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# Worked Examples on Using the Riemann Integral and the Fundamental of Calculus for Integration over a Polygonal Element

*Sulaiman Abo Diab*

*Faculty of Civil Engineering, Tishreen University, Lattakia, Syria*

[sabodiab@tishreen.edu.sy](mailto:sabodiab@tishreen.edu.sy)

**Abstracts:** In this paper, the Riemann integral and the fundamental of calculus will be used to perform double integrals on polygonal domain surrounded by closed curves. In this context, the double integral with two variables over the domain is transformed into sequences of single integrals with one variable of its primitive. The sequence is arranged anti clockwise starting from the minimum value of the variable of integration. Finally, the integration over the closed curve of the domain is performed using only one variable. The way of integration is illustrated by practical examples in which the area and moments of area are found for arbitrary polygons surrounded by closed, straight lines (triangular, quadrilateral, pentagonal, and hexagonal shapes) and compared with the exact values resulting from dividing the polygon into its standard elementary shapes and the parallel axis theorem. The stiffness matrix is derived for an arbitrary quadrilateral finite element for plate bending. The derived element is a generalization of the first finite element used in the analysis of thin plates known as ACM. The results are tested according to program code written in MATLAB. The method is generally applicable and is valid on a domain surrounded not only by straight edges but by closed curves continuous on the partial intervals of the integration domain. The generalization of this technique to volume integrals over polyhedral domains is possible.

**keywords:** Polygonal element, Riemann integral, moments of area, stiffness matrix

## 1 Introduction

The finite element methods among other numerical methods experienced significant developments after the use of the Voronoi Diagram in partitioning of a plane points into convex polygons [1]. Based on Voronoi Diagram, there were several mesh generator for polygonal and polyhedral elements with topology optimization. These offer a general framework for finite element discretization and analysis, see for example [2] and the mesh generators mentioned therein.

In the past fifteen years, many works have been published which operate on polygonal elements in the framework of numerical analysis and used intensively in the fields of applied engineering and physical sciences and even in medical and biological sciences [3, 4, 5, 6, 7, 8, 9, 10,]. An overview of previous developments on conforming polygonal and polyhedral finite elements is included in [9], and an overview on the use of different generalized barycentric coordinates in Galerkin finite element computations is included in [6]. Several other papers that use the polygonal and polyhedral elements in different fields of computational Engineering are listed in [8].

Therefore, it was necessary to use flexible techniques in performing the integrals of the state variables defined on the domains analyzed. These techniques become necessary when changes occur suddenly in the geometry of the domains such as the appearance of cracks or ruptures within them. Some recent publications [11, 12, 13 14, 15] show that the topic is still under study and development; some others [16, 17, 18] reflect its use and wide spreading under different disciplines.

Concurrently, new software languages have been developed that are highly capable of meeting the requirements of researchers in conducting symbolic and arithmetic operations in a built-in software environment, including but not limited to MATLAB, Python, Julia and Octave, etc...[19, 20, 21, 22, 23, 24]

By adopting the Riemann integral, the Green's formula or the Gauss divergence theorem in the calculation of integrals over complex domains by the use of open programming languages that use symbolic operations, young researchers are provided with powerful tools that can be used to address various types of physical and engineering problems.

Integration over a finite element of various shapes is an important part of every finite element code. The numerical integration consumes considerable part of the computational time. Therefore, developing explicit element matrices will reduce the computational costs considerably. One of the great advantages of using the Riemann integral is the possibility to develop explicit expressions of the element matrices at first symbolically and incorporating it after that in a numerical program code.

A study about the history of integration can be found in [25], [26]. Many classical examples exploiting the basic concept of the fundamental of calculus are presented in [27]. According to [28], the most employed technique of integration over polygonal and polyhedral elements is performed by subdividing them into standard-shaped elements and after that applying the corresponding integration rules on each sub-element and summation. [28], itself presents quadrature rules for the numerical approximation of integrals of polynomial functions over general polygonal/polyhedral elements without subdivision exploiting the 'Stokes' Theorem. The surface or volume integral over a polygon is evaluated by computing the integral of the same function over the boundary. Although the integration does not require an explicit construction of a sub-tessellation into standard-shapes, the numerical integration used is complex and a mapping procedure is employed in deriving the stiffness and mass matrix.

This paper demonstrates some worked examples of an integration over arbitrary polygonal domains enclosed by a sequence of edges. The integration is performed in the Cartesian coordinate system using only one variable. There is also no necessity to use any interpolation or to use any natural coordinate system. A mapping procedure is also not necessary. Once, the primitives of the functions to be integrated over the domains are known, the integration can be performed along the boundary between the limits in one direction using one variable. Furthermore, there is no need to use the Gauss divergence theorem, the 'Stokes' Theorem or to integrate in the edge direction. The method of integration is easy to use. It is given in a form such that it can be adopted directly within a computer program for numerical analysis. Areas and moments of areas of a triangle, an arbitrary quadrilateral and a polygon enclosed by six straight edges are calculated using the scheme presented and compared with the results, provided using the combination of these known values for standard shapes and using the parallel axes theorem. In addition, the integration of the stiffness matrix of a generalized version of the well-known ACM plate-bending element of quadrilateral shape is presented. The presented procedure is of general applicability for elements with curved edges and not limited to straight-sided edges in the framework of numerical methods. This work is devoted more to students and young researchers and therefore detailed calculations and program codes are given.

## 2 Integration in the Cartesian coordinate system

Let  $\Omega$  be a polygonal domain related to a Cartesian coordinate system  $(x, y)$  with the origin  $o$ , and the unit vectors  $(\vec{e}_x, \vec{e}_y)$ . Let  $\Omega$  be enclosed in the rectangle  $(x_{(1)} \leq x \leq x_{(2)}; y_{(a)} \leq y \leq y_{(b)})$  and be bounded by  $n$ -edges  $E_{(1)}, E_{(2)}, \dots, E_{(n)}$  described counterclockwise through the explicit sequence of equations

$$E_{(1)}: y = g_1(x) \tag{1.a}$$

$$E_{(2)}: y = g_2(x) \tag{1.b}$$

.....

$$E_{(n)}: y = g_n(x) \tag{1.n}$$

The edges are connected by a sequence of vertices <sup>(1), (2) (...), (n)</sup> (nodal points) with the nodal coordinates  $x_{(p)}$ , see Figure 1.

$$x_{(p)} = \begin{bmatrix} x_{(1)} & y_{(1)} \\ x_{(2)} & y_{(2)} \\ \cdot & \cdot \\ x_{(n+1)} & y_{(n+1)} \end{bmatrix} \tag{2}$$

Let  $p(x, y)$  an arbitrary point of the domain. The position vector of  $p$  reads

$$\vec{r} = x\vec{e}_x + y\vec{e}_y \tag{2}$$

The Cartesian variables of  $p(x, y)$  are connected through the Pythagorean Theorem:

$$r^2 = x^2 + y^2 \tag{3}$$

$$r = \pm \sqrt{x^2 + y^2}$$

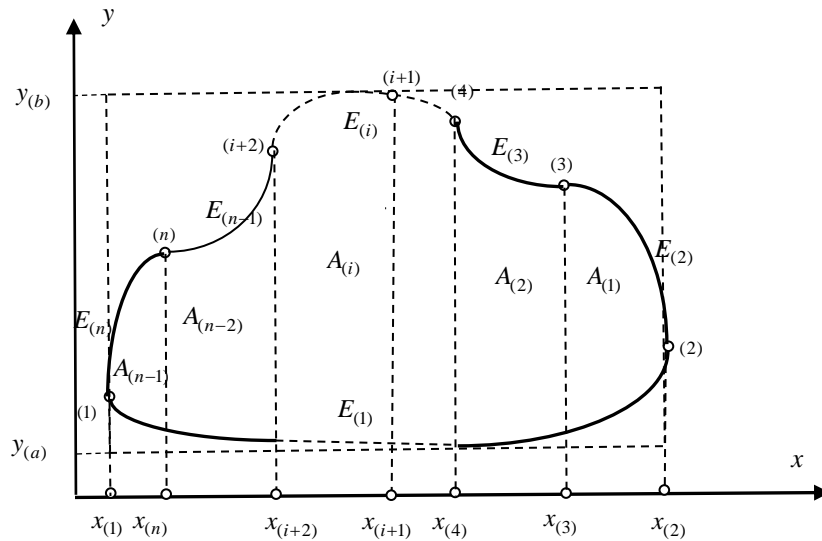


Figure 1: Polygonal domain, Cartesian coordinates of vertices, and curved edges

This fact should not be ignored when we change from a physical domain to a computational domain. For example, the Pythagorean Theorem still holds when changing  $x$  and  $y$  of the physical domain through the polar variables  $r$  and  $\varphi$  as a computational domain.

In Eqn. (3), there are only two independent variables. The variables  $x$  and  $y$  are variant in what concerns axes translation and rotation. The third one  $r$ , the distance between the point  $p$  and the origin  $o$ , is invariant and independent of the coordinate system used. Solving Eqn. (3) for example for  $y$  gives

$$y^2 = r^2 - x^2$$

$$y(r, x) = \pm \sqrt{r^2 - x^2}$$
(4)

Note that Eqn. (4) can be derived infinite number of times with respect to  $x$  under the condition  $r \neq \pm x$  which follow directly from  $y \neq 0$  in the first relation of Eqn. (4).

Let  $d\vec{A}$  the differential element of the area defined by

$$d\vec{A} = dx\vec{e}_x \times dy\vec{e}_y$$
(5)

Denote the scalar value of  $d\vec{A}$  by  $dA = dx dy$ , where  $dx, dy$  are the total differential of  $x, y$  respectively.

Suppose that we want to perform double-integrals such as  $\iint_A dx dy$  or  $\iint_A f(x, y) dx dy$  directly in the  $(x, y)$  system, where  $f(x, y)$  is some function defined on the domain.

The position vector  $\vec{r} = x\vec{e}_x + y\vec{e}_y$  is a sum of two vectors one has the direction of  $x$ ,  $x\vec{e}_x$  and the other has the direction of  $y$ ,  $y\vec{e}_y$ .

Assume that the total differential of  $y(r, x)$  and  $y'$  in Eqn. (4) exist, then the following relation applies

$$dy = y' dx$$
(6)

Now, if the definite integrals of the form  $\iint_A dx dy$  or  $\iint_A f(x, y) dx dy$  depends only on the initial and end-values of the variables  $x$  and  $y$ , then these can be performed using only one variable as states in every encyclopaedia for mathematics, see for example [29, 30]. Using for example  $x$  as a variable, the following integral over the total area of Fig. 1:

$$I = \iint_A f(x, y) dx dy$$
(7)

can be calculated as follows:

Suppose that  $f(x, y)$  is continuous and integrable over the subintervals  $[x_{(1)}, x_{(2)}], \dots, [x_{(n-1)}, x_{(n)}]$ .

The integral  $I_1$  over the subarea  $A_{(1)}$  enclosed between  $E_{(1)}$  and  $E_{(2)}$ , and the coordinate lines  $x = x_{(3)}$   $x = x_{(2)}$  reads

$$I_1 = \int_{x_{(3)}}^{x_{(2)}} \left( \int_{g_1(x)}^{g_2(x)} f(x, y) dy \right) dx = \int_{x_{(3)}}^{x_{(2)}} (F(x, y)|_{E_2} - F(x, y)|_{E_1}) dx = - \int_{x_{(2)}}^{x_{(3)}} F(x, y)|_{E_2} dx - \int_{x_{(3)}}^{x_{(2)}} F(x, y)|_{E_1} dx$$
(8)

The integral  $I_2$  over the subarea  $A_{(2)}$  enclosed between  $E_{(1)}$  and  $E_{(3)}$ , and the coordinate lines  $x = x_{(4)}$   $x = x_{(3)}$  reads

$$I_2 = \int_{x_{(4)}}^{x_{(3)}} \left( \int_{g_1(x)}^{g_3(x)} f(x, y) dy \right) dx = \int_{x_{(4)}}^{x_{(3)}} (F(x, y)|_{E_3} - F(x, y)|_{E_1}) dx = - \int_{x_{(3)}}^{x_{(4)}} F(x, y)|_{E_3} dx - \int_{x_{(4)}}^{x_{(3)}} F(x, y)|_{E_1} dx \quad (9)$$

The integral  $I_i$  over the subarea  $A_{(i)}$  enclosed between  $E_{(1)}$  and  $E_{(i)}$ , and the coordinate lines  $x = x_{(i+2)}$   $x = x_{(i+1)}$  reads

$$I_i = \int_{x_{(i+2)}}^{x_{(i+1)}} \left( \int_{g_1(x)}^{g_{i+2}(x)} f(x, y) dy \right) dx = \int_{x_{(i+2)}}^{x_{(i+1)}} (F(x, y)|_{E_i} - F(x, y)|_{E_1}) dx = - \int_{x_{(i+1)}}^{x_{(i+2)}} F(x, y)|_{E_i} dx - \int_{x_{(i+2)}}^{x_{(i+1)}} F(x, y)|_{E_1} dx \quad (10)$$

And so on. For the Last part of the area  $A_{(n)}$  enclosed between  $E_{(1)}$  and  $E_{(n)}$ , and the coordinate lines  $x = x_{(n)}$   $x = x_{(1)}$

$$I_{(n-1)} = \int_{x_{(1)}}^{x_{(n)}} \left( \int_{g_1(x)}^{g_n(x)} f(x, y) dy \right) dx = \int_{x_{(1)}}^{x_{(n)}} (F(x, y)|_{E_n} - F(x, y)|_{E_1}) dx = - \int_{x_{(n)}}^{x_{(1)}} F(x, y)|_{E_n} dx - \int_{x_{(1)}}^{x_{(n)}} F(x, y)|_{E_1} dx \quad (11)$$

where  $F(x, y)$  denote the  $y$ -primitive of  $f(x, y)$ . The indefinite integral or anti-derivative of  $f(x, y)$  with respect to  $y$  is as follows:

$$\int_y f(x, y) dy = F(x, y) + \varphi(x) \quad (12)$$

In other words,  $f(x, y)$  is the derivative of  $F(x, y)$  with respect to  $y$ .  $\varphi(x)$  is an arbitrary function independent of  $y$ . selecting the integration of  $\varphi(x)$  along the closed interval as zero function and observing that

$$\int_{x_{(1)}}^{x_{(2)}} F(x, y)|_{E_1} dx = - \int_{x_{(2)}}^{x_{(3)}} F(x, y)|_{E_1} dx - \int_{x_{(3)}}^{x_{(4)}} F(x, y)|_{E_1} dx - \dots - \int_{x_{(i+2)}}^{x_{(i+1)}} F(x, y)|_{E_1} dx - \int_{x_{(1)}}^{x_{(n)}} F(x, y)|_{E_1} dx \quad (13)$$

Then, the definite integral can be performed with one variable with respect to  $x$  as a series of integrals over the subintervals counter clockwise using the following relation

$$\iint_A f(x, y) dx dy = - \sum_{i=1}^n \int_{x_{(i)}}^{x_{(i+1)}} F(x, y)|_{E_i} dx \quad (14)$$

$F(x, y)|_{E_i}$  denotes the function  $F(x, y)$  in which  $y$  is replaced by the explicit edge equation  $E_i : y = g_i(x)$  corresponding to Eqns. (1). This means, the edge equation is solved with respect to  $y$  and the expression  $F(x, y)|_{E_i}$  involves terms of  $x$  variables only.

A similar relation applies for the integration in the  $y$ -direction but now with a positive sign.

$$\iint_A f(x, y) dx dy = \sum_{i=1}^n \int_{E_i} F(x, y)|_{E_i} dy \quad (15)$$

Now, let us explain how to obtain this result. For the sake of simplicity, consider the arbitrary polygon depicted in Fig. 2. In case of a domain with straight edges described by their nodal points, the relation between  $x$  and  $y$  becomes linear along the edge and takes the form  $ax + by + c = 0$ . The constants  $a, b, c$  can be uniquely determined depending on the nodal coordinates spanning the edge and the intersection point between the edge-line and the  $x$ -axes.

For a quadrilateral domain for example, the domain is bounded by four edges  $E_{(1)}, E_{(2)}, E_{(3)}, E_{(4)}$ . Every edge-equation can be determined by the two corresponding vertices spanning the edge. The resulting edge-equations are as follows

$$E_{(1)} : a_1x + b_1y + c_1 = 0 \quad (16.a)$$

$$E_{(2)} : a_2x + b_2y + c_2 = 0 \quad (16.b)$$

$$E_{(3)} : a_3x + b_3y + c_3 = 0 \quad (16.c)$$

$$E_{(4)} : a_4x + b_4y + c_4 = 0 \quad (16.d)$$

In the following, integrals for calculating geometrical properties like area and moments of area of a triangle, of an arbitrary quadrilateral and of a polygon enclosed by six straight edges will be demonstrated in details.

### 3 Integration over a triangular element using one variable

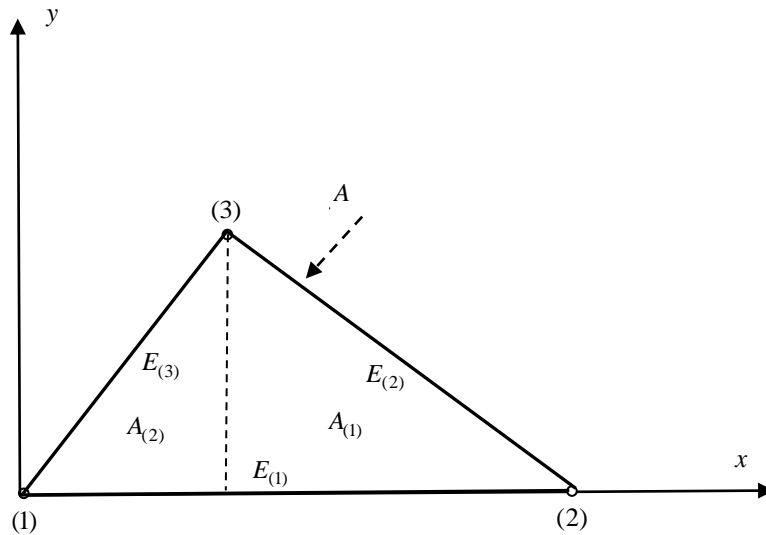


Figure 2: Triangular element, Cartesian coordinates of vertices, and edges

**Example 1:** A triangular domain is defined by its four vertices  $(1), (2), (3)$  (nodal points)

$$\begin{matrix} \text{(p)} \\ \downarrow \\ \tilde{x}_{(p)} \end{matrix} \begin{matrix} \xrightarrow{\tilde{i}} \\ \left[ \begin{array}{cc} 0.0 & 0.0 \\ 5.0 & 0.0 \\ 1.8 & 2.4 \end{array} \right] \end{matrix}$$

The exact values of the area of the triangular element and the moments of area about the axes  $(x, y)$ , computed by the usual way of integration about the axes of a standard shape (here, triangle) as well as the parallel axes theorem [31], are

$$A = \iint_A dA = \frac{(3)(4)}{2} = 6$$

$$I_x = \iint_A y^2 dA = \frac{5(2.4)^3}{12} = 5.76$$

$$I_y = \iint_A x^2 dA = \frac{(2.4)(1.8)^3}{36} + \frac{(2.4)(1.8)}{2} \left( \frac{2}{3}(1.8) \right)^2 + \frac{(2.4)(3.2)^3}{36} + \left( \frac{(2.4)(3.2)}{2} \right) \left( 1.8 + \frac{1}{3}(3.2) \right)^2 = 5.76$$

$$I_{xy} = \iint_A xy dA = \frac{(2.4)^2(1.8)^2}{72} + \frac{(2.4)(1.8)}{2} \left( \frac{2}{3}(1.8) \right) \frac{1}{3}(2.4) - \frac{(2.4)^2(3.2)^2}{72} + \left( \frac{(2.4)(3.2)}{2} \right) \left( 1.8 + \frac{1}{3}(3.2) \right) \frac{1}{3}(2.4) = 10.32$$

Now, the same results can be obtained using the above stated method of integration.

The triangular element is bounded by three edges  $E_1, E_2, E_3$ . Every edge-equation can be determined by the two corresponding vertices spanning the edge. The resulting edge-equations solved with respect to  $y$  are as follows

$$E_1: y = 0$$

$$E_2: y = -\frac{3}{4}x + \frac{15}{4}$$

$$E_3: y = \frac{4}{3}x$$

Dividing the triangle in two subareas  $A_{(1)}, A_{(2)}$  and observing the limits of the integral, the area and the moments of area about  $x$ -axes take the following form:

The area  $A$  of the triangle calculated as a sum of two subareas, the subarea  $A_{(1)}$  enclosed between the two edges  $E_{(1)}$  and  $E_{(2)}$  plus the subarea  $A_{(2)}$  enclosed between the two edges  $E_{(1)}$  and  $E_{(3)}$

$$A = \iint_A dA = \int_0^{1.8E_3} \int_{E_1}^{E_2} (dy) dx + \int_{1.8E_1}^5 \int_{E_1}^{E_2} (dy) dx$$

$$A = -\int_0^{1.8} (y)|_{E_1} dx + \int_0^{1.8} (y)|_{E_3} dx - \int_{1.8}^5 (y)|_{E_1} dx + \int_{1.8}^5 (y)|_{E_2} dx = 6$$

$$A = -\int_{1.8}^5 (0) dx + \int_{1.8}^5 \left( -\frac{3}{4}x + \frac{15}{4} \right) dx - \int_0^{1.8} (0) dx + \int_0^{1.8} \left( \frac{4}{3}x \right) dx = 6$$



The previous integral arranged counter clockwise is then as follows

$$A = -\int_0^5 (0) dx - \int_5^{1.8} \left(-\frac{3}{4}x + \frac{15}{4}\right) dx - \int_{1.8}^0 \left(\frac{4}{3}x\right) dx = 6$$

The moments of area  $I_x$  calculated as the moment of area of the subarea  $A_{(1)}$  enclosed between the two edges  $E_{(1)}$  and  $E_{(2)}$  plus the moment of area of the subarea  $A_{(2)}$  enclosed between the two edges  $E_{(1)}$  and  $E_{(3)}$

$$I_x = \iint_A y^2 dA = \int_0^{1.8} \int_{E_1}^{E_3} (y^2 dy) dx + \int_{1.8}^5 \int_{E_1}^{E_2} (y^2 dy) dx$$

$$I_x = -\int_0^{1.8} \frac{(y^3)|_{E_1}}{3} dx + \int_0^{1.8} \frac{(y^3)|_{E_3}}{3} dx - \int_{1.8}^5 \frac{(y^3)|_{E_1}}{3} dx + \int_{1.8}^5 \frac{(y^3)|_{E_2}}{3} dx = 5.76$$

The previous integral arranged counter clockwise is then as follows

$$I_x = -\int_0^5 \frac{(0)^3}{3} dx - \int_5^{1.8} \frac{\left(-\frac{3}{4}x + \frac{15}{4}\right)^3}{3} dx - \int_{1.8}^0 \frac{\left(\frac{4}{3}x\right)^3}{3} dx = 5.76$$

The moments of area  $I_y$  calculated as the moment of area of the subarea  $A_{(1)}$  enclosed between the two edges  $E_{(1)}$  and  $E_{(2)}$  plus the moment of area of the subarea  $A_{(2)}$  enclosed between the two edges  $E_{(1)}$  and  $E_{(3)}$

$$I_y = \int_0^{1.8} x^2(y)|_{E_1} dx + \int_0^{1.8} x^2(y)|_{E_3} dx - \int_{1.8}^5 x^2(y)|_{E_1} dx + \int_{1.8}^5 x^2(y)|_{E_2} dx$$

$$= -\int_0^{1.8} x^2(0) dx + \int_0^{1.8} x^2\left(\frac{4}{3}x\right) dx - \int_{1.8}^5 x^2(0) dx + \int_{1.8}^5 x^2\left(-\frac{3}{4}x + \frac{15}{4}\right) dx = 37.24$$

The previous integral arranged counter clockwise is then as follows

$$I_y = -\int_0^5 x^2(0) dx - \int_5^{1.8} x^2\left(-\frac{3}{4}x + \frac{15}{4}\right) dx - \int_{1.8}^0 x^2\left(\frac{4}{3}x\right) dx = 37.24$$

And finally, the moment of area  $I_{xy}$

$$I_{xy} = -\int_0^{1.8} \frac{x(y^2)|_{E_1}}{2} dx + \int_0^{1.8} \frac{x(y^2)|_{E_3}}{2} dx - \int_{1.8}^5 \frac{x(y^2)|_{E_1}}{2} dx + \int_{1.8}^5 \frac{x(y^2)|_{E_2}}{2} dx$$

$$= -\int_0^{1.8} \frac{x(0)^2}{2} dx + \int_0^{1.8} \frac{x\left(\frac{4}{3}x\right)^2}{2} dx - \int_{1.8}^5 \frac{x(0)^2}{2} dx + \int_{1.8}^5 \frac{x\left(-\frac{3}{4}x + \frac{15}{4}\right)^2}{2} dx = 10.32$$

Which gives arranged counterclockwise

$$I_{xy} = -\int_0^5 \frac{x(0)^2}{2} dx - \int_5^{1.8} \frac{x(-\frac{3}{4}x + \frac{15}{4})^2}{2} dx - \int_{1.8}^0 \frac{x(\frac{4}{3}x)^2}{2} dx = 10.32$$

#### 4 Integration over the quadrilateral element using one variable

**Example 2:** A quadrilateral domain is defined by its four vertices  $(1), (2), (3), (4)$  (nodal points).

$$x_{\tilde{i}(p)} = \begin{bmatrix} 1. & 1. \\ 8. & 0. \\ 6. & 4. \\ 2. & 5. \end{bmatrix}$$

The exact values of the area of the quadrilateral cross section and the moments of area about the axes  $(x, y)$  can be computed easily as difference between these values of a rectangle  $A_{(1)}$  with the side length  $7 \times 5$  and the summation of them for four rectangular triangles  $A_{(2)}, A_{(3)}, A_{(5)}, A_{(6)}$  with the catheti  $7 \times 1, 4 \times 2, 4 \times 1$  and  $4 \times 1$  as well as of a rectangle  $A_{(4)}$  with the side length  $2 \times 1$ , Fig. 3 ([31])

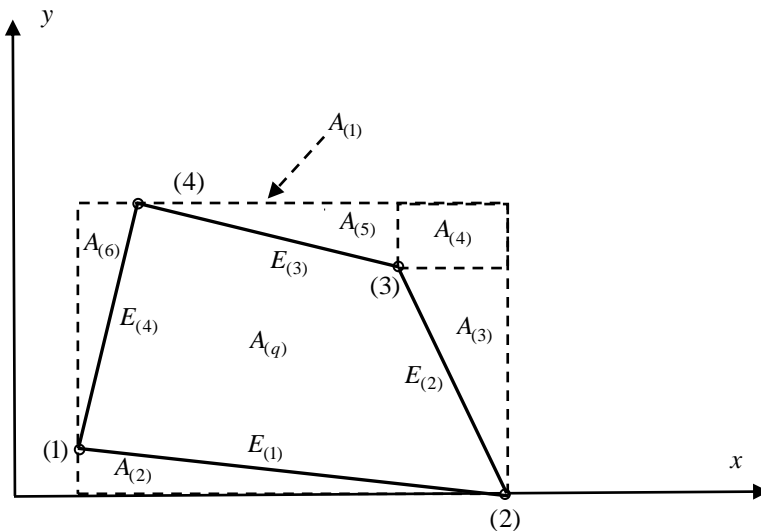


Figure 3: Quadrilateral cross-section, Cartesian coordinates of vertices, and edges

The calculation using the values about the axes of the standard shapes and the parallel axes theorem gives:

$$\begin{aligned} A &= A_{(1)} - (A_{(2)} + A_{(3)} + A_{(4)} + A_{(5)} + A_{(6)}) \\ &= (7)(5) - \left( \frac{(7)(1)}{2} + \frac{(4)(2)}{2} + (2)(1) + \frac{(4)(1)}{2} + \frac{(4)(1)}{2} \right) = 21.5 \end{aligned}$$

$$\begin{aligned}
I_x &= \iint_A y^2 dA = \iint_A y^2 dx dy = \\
&= \iint_{A(1)} y^2 dA - \left\{ \iint_{A(2)} y^2 dA + \iint_{A(3)} y^2 dA + \iint_{A(4)} y^2 dA + \iint_{A(5)} y^2 dA + \iint_{A(6)} y^2 dA \right\} \\
&= \frac{(7)(5)^3}{12} + (7)(5)\left(\frac{5}{2}\right)^2 - \left\{ \left(-\frac{(7)(1)^3}{36} + (7)(1)\left(\frac{1}{3}\right)^2\right) + \frac{(2)(4)^3}{36} + \left(\frac{(2)(4)}{2}\right)\left(\frac{2}{3}(4)\right)^2 - \frac{(2)(1)^3}{12} + (2)(1)\left(4 + \frac{1}{2}\right)^2 \right. \\
&+ \left. \frac{(4)(1)^3}{36} + \left(\frac{(4)(1)}{2}\right)\left(4 + \frac{2}{3}\right)^2 + \frac{(1)(4)^3}{36} + \left(\frac{(1)(4)}{2}\right)\left(1 + \frac{2}{3}(4)\right)^2 \right\} = 146.0833333
\end{aligned}$$

$$\begin{aligned}
I_y &= \iint_A x^2 dA = \iint_A x^2 dx dy \\
&= \iint_{A(1)} x^2 dA - \left\{ \iint_{A(2)} x^2 dA + \iint_{A(3)} x^2 dA + \iint_{A(4)} x^2 dA + \iint_{A(5)} x^2 dA + \iint_{A(6)} x^2 dA \right\} \\
&= \frac{(5)(7)^3}{12} + (5)(7)\left(1 + \frac{7}{2}\right)^2 - \left\{ \left(-\frac{(1)(7)^3}{36} + \frac{(1)(7)}{2}\left(\left(1 + \frac{7}{3}\right)^2\right) + \frac{(4)(2)^3}{36} + \left(\frac{(4)(2)}{2}\right)\left(6 + \frac{2}{3}(2)\right)^2 - \frac{(1)(2)^3}{12} + (1)(2)(6+1)^2 \right. \right. \\
&+ \left. \left. \frac{(1)(4)^3}{36} + \left(\frac{(1)(4)}{2}\right)\left(2 + \frac{2}{3}(4)\right)^2 + \frac{(4)(1)^3}{36} + \left(\frac{(4)(1)}{2}\right)\left(1 + \frac{1}{3}\right)^2 \right\} = 439.5833333
\end{aligned}$$

$$\begin{aligned}
I_{xy} &= \iint_A xy dA = \iint_A xy dx dy \\
&= \iint_{A(1)} xy dA - \left\{ \iint_{A(2)} xy dA + \iint_{A(3)} xy dA + \iint_{A(4)} xy dA + \iint_{A(5)} xy dA + \iint_{A(6)} xy dA \right\} \\
&= 0 + (5)(7)\left(\frac{5}{2}\right)\left(1 + \frac{7}{2}\right) - \left\{ \left(-\frac{(1)^2(7)^2}{72} + \frac{(1)(7)}{2}\left(\left(1 + \frac{7}{3}\right)\left(\frac{1}{3}\right)\right) - \frac{(4^2)(2^3)}{72} + \left(\frac{(4)(2)}{2}\right)\left(6 + \frac{2}{3}(2)\right)\left(\frac{2}{3}(4) - 0 + (1)(2)(6+1)\left(4 + \frac{1}{2}\right)\right) \right. \right. \\
&+ \left. \left. \frac{(1^2)(4^2)}{72} + \left(\frac{(1)(4)}{2}\right)\left(2 + \frac{2}{3}(4)\right)\left(4 + \frac{2}{3}\right) - \frac{(4^2)(1^2)}{72} + \left(\frac{(4)(1)}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{2}{3}(4)\right) \right\} = 196.875000
\end{aligned}$$

The quadrilateral domain is bounded by four edges  $E_1, E_2, E_3, E_4$ . Every edge-equation can be determined by the two corresponding vertices spanning the edge. The resulting edge-equations solved with respect to  $y$  are as follows

$$E1: y = -\frac{1}{7}x + 8$$

$$E2: y = -2x + 16$$

$$E3: y = \frac{1}{4}x + \frac{22}{4}$$

$$E4: y = 4x - 3$$

Observing the limits of the integral, the moments of area about  $x$ -axes take the form

$$\begin{aligned}
I_x &= -\int_1^2 \frac{(y^3)|_{E1}}{3} dx + \int_1^2 \frac{(y^3)|_{E4}}{3} dx - \int_2^6 \frac{(y^3)|_{E1}}{3} dx + \int_2^6 \frac{(y^3)|_{E3}}{3} dx - \int_6^8 \frac{(y^3)|_{E1}}{3} dx + \int_8^6 \frac{(y^3)|_{E2}}{3} dx \\
&= -\int_1^2 \frac{(-x/7+8)^3}{3} dx + \int_1^2 \frac{(4x-3)^3}{3} dx - \int_2^6 \frac{(-x/7+8)^3}{3} dx + \int_2^6 \frac{(-x/4+22/4)^3}{3} dx \\
&\quad - \int_6^8 \frac{(-x/7+8)^3}{3} dx + \int_6^8 \frac{(-2x+16)^3}{3} dx = 146.08333333
\end{aligned}$$

Or calculated counter clockwise

$$I_x = -\int_1^8 \frac{(-x/7+8)^3}{3} dx - \int_8^6 \frac{(-2x+16)^3}{3} dx - \int_6^2 \frac{(-x/4+22/4)^3}{3} dx - \int_2^1 \frac{(4x-3)^3}{3} dx = 146.08333333$$

$$\begin{aligned}
I_y &= -\int_1^2 x^2(y)|_{E1} dx + \int_1^2 x^2(y)|_{E4} dx - \int_2^6 x^2(y)|_{E1} dx + \int_2^6 x^2(y)|_{E3} dx - \int_6^8 x^2(y)|_{E1} dx + \int_8^6 x^2(y)|_{E2} dx \\
&= -\int_1^2 x^2(-x/7+8) dx + \int_1^2 x^2(4x-3) dx - \int_2^6 x^2(-x/7+8) dx + \int_2^6 x^2(-x/4+22/4) dx \\
&\quad - \int_6^8 x^2(-x/7+8) dx + \int_6^8 x^2(-2x+16) dx = 439.58333333
\end{aligned}$$

$$I_y = -\int_1^8 x^2(-x/7+8) dx - \int_8^6 x^2(-2x+16) dx - \int_6^2 x^2(-x/4+22/4) dx - \int_2^1 x^2(4x-3) dx = 439.58333333$$

$$\begin{aligned}
I_{xy} &= -\int_1^2 \frac{x(y^2)|_{E1}}{2} dx + \int_1^2 \frac{x(y^2)|_{E4}}{2} dx - \int_2^6 \frac{x(y^2)|_{E1}}{2} dx + \int_2^6 \frac{x(y^2)|_{E3}}{2} dx - \int_6^8 \frac{x(y^2)|_{E1}}{2} dx + \int_8^6 \frac{x(y^2)|_{E2}}{2} dx \\
&= -\int_1^2 \frac{x(-x/7+8)^2}{2} dx + \int_1^2 \frac{x(4x-3)^2}{2} dx - \int_2^6 \frac{x(-x/7+8)^2}{2} dx + \int_2^6 \frac{x(-x/4+22/4)^2}{2} dx \\
&\quad - \int_6^8 \frac{x(-x/7+8)^2}{2} dx + \int_6^8 \frac{x(-2x+16)^2}{2} dx = 196.875
\end{aligned}$$

$$I_{xy} = -\int_1^8 \frac{x(-x/7+8)^2}{2} dx - \int_8^6 \frac{x(-2x+16)^2}{2} dx - \int_6^2 \frac{x(-x/4+22/4)^2}{2} dx - \int_2^1 \frac{x(4x-3)^2}{2} dx = 196.875$$

## 5 Integration over a polygonal element using one variable

**Example 3:** A polygonal domain is defined by its six vertices <sup>(1)</sup>, <sup>(2)</sup>, <sup>(3)</sup>, <sup>(4)</sup>, <sup>(5)</sup>, <sup>(6)</sup> (nodal points)

$$x_{(p)}^{\tilde{i}} = \begin{matrix} \xrightarrow{\tilde{i}} \\ \downarrow (p) \\ \begin{bmatrix} 0.0 & 2.0 \\ 1.0 & 0.0 \\ 4.0 & 0.0 \\ 8.0 & 3.0 \\ 6.0 & 6.0 \\ 3.0 & 6.0 \end{bmatrix} \end{matrix}$$

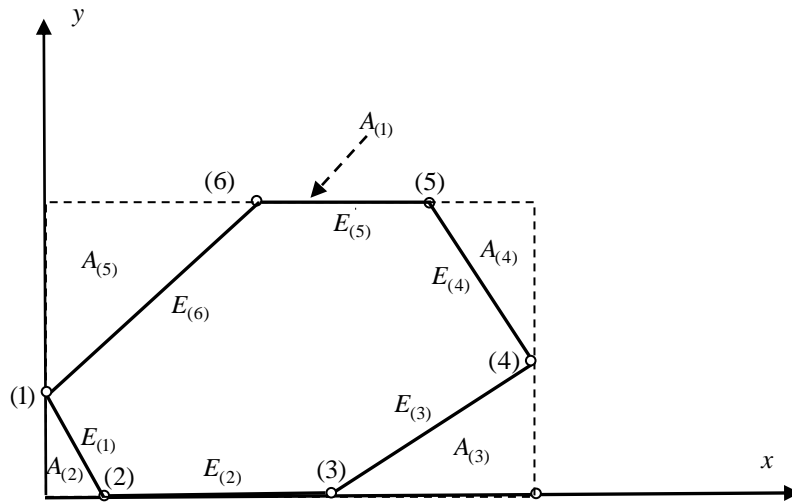


Figure 4: Quadrilateral cross-section, Cartesian coordinates of vertices, and edges

The exact values of the area of the polygonal cross section and the moments of area about the axes  $(x, y)$  can be computed easily as difference between these values of a rectangle  $A_{(1)}$  with the side length  $8 \times 6$  and the summation of them for four rectangled triangles  $A_{(2)}, A_{(3)}, A_{(4)}, A_{(5)}$  with the catheti  $1 \times 2$ ,  $4 \times 3$ ,  $2 \times 3$  and  $3 \times 4$ , Fig. 4 ([31])

$$\begin{aligned} A &= A_{(1)} - (A_{(2)} + A_{(3)} + A_{(4)} + A_{(5)}) \\ &= (8)(6) - \left( \frac{(1)(2)}{2} + \frac{(4)(3)}{2} + \frac{(2)(3)}{2} + \frac{(3)(4)}{2} \right) = 32 \end{aligned}$$

$$\begin{aligned} I_x &= \iint_A y^2 dA = \iint_A y^2 dx dy = \\ &= \iint_{A_{(1)}} y^2 dA - \left\{ \iint_{A_{(2)}} y^2 dA + \iint_{A_{(3)}} y^2 dA + \iint_{A_{(4)}} y^2 dA + \iint_{A_{(5)}} y^2 dA \right\} \\ &= \frac{(8)(6)^3}{12} + (8)(6) \left( \frac{6}{2} \right)^2 - \left\{ \left( \frac{(1)(2)^3}{36} + \frac{(1)(2)}{2} \left( \frac{2}{3} \right)^2 \right) + \frac{(4)(3)^3}{36} + \frac{(4)(3)}{2} \left( \frac{1}{3}(3) \right)^2 - \frac{(2)(3)^3}{36} + \frac{(2)(3)}{2} \left( 3 + \frac{2}{3}(3) \right)^2 \right. \\ &\quad \left. + \frac{(3)(4)^3}{36} + \left( \frac{(3)(4)}{2} \right) \left( 2 + \frac{2}{3}(4) \right)^2 \right\} = 353.833333 \end{aligned}$$

$$\begin{aligned}
I_y &= \iint_A x^2 dA = \iint_A x^2 dx dy \\
&= \iint_{A_{(1)}} x^2 dA - \left\{ \iint_{A_{(2)}} x^2 dA + \iint_{A_{(3)}} x^2 dA + \iint_{A_{(4)}} x^2 dA + \iint_{A_{(5)}} x^2 dA \right\} \\
&= \frac{(6)(8)^3}{12} + (6)(8)\left(\frac{8}{2}\right)^2 - \left\{ \left(\frac{(2)(1)^3}{36} + \frac{(1)(2)}{2} \left(\frac{1}{3}\right)^2 + \frac{(4)(3)^3}{36} + \left(\frac{(4)(3)}{2}\right)\left(4 + \frac{8}{3}\right)^2 + \frac{(2)(3)^3}{36} + \frac{(2)(3)}{2} \left(6 + \frac{4}{3}\right)^2 \right. \right. \\
&\quad \left. \left. + \frac{(4)(3)^3}{36} + \left(\frac{(4)(3)}{2}\right)\left(\frac{1}{3}(3)\right)^2 \right\} = 580.833333
\end{aligned}$$

$$\begin{aligned}
I_{xy} &= \iint_A xy dA = \iint_A xy dx dy \\
&= \iint_{A_{(1)}} xy dA - \left\{ \iint_{A_{(2)}} xy dA + \iint_{A_{(3)}} xy dA + \iint_{A_{(4)}} xy dA + \iint_{A_{(5)}} xy dA \right\} \\
&= 0 + (8)(6)\left(\frac{8}{2}\right)\left(\frac{6}{2}\right) - \left\{ \left(\frac{(1)^2(2)^2}{72} + \frac{(1)(2)}{2} \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \frac{(4^2)(3^2)}{72} + \left(\frac{(4)(2)}{2}\right)\left(4 + \frac{2}{3}(4)\right)\left(\frac{1}{3}(3)\right) \right. \right. \\
&\quad \left. \left. - \frac{(2^2)(3^2)}{72} + \left(\frac{(2)(3)}{2}\right)\left(6 + \frac{2}{3}(2)\right)\left(3 + \frac{2}{3}(3)\right) + \frac{(3^2)(4^2)}{72} + \left(\frac{(3)(4)}{2}\right)\left(\frac{1}{3}(3)\right)\left(2 + \frac{2}{3}(4)\right) \right\} = I_x = 394.333333
\end{aligned}$$

The polygonal domain is bounded by six edges  $E_1, E_2, E_3, E_4, E_5, E_6$ . Every edge-equation can be determined by the two corresponding vertices spanning the edge. The resulting edge-equations solved with respect to  $y$  are as follows

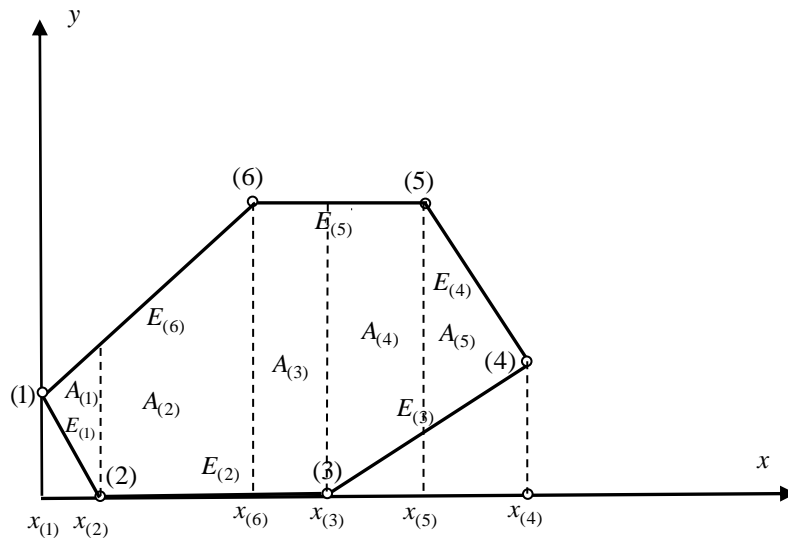


Figure 5: Quadrilateral cross-section, Cartesian coordinates of vertices, and edges

$$E_1: y = -2x + 2$$

$$E_2: y = 0$$

$$E_3: y = \frac{3}{4}x - 3$$

$$E_4: y = -\frac{3}{2}x + 15$$

$$E_5: y = 6$$

$$E_6: y = \frac{4}{3}x + 2$$

Moving in  $x$ -axes positive direction, the area of the polygon can be integrated after dividing the area  $A$  of the polygon in five subareas  $A_{(1)}, A_{(2)}, A_{(3)}, A_{(4)}, A_{(5)}$  corresponding to the discontinuity points of the geometry as follows, see Fig. 5

$$\begin{aligned}
 A &= \iint_A dx dy = \iint_{A_{(1)}} dx dy + \iint_{A_{(2)}} dx dy + \iint_{A_{(3)}} dx dy + \iint_{A_{(4)}} dx dy + \iint_{A_{(5)}} dx dy \\
 &= \int_0^1 \left( \int_{E_1}^{E_6} dy \right) dx + \int_1^3 \left( \int_{E_2}^{E_6} dy \right) dx + \int_3^4 \left( \int_{E_2}^{E_5} dy \right) dx + \int_4^6 \left( \int_{E_2}^{E_5} dy \right) dx + \int_6^8 \left( \int_{E_3}^{E_5} dy \right) dx \\
 &= \int_0^1 (y)|_{E_6} dx - \int_0^1 (y)|_{E_1} dx + \int_1^3 (y)|_{E_6} dx - \int_1^3 (y)|_{E_2} dx + \int_3^4 (y)|_{E_5} dx - \int_3^4 (y)|_{E_2} dx \\
 &\quad + \int_4^6 (y)|_{E_5} dx - \int_4^6 (y)|_{E_3} dx + \int_6^8 (y)|_{E_4} dx - \int_6^8 (y)|_{E_3} dx \\
 &= -\int_0^1 (y)|_{E_1} dx - \int_1^4 (y)|_{E_2} dx - \int_4^8 (y)|_{E_3} dx - \int_8^6 (y)|_{E_4} dx - \int_6^3 (y)|_{E_5} dx - \int_3^0 (y)|_{E_6} dx \\
 &= -\int_0^1 (-2x+2) dx - \int_1^4 (0) dx - \int_4^8 \left(\frac{3}{4}x-3\right) dx - \int_8^6 \left(-\frac{3}{2}x+15\right) dx - \int_6^3 (6) dx - \int_3^0 \left(\frac{4}{3}x+2\right) dx = 32
 \end{aligned}$$

In an analogous way, the moments of area about  $x$ -axes take the form

$$\begin{aligned}
 I_x &= \iint_A y^2 dx dy = \iint_{A_{(1)}} y^2 dx dy + \iint_{A_{(2)}} y^2 dx dy + \iint_{A_{(3)}} y^2 dx dy + \iint_{A_{(4)}} y^2 dx dy + \iint_{A_{(5)}} y^2 dx dy \\
 &= \int_0^1 \left( \int_{E_1}^{E_6} y^2 dy \right) dx + \int_1^3 \left( \int_{E_2}^{E_6} y^2 dy \right) dx + \int_3^4 \left( \int_{E_2}^{E_5} y^2 dy \right) dx + \int_4^6 \left( \int_{E_2}^{E_5} y^2 dy \right) dx + \int_6^8 \left( \int_{E_3}^{E_5} y^2 dy \right) dx \\
 &= \int_0^1 \frac{(y^3)|_{E_6}}{3} dx - \int_0^1 \frac{(y^3)|_{E_1}}{3} dx + \int_1^3 \frac{(y^3)|_{E_6}}{3} dx - \int_1^3 \frac{(y^3)|_{E_2}}{3} dx + \int_3^4 \frac{(y^3)|_{E_5}}{3} dx - \int_3^4 \frac{(y^3)|_{E_2}}{3} dx \\
 &\quad + \int_4^6 \frac{(y^3)|_{E_5}}{3} dx - \int_4^6 \frac{(y^3)|_{E_3}}{3} dx + \int_6^8 \frac{(y^3)|_{E_4}}{3} dx - \int_6^8 \frac{(y^3)|_{E_3}}{3} dx
 \end{aligned}$$

Rearranging the last integral counter clockwise gives:

$$\begin{aligned}
I_x &= -\int_0^1 \frac{(y^3)|_{E_6}}{3} dx - \int_1^4 \frac{(y^3)|_{E_1}}{3} dx - \int_4^8 \frac{(y^3)|_{E_2}}{3} dx - \int_8^6 \frac{(y^3)|_{E_3}}{3} dx - \int_6^3 \frac{(y^3)|_{E_4}}{3} dx - \int_3^0 \frac{(y^3)|_{E_5}}{3} dx \\
&= -\int_0^1 \frac{(-2x+2)^3}{3} dx - \int_1^4 \frac{(0)^3}{3} dx + \int_4^8 \frac{(3x/4-3)^3}{3} dx - \int_8^6 \frac{(-3x/2+15)^3}{3} dx - \int_6^3 \frac{(6)^3}{3} dx \\
&\quad - \int_3^0 \frac{(4x/3+2)^3}{3} dx = 353.83333
\end{aligned}$$

The moment of area about  $y$ -axes can be written as follows:

$$\begin{aligned}
I_y &= \iint_A x^2 dx dy = \iint_{A(1)} x^2 dx dy + \iint_{A(2)} x^2 dx dy + \iint_{A(3)} x^2 dx dy + \iint_{A(4)} x^2 dx dy + \iint_{A(5)} x^2 dx dy \\
&= \int_0^1 x^2 \left( \int_{E_1}^{E_6} dy \right) dx + \int_1^3 x^2 \left( \int_{E_2}^{E_6} dy \right) dx + \int_3^4 x^2 \left( \int_{E_2}^{E_5} dy \right) dx + \int_4^6 x^2 \left( \int_{E_3}^{E_5} dy \right) dx + \int_6^8 x^2 \left( \int_{E_3}^{E_4} dy \right) dx \\
&= \int_0^1 x^2(y)|_{E_6} dx - \int_0^1 x^2(y)|_{E_1} dx + \int_1^3 x^2(y)|_{E_6} dx - \int_1^3 x^2(y)|_{E_2} dx + \int_3^4 x^2(y)|_{E_5} dx - \int_3^4 x^2(y)|_{E_2} dx \\
&\quad + \int_4^6 x^2(y)|_{E_5} dx - \int_4^6 x^2(y)|_{E_3} dx + \int_6^8 x^2(y)|_{E_4} x^2 dx - \int_6^8 x^2(y)|_{E_3} dx
\end{aligned}$$

Arranging this integral counter clockwise gives:

$$\begin{aligned}
I_y &= -\int_0^1 x^2(y)|_{E_1} dx - \int_1^4 x^2(y)|_{E_2} dx - \int_4^8 x^2(y)|_{E_3} dx - \int_8^6 x^2(y)|_{E_4} dx - \int_6^3 x^2(y)|_{E_5} dx - \int_3^0 x^2(y)|_{E_6} dx \\
&= -\int_0^1 x^2(-2x+2) dx - \int_1^4 x^2(0) dx - \int_4^8 x^2(3x/4-3) dx - \int_8^6 x^2(-3x/2+15) dx - \int_6^3 x^2(6) dx \\
&\quad - \int_3^0 x^2(4x/3+2) dx = 580.833333
\end{aligned}$$

Finally, the following relation represents the moment of area  $I_{xy}$

$$\begin{aligned}
I_{xy} &= \iint_A xy dx dy = \iint_{A(1)} xy dx dy + \iint_{A(2)} xy dx dy + \iint_{A(3)} xy dx dy + \iint_{A(4)} xy dx dy + \iint_{A(5)} xy dx dy \\
&= \int_0^1 xy \left( \int_{E_1}^{E_6} y dy \right) dx + \int_1^3 x \left( \int_{E_2}^{E_6} y dy \right) dx + \int_3^4 x \left( \int_{E_3}^{E_5} y dy \right) dx + \int_4^6 x \left( \int_{E_3}^{E_5} y dy \right) dx + \int_6^8 x \left( \int_{E_3}^{E_4} y dy \right) dx \\
&= \int_0^1 \frac{x(y^2)|_{E_6}}{2} dx - \int_0^1 \frac{x(y^2)|_{E_1}}{2} dx + \int_1^3 \frac{x(y^2)|_{E_6}}{2} dx - \int_1^3 \frac{x(y^2)|_{E_2}}{2} dx + \int_3^4 \frac{x(y^2)|_{E_5}}{2} dx - \int_3^4 \frac{x(y^2)|_{E_2}}{2} dx \\
&\quad + \int_4^6 \frac{x(y^2)|_{E_5}}{2} dx - \int_4^6 \frac{x(y^2)|_{E_3}}{2} dx + \int_6^8 \frac{x(y^3)|_{E_4}}{2} x^2 dx - \int_6^8 \frac{x(y^2)|_{E_3}}{2} dx
\end{aligned}$$



This can be written in the following form:

$$\begin{aligned}
 I_{xy} &= -\int_0^1 \frac{x(y^2)|_{E1}}{2} dx - \int_1^4 \frac{x(y^2)|_{E2}}{2} dx - \int_4^8 \frac{x(y^2)|_{E3}}{2} dx - \int_8^6 \frac{x(y^2)|_{E4}}{2} dx - \int_6^3 \frac{x(y^2)|_{E5}}{2} dx - \int_3^0 \frac{x(y^2)|_{E6}}{2} dx \\
 &= -\int_0^1 \frac{x(-2x+2)^2}{2} dx - \int_1^4 \frac{x(0)^2}{2} dx + \int_4^8 \frac{x(3x/4-3)^2}{2} dx - \int_8^6 \frac{x(-3x/2+15)^2}{2} dx - \int_6^3 \frac{x(6)^2}{2} dx \\
 &\quad - \int_3^0 \frac{x(4x/3+2)^2}{2} dx = 394.333333
 \end{aligned}$$

The same results can be obtained by dividing the polygon in two quadrilateral with the following nodal points

$$\tilde{x}_{(p)}^i = \begin{bmatrix} 1. & 0. \\ 4. & 0. \\ 8. & 3. \\ 6. & 6. \end{bmatrix}; \tilde{x}_{(p)}^j = \begin{bmatrix} 1. & 0. \\ 6. & 6. \\ 3. & 6. \\ 0. & 2. \end{bmatrix}$$

and finding the explicit transformation relation between Cartesian variables and natural variables using the standard bilinear approach and integrating the expressions for area and moments of area between -1 and +1.

In the following, a MATLAB function code is given by which the area and moments of area for arbitrary polygon with straight edges can be calculated.

Before using it, some notes on using formula (12) within a finite element program code must be considered. The vertices of the polygonal domain must be ordered counterclockwise starting with the minimal  $x$ -value when integrating over  $x$  or the smallest  $y$ -value when integrating over  $y$  in order to avoid any sign confusion which leads to incorrect results. Furthermore, an edge equation of the form  $x=const$ , when integrating over  $x$  or of the form  $y=const$ , when integrating over  $y$  must be avoided. In case of a parallel edge to  $x$ -axes the program will stop running from itself (the code integrate over  $x$ ). The given code is not intended for a general use in a finite element program because it needs further editing to account for some special geometry cases of a polygon. It should serve understanding and investigating the use of the Riemann formula in the current paper.

In order to call the code, the following script file must be saved under the name moa.m in the working directory.

```

=====moa.m=====
% function for calculating moments of area of a Polygon with straight edges
%=====
function [A, Ix, Iy, Ixy]=moa(xip)
n=size(xip);
if (xip(1:1,1))< min(xip(2:n(1)-1,1))
syms x y;
for i=1:n(1)-1
E(i)=det([ x y 1; xip(i,1) xip(i,2) 1; xip(i+1,1) xip(i+1,2) 1]);
E(i)=solve(E(i), y);
A(i)=-vpa(int(E(i), xip(i,1), xip(i+1,1)));
Ix(i)=-vpa(int((E(i))^3/3, xip(i,1), xip(i+1,1)));
Iy(i)=-vpa(int(x^2*E(i), xip(i,1), xip(i+1,1)));
Ixy(i)=-vpa(int(x*(E(i))^2/2, xip(i,1), xip(i+1,1)));
end

E(n(1))=det([ x y 1; xip(n(1),1) xip(n(1),2) 1; xip(1,1) xip(1,2) 1]);

```

```

E(n(1))=solve(E(n(1)),y);
A(n(1))=-vpa(int(E(n(1)),xip(n(1),1),xip(1,1)));

Ix(n(1))=-vpa(int((E(n(1)))^3/3,xip(n(1),1),xip(1,1)));
Iy(n(1))=-vpa(int(x^2*E(n(1)),xip(n(1),1),xip(1,1)));
Ixy(n(1))=-vpa(int(x*(E(n(1)))^2/2,xip(n(1),1),xip(1,1)));

A=sum(transpose(A));
Ix=sum(transpose(Ix));
Iy=sum(transpose(Iy));
Ixy=sum(transpose(Ixy));
else
fprintf('Please order the coordinate counter clockwise starting with the
smallest xip-one\n');
end
end
%=====

```

Calling the function for the triangle by the two following command lines

```

xip=[0 0 ; 5 0 ;1.8 2.4]
[A,Ix,Iy,Ixy]=moa(xip)

```

, for the quadrilateral by the two following command lines

```

xip=[1 1 ; 8 0 ;6 4 ;2 5]
[A,Ix,Iy,Ixy]=moa(xip)

```

and finally for the polygon by the two following command lines

```

xip=[0 2;1 0;4 0;8 3;6 6;3 6]
[A,Ix,Iy,Ixy]=moa(xip)

```

give the results obtained in the three examples.

**Example 4:** Finally, a polygonal domain with five edges (Fig. 6) with the following Cartesian coordinates of vertices

$$x_{(p)}^{\tilde{i}} = \begin{bmatrix} 1.0 & 1.0 \\ 6.0 & 0.0 \\ 3.0 & 3.0 \\ 8.0 & 6.0 \\ 2.0 & 5.0 \end{bmatrix}$$

is computed using the above given program function called by the two following command lines

```

xip=[1 1;6 0;3 3;8 6;2 5]
[A,Ix,Iy,Ixy]=moa(xip)

```

The output gives the following results for area and moments of area

A =15.5; Ix =186.083333333; Iy =218.75; Ixy =178.375

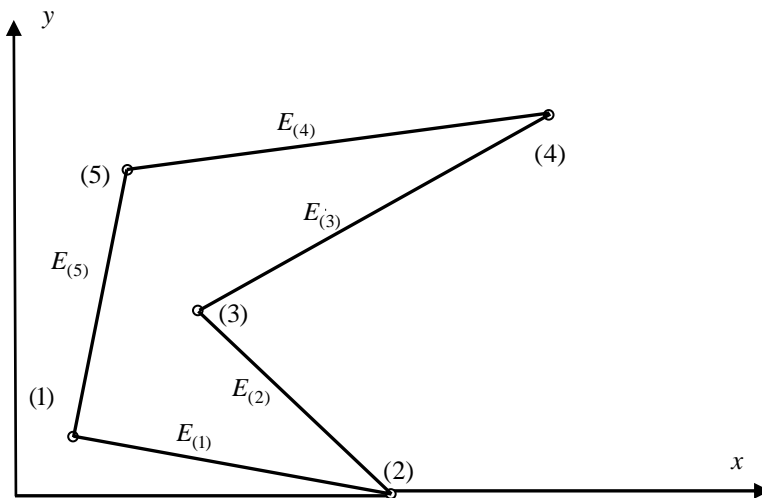


Figure 6: Polygonal domain with five straight edges, Cartesian coordinates of vertices, and edges

The same results can be obtained by dividing the polygon in two quadrilateral with the following nodal points

$$x_{(p)}^{\tilde{i}} = \begin{bmatrix} 1.0 & 1.0 \\ 6.0 & 0.0 \\ 3.0 & 3.0 \\ 1.5 & 3.0 \end{bmatrix}; x_{(p)}^{\tilde{i}} = \begin{bmatrix} 1.5 & 2.0 \\ 3.0 & 3.0 \\ 8.0 & 6.0 \\ 2.0 & 5.0 \end{bmatrix}$$

and performing the integration using the standard bilinear approach.

Here, the results are exact also in case of non-convex polygon.

**Example 5:** The last assessment in this section is on a circular disc with a diameter of two units of length. The circumference is divided into hundred segments. The above function moa(xip) is used along with the following code:

```
% area and moments of area for a circular disc with a diameter of two units of length
i=0;
for f=-pi:pi/100:pi-pi/100
i=i+1
xip(i,1)=cos(f);xip(i,2)=sin(f);
end
xip
[A,Ix,Iy,Ixy]=moa(xip)
```

The output gives the following results

A =3.1410759078128293940787214943951;Ix =0.78513981600761063777789906861322

Iy =0.78513981600761063432172486883439; Ixy =-0.0000000000000000116128775033486671270401753.

These results are quasi correct ( $A=\pi$ ;  $I_x=I_y=\pi/4$  ;  $I_{xy}=0$ ). It is worth mentioning that the convergence to the exact results occurs very slowly especially for the quadratic form of  $I_x$  or  $I_y$ . One can easily discover this fact by changing the number of segments in the for loop.

Two reasons may delay the use of Riemann integral. Firstly, one needs to know the primitive of the function to be integrated inside the domain. Secondly one needs also to express one variable using the other one explicitly at the boundaries of the polygon, which is eventually not possible in case of a complex geometry bounded by curves with complex implicit expressions in the two variables. In the second case, one can use one-dimensional curve fitting techniques to find explicit expressions at the domain boundaries.

## 6 Deriving the Stiffness matrix of a generalized ACM-plate bending element

The finite Element approximation is based on Hamilton's Principle. The 2D expression for the special case of the thin plate considered can be written in the absence of the prescribed boundary displacements relating to a Cartesian coordinate system in the following form:

$$\delta \int_{t_1}^{t_2} \left\{ \int_A \frac{1}{2} \chi_{ij} E^{ijkl} \chi_{kl} dA - \int_A \bar{q} u_{x^3}^0 dA - \frac{1}{2} \int_A \dot{u}_i \rho^{ij} \dot{u}_j dA - \sum_{i=1}^n \bar{F}^{(i)} u_{x^3}^0 \right\} dt = 0 \quad (17)$$

where

$t_1$  and  $t_2$  are two fixed time points of the vibration process,  $\delta$  is the first variation,  $A$  is the area and  $dA$  its differential element,  $\dot{u}_i$  is the velocity vector in which both displacement and rotation components are included,  $\rho^{ij}$  is the corresponding mass density matrix,  $\bar{F}^{(i)}$  is the concentrated load applied at the point  $(i)$ .

$\chi_{ij}$  is the curvature tensor, which reads expressed in terms of the deflection  $u_{x^3}^0(x^1, x^2)$ :

$$\begin{bmatrix} \chi_{x^1 x^1} \\ \chi_{x^2 x^1} \\ \chi_{x^1 x^2} \\ \chi_{x^2 x^2} \end{bmatrix} = - \begin{bmatrix} u_{x^3, x^1 x^1}^0 \\ u_{x^3, x^1 x^2}^0 \\ u_{x^3, x^2 x^1}^0 \\ u_{x^3, x^2 x^2}^0 \end{bmatrix} \quad (18)$$

$E^{ijkl}$  is the matrix of the force-curvature dependency given in a matrix form as follows:

$$E^{ijkl} = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & 0 & 0 & \nu \\ 0 & (1-\nu)/2 & (1-\nu)/2 & 0 \\ 0 & (1-\nu)/2 & (1-\nu)/2 & 0 \\ \nu & 0 & 0 & 0 \end{bmatrix} \quad (19)$$

$\rho^{ij}$  is defined by the following matrix:

$$\rho^{ij} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho h^3/12 & 0 \\ 0 & 0 & \rho h^3/12 \end{bmatrix} \quad (20)$$

where  $\rho$  is the material density,  $E$  is the modulus of elasticity,  $h$  is the plate thickness and  $\nu$  the Poisson's ratio. The indicial notation to indicate the Cartesian variables  $x^1, x^2$  is used instead of the  $x, y$ - frame and indices between brackets range over the nodal points.

In Eqn. (17), the internal work associated with the bending and twist moments is only considered.

The plate finite element with the nodal points  $(i), (j), (k), (l)$  has three degrees of freedom each node. These are the displacement normal to the plate surface in  $x^3$ -direction and the two rotations about  $x^1$  and  $x^2$ -axes. The total number of degrees of freedom each element is then represented by the element nodal displacement vector with 12 degrees of freedom

$$\mathbf{u}_{n(n)} = \{ \mathbf{u}_{x^3(i)}^0, \varphi_{x^1(i)}, \varphi_{x^2(i)}, \mathbf{u}_{x^3(j)}^0, \varphi_{x^1(j)}, \varphi_{x^2(j)}, \mathbf{u}_{x^3(k)}^0, \varphi_{x^1(k)}, \varphi_{x^2(k)}, \mathbf{u}_{x^3(l)}^0, \varphi_{x^1(l)}, \varphi_{x^2(l)} \} \quad (21)$$

The local axes  $x^1, x^2$  are defined using the directions of the element base vectors in an analogous procedure used in section 4 in [32, 33]. The approximation basis is constructed using the defined local Cartesian variables  $x^1, x^2$  in the usual parametric form [34, 35]:

$$\begin{aligned} \mathbf{u}_{x^3}^0(x^1, x^2) &= \mathbf{M}^{m(m)} \mathbf{c}_{m(m)} \\ \mathbf{M} &= \begin{bmatrix} 1 & x^1 & x^2 & (x^1)^2 & x^1 x^2 & (x^2)^2 & (x^1)^3 & (x^1)^2 x^2 & x^1 (x^2)^2 & (x^2)^3 & (x^1)^3 x^2 & x^1 (x^2)^3 \end{bmatrix} \\ \mathbf{c} &= \{ c_{1(1)} \ c_{1(2)} \ \dots \dots \ c_{3(4)} \} \end{aligned} \quad (22)$$

Linking the free parameters  $\mathbf{c}_{m(m)}$  to the nodal degrees of freedom using the essential boundary conditions at the finite element level yield:

$$\mathbf{u}_{r(r)} = \mathbf{A}_{r(r)}^{m(m)} \mathbf{c}_{m(m)} \quad (23)$$

Eliminating the free parameters from Eqn. (22) by solving the linear system of equations (23) and substituting the result into equation (22), the following relationship between the internal displacements  $\mathbf{u}_{x^3}^0(x^1, x^2)$  and the nodal degrees of freedoms  $\mathbf{u}_{r(r)}$  is obtained

$$\mathbf{u}_{x^3}^0(x^1, x^2) = \mathbf{M}^{m(m)} (\mathbf{A}_{r(r)}^{m(m)})^{-1} \mathbf{u}_{r(r)} \quad (24)$$

In Eqn. (23),  $\mathbf{A}_{r(r)}^{m(m)}$  is a  $12 \times 12$  matrix derived from  $\mathbf{M}^{m(m)}$  by substituting the coordinates of the element nodes and  $(\mathbf{A}_{r(r)}^{m(m)})^{-1}$  is the inverse matrix of  $\mathbf{A}_{r(r)}^{m(m)}$ .

Deriving the curvature tensor  $\chi_{ij}$  using Eqn. (24) yield

$$\chi_{ij} = p_{ij}^{m(m)} (A_{r(r)}^{m(m)})^{-1} u_{r(r)}$$

$$p_{ij}^{m(m)} = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x^1 & 2x^2 & 0 & 0 & 6x^1x^2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2x^1 & 2x^2 & 0 & 3(x^1)^2 & 3(x^2)^2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2x^1 & 2x^2 & 0 & 3(x^1)^2 & 3(x^2)^2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x^1 & x^2 & 0 & 6x^1x^2 \end{bmatrix} \quad (25)$$

Applying the expressions of  $\chi_{ij}$  in the first term of Eqn. (17) gives

$$\frac{1}{2} \int_A \chi_{ij} E^{ijkl} \chi_{kl} dA = \frac{1}{2} u_{r(r)} (A_{r(r)}^{m(m)})^{-1} \left( \int_A p_{ij}^{m(m)} E^{ijkl} p_{kl}^{n(n)} dA \right) (A_{q(q)}^{n(n)})^{-1} u_{q(q)} = \frac{1}{2} u_{r(r)} k^{r(r)q(q)} u_{q(q)} \quad (26)$$

where

$$k^{r(r)q(q)} = (A_{r(r)}^{m(m)})^{-1} \left( \int_A p_{ij}^{m(m)} E^{ijkl} p_{kl}^{n(n)} dA \right) (A_{q(q)}^{n(n)})^{-1} \quad (27)$$

is the element stiffness matrix related to the Cartesian coordinate system.

The integration over the area can be performed using the scheme presented above as follow:

$$I = \left( \int_A p_{ij}^{m(m)} E^{ijkl} p_{kl}^{n(n)} dA \right) = \int_{x^1} \left( \int_{x^2} p_{ij}^{m(m)} E^{ijkl} p_{kl}^{n(n)} dx^2 \right) dx^1 = - \sum_{i=1}^n \int_{x(i)}^{x(i+1)} P_r^{m(m)n(n)}|_{E_i} dx^1 \quad (28)$$

where  $P_r^{m(m)n(n)}$  is the primitive matrix of the  $12 \times 12$  matrix  $p_{ij}^{m(m)} E^{ijkl} p_{kl}^{n(n)}$  with respect to  $x^2$

$$P_r^{m(m)n(n)} = \int_{x^2} p_{ij}^{m(m)} E^{ijkl} p_{kl}^{n(n)} dx^2 \quad (29)$$

The stiffness matrix of the quadrilateral element of sec. 4 with the nodal coordinates

$$x_{(p)}^{\tilde{i}} = \begin{bmatrix} 1. & 1. \\ 8. & 0. \\ 6. & 4. \\ 2. & 5. \end{bmatrix}$$

is derived. Defining a local coordinate system from the directions of the base vectors corresponding to [30], [31], the new local coordinates read

$$x_{(p)}^i = \begin{bmatrix} -3.2071 & -1.5896 \\ 3.8179 & -2.3950 \\ 1.7077 & 1.5480 \\ -2.3185 & 2.4366 \end{bmatrix}$$

The following two function code generate the stiffness matrix of the quadrilateral element

```

%=====Ekr.m=====
function [k]=Ekr(xip,E,m,t)
%xip: local cartesian coordinates of the quadrilateral
%E: Young modulus m Poissons ratio t:thickness
%=====

```

```

syms x y ;
M=[ 1 x y x^2 x*y y^2 x^3 x^2*y x*y^2 y^3 x^3*y x*y^3];
Mxx=diff(diff(M,x),x);
Mxy=diff(diff(M,x),y);
Myy=diff(diff(M,y),y);
%c: elasticity tensor in the x y-system
c=((E*t^3)/(12.*(1-m^2)))*[1 0 0 m; 0 (1-m)/2. (1-m)/2. 0; 0 (1-m)/2.
(1-m)/2. 0;m 0 0 1];
p=[Mxx;Mxy;Mxy;Myy];
Mi=[M;diff(M,y);-diff(M,x)];
A=[subs(subs(Mi,'x',xip(1,1)),'y',xip(1,2));
subs(subs(Mi,'x',xip(2,1)),'y',xip(2,2));subs(subs(Mi,'x',xip(3,1)),'y',
' ,xip(3,2));subs(subs(Mi,'x',xip(4,1)),'y',xip(4,2))];
a=inv(A);
k=transpose(p)*c*p
%Primitive matrix of the matrix pT*c*p
k=int(k,y)
%integration over the boundary using x as variable
[kr]=Rk(k,xip)
k=transpose(a)*kr*a
eig(k)
%=====EOF=====

%=====Rk.m=====
% function for calculating stiffness matrix of a Polygon with straight
% edges using Riemann integral
% before call order the x-coordinate such that x(1)=min
%=====

function [kr]=Rk(k,xip)
n=size(xip);
if (xip(1:1,1))< min(xip(2:n(1)-1,1))
kr=zeros(12);
syms x y
for i=1:n(1)-1
E(i)=det([ x y 1; xip(i,1) xip(i,2) 1; xip(i+1,1) xip(i+1,2) 1]);
E(i)=solve(E(i),y);
kE=subs(k,y,E(i));
kr=kr-vpa(int(kE,xip(i,1),xip(i+1,1)));
end
E(i+1)=det([ x y 1; xip(n(1),1) xip(n(1),2) 1; xip(1,1) xip(1,2) 1]);
E(i+1)=solve(E(n(1)),y);
kE=subs(k,y,E(i));
Kr=kr-vpa(int(kE,xip(n(1),1),xip(1,1)));
vpa(kr,6);
else
fprintf('Please order the cordinate counter clockwise starting with
the smallest xip-one');
end
end
end
%=====#

```

Calling the above two functions using the following three commands

```
xip=[ -3.2071 -1.5896; 3.8179 -2.3950; 1.7077 1.5480; -2.3185 2.4366]
```

```
EKr(xip,1365.0,0.3,0.2)
```

```
[kr]=Rk(k,xip)
```

produces the stiffness matrix and its eigenvalues listed in the first line of Tab. 1.

Table 1: Eigen values of the stiffness matrix of the quadrilateral element of sec. 4 related to a local Cartesian coordinate system (integrated exactly using the Riemann integral)

$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_4$	$\lambda_5$	$\lambda_6$	$\lambda_7$	$\lambda_8$	$\lambda_9$
10.77862	4.538367	3.277139	1.648885	0.837864	0.691052	0.629379	0.427979	0.162471
<i>11.08210</i>	<i>4.542962</i>	<i>3.265729</i>	<i>1.804597</i>	<i>0.817196</i>	<i>0.730516</i>	<i>0.650880</i>	<i>0.432157</i>	<i>0.165944</i>

Similar results can be obtained by substituting the expressions for  $x^1, x^2$  derived using the standard bilinear approach into Eqn. (27) between the limits -1, +1. The results of such procedure are listed cursive in the second line of Tab.1.

Transforming the stiffness matrix into the global coordinate system gives the same results listed in Tab.1.

Performing the integration directly in the global coordinate system with the coordinates

```
xip=[1 1 ; 8 0 ;6 4 ;2 5 ]
```

gives a very different results, which are lower by about 16%.

This example shows the necessity to define the proposed local element coordinate system and to locate it at the geometric center of the element.

Similar procedure can be adopted to evaluate the element mass matrix resulting in from evaluating the third term of Eqn. (17).

Note that the results in the first five examples using either the Riemann formula or the standard bilinear approach are exact but the results of the stiffness matrix are different. In such cases, it is difficult to make right conclusions. A study on benchmark of polygon quality metrics for polytopal element methods is still at the beginning [36]. However, one can observe that the characters of the integrals in the first five examples are additive, whilst they are not additive when deriving the stiffness matrix, which include the rotations and their derivatives. In addition the integrated shape functions are discontinuous and non-conform. Therefore, we found ourselves confronted with similar problems that appear during the discretization. Therefore, it is necessary to subject the element matrices to the same tests required for a finite element application especially when the integrated function are discontinuous.



## 7 Changing the variable of integration

### 7.1. Using a polar coordinate system

An example of changing the variable of integration (i. e.  $x$  or  $y$ ) is the use of a polar coordinate system. In polar coordinate system the following exact relations apply

$$\begin{aligned}x &= r \cos \varphi \\y &= r \sin \varphi\end{aligned}\quad ; \quad r = \sqrt{x^2 + y^2}; \varphi = \arccos(x/\sqrt{x^2 + y^2}) \quad (30)$$

Note that  $x$  and  $y$  are replaced each by other two variables, namely  $r, \varphi$  such that the Pythagorean Theorem applies. In this sense, the transformation is exact and there is no approximation.

The position vector of an arbitrary point  $\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$  formulated in  $r, \varphi$  terms reads

$$\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z = r \cos \varphi \vec{e}_x + r \sin \varphi \vec{e}_y + 0 \vec{e}_z \quad (31)$$

and the changes of it in  $r, \varphi$  directions are as follows:

$$\vec{r}_r dr = \cos \varphi \vec{e}_x + \sin \varphi \vec{e}_y + 0 \vec{e}_z \quad (32)$$

$$\vec{r}_\varphi d\varphi = -r \sin \varphi \vec{e}_x + r \cos \varphi \vec{e}_y + 0 \vec{e}_z$$

The differential element of the area can now be calculated as a vector product of these changes

$$d\vec{A} = \vec{r}_r \times \vec{r}_\varphi dr d\varphi = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \cos \varphi & \sin \varphi & 0 \\ -r \sin \varphi & r \cos \varphi & 0 \end{vmatrix} = r dr d\varphi \vec{e}_z \quad (34)$$

The integral over the area is now to be performed over the new variables

$$I = \iint_A f(x, y) dx dy = \iint_{\varphi r} f(r, \varphi) r dr d\varphi = \int_{\varphi_1}^{\varphi_2} \int_{r=r_1(\varphi)}^{r=r_2(\varphi)} f(r, \varphi) r dr d\varphi \quad (35)$$

Usually, integration is first performed over the variable limits and after that over the constant limits.

### 7.2. Using a $\rho, s$ system

In the previous examples both limits of the integral change in  $x$  and  $y$ -direction even if the domain is bounded by straight edges. Sometimes it is useful to use another coordinate system with one variant variable instead of using the Cartesian coordinate system or a polar coordinate system. Therefore, it is more reliable to find other two variables instead of  $x$  and  $y$  such that at least one of them has a constant value at the edges of the domain. Along the boundary with straight edges both  $x$  and  $y$  change, but  $y'$  has a constant value.

The idea now is to decompose the position vector  $\vec{r}$  into two vectors  $\vec{\rho}$  and  $\vec{s}$ .

Let us define the position vector  $\vec{r}$  corresponding to the Chasles relation of vector addition as a sum of two vectors  $\vec{\rho}, \vec{s}$

$$\vec{\rho} + \vec{s} = \vec{r} \quad (36)$$

where,  $\vec{\rho}$  is the position vector of an arbitrary point  $o'$  of the plane  $z=0$  and  $\vec{s}$  the vector from  $o'$  to  $p$ .  $o'$  can be located everywhere in the plane  $z=0$ .

An exact relation between the magnitude of  $\vec{\rho}, \vec{s}$  and of that of the position vector is then given by

$$\rho^2 + s^2 + 2\rho s \cos(\vec{\rho}, \vec{s}) = x^2 + y^2 = r^2 \quad (37)$$

where  $\rho = \|\vec{\rho}\|, s = \|\vec{s}\|$  and  $r = \|\vec{r}\|$ .

This relation is valid everywhere in the domain and along the boundaries with  $o'$  as free selectable point.

In order to evaluate integrals such as  $\iint_A dx dy$  or  $\iint_A f(x, y) dx dy$ ,  $x, y$  must be expressed in terms of  $\rho, s$ .

In case of straight edges, after suitable selection of  $o'$ , the relation between  $\rho, s$  and  $x, y$  can be formulated at the boundary. The integral can be performed in the  $\rho, s$  instead of  $x, y$ .

The Wachspress space [37] offered an approximation basis for many publications dealing with polygonal and polyhedral elements.

This idea can be extended by decomposing the position vector  $\vec{r}$  into more than two vectors  $\vec{\rho}$  and  $\vec{s}$  when necessary and useful. In any case the connection between the origin of the coordinate system  $x, y$  and  $o'$  should not be lost.

## 8 Conclusion

This paper shows the power of the Fundamental of Calculus in dealing with complex geometry-domains for some practical Engineering problems. The integration over a polygonal domain enclosed by a sequence of edges is performed exactly using the Fundamental of Calculus without sub-division. Double integrals are transformed into sequences of single integrals in one direction using the Fundamental of Calculus. There is also no need to use the Gauss divergence theorem, the 'Stokes' Theorem or to integrate in the edge direction. It is not necessary to use any mapping procedure in order to integrate over a polygonal element. Some notes on changing the variables of the integration for example in case of using polar coordinate system or non-orthogonal coordinate system are discussed.

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*Address:* Sulaiman Abo Diab, Syria, Tartous, Hussain Al Baher  
*email:* [sabodiab@tishreen.edu.sy](mailto:sabodiab@tishreen.edu.sy) or [sabodiab@hotmail.com](mailto:sabodiab@hotmail.com)

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