. I, chap. IX, Theorem 6.18 (the proof, not obvious, is in Vol. II, chap. XVI, 10.25, and relies on Fourier integrals). Hence, $F = bG + c$ is also a U-set, as any translate on $\mathbb{T}$ of a U-set is a U-set. This ends the proof of the theorem.

□

Remark. — As a general fact, the hypothesis $1/\lambda > N + 2$ ensures that H and F have zero Lebesgue measure, which is a necessary condition for a set to be a U-set. If overlaps happen in H, it would be interesting to consider extensions of the previous theorem, when the above condition on λ not necessarily holds.

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