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A short note on the weak Lefschetz property for Chow groups

Robert Laterveer

Abstract Motivated by the Bloch–Beilinson conjectures, we formulate a certain covariant weak Lefschetz property for Chow groups. We prove this property in some special cases, using Kimura’s nilpotence theorem.

Keywords Algebraic cycles · Chow groups · Finite–dimensional motives · Weak Lefschetz

Mathematics Subject Classification (2010) 14C15 · 14C25 · 14C30

1 Introduction

Let X be a smooth projective variety over \mathbb{C} of dimension n . The Chow groups $A^i X$ (of codimension i algebraic cycles modulo rational equivalence) are notoriously hard to understand. For instance, the following conjecture dating from 1974 is still completely open for $i > 1$:

Conjecture 1 (Hartshorne [9]) If $Y \subset X$ is a smooth hyperplane section, restriction induces isomorphisms

$$A^i X_{\mathbb{Q}} \xrightarrow{\cong} A^i Y_{\mathbb{Q}}$$

for $2i < n - 1$.

Since this seems a very difficult problem, in this note we try and formulate a covariant weak Lefschetz property for Chow groups and hope this is easier. To emphasize that we consider the Chow groups as a homology theory, we now switch to the notation $A_i X = A^{n-i} X$. Let A_i^{hom} and A_i^{AJ} denote the subgroup of homologically trivial resp. Abel–Jacobi trivial cycles.

To fix ideas, let’s now consider $A_0 X$, the Chow group of 0–cycles. Since $H^{2n}(X, \mathbb{Q})$ is one–dimensional, obviously

$$A_0 Y_{\mathbb{Q}} \rightarrow A_0 X_{\mathbb{Q}} / A_0^{hom} X_{\mathbb{Q}}$$

is surjective for any point Y of X —and in particular, for a 0–dimensional complete intersection $Y \subset X$. The next step is that (by weak Lefschetz applied to $H^{2n-1}(X, \mathbb{Q})$)

$$A_0 Y_{\mathbb{Q}} \rightarrow A_0 X_{\mathbb{Q}} / A_0^{AJ} X_{\mathbb{Q}}$$

is surjective, for any smooth complete intersection curve $Y \subset X$. Going beyond the Abel–Jacobi map, it is conjectured there is a filtration F^* on A_0 , of which the first two steps are $F^1 = A_0^{hom}$ and $F^2 = A_0^{AJ}$ (cf. [12], [19], [20], [26]). One can then ask:

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Question 1 Is it true that

$$A_0 Y_{\mathbb{Q}} \rightarrow A_0 X_{\mathbb{Q}} / F^{\ell+1}$$

is surjective, for any smooth complete intersection $Y \subset X$ of dimension ℓ ?

This question is motivated (pun intended) by the expectation that the quotient $A_0 X_{\mathbb{Q}} / F^{\ell+1}$ is determined by the cohomology groups $H^{2n} X, H^{2n-1} X, \dots, H^{2n-\ell} X$. Since the filtration F^* only exists conjecturally, this question is not falsifiable. However, it is expected that $F^{\ell+1}$ vanishes exactly when $H^n X, \dots, H^{\ell+1} X$ are supported in codimension 1. This gives the following conjecture, in which F^* does not appear:

Conjecture 2 Let X be a smooth projective variety, and suppose $H^i(X, \mathbb{Q}) = N^1 H^i(X, \mathbb{Q})$ for $i \in [\ell + 1, n]$. Then

$$A_0 Y_{\mathbb{Q}} \rightarrow A_0 X_{\mathbb{Q}}$$

is surjective, for any smooth complete intersection $Y \subset X$ of dimension ℓ .

The main result of this note provides a verification of this conjecture in some special cases. As a by-product, we also get the injectivity part of conjecture 1 in these special cases:

Theorem (=Theorem 3) *Suppose the Voisin standard conjecture (conjecture 4) holds. Let X be a smooth projective variety of dimension n , and suppose*

(i) *Either the motive of X is finite-dimensional, or $\text{Griff}^n(X \times X)_{\mathbb{Q}} = 0$;*

(ii) *The Lefschetz standard conjecture $B(X)$ holds;*

(iii) *There exists r such that $H^i(X, \mathbb{Q}) = N^r H^i(X, \mathbb{Q})$ for all $i \in [n - r + 1, n]$.*

Then for any codimension r smooth complete intersection $Y \subset X$ of class $[Y] = L^r \in H^{2r}(X, \mathbb{Q})$ with L ample, push-forward maps

$$A_i(Y)_{\mathbb{Q}} \rightarrow A_i(X)_{\mathbb{Q}}$$

are surjective for $i < r$. Moreover, restriction maps

$$A_{AJ}^i(X)_{\mathbb{Q}} \rightarrow A_{AJ}^i(Y)_{\mathbb{Q}}$$

are injective for $i \leq r + 1$.

In certain cases some of the hypotheses are automatically satisfied, and the statement simplifies:

Corollary (cf. Corollary 1) *Let X be a smooth projective 3fold which is dominated by a product of curves. Suppose*

$$H^3(X, \mathbb{Q}) = N^1 H^3(X, \mathbb{Q}).$$

Then for any smooth ample hypersurface $Y \subset X$, the push-forward map

$$A_0(Y)_{\mathbb{Q}} \rightarrow A_0(X)_{\mathbb{Q}}$$

is surjective, and

$$A_{AJ}^2(X)_{\mathbb{Q}} \rightarrow A_{AJ}^2(Y)_{\mathbb{Q}}$$

is injective.

Corollary (=Corollary 2) *Let X be a product of smooth projective surfaces*

$$X = S_1 \times \dots \times S_m,$$

where each S_j is either a K3 surface of Picard number 19 or 20, or has $A_0^{AJ}(S_j)_{\mathbb{Q}} = 0$. Suppose at least one S_j has $A_0^{AJ}(S_j)_{\mathbb{Q}} = 0$. Then for any smooth ample hypersurface $Y \subset X$, the push-forward map

$$A_0(Y)_{\mathbb{Q}} \rightarrow A_0(X)_{\mathbb{Q}}$$

is surjective, and

$$A_{AJ}^2(X)_{\mathbb{Q}} \rightarrow A_{AJ}^2(Y)_{\mathbb{Q}}$$

is injective.

It was already known that in situations like these two corollaries, $A_0X_{\mathbb{Q}}$ is supported on *some* divisor (this follows for instance from [26, Theorem 3.32]); thus, our only contribution is the precision that any ample hyper-surface does the job. The injectivity statement, on the other hand, seems to be genuinely new: as far as we know, these are the first examples of varieties with non-trivial $A_{A,J}^2$ for which this injectivity is known to hold.¹ The proof of the theorem is an easy exercise in using the meccano of correspondences; the only “deep” ingredient is Kimura’s nilpotence theorem [16].

We end this introduction with a challenge. As is well-known [5], the hypothesis of conjecture 2 is verified when $A_0X_{\mathbb{Q}}$ is supported in dimension ℓ . This gives the following special case of conjecture 2:

Conjecture 3 Let X be a smooth projective variety, and suppose $A_0X_{\mathbb{Q}}$ is supported on a closed subvariety of dimension ℓ . Then any smooth complete intersection $Y \subset X$ of dimension ℓ supports $A_0X_{\mathbb{Q}}$.

This is true for $\ell \leq 1$, but for $\ell > 1$ I have no idea how to prove this...

Conventions In this note, the word *variety* refers to a quasi-projective algebraic variety over \mathbb{C} . A subvariety will be a (possibly reducible) reduced subscheme which is equidimensional. The Chow group of i -dimensional cycles on X is denoted A_iX ; for X smooth of dimension n the notations A_iX and $A^{n-i}X$ will be used interchangeably. The Griffiths group Griff_i is the group of i -dimensional cycles that are homologically trivial modulo algebraic equivalence. In diagrams, we will sometimes write H^jX or H_jX to designate singular cohomology $H^j(X, \mathbb{Q})$ resp. Borel–Moore homology $H_j(X, \mathbb{Q})$.

2 Preliminary

Definition 1 (Coniveau filtration [4]) Let X be a quasi-projective variety. The coniveau filtration on cohomology and on homology is defined as

$$\begin{aligned} N^c H^i(X, \mathbb{Q}) &= \sum \text{Im}(H_Y^i(X, \mathbb{Q}) \rightarrow H^i(X, \mathbb{Q})); \\ N_c H_i(X, \mathbb{Q}) &= \sum \text{Im}(H_i(Z, \mathbb{Q}) \rightarrow H_i(X, \mathbb{Q})), \end{aligned}$$

where Y runs over codimension $\geq c$ subvarieties of X , and Z over dimension $\leq c$ subvarieties.

We recall the statement of the “Voisin standard conjecture”:

Conjecture 4 (Voisin standard conjecture [25]) Let X be a smooth projective variety, and $Y \subset X$ closed with complement U . Then the natural sequence

$$N_i H_{2i}(Y, \mathbb{Q}) \rightarrow N_i H_{2i}(X, \mathbb{Q}) \rightarrow N_i H_{2i}(U, \mathbb{Q}) \rightarrow 0$$

is exact for any i .

Remark 1 Hodge theory gives an exact sequence

$$\text{Gr}_{-2i}^W H_{2i} Y \cap F^{-i} \rightarrow H_{2i} X \cap F^{-i} \rightarrow \text{Gr}_{-2i}^W H_{2i} U \cap F^{-i} \rightarrow 0,$$

where W denotes Deligne’s weight filtration, and F the Hodge filtration on $H_*(-, \mathbb{C})$. Hence if the Hodge conjecture (that is, its homology version for singular varieties [11]) is true, then conjecture 4 is true.

What’s more, this conjecture fits in very neatly with the classical standard conjectures: Voisin shows that conjecture 4 plus the algebraicity of the Künneth components of the diagonal is equivalent to the Lefschetz standard conjecture [25, Proposition 1.6].

Remark 2 Conjecture 4 is obviously true for $i \leq 1$ (this follows from the truth of Hodge conjecture for curve classes), and for $i \geq \dim Y - 1$ (where it follows from the Hodge conjecture for divisors).

The main ingredient used in this note is Kimura’s nilpotence theorem:

¹ This is not strictly true: indeed, [7, Corollary 5] gives non-trivial examples of varieties where the injectivity part of conjecture 1 is verified.

Theorem 1 (Kimura [14]) *Let X be a smooth projective variety of dimension n with finite-dimensional motive. Let $\Gamma \in A^n(X \times X)_{\mathbb{Q}}$ be a correspondence which is homologically trivial. Then there is $N \in \mathbb{N}$ such that*

$$\Gamma^{\circ N} = 0 \quad \in A^n(X \times X)_{\mathbb{Q}}.$$

Remark 3 We refer to [14], [1], [20] for the definition of finite-dimensional motive. Conjecturally, any variety has finite-dimensional motive [14]. What mainly concerns us in the scope of this note, is that there are quite a few examples which are known to have finite-dimensional motive: varieties dominated by products of curves [14], K3 surfaces with Picard number 19 or 20 [21], any surface with vanishing geometric genus for which Bloch's conjecture has been verified [8, Theorem 2.11], 3folds with nef tangent bundle [10], certain 3folds of general type [22, Section 8].

There is also the following nilpotence result, which predates Kimura's theorem:

Theorem 2 (Voisin [24], Voevodsky [23]) *Let X be a smooth projective algebraic variety of dimension n , and $\Gamma \in A^n(X \times X)_{\mathbb{Q}}$ a correspondence which is algebraically trivial. Then there is $N \in \mathbb{N}$ such that*

$$\Gamma^{\circ N} = 0 \quad \in A^n(X \times X)_{\mathbb{Q}}.$$

3 Main

We now proceed with the proof of the main result of this note:

Theorem 3 *Suppose the Voisin standard conjecture holds. Let X be a smooth projective variety of dimension n , and suppose*

- (i) *Either the motive of X is finite-dimensional, or $\text{Griff}^n(X \times X)_{\mathbb{Q}} = 0$;*
- (ii) *The Lefschetz standard conjecture $B(X)$ holds;*
- (iii) *$H^i(X, \mathbb{Q}) = N^r H^i(X, \mathbb{Q})$ for all $i \in [n - r + 1, n]$.*

Then for any codimension r smooth complete intersection $Y \subset X$ of class $[Y] = L^r \in H^{2r}(X, \mathbb{Q})$ with L ample, push-forward maps

$$A_i(Y)_{\mathbb{Q}} \rightarrow A_i(X)_{\mathbb{Q}}$$

are surjective for $i < r$. Moreover, restriction maps

$$A_{AJ}^i(X)_{\mathbb{Q}} \rightarrow A_{AJ}^i(Y)_{\mathbb{Q}}$$

are injective for $i \leq r + 1$.

In certain cases, some of the hypotheses can be removed:

Corollary 1 *Let X be a smooth projective variety of dimension $n \leq 3$, and suppose*

- (i) *Either the motive of X is finite-dimensional, or $\text{Griff}^n(X \times X)_{\mathbb{Q}} = 0$;*
- (ii) *$H^n(X, \mathbb{Q}) = N^1 H^n(X, \mathbb{Q})$.*

Then for any smooth ample hypersurface $Y \subset X$, push-forward maps

$$A_0(Y)_{\mathbb{Q}} \rightarrow A_0(X)_{\mathbb{Q}}$$

are surjective, and restriction

$$A_{AJ}^2(X)_{\mathbb{Q}} \rightarrow A_{AJ}^2(Y)_{\mathbb{Q}}$$

is injective.

Corollary 2 *Let X be a product of smooth projective surfaces*

$$X = S_1 \times \cdots \times S_m,$$

where each S_j is either a K3 surface of Picard number 19 or 20, or has $A_0^{AJ}(S_j)_{\mathbb{Q}} = 0$.

(i) Suppose at least one S_j has $A_0^{AJ}(S_j)_{\mathbb{Q}} = 0$. Then for any smooth ample hypersurface $Y \subset X$,

$$A_0(Y)_{\mathbb{Q}} \rightarrow A_0(X)_{\mathbb{Q}}$$

is surjective, and

$$A_{AJ}^2(X)_{\mathbb{Q}} \rightarrow A_{AJ}^2(Y)_{\mathbb{Q}}$$

is injective.

(ii) Suppose there are at least 4 surfaces S_j with $A_0^{AJ}(S_j)_{\mathbb{Q}} = 0$. Let $Y \subset X$ be a codimension 2 complete intersection of class $[Y] = L^2 \in H^4(X, \mathbb{Q})$ with L ample. Then

$$A_i(Y)_{\mathbb{Q}} \rightarrow A_i(X)_{\mathbb{Q}}$$

is surjective for $i \leq 1$, and

$$A_{AJ}^i(X)_{\mathbb{Q}} \rightarrow A_{AJ}^i(Y)_{\mathbb{Q}}$$

is injective for $i \leq 3$.

Proof (of theorem 3) Let $\tau: Y \hookrightarrow X$ be a smooth complete intersection of class L^r as in the statement of the theorem. Let

$$L^j: H^i X(\mathbb{Q}) \rightarrow H^{i+2j}(X, \mathbb{Q})$$

denote the result of cupping with a power of L ; we use the same notation L^j for the correspondence inducing this action. Since $B(X)$ is true, for any $i < n$ there exists a correspondence $C_i \in A^i(X \times X)_{\mathbb{Q}}$ inducing an isomorphism

$$(C_i)_*: H^{2n-i}(X, \mathbb{Q}) \xrightarrow{\cong} H^i(X, \mathbb{Q})$$

that is inverse to L^{n-i} .

$B(X)$ being true, the Künneth components π_i of the diagonal of X are algebraic [16]. Since $B(X)$ implies $B(Y)$ [16], the same holds for the Künneth components π_i^Y of Y . We now proceed to relate them:

Lemma 1 For each $i \leq n - r$, define

$$\Pi_i := (C_i) \circ (L^{n-i-r}) \circ ((\tau \times \tau)_*(\pi_i^Y)) \in A^n(X \times X, \mathbb{Q}).$$

Then for each $i \leq n - r$, we have equality

$$\Pi_i = \pi_i \in H^{2n}(X \times X, \mathbb{Q}).$$

Proof We consider the action on $H^j(X, \mathbb{Q})$. There is a factorization

$$\begin{array}{ccccc} H^j X & \xrightarrow{((\tau \times \tau)_*(\pi_i^Y))_*} & H^{j+2r} X & \xrightarrow{L^{n-i-r}} & H^{2n-2i+j} X & \xrightarrow{(C_i)_*} & H^j X \\ \downarrow & & \uparrow & & & & \\ H^j Y & \xrightarrow{(\pi_i^Y)_*} & H^j Y & & & & \end{array}$$

Hence, if $j \neq i$ then

$$(\Pi_i)_* H^j X = 0,$$

and for $j = i$ we have

$$\Pi_i = (C_i) \circ (L^{n-i-r}) \circ (L^r) = \text{id} : H^i X \rightarrow H^i X.$$

It follows that for any variety Z , the action of Π_i on $H^j(X \times Z)$ is projection on $H^i X \otimes H^{j-i} Z$; thus by Manin's identity principle, Π_i and π_i coincide as homological correspondences. \square

Lemma 2 For each $i \leq n - r$, and each $j < r$, we have

$$(\Pi_i)_* A_j X_{\mathbb{Q}} = 0.$$

Proof For any correspondence $C \in A^{n-r}(Y \times Y)_{\mathbb{Q}}$, there is a factorization

$$\begin{array}{ccc} A_j X_{\mathbb{Q}} & \xrightarrow{((\tau \times \tau)_* C)_*} & A_{j-r} X_{\mathbb{Q}} \\ \downarrow & & \uparrow \\ A_{j-r} Y_{\mathbb{Q}} & \xrightarrow{C_*} & A_{j-r} Y_{\mathbb{Q}} \end{array}$$

In particular, taking $C = \pi_i^Y$, we see that the action of $(\tau \times \tau)_*(\pi_i^Y)$ on $A_j X_{\mathbb{Q}}$ factors over $A_{j-r} Y_{\mathbb{Q}}$, hence is 0 for $j < r$. \square

Lemma 3 *Let ${}^t\Pi_i$ denote the transpose of Π_i . For each $i \leq n - r$, and each j , we have*

$$({}^t\Pi_i)_* A_j X_{\mathbb{Q}} \subset \text{Im}(A_j Y_{\mathbb{Q}} \rightarrow A_j X_{\mathbb{Q}}).$$

Moreover, for each $j \leq r + 1$, we have

$$({}^t\Pi_i)_* A_{AJ}^j X_{\mathbb{Q}} = 0.$$

Proof It is immediate from the definition that

$${}^t\Pi_i = ((\tau \times \tau)_*({}^t\pi_i^Y)) \circ {}^t(L^{n-i-r}) \circ {}^t C_i \in A^n(X \times X)_{\mathbb{Q}}.$$

Using the same diagram as in the proof of lemma 2, one can find a factorization

$$\begin{array}{ccccc} A_j X_{\mathbb{Q}} & \xrightarrow{({}^t(L^{n-i-r}) \circ {}^t C_i)_*} & A_{j+r} X_{\mathbb{Q}} & \xrightarrow{((\tau \times \tau)_*({}^t\pi_i^Y))_*} & A_j X_{\mathbb{Q}} \\ & & \downarrow & & \uparrow \\ & & A_j Y_{\mathbb{Q}} & \xrightarrow{{}^t(\pi_i^Y)_*} & A_j Y_{\mathbb{Q}}, \end{array}$$

and the lemma is proven. \square

By hypothesis (iii), we have

$$H^i(X, \mathbb{Q}) = N^r H^i(X, \mathbb{Q}) \quad \forall n - r < i \leq n.$$

Applying hard Lefschetz, one finds

$$H^i(X, \mathbb{Q}) = N^r H^i(X, \mathbb{Q}) \quad \forall n - r < i < n + r.$$

This means that in the range $n - r < i < n + r$, the Künneth component π_i is supported in codimension r . That is, there exists a subvariety $Z \subset X$ of codimension r , such that for each $n - r < i < n + r$, π_i goes to 0 under the restriction

$$H^{2n}(X \times X, \mathbb{Q}) \rightarrow H^{2n}((X \times X) \setminus (Z \times Z), \mathbb{Q}).$$

Using the Voisin standard conjecture (conjecture 4), this implies the existence of an algebraic cycle $P'_i \in A_n(Z \times Z)_{\mathbb{Q}}$ such that (denoting by P_i the push-forward of P'_i to $X \times X$) we have

$$P_i = \pi_i \in H^{2n}(X \times X, \mathbb{Q}) \quad \forall n - r < i < n + r.$$

Lemma 4 *For any $i \in [n - r + 1, n + r - 1]$, and any $j < r$, we have*

$$(P_i)_* A_j X_{\mathbb{Q}} = 0.$$

Moreover, for any $j \leq r + 1$, we have

$$(P_i)_* A_{AJ}^j X_{\mathbb{Q}} = 0.$$

Proof Let $\psi: Z \rightarrow X$ denote the inclusion, so $P_i = (\psi \times \psi)_*(P'_i)$. Similar to lemma 2, there is a factorization

$$\begin{array}{ccc} A_j X_{\mathbb{Q}} & \xrightarrow{(P_i)_*} & A_j X_{\mathbb{Q}} \\ \downarrow & & \uparrow \\ A_{j-r} Z_{\mathbb{Q}} & \xrightarrow{(P'_i)_*} & A_j Z_{\mathbb{Q}}. \end{array}$$

That is, the action of P_i in the indicated range factors over groups that vanish for dimension reasons and the lemma follows. \square

Putting together the various parts, we find a decomposition of the diagonal

$$\Delta = \sum_{i=0}^{n-r} \Pi_i + \sum_{i=n-r+1}^{n+r-1} P_i + \sum_{i=0}^{n-r} {}^t \Pi_i \in H^{2n}(X \times X, \mathbb{Q}).$$

This is an equality of cycles modulo homological equivalence. Now, applying Kimura's nilpotence theorem (theorem 1), we get that there exists N such that

$$\left(\Delta - \sum_{i=1}^{n-r} \Pi_i - \sum_{i=n-r+1}^{n+r-1} P_i - \sum_{i=1}^{n-r} {}^t \Pi_i \right)^{\circ N} = 0 \in A^n(X \times X)_{\mathbb{Q}}.$$

Developing this expression (and noting that $\Delta^{\circ N} = \Delta$), we find

$$\Delta = \sum_j Q_j \in A^n(X \times X)_{\mathbb{Q}},$$

where each Q_j is a composition of elements Π_ℓ and $P_{\ell'}$ and ${}^t \Pi_{\ell''}$. Let Q_j^0 denote the ‘‘tail element’’ of Q_j , i.e. we write

$$Q_j = Q_j^0 \circ Q_j^1 \circ \cdots \circ Q_j^{N'} \in A^n(X \times X)_{\mathbb{Q}},$$

with $Q_j^0 \neq \Delta$ (so that $N' \leq N$).

Let's consider the action of Q_j on $A_i X_{\mathbb{Q}}$, for $i < r$:

If Q_j^0 is a Π_ℓ (for some $\ell \in [0, n-r]$), it follows from lemma 2 that

$$(Q_j)_*(A_i X_{\mathbb{Q}}) = 0.$$

Likewise, if Q_j^0 is of the form P_ℓ (for some $n-r < \ell < n+r$), then applying lemma 4, we find again

$$(Q_j)_*(A_i X_{\mathbb{Q}}) = 0.$$

Finally, if Q_j^0 is of the form ${}^t \Pi_\ell$ (for some $\ell \in [0, n-r]$), it follows from lemma 3 that

$$(Q_j)_*(A_i X_{\mathbb{Q}}) \subset \text{Im}(A_i Y_{\mathbb{Q}} \rightarrow A_i X_{\mathbb{Q}}).$$

Since Δ acts as the identity, we conclude that for $i < r$, push-forward

$$A_i Y_{\mathbb{Q}} \rightarrow A_i X_{\mathbb{Q}}$$

is surjective.

The argument for the injectivity statement is similar: we consider the action of $\Delta = \sum_j Q_j$ on $A_{AJ}^i X_{\mathbb{Q}}$ for $i \leq r+1$. If Q_j is such that its ‘‘head’’ $Q_j^{N'}$ is of type ${}^t \Pi_\ell$ or P_ℓ , then Q_j does not act (by lemma 3 resp. lemma 4). It follows that we can write

$$A_{AJ}^i X_{\mathbb{Q}} = \Delta_* A_{AJ}^i X_{\mathbb{Q}} = \left(\sum \text{something} \circ (\tau \times \tau)_* (\text{something}) \right)_* A_{AJ}^i X_{\mathbb{Q}};$$

the injectivity is then obvious.

Finally, if the hypothesis in (i) of the theorem is that

$$\mathrm{Griff}^n(X \times X)_{\mathbb{Q}} = 0,$$

the proof goes as follows: the decomposition of Δ is now an equality modulo algebraic equivalence (since by hypothesis, algebraic and homological equivalence coincide on $X \times X$). Then, instead of applying Kimura’s theorem, we apply the Voisin/Voevodsky nilpotence theorem (theorem 2). The rest of the proof is verbatim the same. \square

Proof (of corollary 1) In case $n = 2$, we know $B(X)$ holds since it holds for any surface [16]. The Voisin standard conjecture is used to get that some Hodge classes in $H_4(Z \times Z, \mathbb{Q})$ are algebraic, where $\dim Z = 1$; this is trivially true.

Next, the case $n = 3$. Under the hypothesis $H^3 X = N^1 H^3 X$, X is “motivated by a surface” in the sense of [2], so $B(X)$ is known to hold [2]. The Voisin standard conjecture is only used to get that some Hodge classes in $H_6(Z \times Z, \mathbb{Q})$ are algebraic, where $\dim Z = 2$; this is OK by the Hodge conjecture for divisors (remark 2). \square

Proof (of corollary 2) As we noted in remark 3, it follows from work of Pedrini [21] and Guletskiĭ–Pedrini [8] that the S_j have finite–dimensional motive. Hence X has finite–dimensional motive. We also know $B(X)$ is true since the Lefschetz standard conjecture is true for all surfaces [16].

In case (i), since there is at least one surface with $H^2(S_j) = N^1$, we obviously have

$$H^{2m}(X, \mathbb{Q}) = N^1 H^{2m}(X, \mathbb{Q}).$$

The corollary now follows from theorem 3; note that we don’t need to assume the Voisin standard conjecture, since we can find cycles P'_i by using the Hodge conjecture on the surfaces with vanishing geometric genus.

In case (ii), the assumptions imply

$$\begin{aligned} H^{2m}(X, \mathbb{Q}) &= N^2 H^{2m}(X, \mathbb{Q}); \\ H^{2m-1}(X, \mathbb{Q}) &= N^2 H^{2m-1}(X, \mathbb{Q}), \end{aligned}$$

and we again apply theorem 3.

Remark 4 The hypothesis on $\mathrm{Griff}^n(X \times X)$ in theorem 3 is mainly of theoretical interest, and not practically useful. Indeed, there are precise conjectures predicting when Griffiths groups should vanish [13]; for instance, if X is a 4fold with $h^{2,0} = h^{4,0} = h^{3,0} = h^{2,1} = 0$, [13, Corollary 6.8] implies that if the Bloch–Beilinson conjectures are true then

$$\mathrm{Griff}^4(X \times X)_{\mathbb{Q}} = 0.$$

Unfortunately, no non–trivial examples seem to be known. Specifically, I am not aware of any example of a variety X of dimension n that satisfies $\mathrm{Griff}^n(X \times X)_{\mathbb{Q}} = 0$, but not $A_{AJ}^i X_{\mathbb{Q}} = 0 \forall i$.

Remark 5 In [17], I study a certain hard Lefschetz property for Chow groups. Using arguments similar to the present note, this hard Lefschetz property can be proven in some special cases [17].

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