A Versatile Six-wing 3D Strange Attractor

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Abstract

To create additional wings to a given strange attractor, several methods based on the heteroclinic loop or switching controls for example are applied, but complicate the approach and require the extension of the system to one or more other dimensions of the phase space. This deflects us from the objectives of research on low-dimensional chaotic systems.
Remaining in this narrow area of 3D phase spaces to invent multi-wing attractors constitutes the main scope of the present paper. Indeed, we present a rapid investigation of a very simple autonomous 3D system of firts-order differential equations with a rich variety of phase portraits. This new intentionally constructed model exhibits double, four- or even six-wing strange attractors. We point out that under the influence of the scalar parameters, such versatile chaotic attractors are obtained. A similar sequence was likewise observed for the periodic behaviors. Besides, both chaotic or regular featured trajectories are found to be in bilateral agreement even when the morphology of the portrait changes. Obviously, we present the basic attributes of the system and its bifurcation diagram.
Eventually, we emphasize that the study of the relationship between the written differential equations and the observed characteristics of attractors remains undervalued.

I. Introduction

Techniques of mutating known double-wing attractors into multi-scroll shapes are still experienced at least from 20 years [Aziz-Alaoui, 1999], and recent papers reported that such theoretical (and experiemental) field remains a challenging task [Tahir *et al.*, 2015; Chen *et al.*, 2016; Wang *et al.*, 2017; Zhang *et al.*, 2018]. Having in mind that it is possible to achieve multi-scroll chaotic attractors by establishing unstable points in a chaotic system [Elwakil *et al.*, 2003], several other methods to increase the number of wings were accurately analyzed [Lü and Chen, 2006].

Among them, the piecewise-linear switching system yielding to multi-scroll attractors appears as the most applied framework [Han *et al.*, 2015].

This paper will also introduce a chaotic model with an increasing number of wings, however *without* these sophisticated techniques.

Indeed, our main concern is to build a minimalist model related to the number of terms in the right-hand side of the equations with the lowest nonlinearity. Known as the parcimony principle, such goal can lead to a non-wasteful 3D system in the sense of economic use of mathematical relations. The ultimate purpose of this theoretical exploration is to determine a simple 3D strange attractor with different number of rolls in relation to scalar parameters rejecting obviously the bias of additional differential equations or artifacts (wave functions, homoclinic loops, ...).
Versatile number of wings and more complexity could be expected from the designing our mathematical application.

II. The 3D Chaotic System

Our new system written in three first-order differential equations modifies the model of Bouali (2013) by adding a nonlinear term in the right-hand side of the third equation:

$$\begin{cases} \dfrac{dx}{dt} = \alpha z - x(1 - y) \\ \dfrac{dy}{dt} = -y(1 - x^2) \\ \dfrac{dz}{dt} = -\beta x - \mu z(1 - y) \end{cases}$$

where x, y, and z are the state variables of the system, and P (α, β, μ), the set of parameters. The model embeds two quadratic nonlinearities and only one cubic term, respectively, xy, yz, and yx^2.

II.1. Basic mathematical attributes

The equilibria coordinates could be found by setting:

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0.$$

Thus, the following system leads to the solutions:

$$\begin{cases} \alpha z - x(1 - y) & = 0 \\ -y(1 - x^2) & = 0 \\ -\beta x - \mu z(1 - y) & = 0 \end{cases}$$

For the set of the parameters P (α, β, μ) = (0.1, 0.1, 1), the coordinates of the three equilibria are:

S_0 (x_0, y_0, z_0) = (0, 0, 0),
S_1 (x_1, y_1, z_1) = (1, 1-(0.1) *i*, *i*),
and **S_2 (x_2, y_2, z_2) = (-1, 1+ (0.1) *i*, *i*).**

Besides, in order to investigate the stability of S_0 (x_0, y_0, z_0), J the Jacobian is:

$$J = \begin{bmatrix} (1 - y) & x & \alpha \\ 2xy & (1 - x^2) & 0 \\ -\beta & \mu z & -\mu(1 - y) \end{bmatrix}$$

It is easy to found that the corresponding characteristic equation, $|J - \lambda I| = 0$ at S_0 is:

$(1-\lambda)(\lambda^2 + 2\lambda + 1.01) = 0$

Thus, the eigenvalues are :

$\lambda_1 = 1$, $\lambda_2 = -1 + 0.1\,i$, and $\lambda_3 = -1 - 0.1\,i$

We notice that λ_1, the real eigenvalue, is positive reporting that S_0 is unstable with Index-1. Therefore, the emergence of chaos becomes confirmed knowing that the Shilnikov criteria [Shilnikov *et al.*, 1998] requiring at least one unstable equilibrium are fulfilled.

II.2. The dissipativity

The dissipative nature of the 3D system could be derived from the divergence nature of the whole vector field:

$$\boldsymbol{div.}(\boldsymbol{Volume}) = \frac{\partial \boldsymbol{Volume}/_{\partial \boldsymbol{t}}}{\boldsymbol{Volume}} = \boldsymbol{Tr}(\boldsymbol{J})$$

To this end, *Tr (J)*, the sum of the diagonal terms of the Jacobian should be negative to attest the dissipativity of the flow:

$$\boldsymbol{Tr}(\boldsymbol{J}) = -(\mathbf{1} - \mathbf{y})(\mathbf{1} + \mu) + (\mathbf{1} - \mathbf{x}^2) < 0$$

where **μ** real value.

Dissipativity and volume contraction of the flow are accurately identified when these state variables of the flow, x, and y (and not including z) satisfy the required condition:

$$(\mathbf{1} - \mathbf{x}^2) < (\mathbf{1} - \mathbf{y})(\mathbf{1} + \mu)$$

In the specific case where the condition is actually fulfilled, orbits are ultimately limited in a specific fractal-dimensional subspace of zero volume. That is the case at the neighborhood of $\mathbf{S_0}$.

On the other hand, the Lyapunov exponents spectrum shows the chaotic nature of the system since for **μ**, varying from 0.1 to 1.4, the dominant Lyapunov exponent reaches a positive value (Fig. 1).

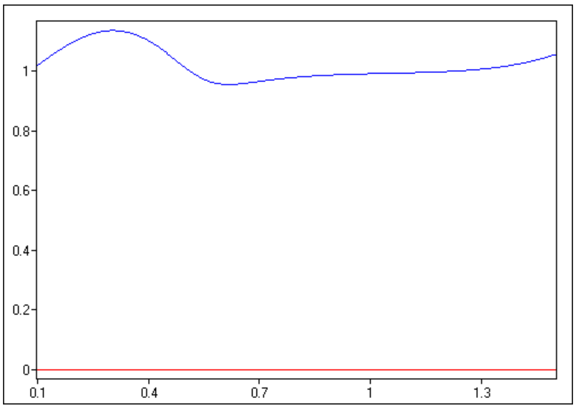

Fig.1. Spectrum of the largest Lyapunov exponent. Its value varies in the vicinity of the unit, which testifies to the chaoticity *of the 3D model.*

III. Strange Attractors Evolving from Two- to Four or Six-wing

We simulate the system with the parameter set P (α, β, μ)= (1, -0.1, 1) that lead to an unconventional double-wing attractor (fig. 2a). Surprisingly, the modification of the parameter set to P (α, β, μ)= (0.1, 0.2, 1) displays a portrait of four-wing attractor (fig. 2b). Furthermore, for P (α, β, μ)= (0.1, 0.1, 1), the third representation depicts an identifiable six-wing strange attractor having a very complex dynamics (fig. 2c). We can notice that the rolls could be gathered in non identical pairs exhibiting a certain degre of symmetry. We can also indicate that almost all dynamics are highly volatile in relation the initial conditions. A small gap can project any simulation to a vanishing path.

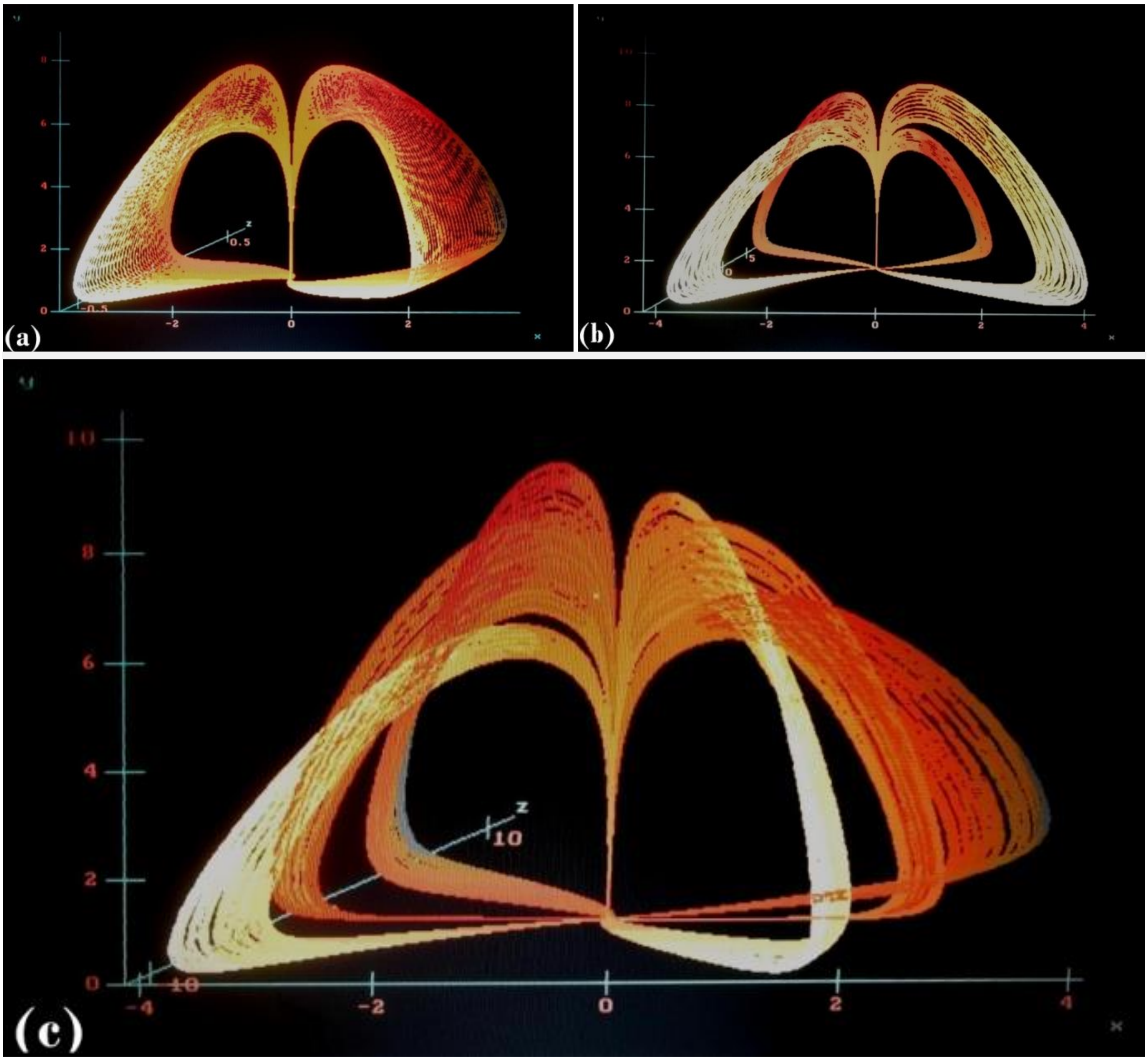

Fig. 2. Multiwing strange attractors.
(a) 2-wing attractor of the 3D system with P(α, β, μ)= (1, -0.1, 1) and IC (x_0, y_0, z_0)= (1.9053, 0.0038, -0.2332), (b) 4-wing attractor of the 3D system with P(α, β, μ)= (0.1, 0.2, 1) and IC_b (x_0, y_0, z_0)= (0.01, 2.734, 0.0456), and (c) 6-wing attractor of the 3D system with P(α, β, μ)= (0.1, 0.1, 1) and IC_c (x_0, y_0, z_0)= (-0.794, 0.00132, -0.941).

The volatility of the number of wings in the narrow chaos bubbles is also observed for the regular movements in the wide periodicity windows, as depicted in the bifurcation diagrams of x and z (Fig. 3).

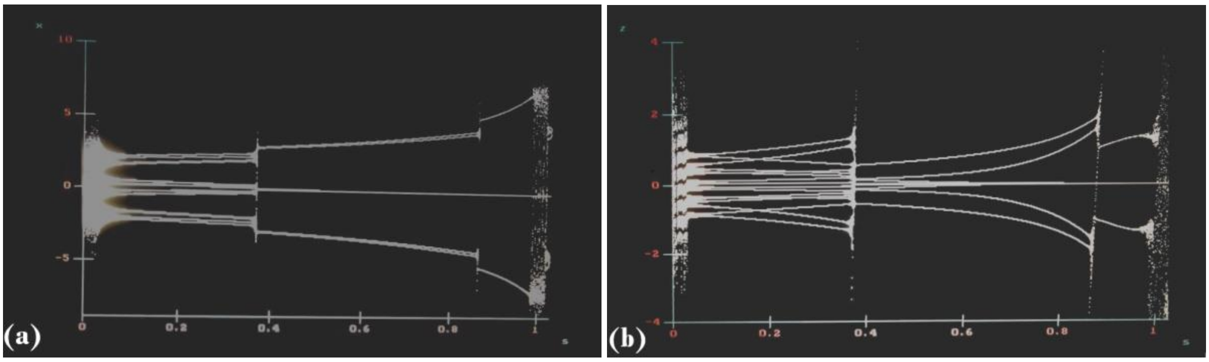

Fig. 3. The bifurcation diagrams of x and z show wide windows of periodicity. The system parametrized with P (α, β, μ)= (0.1, 0.1, s), and s the control parameter ϵ]0, 1.1 [leads to large bubbles of stability when the trajectories of (a) the variable x, and (b) the variable z cross the plane $\bar{y} = 3$.

The variation of the parameter set P leads to a sequence of periodic cycles strongly similar to that shown in chaotic patterns. The sample of periodic behaviors in Figure 4 displays even and likewise odd rolls. For instance, the featured regular dynamics are obtained by the modification of β and μ, when α keeps the value 0.1.

It is worth mentioning that further analysis should be done to encompass the whole characteristics of this new 3D dynamical system.

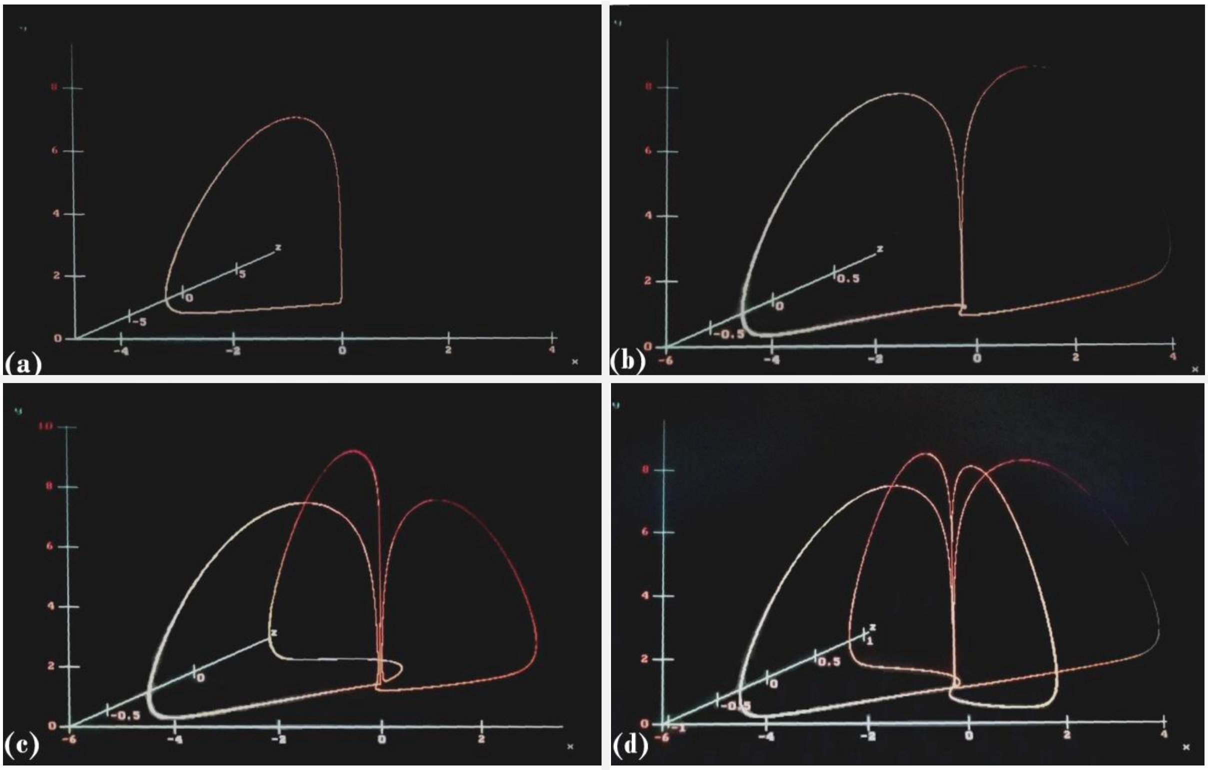

Fig. 4. A Sample of the Periodic Motions.
(a) A basic periodicity observed for P(α, β, μ)= (0.1, -0.1, 1), (b) period-4 dynamics for P(α, β, μ)= (0.1, 0.2, 0.7), (c) period-6 for P(α, β, μ)= (0.1, 0.1, 0.5), and (d) period-8 for P(α, β, μ)= (0.1, 0.1, 0.7).

Despite its apparently simple formulation, the model leads also toward typical solenoid-like dynamics for particular parameter sets (figure 5). Whether chaotic or regular, the dynamics reported illustrate the ability of 3D systems to adopt a wide range of patterns and shapes.

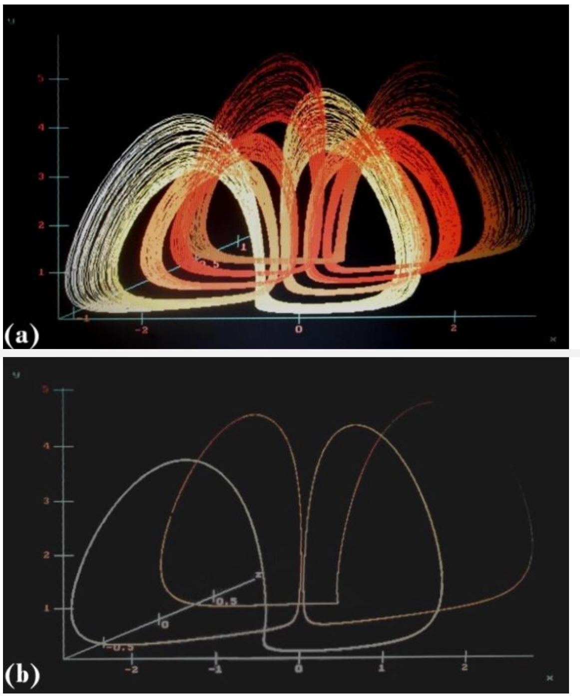

Fig.5. The 3D system exhibits solenoid-like phase portraits.
(a) A chaotic attractor is displayed for P(α, β, μ)= (0.1, 0.1, 0.2) and IC (x_0, y_0, z_0)= (-0.446, 2.826, -0.391,and (b) a periodic behavior for P(α, β, μ)= (0.1, 0.2, 0.2).

Our model could be described as *versatile* not only for its sensitive dependence of the number of wings on the variation of the parameters but also for the extreme volatility of the dynamics in relation with the asymptotic stability of the attractors. Indeed, small gaps of the initial conditions are omitted and the attractors disappear. Thus, for the attractor with six wings, starting a simulation with IC (-0.794, 0.001, -0.941) and not IC (-0.794, 0.00132, -0.941) does not replicate the related dynamics.

This tiny difference $\Delta y = 0.00032$ is sufficient to deflect the trajectory out from the basin of attraction.

IV. Concluding Remarks

Generating grid multi-wing attractors throught heteroclinic loops, by adding a piecewise linear function, and other advanced methods as coupling 3D systems with additional first-order differential equations were widely investigated.

However, our research does not lie into such mainstream, but explores directly what system allows the variation of the number of wings without heavy techniques. To this end, generating multi-wing chaotic attractors without complicatedness can be achieved.

We have shown that the multiplication of wings to attractors, relevant for research purposes, could be controled by scalar parameters. The carefully selection of the equations governing the dynamics deserves to be practiced in this direction.

Our results are expected to develop further experimentations. Indeed, the discovery of such new system contributing towards knowledge of the relation between the choice of right-hand terms of the system equations and global complex dynamical behaviors are quite subtle and deserve to be deeply studied.

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