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Schauder estimates for degenerate stable Kolmogorov equations

Lorenzo Marino^{*†}

4th October, 2019

Abstract

We provide here global Schauder-type estimates for a chain of integro-partial differential equations (IPDE) driven by a degenerate stable Ornstein-Uhlenbeck operator possibly perturbed by a deterministic drift, when the coefficients lie in some suitable anisotropic Hölder spaces. Our approach mainly relies on a perturbative method based on forward parametrix expansions and, due to the low regularizing properties on the degenerate variables and to some integrability constraints linked to the stability index, it also exploits duality results between appropriate Besov Spaces. In particular, our method also applies in some super-critical cases. Thanks to these estimates, we show in addition the well-posedness of the considered IPDE in a suitable functional space.

Keywords: Schauder estimates, degenerate IPDEs, perturbation techniques, parametrix, Besov spaces.

MSC: Primary: 35K65, 35R09, 35B45; Secondary: 60H30.

1 Introduction

For a fixed time horizon $T > 0$ and two integers n, d in \mathbb{N} , we are interested in proving global Schauder estimates for the following parabolic integro-partial differential equation (IPDE):

$$\begin{cases} \partial_t u(t, \mathbf{x}) + \langle A\mathbf{x} + \mathbf{F}(t, \mathbf{x}), D_{\mathbf{x}}u(t, \mathbf{x}) \rangle + L_{\alpha}u(t, \mathbf{x}) = -f(t, \mathbf{x}), & \text{on } [0, T] \times \mathbb{R}^{nd} \\ u(T, \mathbf{x}) = g(\mathbf{x}) & \text{on } \mathbb{R}^{nd}. \end{cases} \quad (1.1)$$

where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ is in \mathbb{R}^{nd} with each \mathbf{x}_i in \mathbb{R}^d and $\langle \cdot, \cdot \rangle$ represents the inner product on \mathbb{R}^{nd} . We consider a symmetric non-local α -stable operator L_{α} acting non-degenerately only on the first d variables and a matrix A in $\mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$ with the following sub-diagonal structure:

$$A := \begin{pmatrix} 0_{d \times d} & \dots & \dots & \dots & 0_{d \times d} \\ A_{2,1} & 0_{d \times d} & \dots & \dots & 0_{d \times d} \\ 0_{d \times d} & A_{3,2} & 0_{d \times d} & \dots & 0_{d \times d} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{d \times d} & \dots & 0_{d \times d} & A_{n,n-1} & 0_{d \times d} \end{pmatrix}. \quad (1.2)$$

We will assume moreover that it satisfies a Hörmander-like condition, allowing the smoothing effect of L_{α} to propagate into the system.

Above, the source $f: [0, T] \times \mathbb{R}^{nd} \rightarrow \mathbb{R}$ and the terminal condition $g: \mathbb{R}^{nd} \rightarrow \mathbb{R}$ are assumed to be bounded and to belong to some suitable anisotropic Hölder space.

The additional drift term $\mathbf{F}(t, \mathbf{x}) = (F_1(t, \mathbf{x}), \dots, F_n(t, \mathbf{x}))$ can be seen as a perturbation of the Ornstein-Uhlenbeck operator $L_{\alpha} + \langle A\mathbf{x}, D_{\mathbf{x}} \rangle$ and it has structure "compatible" with A , i.e. at level i , it depends only on the super diagonal entries:

$$F_i(t, \mathbf{x}) := F_i(t, \mathbf{x}_i, \dots, \mathbf{x}_n).$$

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It may be unbounded but we assume it to be Hölder continuous with an index depending on the level of the chain.

Related Results. A large literature on the topic of Schauder estimates in the α -stable non-local framework has been developed in the recent years (see e.g. Lunardi and Röckner [LR19] for an overview of the field), mainly in the non-degenerate setting and assuming that $\alpha \geq 1$, the so called sub-critical case. We mention for instance the stable-like setting, corresponding to time-inhomogeneous operators of the form

$$L_t \phi(\mathbf{x}) = \int_{\mathbb{R}^{nd}} [\phi(\mathbf{x} + \mathbf{y}) - \phi(\mathbf{x}) - \mathbb{1}_{1 \leq \alpha < 2} \langle \mathbf{y}, D_{\mathbf{x}} \rangle] m(t, \mathbf{x}, \mathbf{y}) \frac{d\mathbf{y}}{|\mathbf{y}|^{d+\alpha}} + \mathbb{1}_{1 \leq \alpha < 2} \langle \mathbf{F}(t, \mathbf{x}), D_{\mathbf{x}} u(t, \mathbf{x}) \rangle \quad (1.3)$$

where the diffusion coefficient m is bounded from above and below, Hölder continuous in the spatial variable \mathbf{x} and even in \mathbf{y} if $\alpha = 1$. Under these conditions and assuming the drift \mathbf{F} to be bounded and Hölder continuous in space, Mikulevicius and Pragarauskas in [MP14] obtained parabolic Schauder type bounds on the whole space and derived from those estimates the well-posedness of the corresponding martingale problem. We notice however that for the super-critical case (when $\alpha < 1$), the drift term in (1.3) is set to zero. This is mainly due to the fact that in the super-critical case, L_{α} is of order α (in the Fourier space) and does not dominate the drift term \mathbf{F} which is roughly speaking of order one.

In the non-degenerate, driftless framework (i.e. when $A\mathbf{x} + \mathbf{F} = 0$ and $n = 1$ in (1.1)), Bass [Bas09] was the first to derive elliptic Schauder estimates for stable like operators. We can refer as well to the recent work of Imbert and collaborators [IJS18] concerning Schauder estimates for stable-like operator (1.3) with $\alpha = 1$ and some related applications to non-local Burgers equations. Eventually, still in the driftless case, Ros-Oton and Serra worked in [ROS16] for interior and boundary elliptic-regularity in a general, symmetric α -stable setting, assuming that the Lévy measure ν_{α} associated with L_{α} writes in polar coordinates $y = \rho s$, $(\rho, s) \in [0, \infty) \times \mathbb{S}^{d-1}$ as

$$\nu_{\alpha}(dy) = \tilde{\mu}(ds) \frac{d\rho}{\rho^{1+\alpha}}$$

where $\tilde{\mu}$ is a non-degenerate, symmetric measure on the sphere \mathbb{S}^{d-1} . Related to the above, we can mention also the associated work of Fernandez-Real and Ros-Oton [FRRO17] for parabolic equations.

In the elliptic setting, when $\alpha \in [1, 2)$ and L_{α} is a non-degenerate, symmetric α -stable operator and for bounded Hölder drifts, global Schauder estimates were obtained by Priola in [Pri12] or in [Pri18] for respective applications to the strong well-posedness and Davie's uniqueness for the corresponding SDE. We notice furthermore that in the sub-critical case, elliptic Schauder estimates can be proven for more general, translation invariant, Lévy-type generators for following [Pri18] (see Section 6, and Remark 5 therein).

In the super-critical case, parabolic Schauder estimates were established by Chaudru de Raynal, Menozzi and Priola in [CdRMP19] under similar assumptions to [ROS16]. An existence result is also provided therein. We mention as well the work of Zhang and Zhao [ZZ18] who address through probabilistic arguments the parabolic Dirichlet problem for stable-like operators of the form (1.3) with a non-trivial bounded drift, i.e. getting rid of the indicator function for the drift. They also obtain interior Schauder estimates and some boundary decay estimates (see e.g. Theorem 1.5 therein).

As we have seen, most of the literature is focused on the non-degenerate case. In the degenerate diffusive setting, Lunardi [Lun97] was the first one to prove Schauder estimates for linear Kolmogorov equations under weak Hörmander assumptions, exploiting anisotropic Hölder spaces (where the Hölder index depends on the variable considered), in order exactly to control the multiple scales appearing in the different directions, due to the degeneracy of the system.

After, in [Lor05] and [Pri09], the authors established Schauder-like estimates for hypoelliptic Kolmogorov equations driven by partially nonlinear smooth drifts. On the other hand, let us also mention [CdRHM18] where the authors first establish Schauder estimates for nonlinear Kolmogorov equations under some weak Hörmander-type assumption. Their method is based on a perturbative approach through proxies that we here adapt and exploit. In the degenerate, stable setting, we have to refer also to a recent work of Zhang and collaborators [HWZ19] who show Schauder estimates for the degenerate kinetic dynamics ($n = 2$ above) extending a method based on Littlewood-Paley decompositions already used in other works by Zhang (see e.g. [ZZ18]), to the degenerate, multi-scaled framework. Even with different approaches and frameworks, we consider here a generic d -level chain and we exploit thermic characterizations of Besov norms, our and

their works bring to the same results in the intersecting cases, at least to the best of our knowledge. About a different but correlated argument, we mention that the L^p -maximal regularity for degenerate non-local Kolmogorov equations with constant coefficients was also obtained in [CZ18] for the kinetic dynamics ($n = 2$ above) and in [HMP19] for the general n -levels chain.

In the diffusive setting, Equation (1.1) appears naturally as a microscopic model for heat diffusion phenomena (see [RBT00]) or, in the kinetic case ($n = 2$), it can be naturally associated with speed/position (or Hamiltonian) dynamics where the speed component is noisy. It can be found in many fields of application from physics to finance, see for example [HN04] or [BPV01]. When noised by stable processes, it can be used to model the appearance of turbulence (cf. [CPKM05]) or some abnormal diffusion phenomena.

Moreover, the Schauder estimates will be a fundamental first step in order to study the weak and strong well-posedness for the following stochastic differential equation (SDE):

$$\begin{cases} d\mathbf{X}_t^1 = \mathbf{F}_1(t, \mathbf{X}_t^1, \dots, \mathbf{X}_t^n)dt + dZ_t \\ d\mathbf{X}_t^2 = A_{2,1}\mathbf{X}_t^1 + \mathbf{F}_2(t, \mathbf{X}_t^2, \dots, \mathbf{X}_t^n)dt \\ \vdots \\ d\mathbf{X}_t^n = A_{n,n-1}\mathbf{X}_t^{n-1} + \mathbf{F}_n(t, \mathbf{X}_t^n)dt \end{cases} \quad (1.4)$$

where Z_t is a symmetric, \mathbb{R}^d -valued α -stable process with non-degenerate Lévy measure ν_α on some filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. The complete operator $L_\alpha + \langle A\mathbf{x} + \mathbf{F}(t, \mathbf{x}), D\mathbf{x} \rangle$ then corresponds to the infinitesimal generator of the process $(\mathbf{X})_{t \geq 0}$, solution of Equation (1.4).

Mathematical Outline. In this work, we will establish global Schauder estimates for the solution of the IPDE (1.1) exploiting the perturbative approach firstly introduced in [CdRHM18] to derive such estimates for degenerate Kolmogorov equations. Roughly speaking, the idea is to perform a first order parametrix expansion, such as a Duhamel-type representation, to a solution of the IPDE (1.1) around a suitable proxy. The main idea behind consists in exploiting this easier framework in order to subsequently obtain a tractable control on the error expansion. When applying such a strategy, we basically have two ways to proceed.

On the one hand, one can adopt a backward parametrix approach, as introduced by McKean and Singer [MS67] in the non-degenerate, diffusive setting. This technique has been extended to the degenerate Brownian case involving unbounded perturbation, and successfully exploited for handling the corresponding martingale problem in [CM17]. Anyway, this approach does not seem very adapted to our framework especially because it does not allow to deal easily with point-wise gradient estimates which will, at least along the non-degenerate variable \mathbf{x}_1 , be fundamental to establish our result.

On the other hand, the so-called forward parametrix approach has been successfully used by Friedman [Fri64] or Il'in et al. [IKO62] in the non-degenerate, diffusive setting to obtain point-wise bounds on the fundamental solution and its derivatives for the corresponding heat-type equation or in [CdR17] to derive strong uniqueness for the associated SDE (1.4) (i.e. $n = 2$ with the previous notations). Especially, this approach is better tailored to exploit cancellation techniques that are crucial when derivatives come in, as opposed to the backward one.

The main difficulties to overcome in order to prove Schauder estimates in our framework will be linked to the degeneracy of the operator L_α that acts only on the first d variables, as well as the unboundedness of the perturbation \mathbf{F} . Concerning this second issue, let us also mention that Schauder estimates for unbounded non-linear drift coefficients in the non-degenerate diffusive setting were obtained under mild smoothness assumptions by Krylov and Priola [KP10] who heavily used an auxiliary, deterministic flow associated with the transport term in (1.1), i.e. for a fixed couple (t, x) ,

$$\begin{cases} \partial_s \boldsymbol{\theta}_s(\mathbf{x}) = A\boldsymbol{\theta}_s(\mathbf{x}) + \mathbf{F}(s, \boldsymbol{\theta}_s(\mathbf{x})); & \text{if } s > t \\ \boldsymbol{\theta}_t(\mathbf{x}) = \mathbf{x}, \end{cases} \quad (1.5)$$

to precisely get rid of the unbounded terms.

The drawback of this approach is that we will need at first to establish Schauder estimates in a small time interval. This seems quite intuitive since the expansion along the chosen proxy on which the method relies is precisely designed for small times because it requires that the original operator and the proxy are "close"

enough in a suitable sense. To obtain the result for an arbitrary but finite time, we will then iterate the reasoning, which is quite natural since Schauder estimates provide a sort of stability in the considered functional space. We are therefore far from the optimal constants for the Schauder estimates established in the non-degenerate, diffusive setting for time dependent coefficients by Krylov and Priola [KP17].

On the other hand, we want to establish the Schauder estimates in the sharpest possible Hölder setting for the coefficients of the IPDE (1.1). To do so, we will need to establish some subtle controls, in particular we have no true derivatives of the coefficients. This is the reason why we will heavily rely on duality results on Besov spaces (see Section 4.1 below, Chapter 3 in [LR02] or [Tri83] for a more complete survey of the argument). However, in contrast with the non-degenerate case (cf. [CdRMP19]), we will need to ask for the perturbation \mathbf{F} some additional regularity, represented by parameter γ_i in assumption **(R)** below, on the degenerate entries \mathbf{F}_i ($i > 1$). This assumption seems quite natural if we think that, due to the degenerate structure of the system (cf. Section 2.2 below), the more we descend on the chain, the lower the smoothing effect of L_α will be. The additional smoothness on \mathbf{F} can be then seen as the "price" to pay to re-equilibrate the increasing time singularities appearing along the chain.

Organization of the paper. The article is organized as follows. We state our precise framework and give our main results in the following Section 2. Section 3 is then dedicated to the perturbative approach which is the central argument to derive our estimates. In particular, we obtain therein some Schauder estimates for drifted operators along the inhomogeneous flow $\theta_{t,s}$ defined above in (1.5), as well as the key Duhamel representation for solutions. Since the arguments to show the Schauder estimates will be quite long and involved, we postpone the proofs of these results in the next Sections 4 and 5. The existence results are then established in Section 6. In the last Section 7, we are going to explain briefly how the perturbative approach presented before could be applied with slight modifications to prove Schauder-type estimates for a class of completely non-linear, locally Hölder continuous drifts with an additional "diffusion" coefficient. Finally, the proof of some technical results concerning the stability properties of Hölder flows are postponed to the Appendix.

2 Setting and Main Results

2.1 Considered Operators

The operator L_α we consider is the generator of a non-degenerate, symmetric, stable process and it acts only on the first d coordinates of the system. More precisely, L_α can be represented for any sufficiently regular $\phi: [0, T] \times \mathbb{R}^{nd} \rightarrow \mathbb{R}$ as

$$L_\alpha \phi(t, \mathbf{x}) := \text{p.v.} \int_{\mathbb{R}^d} [\phi(t, \mathbf{x} + B\mathbf{y}) - \phi(t, \mathbf{x})] \nu_\alpha(d\mathbf{y}) \quad \text{where} \quad B := \begin{bmatrix} I_{d \times d} \\ 0_{d \times d} \\ \vdots \\ 0_{d \times d} \end{bmatrix}$$

and ν_α is a symmetric, stable Lévy measure on \mathbb{R}^d of order α that we assume to be non-degenerate in a sense that we are going to specify below.

Passing to polar coordinates $\mathbf{y} = \rho s$ where $(\rho, s) \in [0, \infty) \times \mathbb{S}^{d-1}$, it is well-known (see for example Chapter 3 in [Sat99]) that the stable Lévy measure ν_α can be decomposed as

$$\nu_\alpha(d\mathbf{y}) := \frac{d\rho \tilde{\mu}(ds)}{\rho^{1+\alpha}} \tag{2.1}$$

where $\tilde{\mu}$ is a symmetric measure on \mathbb{S}^{d-1} which represents the spherical part of ν_α .

We remember now that the Lévy symbol associated with L_α is defined through the Levy-Khitchine formula (see, for instance [Jac05]) as:

$$\Psi(p) := \int_{\mathbb{R}^d} [e^{ip \cdot \mathbf{y}} - 1] \nu_\alpha(d\mathbf{y}) \quad \text{for any } p \text{ in } \mathbb{R}^d,$$

where " \cdot " represents the inner product on the smaller space \mathbb{R}^d . In the current symmetric setting, it can be rewritten (cf. Theorem 14.10 in [Sat99]) as

$$\Psi(p) = - \int_{\mathbb{S}^{d-1}} |p \cdot s|^\alpha \mu(ds) \quad (2.2)$$

where $\mu = C_{\alpha,d} \tilde{\mu}$ is usually called the spherical measure associated with ν_α . Following [Kol00], we then say that ν_α is non-degenerate if the associated Lévy symbol Ψ is equivalent, up to some multiplicative constant, to $|p|^\alpha$. More precisely, we suppose that μ is non-degenerate if

(ND) there exists a constant $\eta \geq 1$ such that for any p in \mathbb{R}^d .

$$\eta^{-1} |p|^\alpha \leq \int_{\mathbb{S}^{d-1}} |p \cdot s|^\alpha \mu(ds) \leq \eta |p|^\alpha \quad (2.3)$$

It is important to remark that such a condition does not restrict our model too much. Indeed, there are many different kind of spherical measures μ that are non-degenerate in the above sense, from the stable-like case, i.e. measures that are absolutely continuous with respect to the Lebesgue measure on \mathbb{S}^{d-1} , to very singular ones such that the spherical measure induced by the sum of Dirac masses along the canonical directions:

$$\sum_{i=1}^d (\partial_{x_k}^2)^{\alpha/2}.$$

We can introduce now the complete Ornstein-Uhlenbeck operator L_{ou} , defined for any sufficiently regular $\phi: \mathbb{R}^{nd} \rightarrow \mathbb{R}$ as

$$L^{ou} \phi(\mathbf{x}) := \langle A\mathbf{x}, D_{\mathbf{x}} \phi(\mathbf{x}) \rangle + L_\alpha \phi(\mathbf{x}) \quad (2.4)$$

where A is the matrix in $\mathbb{R}^{nd} \times \mathbb{R}^{nd}$ defined in Equation (1.2). We assume that A satisfies the following Hörmander-like condition of non-degeneracy:

(H) $A_{i,i-1}$ is non-degenerate (i.e. it has full rank d) for any i in $\llbracket 2, n \rrbracket$.

Above, $\llbracket 2, n \rrbracket$ denotes the set of all the integers in the interval. It is well known (see for example [Sat99]) that under these assumptions, the operator L^{ou} generates a convolution Markov semigroup $(P_t^{ou})_{t \geq 0}$ on $B_b(\mathbb{R}^{nd})$, the family of all the bounded and Borel measurable functions on \mathbb{R}^{nd} , defined by

$$\begin{cases} P_t^{ou} \phi(\mathbf{x}) = \int_{\mathbb{R}^{nd}} \phi(\mathbf{x} + \mathbf{y}) \mu_t(d\mathbf{y}) & \text{for } t > 0, \\ P_0^{ou} \phi(\mathbf{x}) = \phi(\mathbf{x}). \end{cases}$$

where $(\mu_t)_{t \geq 0}$ is a family of Borel probability measures on \mathbb{R}^{nd} . In particular, the function $P_t^{ou} \phi(\mathbf{x})$ provides the classical solution to the Cauchy problem

$$\begin{cases} \partial_t u(t, \mathbf{x}) + L_\alpha u(t, \mathbf{x}) + \langle A\mathbf{x}, D_{\mathbf{x}} u(t, \mathbf{x}) \rangle = 0 & \text{on } (0, \infty) \times \mathbb{R}^{nd}, \\ u(0, \mathbf{x}) = \phi(\mathbf{x}) & \text{on } \mathbb{R}^{nd}. \end{cases} \quad (2.5)$$

Moving to the stochastic counterpart if necessary, it is readily derived from [PZ09] that the semigroup $(P_t^{ou})_{t \geq 0}$ admits a smooth density $p^{ou}(t, \cdot)$ with respect to the Lebesgue measure on \mathbb{R}^{nd} . Moreover, such a density p^{ou} has the following useful representation:

$$p^{ou}(t, \mathbf{x}, \mathbf{y}) = \frac{1}{\det \mathbb{M}_t} p_S(t, \mathbb{M}_t^{-1}(e^{At} \mathbf{x} - \mathbf{y})) \quad (2.6)$$

where p_S is the density of $(S_t)_{t \geq 0}$, a stable process in \mathbb{R}^{nd} whose Lévy measure satisfies the assumption **(ND)** above on \mathbb{R}^{nd} and \mathbb{M}_t is a diagonal matrix on $\mathbb{R}^{nd} \times \mathbb{R}^{nd}$ given by

$$[\mathbb{M}_t]_{i,j} := \begin{cases} t^{i-1} I_{d \times d}, & \text{if } i = j \\ 0_{d \times d}, & \text{otherwise.} \end{cases} \quad (2.7)$$

We remark already that the appearance of the matrix \mathbb{M}_t in Equation (2.6) and its particular structure reflect the multi-scaled structure of the dynamics considered (cf. Paragraph below for a more precise explanation). Moreover, the density p_S shows a useful property we will call the smoothing effect since it will be fundamental to reduce the singularities appearing when working with time integrals. Fixed γ in $[0, \alpha]$, there exists a constant $C := C(\gamma)$ such that for any l in $\llbracket 0, 3 \rrbracket$,

$$\int_{\mathbb{R}^{nd}} |\mathbf{y}|^\gamma |D_{\mathbf{y}}^l p_S(t, \mathbf{y})| d\mathbf{y} \leq Ct^{\frac{\gamma-l}{\alpha}} \quad \text{for any } t > 0. \quad (2.8)$$

These results can be proven following the arguments of Proposition 2.3 and Lemma 4.3 in [HMP19]. We will provide however a complete proof in the Appendix for the sake of completeness.

2.2 Intrinsic Time Scale and Associated Hölder spaces

In this section, we are going to choose which is the most suitable functional space in which to state our Schauder estimates.

To answer this question, we need firstly to understand how the system typically behaves. We focus for the moment on the Ornstein-Uhlenbeck case:

$$(\partial_t + L^{ou})u(t, \mathbf{x}) = -f(t, \mathbf{x}) \quad \text{on } (0, \infty) \times \mathbb{R}^{nd}$$

and search for a dilation operator $\delta_\lambda: (0, \infty) \times \mathbb{R}^{nd} \rightarrow (0, \infty) \times \mathbb{R}^{nd}$ that is invariant for the considered dynamics, i.e. a dilation that transforms solutions of the above equation into other solutions of the same equation.

Due to the structure of A and the α -stability of ν , we can consider for any fixed $\lambda > 0$, the following

$$\delta_\lambda(t, \mathbf{x}) := (\lambda^\alpha t, \lambda \mathbf{x}_1, \lambda^{1+\alpha} \mathbf{x}_2, \dots, \lambda^{1+\alpha(n-1)} \mathbf{x}_n),$$

i.e. with a slight abuse of notation, $(\delta_\lambda(t, \mathbf{x}))_0 := \lambda^\alpha t$ and for any i in $\llbracket 1, n \rrbracket$, $(\delta_\lambda(t, \mathbf{x}))_i := \lambda^{1+\alpha(i-1)} \mathbf{x}_i$. It then holds that

$$(\partial_t + L^{ou})u = 0 \implies (\partial_t + L^{ou})(u \circ \delta_\lambda) = 0.$$

The previous reasoning suggests us to introduce a parabolic distance d_P that is homogenous with respect to the dilation δ_λ , so that $d_P(\delta_\lambda(t, \mathbf{x}); \delta_\lambda(s, \mathbf{x}')) = \lambda d_P((t, \mathbf{x}); (s, \mathbf{x}'))$. Precisely, following the notations in [HMP19], we set for any s, t in $[0, T]$ and any \mathbf{x}, \mathbf{x}' in \mathbb{R}^{nd} ,

$$d_P((t, \mathbf{x}), (s, \mathbf{x}')) := |s - t|^{\frac{1}{\alpha}} + \sum_{j=1}^n |(\mathbf{x} - \mathbf{x}')_j|^{\frac{1}{1+\alpha(j-1)}}. \quad (2.9)$$

The idea of a dilation δ_λ that summarizes the multi-scaled behaviour of the dynamics was firstly introduced by Lanconelli and Polidoro in [LP94] for degenerate Kolmogorov equations in the diffusive setting. Since then, it has become a "standard" tool in the analysis of degenerate equations (see for example [Lun97], [HMP19] or [HWZ19]).

Since we will quite always use only the spatial part of the distance d_P , we denote for simplicity

$$d(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^n |(\mathbf{x} - \mathbf{y})_j|^{\frac{1}{1+\alpha(j-1)}}. \quad (2.10)$$

Technically speaking, d_P (and thus, d) does not however induce a norm on $[0, T] \times \mathbb{R}^{nd}$ in the usual sense since it lacks of linear homogeneity. We remark anyhow again that for any $\lambda > 0$, it precisely holds that $d(\delta_\lambda(t, \mathbf{x}); \delta_\lambda(s, \mathbf{x}')) = \lambda d((t, \mathbf{x}); (s, \mathbf{x}'))$. As it can be seen, d_P is an extension of the standard parabolic distance in the stable case, adapted to respect the multi-scaled nature of our dynamics. Indeed, the exponents appearing in (2.9) are those which make each space component homogeneous to the characteristic time scale $t^{1/\alpha}$.

The appearance of this kind of phenomena is due essentially by the particular structure of the matrix A (cf. Equation (1.1)) that allows the smoothing effect of L_α , acting only on the first variable, to propagate in the system, as it can be seen in the following lemma:

Lemma 1 (Scaling Lemma). *Let i be in $\llbracket 1, n \rrbracket$. Then, there exist $\{C_j\}_{j \in \llbracket 1, n \rrbracket}$ positive constants, depending only from A and i , such that*

$$D_{\mathbf{x}_i} p^{ou}(t, \mathbf{x}, \mathbf{y}) = - \sum_{j=i}^n C_j t^{j-i} D_{\mathbf{y}_j} p^{ou}(t, \mathbf{x}, \mathbf{y})$$

for any $t > 0$ and any \mathbf{x}, \mathbf{y} in \mathbb{R}^{nd} .

Proof. Recalling the representation of p^{ou} in Equation (2.6), it is easy to see that

$$D_{\mathbf{x}_i} p^{ou}(t, \mathbf{x}, \mathbf{y}) = \frac{1}{\det \mathbb{M}_t} D_{\mathbf{z}} p_S(t, \cdot) (\mathbb{M}_t^{-1} (e^{At} \mathbf{x} - \mathbf{y})) \mathbb{M}_t^{-1} D_{\mathbf{x}_i} [e^{At} \mathbf{x} - \mathbf{y}].$$

Hence, in order to conclude, we need to show that

$$D_{\mathbf{x}_i} [e^{At} \mathbf{x} - \mathbf{y}] = - \sum_{j=i}^n C_j t^{j-i} D_{\mathbf{y}_j} [e^{At} \mathbf{x} - \mathbf{y}]. \quad (2.11)$$

To prove the above equality, we need to analyze more in depth the structure of the resolvent e^{At} . Recalling from Equation (1.2) that A has a sub-diagonal structure, we notice that for any i, j in $\llbracket 1, n \rrbracket$,

$$[e^{At}]_{i,j} = \begin{cases} C_{i,j} t^{j-i}, & \text{if } j \geq i; \\ 0, & \text{otherwise,} \end{cases} \quad (2.12)$$

for a family of constants $\{C_{i,j}\}_{i,j \in \llbracket 1, n \rrbracket}$ depending only from A . It then follows that for any \mathbf{x}, \mathbf{y} in \mathbb{R}^{nd} , it holds that

$$[e^{At} \mathbf{x} - \mathbf{y}]_i = \sum_{k=1}^i C_{i,k} t^{i-k} \mathbf{x}_k - \mathbf{y}_i. \quad (2.13)$$

Equation (2.11) then follows immediately. For a more detailed proof of this result, see also [HM16] or [HMP19]. □

We finally remark the link with the stochastic counterpart of equation (1.1). From a more probabilistic point of view, the exponents in equation (2.9), can be related to the characteristic time scales of the iterated integrals of an α -stable process.

We are now ready to define the suitable Hölder spaces for our estimates. We start recalling some useful notations we will need below. Fixed k in $\mathbb{N} \cup \{0\}$ and β in $(0, 1)$, we follow Krylov [Kry96], denoting the usual *homogeneous* Hölder space $C^{k+\beta}(\mathbb{R}^d)$ as the family of functions $\phi: \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\|\phi\|_{C^{k+\beta}} := \sum_{i=1}^k \sup_{|\vartheta|=i} \|D^{\vartheta} \phi\|_{L^\infty} + \sup_{|\vartheta|=k} [D^{\vartheta} \phi]_\beta < \infty$$

where

$$[D^{\vartheta} \phi]_\beta := \sup_{\mathbf{x} \neq \mathbf{y}} \frac{|D^{\vartheta} \phi(\mathbf{x}) - D^{\vartheta} \phi(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\beta}.$$

Additionally, we are going to need the associated subspace $C_b^{k+\beta}(\mathbb{R}^d)$ of bounded functions in $C^{k+\beta}(\mathbb{R}^d)$, equipped with the norm

$$\|\cdot\|_{C_b^{k+\beta}} = \|\cdot\|_{L^\infty} + \|\cdot\|_{C^{k+\beta}}.$$

We can now define the anisotropic Hölder space with multi-index of regularity associated with the distance d . For sake of brevity and readability, we firstly define for a function $\phi: \mathbb{R}^{nd} \rightarrow \mathbb{R}$, a point z in $\mathbb{R}^{d(n-1)}$ and i in $\llbracket 1, n \rrbracket$, the function

$$\Pi_z^i \phi: x \in \mathbb{R}^d \rightarrow \phi(z_1, \dots, z_{i-1}, x, z_{i+1}, z_n)$$

with the obvious modifications if $i = 1$ or $i = n$. Intuitively speaking, the function $\Pi_z^i \phi$ is the restriction of ϕ on its i -th d -dimensional variable while fixing all the other coordinates in z . The space $C_d^{k+\beta}(\mathbb{R}^{nd})$ is then defined as the family of all the function $\phi: \mathbb{R}^{nd} \rightarrow \mathbb{R}$ such that

$$\|\phi\|_{C_d^{k+\beta}} := \sum_{i=1}^n \sup_{z \in \mathbb{R}^{d(n-1)}} \|\Pi_z^i \phi(x)\|_{C^{\frac{k+\beta}{1+\alpha(i-1)}}}.$$

The modification to the bounded subspace $C_{b,d}^{k+\beta}$ is straightforward.

Roughly speaking, the anisotropic norm works component-wise, i.e. we firstly fix a coordinate and then calculate the standard Hölder norm along that particular direction, but with index scaled according to the dilation of the system in that direction, uniformly over time and the other space components. We conclude summing the contributions associated with each component.

We highlight however that it is possible to recover the expected joint regularity for the partial derivatives, when they exist. In such a case, they actually turn out to be Hölder continuous in the pseudo-metric d with order one less than the function. (cf. Lemma 23 in the Appendix for the case $i = 1$).

Since we are working with evolution equations, the functions we consider will quite often depend on time, too. For this reason, we denote by $L^\infty(0, T, C_d^{k+\beta}(\mathbb{R}^{nd}))$ (respectively, $L^\infty(0, T, C_{b,d}^{k+\beta}(\mathbb{R}^{nd}))$) the family of functions $\psi: [0, T] \times \mathbb{R}^{nd} \rightarrow \mathbb{R}$ with finite $C_d^{k+\beta}$ -norm (respectively, $C_{b,d}^{k+\beta}$ -norm), uniformly in time.

2.3 Assumptions and Main Results

From this point further, we consider two fixed numbers α in $(0, 2)$ and β in $(0, 1)$ such that α will represent the index of stability of the operator L_α while β will stand for the index of Hölder regularity of the coefficients.

From this point further, we assume the following:

- (S) assumptions (ND) and (H) are satisfied and the drift $\mathbf{F} = (\mathbf{F}_1, \dots, \mathbf{F}_n)$ is such that for any i in $\llbracket 1, n \rrbracket$, \mathbf{F}_i depends only on time and on the last $n - (i - 1)$ components, i.e. $\mathbf{F}_i(t, \mathbf{x}_i, \dots, \mathbf{x}_n)$;
- (P) α is a number in $(0, 2)$, β is in $(0, 1)$ such that $\alpha + \beta \in (1, 2)$ and if $\alpha < 1$ (super-critical case),

$$\beta < \alpha, \quad 1 - \alpha < \frac{\alpha - \beta}{1 + \alpha(n - 1)};$$

- (R) Recalling the notations in Section 2.2, the source f is in $L^\infty(0, T; C_{b,d}^\beta(\mathbb{R}^{nd}))$, the terminal condition g is in $C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd})$ and for any i in $\llbracket 1, n \rrbracket$, \mathbf{F}_i belongs to $L^\infty(0, T; C_d^{\gamma_i+\beta}(\mathbb{R}^{nd}))$ where

$$\gamma_i := \begin{cases} 1 + \alpha(i - 2), & \text{if } i > 1; \\ 0, & \text{if } i = 1. \end{cases} \quad (2.14)$$

From now on, we will say that assumption (A) holds when the above conditions (S), (P) and (R) are in force.

Remark (About the Assumptions). We remark that the constraints (P) we are imposing in the super-critical case ($\alpha < 1$) seem quite natural for our system. The condition $\beta < \alpha$ reflects essentially the low integrability properties of the stable density p_S (cf. Equation (2.8)). Even if one is interested only on the fractional Laplacian case, i.e. $L_\alpha = \Delta^{\alpha/2}$, such a condition cannot be dropped in general, since it does not refer to the integrability property of p_α and its derivatives but instead to those of its "projection" p_S on the bigger space \mathbb{R}^{nd} (cf. Equation (2.6)).

About the second condition $\alpha + \beta > 1$, it is necessary to give a point-wise definition of the gradient of a solution u with respect to the non-degenerate variable x_1 . Moreover, there is a famous counterexample of Tanaka and his collaborators [TTW74] that shows that even in the scalar case, weak uniqueness (a direct consequence of Schauder estimates) may fail for the associated SDE if $\alpha + \beta$ is smaller than one.

The last assumption is indeed a technical constraint and it is necessary to work properly with the perturbation \mathbf{F} at any level $i = 1, \dots, n$. In particular, it seems the minimal threshold that allows us to exploit the smoothing effect of the density (see for example Equation (5.31) in the proof of Lemma 12 for more details). We conclude highlighting that these assumptions are always fulfilled if $\alpha \geq 1$ (sub-critical case).

At this stage, it should be clear that under our assumptions **(A)**, the IPDE (1.1) will be understood in a *distributional* sense. Indeed, we cannot hope to find a "classical" solution for (1.1), since for such a function u in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$, the total gradient $D_{\mathbf{x}}u$ is not defined point-wise.

Let us denote for any function $\phi: [0, T] \rightarrow \mathbb{R}^{nd}$ regular enough, the complete operator \mathcal{L}_α as

$$\mathcal{L}_\alpha \phi(t, \mathbf{x}) := \langle A\mathbf{x} + \mathbf{F}(t, \mathbf{x}), D_{\mathbf{x}}u(t, \mathbf{x}) \rangle + L_\alpha u(t, \mathbf{x}). \quad (2.15)$$

We will say that a function u in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ is a distributional (or weak) solution of the IPDE (1.1) if for any ϕ in $C_0^\infty((0, T] \times \mathbb{R}^{nd})$, it holds that

$$\int_0^T \int_{\mathbb{R}^{nd}} \left(-\partial_t + \mathcal{L}_\alpha^* \right) \phi(t, \mathbf{y}) u(t, \mathbf{y}) d\mathbf{y} + \int_{\mathbb{R}^{nd}} g(\mathbf{y}) \phi(T, \mathbf{y}) d\mathbf{y} = - \int_0^T \int_{\mathbb{R}^{nd}} \phi(t, \mathbf{y}) f(t, \mathbf{y}) d\mathbf{y} \quad (2.16)$$

where \mathcal{L}_α^* denotes the formal adjoint of \mathcal{L}_α . On the other hand, denoting from now on,

$$\|\mathbf{F}\|_H := \sup_{i \in \llbracket 1, n \rrbracket} \|\mathbf{F}_i\|_{L^\infty(C_d^{\gamma_i+\beta})}, \quad (2.17)$$

we will quite often use the following other notion of solution:

Definition 1. A function u is a mild solution in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ of Equation (1.1) if for any triple of sequences $\{f_m\}_{m \in \mathbb{N}}$, $\{g_m\}_{m \in \mathbb{N}}$ and $\{\mathbf{F}_m\}_{m \in \mathbb{N}}$ such that

- $\{f_m\}_{m \in \mathbb{N}}$ is in $C_b^\infty((0, T) \times \mathbb{R}^{nd})$ and f_m converges to f in $L^\infty(0, T; C_{b,d}^\beta(\mathbb{R}^{nd}))$;
- $\{g_m\}_{m \in \mathbb{N}}$ is in $C_b^\infty(\mathbb{R}^{nd})$ and g_m converges to g in $C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd})$;
- $\{\mathbf{F}_m\}_{m \in \mathbb{N}}$ is in $C_b^\infty((0, T) \times \mathbb{R}^{nd}; \mathbb{R}^{nd})$ and $\|\mathbf{F}_m - \mathbf{F}\|_H$ converges to 0,

there exists a sub-sequence $\{u_m\}_{m \in \mathbb{N}}$ in $C_b^\infty((0, T) \times \mathbb{R}^{nd})$ such that

- u_m converges to u in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$;
- for any fixed m in \mathbb{N} , u_m is a classical solution of the following "regularized" IPDE:

$$\begin{cases} \partial_t u_m(t, \mathbf{x}) + L_\alpha u_m(t, \mathbf{x}) + \langle A\mathbf{x} + \mathbf{F}_m(t, \mathbf{x}), D_{\mathbf{x}}u_m(t, \mathbf{x}) \rangle = -f_m(t, \mathbf{x}) & \text{on } (0, T) \times \mathbb{R}^{nd}, \\ u_m(T, \mathbf{x}) = g_m(\mathbf{x}) & \text{on } \mathbb{R}^{nd}. \end{cases} \quad (2.18)$$

We can now state our main result:

Theorem 1. (*Schauder Estimates*) Let u be a mild solution in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ of Equation (1.1). Under **(A)**, there exists a constant $C := C(T, (\mathbf{A}))$ such that

$$\|u\|_{L^\infty(C_d^{\alpha+\beta})} \leq C[\|f\|_{L^\infty(C_{b,d}^\beta)} + \|g\|_{C_{b,d}^{\alpha+\beta}}]. \quad (2.19)$$

Associated with an existence result we will exhibit in Section 6, we will eventually derive the well-posedness for Equation (1.1).

Theorem 2. Under **(A)**, there exists a unique mild solution u in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ of the IPDE (1.1). Moreover, such a function u is a weak solution, too.

In the following, we will denote for sake of brevity

$$\alpha_i := \frac{\alpha}{1 + \alpha(i-1)} \quad \text{and} \quad \beta_i := \frac{\beta}{1 + \alpha(i-1)} \quad \text{for any } i \text{ in } \llbracket 1, n \rrbracket. \quad (2.20)$$

Clearly, these quantities were introduced to reflect exactly the relative scale of the system at every considered level i (cf. Section 2.2 above).

In the following, as well as in Theorem 1 above, C denotes a generic constant that may change from line to line but depending only on the parameters in assumption **(A)**. Other dependencies that may occur are explicitly specified.

3 Proof through Perturbative Approach

As already said in the Introduction, our method of proof relies on a perturbative approach introduced in [CdRHM18] for the degenerate, Kolmogorov, diffusive setting.

Roughly speaking, we will firstly choose a suitable proxy for the equation of interest, i.e. an operator whose associated semigroup and density are known and that is close enough to the original one:

$$L_\alpha + \langle A\mathbf{x} + \mathbf{F}(t, \mathbf{x}), D\mathbf{x} \rangle.$$

Furthermore, we will exhibit suitable regularization properties for the proxy and in particular, we will show that it satisfies the Schauder estimates (2.19). This will be the purpose of the Sub-section 3.1.

In Sub-section 3.2 below, we will then expand a solution u of the IPDE (1.1) along the chosen proxy through a Duhamel-type formula and eventually show that the expansion error only brings a negligible contribution so that the Schauder estimates still holds for u . Due to our choice of method, this will be possible only adding some more assumptions on the system. Namely, we will assume in addition to be in a small time interval, so that the proxy and the original operator do not differ too much.

The last Sub-section 3.3 will finally show how to remove the additional assumption in order to prove the Schauder estimates (Theorem 1) through a scaling argument.

3.1 Frozen Semigroup

The crucial element in our approach consists in choosing wisely a suitable proxy operator along which to expand a solution u in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ of Equation (1.1). In order to deal with potentially unbounded perturbations \mathbf{F} , it is natural to use a proxy involving a non-zero first order term associated with a flow representing the dynamics driven by $A\mathbf{x} + \mathbf{F}$, the transport part of Equation (1.1) (see e.g. [KP10] or [CdRMP19]).

Remembering that we assume \mathbf{F} to be Hölder continuous, we know that there exists a solution of

$$\begin{cases} d\boldsymbol{\theta}_{\tau,s}(\boldsymbol{\xi}) = [A\boldsymbol{\theta}_{\tau,s}(\boldsymbol{\xi}) + \mathbf{F}(s, \boldsymbol{\theta}_{\tau,s}(\boldsymbol{\xi}))] ds \text{ on } [\tau, T], \\ \boldsymbol{\theta}_{\tau,\tau}(\boldsymbol{\xi}) = \boldsymbol{\xi}, \end{cases}$$

even if it may be not unique. For this reason, we are going to choose one particular flow, denoted by $\boldsymbol{\theta}_{\tau,s}(\boldsymbol{\xi})$, and consider it fixed throughout the work.

More precisely, given a freezing couple $(\tau, \boldsymbol{\xi})$ in $[0, T] \times \mathbb{R}^{nd}$, the flow will be defined on $[\tau, T]$ as

$$\boldsymbol{\theta}_{\tau,s}(\boldsymbol{\xi}) = \boldsymbol{\xi} + \int_\tau^s [A\boldsymbol{\theta}_{\tau,v}(\boldsymbol{\xi}) + \mathbf{F}(v, \boldsymbol{\theta}_{\tau,v}(\boldsymbol{\xi}))] dv. \quad (3.1)$$

We can now introduce the "frozen" IPDE associated with the chosen proxy:

$$\begin{cases} \partial_t \tilde{u}^{\tau,\boldsymbol{\xi}}(t, \mathbf{x}) + L_\alpha \tilde{u}^{\tau,\boldsymbol{\xi}}(t, \mathbf{x}) + \langle A\mathbf{x} + \mathbf{F}(t, \boldsymbol{\theta}_{\tau,t}(\boldsymbol{\xi})), D\mathbf{x} \tilde{u}^{\tau,\boldsymbol{\xi}}(t, \mathbf{x}) \rangle = -f(t, \mathbf{x}) & \text{on } (0, T) \times \mathbb{R}^{nd}, \\ \tilde{u}^{\tau,\boldsymbol{\xi}}(T, \mathbf{x}) = g(\mathbf{x}) & \text{on } \mathbb{R}^{nd}. \end{cases} \quad (3.2)$$

Remarking that the proxy operator $L_\alpha + \langle A\mathbf{x} + \mathbf{F}(t, \boldsymbol{\theta}_{\tau,t}(\boldsymbol{\xi})), D\mathbf{x} \rangle$ can be seen as an Ornstein-Uhlenbeck operator with an additional time-dependent component $\mathbf{F}(t, \boldsymbol{\theta}_{\tau,t}(\boldsymbol{\xi}))$, it is clear that under assumption (A), it generates a two parameters semigroups we will denote by $(\tilde{P}_{t,s}^{\tau,\boldsymbol{\xi}})_{t \leq s}$. Moreover, it admits a density given by

$$\tilde{p}^{\tau,\boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y}) = \frac{1}{\det(\mathbb{M}_{s-t})} p_S(s-t, \mathbb{M}_{s-t}^{-1}(\mathbf{y} - \tilde{\mathbf{m}}_{t,s}^{\tau,\boldsymbol{\xi}}(\mathbf{x}))), \quad (3.3)$$

remembering Equation (2.6) for the definition of p_S and with the following notation for the "frozen shift" $\tilde{\mathbf{m}}_{t,s}^{\tau,\boldsymbol{\xi}}(\mathbf{x})$:

$$\tilde{\mathbf{m}}_{t,s}^{\tau,\boldsymbol{\xi}}(\mathbf{x}) = e^{A(s-t)} \mathbf{x} + \int_t^s e^{A(s-v)} \mathbf{F}(v, \boldsymbol{\theta}_{\tau,v}(\boldsymbol{\xi})) dv. \quad (3.4)$$

We point out already the following important property of the shift $\tilde{\mathbf{m}}_{t,s}^{\tau,\boldsymbol{\xi}}(\mathbf{x})$:

Lemma 2. Let $t < s$ in $[0, T]$ and \mathbf{x} a point in \mathbb{R}^{nd} . Then,

$$\tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x}) = \boldsymbol{\theta}_{\tau,s}(\xi), \quad (3.5)$$

taking $\tau = t$ and $\xi = \mathbf{x}$.

Proof. We start noticing that by construction, $\tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x})$ satisfies

$$\tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x}) = \mathbf{x} + \int_t^s [A\tilde{\mathbf{m}}_{t,v}^{\tau,\xi}(\mathbf{x}) + \mathbf{F}(v, \boldsymbol{\theta}_{\tau,v}(\xi))] dv.$$

It then holds that

$$|\tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x})| \leq \int_t^s A|\tilde{\mathbf{m}}_{t,v}^{\tau,\xi}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x})| dv.$$

The above Equation (3.5) then follows immediately applying the Grönwall lemma. \square

Moreover, we can extend the smoothing effect (2.8) of p_S to the frozen density $\tilde{p}^{\tau,\xi}$ through the representation (3.3):

Lemma 3 (Smoothing effects of the frozen density). Let ϑ, ϱ be two multi-indexes in \mathbb{N}^n such that $|\varrho + \vartheta| \leq 3$ and γ in $[0, \alpha)$. Under (A), there exists a constant $C := C(\vartheta, \varrho, \gamma)$ such that

$$\int_{\mathbb{R}^{nd}} |D_{\mathbf{y}}^{\vartheta} D_{\mathbf{x}}^{\varrho} \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y})| d^\gamma(\mathbf{y}, \tilde{\mathbf{m}}_{\tau,s}^{\tau,\xi}(\mathbf{x})) d\mathbf{y} \leq C(s-t)^{\frac{\gamma}{\alpha} - \sum_{i=k}^n \frac{\vartheta_k + \varrho_k}{\alpha_k}} \quad (3.6)$$

for any $t < s$ in $[0, T]$, any \mathbf{x} in \mathbb{R}^{nd} and any frozen couple (τ, ξ) in $[0, T] \times \mathbb{R}^{nd}$. In particular, if $|\vartheta| \neq 0$, it holds for any ϕ in $C_d^\gamma(\mathbb{R}^{nd})$ that

$$|D_{\mathbf{x}}^{\vartheta} \tilde{P}_{t,s}^{\tau,\xi} \phi(\mathbf{x})| \leq C \|\phi\|_{C_d^\gamma} (s-t)^{\frac{\gamma}{\alpha} - \sum_{k=1}^n \frac{\vartheta_k}{\alpha_k}}. \quad (3.7)$$

Proof. Let us start assuming that $|\vartheta| = 1$ and $|\varrho| = 1$. The other cases can be treated in a similar way. Since p_S is the density of an α -stable process, we remember that the following α -scaling property

$$p_S(t, \mathbf{y}) = t^{-\frac{nd}{\alpha}} p_S(1, t^{-\frac{1}{\alpha}} \mathbf{y}) \quad (3.8)$$

holds for any $t > 0$ and any \mathbf{y} in \mathbb{R}^{nd} . Fixed i in $\llbracket 1, n \rrbracket$, we then denote for simplicity

$$\mathbb{T}_{s-t} := (s-t)^{\frac{1}{\alpha}} \mathbb{M}_{s-t}$$

and we calculate the derivative of $\tilde{p}^{\tau,\xi}$ with respect to \mathbf{x}_i through

$$\begin{aligned} |D_{\mathbf{x}_i} \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y})| &= \left| \frac{1}{\det(\mathbb{M}_{s-t})} D_{\mathbf{x}_i} [p_S(s-t, \mathbb{M}_{s-t}^{-1}(\tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x}) - \mathbf{y}))] \right| \\ &= \left| \frac{1}{\det(\mathbb{T}_{s-t})} D_{\mathbf{x}_i} [p_S(1, \mathbb{T}_{s-t}^{-1}(\tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x}) - \mathbf{y}))] \right| \\ &= \left| \frac{1}{\det(\mathbb{T}_{s-t})} D_{\mathbf{z}} p_S(1, \cdot) (\mathbb{T}_{s-t}^{-1}(\tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x}) - \mathbf{y})) \mathbb{T}_{s-t}^{-1} D_{\mathbf{x}_i}(\tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x})) \right|. \end{aligned}$$

where in the second equality we exploited the α -scaling property (3.8). From Equation (2.12) in the Scaling Lemma 1, we notice now that

$$|\mathbb{T}_{s-t}^{-1} D_{\mathbf{x}_i}(\tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x}))| = |\mathbb{T}_{s-t}^{-1} D_{\mathbf{x}_i}(e^{A(t-s)}(\mathbf{x}))| = (s-t)^{-\frac{1}{\alpha}} \sum_{k=i}^n C_k (s-t)^{-(k-1)} (s-t)^{k-i} \leq C(s-t)^{-\frac{1+\alpha(i-1)}{\alpha}}$$

and we use it to show that

$$|D_{\mathbf{x}_i} \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y})| \leq C(s-t)^{-\frac{1+\alpha(i-1)}{\alpha}} \frac{1}{\det(\mathbb{T}_{s-t})} |D_{\mathbf{z}} p_S(1, \cdot) (\mathbb{T}_{s-t}^{-1}(\tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x}) - \mathbf{y}))|.$$

Similarly, if we fix j in $\llbracket 1, n \rrbracket$, it holds that

$$|D_{\mathbf{y}_j} D_{\mathbf{x}_i} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| \leq C(s-t)^{-\frac{1}{\alpha_i} - \frac{1}{\alpha_j}} \frac{1}{\det(\mathbb{T}_{s-t})} |D_{\mathbf{z}}^2 p_S(1, \cdot)(\mathbb{T}_{s-t}^{-1}(\tilde{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x}) - \mathbf{y}))|.$$

It is then easy to show by iteration of the same argument that

$$|D_{\mathbf{y}}^{\vartheta} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| \leq C(s-t)^{-\sum_{k=1}^n \frac{\vartheta_k + \vartheta_k}{\alpha_k}} \frac{1}{\det(\mathbb{T}_{s-t})} |D_{\mathbf{z}}^{|\vartheta|} p_S(1, \cdot)(\mathbb{T}_{s-t}^{-1}(\tilde{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x}) - \mathbf{y}))|. \quad (3.9)$$

Control (3.6) follows immediately from the analogous smoothing effect for p_S (cf. Equation (2.8)) and the change of variables $\mathbf{z} = \mathbb{T}_{s-t}^{-1}(\tilde{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x}) - \mathbf{y})$. Indeed,

$$\begin{aligned} & \int_{\mathbb{R}^{nd}} |D_{\mathbf{y}}^{\vartheta} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| d^{\gamma}(\mathbf{y}, \tilde{\mathbf{m}}_{\tau, s}^{\tau, \xi}(\mathbf{x})) d\mathbf{y} \\ & \leq C(s-t)^{-\sum_{k=1}^n \frac{\vartheta_k + \vartheta_k}{\alpha_k}} \int_{\mathbb{R}^{nd}} \frac{1}{\det(\mathbb{T}_{s-t})} |D_{\mathbf{z}}^{|\vartheta|} p_S(1, \cdot)(\mathbb{T}_{s-t}^{-1}(\tilde{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x}) - \mathbf{y}))| d^{\gamma}(\mathbf{y}, \tilde{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x})) d\mathbf{y} \\ & = (s-t)^{-\sum_{k=1}^n \frac{\vartheta_k + \vartheta_k}{\alpha_k}} \int_{\mathbb{R}^{nd}} |D_{\mathbf{z}}^{|\vartheta|} p_S(1, \mathbf{z})| d^{\gamma}(\mathbb{T}_{s-t}(\mathbf{z}) + \tilde{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x}), \tilde{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x})) d\mathbf{y} \end{aligned}$$

To conclude, we notice that

$$d^{\gamma}(\mathbb{T}_{s-t}(\mathbf{z}) + \tilde{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x}), \tilde{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x})) \leq C \sum_{i=1}^n |(s-t)^{\frac{1+\alpha(i-1)}{\alpha}} \mathbf{z}_i|^{\frac{\gamma}{1+\alpha(i-1)}} = (s-t)^{\frac{\gamma}{\alpha}} \sum_{i=1}^n |\mathbf{z}_i|^{\frac{\gamma}{1+\alpha(i-1)}}$$

and use it to write that

$$\begin{aligned} & \int_{\mathbb{R}^{nd}} |D_{\mathbf{y}}^{\vartheta} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| d^{\gamma}(\mathbf{y}, \tilde{\mathbf{m}}_{\tau, s}^{\tau, \xi}(\mathbf{x})) d\mathbf{y} \\ & \leq C(s-t)^{\frac{\gamma}{\alpha} - \sum_{k=1}^n \frac{\vartheta_k + \vartheta_k}{\alpha_k}} \sum_{i=1}^n \int_{\mathbb{R}^{nd}} |D_{\mathbf{z}}^{|\vartheta|} p_S(1, \mathbf{z})| |\mathbf{z}_i|^{\frac{\gamma}{1+\alpha(i-1)}} d\mathbf{y} \leq C(s-t)^{\frac{\gamma}{\alpha} - \sum_{k=1}^n \frac{\vartheta_k + \vartheta_k}{\alpha_k}} \end{aligned}$$

where in the last passage we used the smoothing effect for p_S (Equation (2.8)), recalling that for any i in $\llbracket 1, n \rrbracket$, it holds that

$$\frac{\gamma}{1+\alpha(i-1)} \leq \gamma < \alpha$$

and we have thus the required integrability.

To prove instead the second inequality (3.7), we use a cancellation argument to write

$$\begin{aligned} |D_{\mathbf{x}}^{\vartheta} \tilde{P}_{t,s}^{\tau, \xi} \phi(\mathbf{x})| &= \left| \int_{\mathbb{R}^{nd}} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) [\phi(\mathbf{y}) - \phi(\tilde{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x}))] d\mathbf{y} \right| \\ &\leq \int_{\mathbb{R}^{nd}} |D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| |\phi(\mathbf{y}) - \phi(\tilde{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x}))| d\mathbf{y}. \end{aligned}$$

But since we assume ϕ to be in $C_d^{\gamma}(\mathbb{R}^{nd})$, we can control the last expression as

$$|D_{\mathbf{x}}^{\vartheta} \tilde{P}_{t,s}^{\tau, \xi} \phi(\mathbf{x})| \leq \|\phi\|_{C_d^{\gamma}} \int_{\mathbb{R}^{nd}} d^{\gamma}(\mathbf{y}, \tilde{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x})) |D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| d\mathbf{y} \leq C\|\phi\|_{C_d^{\gamma}} (s-t)^{\frac{\gamma}{\alpha} - \sum_{k=1}^n \frac{\vartheta_k}{\alpha_k}}$$

where in the last passage we used Equation (3.6). \square

We can define now our candidate to be the mild solution of the "frozen" IPDE. If it exists and it is smooth enough, such a candidate appears to be the representation of the solution of (3.2) obtained through the Duhamel principle. For this reason, the following expression:

$$\tilde{u}^{\tau, \xi}(t, \mathbf{x}) := \tilde{P}_{t,T}^{\tau, \xi} g(\mathbf{x}) + \int_t^T \tilde{P}_{t,s}^{\tau, \xi} f(s, \mathbf{x}) ds \quad \text{for any } (t, \mathbf{x}) \text{ in } [0, T] \times \mathbb{R}^{nd}, \quad (3.10)$$

will be called the Duhamel representation of the proxy. As it seems, under our assumption **(A)** such a representation is robust enough to satisfy Schauder estimates similar to (2.19). Since the proof of this result is quite long, we will postpone it to Section 4.2 for clarity.

Proposition 1. (Schauder Estimates for the Proxy) Under **(A)**, there exists a constant $C := C(T)$ such that

$$\|\tilde{u}^{\tau,\xi}\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \leq C[\|g\|_{C_{b,d}^{\alpha+\beta}} + \|f\|_{L^\infty(C_{b,d}^\beta)}] \quad (3.11)$$

for any freezing couple (τ, ξ) in $[0, T] \times \mathbb{R}^{nd}$.

We conclude this section showing that $\tilde{u}^{\tau,\xi}$ is indeed a mild solution in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ of the "frozen" IPDE (3.2). Moreover, the converse statement is also true. If regular enough, any solution of (3.2) corresponds to the Duhamel representation (3.10).

Proposition 2. Let us assume to be under assumption **(A)**. Then,

- the function $\tilde{u}^{\tau,\xi}$ defined in (3.10) is a mild solution in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ of the "frozen" IPDE (3.2) for any freezing couple (τ, ξ) in $[0, T] \times \mathbb{R}^{nd}$;
- Fixed a freezing couple (τ, ξ) in $[0, T] \times \mathbb{R}^{nd}$, let $\tilde{v}^{\tau,\xi}$ be a mild solution in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ of the IPDE (3.2). Then,

$$\tilde{v}^{\tau,\xi}(t, x) = \tilde{P}_{t,T}^{\tau,\xi} g(x) + \int_t^T \tilde{P}_{t,s}^{\tau,\xi} f(s, x) ds.$$

Proof. The first assertion is quite straightforward. Let $\{f_m\}_{m \in \mathbb{N}}$, $\{g_m\}_{m \in \mathbb{N}}$ and $\{\mathbf{F}_m\}_{m \in \mathbb{N}}$ be three sequences of smooth and bounded coefficients such that $f_m \rightarrow f$ in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$, $g_m \rightarrow g$ in $C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd})$ and $\|\mathbf{F}_m - \mathbf{F}\|_H \rightarrow 0$. Denoting now by $\left(\tilde{P}_{t,s}^{m,\tau,\xi}\right)_{t \leq s}$ the semigroup associated with the "regularized" operator

$$L_\alpha + \langle A\mathbf{x} + \mathbf{F}_m(t, \boldsymbol{\theta}_{\tau,t}(\xi)), D\mathbf{x} \rangle,$$

it is not difficult to show that for any fixed m in \mathbb{N} , the following

$$\tilde{u}_m^{\tau,\xi} := \tilde{P}_{t,T}^{m,\tau,\xi} g_m(x) + \int_t^T \tilde{P}_{t,s}^{m,\tau,\xi} f_m(s, x) ds$$

is a classical solution of the "frozen" IPDE (3.2) with regularized coefficients f_m, g_m and \mathbf{F}_m . A detailed guide of this result can be found, even if in the diffusive setting, in Lemma 3.3 in [KP10]. Using now the Schauder Estimates (3.11) for the regularized solutions $\tilde{u}_m^{\tau,\xi}$, it follows immediately that $\tilde{u}_m^{\tau,\xi} \rightarrow \tilde{u}^{\tau,\xi}$ in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ and thus, that $\tilde{u}^{\tau,\xi}$ is a mild solution of (3.2) in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$.

To prove the second statement, we start fixing a freezing couple (τ, ξ) in $[0, T] \times \mathbb{R}^{nd}$ and consider three sequences $\{f_m\}_{m \in \mathbb{N}}$, $\{g_m\}_{m \in \mathbb{N}}$ and $\{\mathbf{F}_m\}_{m \in \mathbb{N}}$ of bounded and smooth coefficients such that $f_m \rightarrow f$ in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$, $g_m \rightarrow g$ in $C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd})$ and $\|\mathbf{F}_m - \mathbf{F}\|_H \rightarrow 0$. They can be constructed through mollification.

Since $\tilde{v}^{\tau,\xi}$ is a mild solution of the "frozen" IPDE (3.2), we know that there exists a sequence $\{\tilde{v}_m^{\tau,\xi}\}_{m \in \mathbb{N}}$ of classical solutions of the "regularized frozen" IPDE (3.2) with coefficients f_m, g_m and \mathbf{F}_m such that $\tilde{v}_m^{\tau,\xi} \rightarrow \tilde{v}^{\tau,\xi}$ in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$. Fixed m in \mathbb{N} , we then denote

$$h_m(t, x) := \tilde{v}_m^{\tau,\xi}(t, x - \int_t^T e^{A(t-s)} \mathbf{F}_m(s, \boldsymbol{\theta}_{\tau,s}(\xi)) ds)$$

for any t in $[0, T]$ and any x in \mathbb{R}^{nd} . Direct calculations imply that

$$\begin{aligned} D_{\mathbf{x}} h_m(t, x) &= D_{\mathbf{x}} \tilde{v}_m^{\tau,\xi}(t, x - \int_t^T e^{A(t-s)} \mathbf{F}_m(s, \boldsymbol{\theta}_{\tau,s}(\xi)) ds) \\ L_\alpha h_m(t, x) &= L_\alpha \tilde{v}_m^{\tau,\xi}(t, x - \int_t^T e^{A(t-s)} \mathbf{F}_m(s, \boldsymbol{\theta}_{\tau,s}(\xi)) ds) \end{aligned}$$

and

$$\begin{aligned} \partial_t h_m(t, x) &= \partial_t \tilde{v}_m^{\tau,\xi}(t, x - \int_t^T e^{A(t-s)} \mathbf{F}_m(s, \boldsymbol{\theta}_{\tau,s}(\xi)) ds) \\ &\quad + \langle \mathbf{F}_m(t, \boldsymbol{\theta}_{\tau,t}(\xi)) - A \int_t^T e^{A(t-s)} \mathbf{F}_m(s, \boldsymbol{\theta}_{\tau,s}(\xi)) ds, D_{\mathbf{x}} \tilde{v}_m^{\tau,\xi}(t, x - \int_t^T e^{A(t-s)} \mathbf{F}_m(s, \boldsymbol{\theta}_{\tau,s}(\xi)) ds) \rangle. \end{aligned}$$

Remembering that $\tilde{v}_m^{\tau, \xi}$ is a classical solution of Equation (3.2) replacing therein f , g and \mathbf{F} with coefficients f_m, g_m and \mathbf{F}_m , it follows immediately that the function h_m solves for any m in \mathbb{N} the following:

$$\begin{cases} \partial_t h_m(t, \mathbf{x}) + L_\alpha h_m(t, \mathbf{x}) + \langle A\mathbf{x}, D_{\mathbf{x}} h_m(t, \mathbf{x}) \rangle = -l_m(t, \mathbf{x}), \\ h_m(T, \mathbf{x}) = g_m(\mathbf{x}) \end{cases} \quad (3.12)$$

where $l_m(t, \mathbf{x}) := f_m(t, \mathbf{x} - \int_t^T e^{A(t-s)} \mathbf{F}_m(s, \boldsymbol{\theta}_{\tau, s}(\xi)) ds)$.

Since we are going to exploit reasonings in Fourier spaces, we need however to have integrability properties on the solution h_m . For this reason, we introduce now a family $\{\rho_R\}_{R>0}$ of smooth functions such that any ρ_R is equal to 1 in $B(0, R)$ and vanishes outside $B(0, R+1)$. We then denote for any $R > 0$,

$$h_{m,R}(t, \mathbf{x}) := h_m(t, \mathbf{x}) \rho_R(\mathbf{x}).$$

It is then straightforward that $h_{m,R}$ solves

$$\begin{cases} \partial_t h_{m,R}(t, \mathbf{x}) + L_\alpha h_{m,R}(t, \mathbf{x}) + \langle A\mathbf{x}, D_{\mathbf{x}} h_{m,R}(t, \mathbf{x}) \rangle = -\tilde{l}_{m,R}(t, \mathbf{x}), \\ h_{m,R}(T, \mathbf{x}) = g_{m,R}(\mathbf{x}) \end{cases} \quad (3.13)$$

where $g_{m,R}(\mathbf{x}) = g_m(\mathbf{x}) \rho_R(\mathbf{x})$ and

$$\tilde{l}_{m,R}(t, \mathbf{x}) = \rho_R(\mathbf{x}) l_m(t, \mathbf{x}) + h_m(t, \mathbf{x}) L_\alpha \rho_R(\mathbf{x}) + \int_{\mathbb{R}^d} [h_m(t, \mathbf{x} + B\mathbf{y}) - h_m(t, \mathbf{x})] [\rho_R(\mathbf{x} + B\mathbf{y}) - \rho_R(\mathbf{x})] \nu_\alpha(d\mathbf{y}).$$

Noticing now that $\tilde{l}_{m,R}$ is integrable with integrable Fourier transform, we can apply the Fourier transform in space to equation (3.13) in order to write that

$$\begin{cases} \partial_t \hat{h}_{m,R}(t, \mathbf{p}) + \mathcal{F}_x([L_\alpha + \langle A\mathbf{x}, D_{\mathbf{x}} \rangle] h_{m,R})(t, \mathbf{p}) = -\hat{\tilde{l}}_{m,R}(t, \mathbf{p}), \\ \hat{h}_{m,R}(T, \mathbf{p}) = \hat{g}_{m,R}(\mathbf{p}). \end{cases}$$

We remember in particular that the above operator $L_\alpha + \langle A\mathbf{x}, D_{\mathbf{x}} \rangle$ has an associated Lévy symbol $\Psi^{ou}(\mathbf{p})$ and, following Section 3.3.2 in [App09], it holds that

$$\mathcal{F}_x([L_\alpha + \langle A\mathbf{x}, D_{\mathbf{x}} \rangle] h_{m,R})(t, \mathbf{p}) = \Psi^{ou}(\mathbf{p}) \hat{h}_{m,R}(t, \mathbf{p}).$$

We can then use it to show that $\hat{h}_{m,R}$ is a classical solution of the following equation:

$$\begin{cases} \partial_t \hat{h}_{m,R}(t, \mathbf{p}) + \Psi^{ou}(\mathbf{p}) \hat{h}_{m,R}(t, \mathbf{p}) = -\hat{\tilde{l}}_{m,R}(t, \mathbf{p}), \\ \hat{h}_{m,R}(T, \mathbf{p}) = \hat{g}_{m,R}(\mathbf{p}). \end{cases}$$

The above equation can be easily solved by integration in time, giving the following representation of $\hat{h}_{m,R}(t, \mathbf{p})$:

$$\hat{h}_{m,R}(t, \mathbf{p}) = e^{(T-t)\Psi^{ou}(\mathbf{p})} \hat{g}_{m,R}(\mathbf{p}) + \int_t^T e^{(s-t)\Psi^{ou}(\mathbf{p})} \hat{\tilde{l}}_{m,R}(s, \mathbf{p}) ds.$$

In order to go back to $\tilde{v}_m^{\tau, \xi}$, we apply now the inverse Fourier transform to write that

$$h_{m,R}(t, \mathbf{x}) = P_{T-t}^{ou} g_{m,R}(\mathbf{x}) + \int_t^T P_{s-t}^{ou} \tilde{l}_{m,R}(s, \mathbf{x}) ds,$$

remembering that $(P_t^{ou})_{t \geq 0}$ is the convolution Markov semigroup associated with the Ornstein-Uhlenbeck operator $L_\alpha + \langle A\mathbf{x}, D_{\mathbf{x}} \rangle$. Letting m go to ∞ , it then follows immediately that $g_{m,R} \rightarrow g_m$, $h_{m,R} \rightarrow h_m$ and $\tilde{l}_{m,R} \rightarrow l_m$. A change of variable allows us to show the Duhamel representation, at least in the regularized setting:

$$\tilde{v}_m^{\tau, \xi}(t, \mathbf{y}) = P_{T-t}^{ou} g_m\left(\mathbf{y} + \int_t^T e^{A(t-s)} \mathbf{F}_m(s, \boldsymbol{\theta}_{\tau, s}(\xi)) ds\right) + \int_t^T P_{s-t}^{ou} f_m\left(s, \mathbf{y} + \int_t^s e^{A(t-u)} \mathbf{F}_m(u, \boldsymbol{\theta}_{\tau, u}(\xi)) du\right) ds.$$

Letting m goes to zero and remembering that $\tilde{v}_m^{\tau, \xi} \rightarrow \tilde{v}^{\tau, \xi}$, $f_m \rightarrow f$, $g_m \rightarrow g$ and $\mathbf{F}_m \rightarrow \mathbf{F}$ in the right functional spaces, we can conclude that $\tilde{v}^{\tau, \xi} = \tilde{u}^{\tau, \xi}$. \square

3.2 Expansion along the Proxy

We are going to use now the "frozen" IPDE (3.2) in order to derive appropriate quantitative controls of a solution u of Equation (1.1). Up to now, the freezing parameters (τ, ξ) were set free but they will be later chosen appropriately depending on the control we aim to establish.

The main idea is to exploit the Duhamel formula (Proposition 2) for the proxy to expand any solution u of the original IPDE (1.1) along the proxy. To make things more precise, let u be a mild solution in $L(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ of the IPDE (1.1). Mollifying if necessary, it is possible to construct three sequences $\{f_m\}_{m \in \mathbb{N}}$, $\{g_m\}_{m \in \mathbb{N}}$ and $\{F_m\}_{m \in \mathbb{N}}$ of bounded and smooth functions with bounded derivatives such that $f_m \rightarrow f$ in $L^\infty(0, T; C_{b,d}^\beta(\mathbb{R}^{nd}))$, $g_m \rightarrow g$ in $C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd})$ and $\|F_m - F\|_H \rightarrow 0$. Since u is a mild solution of (1.1), we know that there exists a smooth sequence $\{u_m\}_{m \in \mathbb{N}}$ converging to u in $L(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ and such that for any fixed m in \mathbb{N} , u_m solves in a classical sense the "regularized" IPDE (2.18).

Exploiting now that F_m is bounded and smooth, we can define the "regularized" flow $\theta_{\tau,t}^m(\xi)$ as the *unique* flow satisfying

$$\theta_{\tau,t}^m(\xi) = \xi + \int_\tau^t [A\theta_{\tau,s}^m(\xi) + F_m(s, \theta_{\tau,s}^m(\xi))] ds, \quad t \in [\tau, T]. \quad (3.14)$$

It is then easy to notice that u_m is also a classical solution in $L(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ of

$$\partial_t u_m(t, \mathbf{x}) + L_\alpha u_m(t, \mathbf{x}) + \langle A\mathbf{x} + F_m(t, \theta_{\tau,t}^m(\xi)), D_{\mathbf{x}} u_m(t, \mathbf{x}) \rangle = -[f_m(t, \mathbf{x}) + R_m^{\tau, \xi}(s, \mathbf{x})]$$

on $(0, T) \times \mathbb{R}^{nd}$ with terminal condition g_m . Above, we have denoted

$$R_m^{\tau, \xi}(t, \mathbf{x}) := \langle F_m(t, \mathbf{x}) - F_m(t, \theta_{\tau,t}^m(\xi)), D_{\mathbf{x}} u_m(t, \mathbf{x}) \rangle. \quad (3.15)$$

Since clearly, $R_m^{\tau, \xi}$ is in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$, we can use the Duhamel Formula (Proposition 2) for the proxy to write that

$$u_m(t, \mathbf{x}) = \tilde{P}_{t,T}^{m, \tau, \xi} g_m(\mathbf{x}) + \int_t^T \tilde{P}_{t,s}^{m, \tau, \xi} [f_m(s, \mathbf{x}) + R_m^{\tau, \xi}(s, \mathbf{x})] ds, \quad (t, \mathbf{x}) \in (0, T) \times \mathbb{R}^{nd}$$

where $(\tilde{P}_{t,s}^{m, \tau, \xi})_{t \leq s}$ is the semigroup associated with the operator $L_\alpha + \langle A\mathbf{x} + F_m(t, \theta_{\tau,t}^m(\xi)), D_{\mathbf{x}} \rangle$.

The reasoning above is summarized in the following Duhamel-type formula that allows to expand any classical solution u_m of the "regularized" IPDE (2.18) along the "regularized frozen" proxy.

Proposition 3 (Duhamel Type Formula). *Let (τ, ξ) a freezing couple in $[0, T] \times \mathbb{R}^{nd}$. Under (A), any classical solution u_m of the "regularized" IPDE (2.18) can be represented as*

$$u_m(t, \mathbf{x}) = \tilde{u}_m^{\tau, \xi}(t, \mathbf{x}) + \int_t^T \tilde{P}_{t,s}^{m, \tau, \xi} R_m^{\tau, \xi}(s, \mathbf{x}) ds, \quad (t, \mathbf{x}) \in (0, T) \times \mathbb{R}^{nd} \quad (3.16)$$

where $R_m^{\tau, \xi}$ is as in (3.15) and $\tilde{u}_m^{\tau, \xi}$ is defined through the Duhamel representation (3.10) with the "regularized" coefficients f_m, g_m .

Thanks to the above representation (Equation (3.16)), we know that, since we have already shown the suitable control for the frozen solution $u_m^{\tau, \xi}$ (namely, Proposition 1 with f_m, g_m), the main term which remains to be investigated in order to show the Schauder Estimates (Theorem 1) is the remainder

$$\int_t^T \tilde{P}_{t,s}^{m, \tau, \xi} R_m^{\tau, \xi}(s, \mathbf{x}) ds, \quad (3.17)$$

that represents exactly the error in the expansion along the proxy.

To be precise, we could have passed to the limit in Equation (3.16) in order to obtain a similar Duhamel-type formula for a mild solution u in $L^\infty([0, T]; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$. However, a problem appears when trying to give a precise meaning at the limit for the remainder contribution (3.17). We already know that the limit exists point-wise by difference, but for our approach to work, we need to establish precise quantitative controls on this term. Such estimates could be obtained through duality techniques in Besov spaces (cf. Section 5.1) but

only at the expense of fixing already the freezing couple as $(\tau, \xi) = (t, \mathbf{x})$. The drawback of this method is that it does not allow to differentiate Equation (3.16), which is needed to estimate $D_{\mathbf{x}_1} u$.

In order to show the suitable estimates for (3.17), we will need at first an additional constraint on the behaviour of the system. In particular, we will say to be under assumption (\mathbf{A}') when assumption (\mathbf{A}) is considered and if moreover,

(ST) we assume to be in a small time interval, i.e. $T \leq 1$.

Under these stronger assumptions, we will then be able to show in Section 5 below that the following control holds:

Proposition 4 (A Priori Estimates). *Let u be a mild solution in $L^\infty([0, T]; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ of IPDE (1.1). Under (\mathbf{A}') , there exists a constant $C \geq 1$ such that*

$$\|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \leq C c_0^{\frac{\beta-\gamma n}{\alpha}} [\|g\|_{C_{b,d}^{\alpha+\beta}} + \|f\|_{L^\infty(C_{b,d}^\beta)}] + C (c_0^{\frac{\beta-\gamma n}{\alpha}} \|\mathbf{F}\|_H + c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}) \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \quad (3.18)$$

where $c_0 \in (0, 1)$ is assumed to be fixed but chosen later.

We remark already that in the above control, the constants multiplying $\|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})}$ have to be small if one wants to derive the expected Schauder estimates. If c_0 is small enough, then $C c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}$ can be made smaller than $1/4$. Anyhow, for this chosen small c_0 , the quantity $c_0^{\frac{\beta-\gamma n}{\alpha}}$ becomes large and therefore, it needs to be balanced with $C \|\mathbf{F}\|_H$. Namely, we can conclude if for instance, $C c_0^{\frac{\beta-\gamma n}{\alpha}} \|\mathbf{F}\|_H < 1/4$ that implies in particular that $\|\mathbf{F}\|_H$ has to be small with respect to c_0 . bbbbbbbbbb

3.3 Conclusion of Proof

In the first part of this section, we prove the Schauder estimates (Theorem 1) from the A Priori estimates (Proposition 4) through a suitable scaling procedure. Roughly speaking, the idea is to start from a general dynamics and then use the scaling procedure to make the Hölder norm $\|\mathbf{F}\|_H$ small enough in order to make a *circular* argument work. Again, if c_0 and $\|\mathbf{F}\|_H$ are small enough in (3.18), the $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ -norm of u on the right-hand side of (3.18) can be absorbed by the left-hand one. Once the Schauder estimates (2.19) holds in the scaled dynamics, we will conclude going back to the original IPDE through the inverse scaling procedure, even if for a small final time horizon T .

The second part of the section focuses on showing how to drop the additional assumption (\mathbf{A}') . The key point here is to proceed through iteration up to an arbitrary, but finite, given time T thanks to the stability of a solution u in the space $L^\infty([0, T], C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$.

3.3.1 Scaling Argument

Under (\mathbf{A}) , we start considering a mild solution u of the IPDE (1.1) on $[0, T]$ for some final time $T \leq 1$ to be fixed later. For a scaling parameter λ in $(0, 1]$ to be chosen later, we would like to analyze the IPDE (1.1) under the change of variables

$$(t, \mathbf{x}) \mapsto (\lambda t, \mathbb{T}_\lambda \mathbf{x}) \quad (3.19)$$

where $\mathbb{T}_\lambda := \lambda^{1/\alpha} \mathbb{M}_\lambda$. Again, the scaling is performed accordingly to the homogeneity induced by the distance d_P in (2.9).

To this purpose, we firstly introduce the scaled solution u_λ defined by

$$u_\lambda(t, \mathbf{x}) := u(\lambda t, \mathbb{T}_\lambda \mathbf{x}).$$

It then follows immediately that this function u_λ is a mild solution of

$$\begin{cases} \lambda^{-1} \partial_t u_\lambda(t, \mathbf{x}) + \lambda^{-1} L_\alpha u_\lambda + \langle A \mathbb{T}_\lambda \mathbf{x} + \mathbf{F}(\lambda t, \mathbb{T}_\lambda \mathbf{x}), \mathbb{T}_\lambda^{-1} D_{\mathbf{x}} u_\lambda(t, \mathbf{x}) \rangle = -f(\lambda t, \mathbb{T}_\lambda \mathbf{x}), & \text{on } (0, T_\lambda) \times \mathbb{R}^{nd}, \\ u_\lambda(T_\lambda, \mathbf{x}) = g(\mathbb{T}_\lambda \mathbf{x}) & \text{on } \mathbb{R}^{nd}, \end{cases}$$

where $T_\lambda := T/\lambda$. Since we want the scaled dynamics to satisfy assumption (\mathbf{A}') , we choose now T so that $T_\lambda \leq 1$. It is important to notice that this is possible since we assumed λ to be fixed, even if we have not

chosen it yet. Denoting now

$$\begin{aligned} f_\lambda(t, \mathbf{x}) &:= \lambda f(\lambda t, \mathbb{T}_\lambda \mathbf{x}); \\ g_\lambda(\mathbf{x}) &:= g(\mathbb{T}_\lambda \mathbf{x}); \\ A_\lambda &:= \lambda \mathbb{T}_\lambda^{-1} A \mathbb{T}_\lambda; \\ \mathbf{F}_\lambda(t, \mathbf{x}) &:= \lambda \mathbb{T}_\lambda^{-1} \mathbf{F}(\lambda t, \mathbb{T}_\lambda \mathbf{x}), \end{aligned}$$

we can rewrite the scaled dynamics as:

$$\begin{cases} \partial_t u_\lambda(t\mathbf{x}) + \langle A_\lambda \mathbf{x} + \mathbf{F}_\lambda(t\mathbf{x}), D_\mathbf{x} u_\lambda(t\mathbf{x}) \rangle + L_\alpha u_\lambda(t\mathbf{x}) = -f_\lambda(t\mathbf{x}), & \text{on } (0, T_\lambda) \times \mathbb{R}^{nd}, \\ u_\lambda(T_\lambda, \mathbf{x}) = g_\lambda(\mathbf{x}) & \text{on } \mathbb{R}^{nd}. \end{cases} \quad (3.20)$$

To continue, we need now the following lemma that exploits how the scaling procedure reflects on the norms of the coefficients. Recalling Equation (2.17) for the definition of $\|\cdot\|_H$, a direct calculation on the norms leads to the following result:

Lemma 4 (Scaling Homogeneity of Norms). *Under (A), it holds that*

$$\begin{aligned} \|\mathbf{F}_\lambda\|_H &= \lambda^{\beta/\alpha} \|\mathbf{F}\|_H; \\ \lambda^{\frac{\alpha+\beta}{\alpha}} \|f\|_{L^\infty(C_{b,d}^\beta)} &\leq \|f_\lambda\|_{L^\infty(C_{b,d}^\beta)} \leq \|f\|_{L^\infty(C_{b,d}^\beta)} \\ \lambda^{\frac{\alpha+\beta}{\alpha}} \|g\|_{C_{b,d}^{\alpha+\beta}} &\leq \|g_\lambda\|_{C_{b,d}^{\alpha+\beta}} \leq \|g\|_{C_{b,d}^{\alpha+\beta}}; \\ \lambda^{\frac{\alpha+\beta}{\alpha}} \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} &\leq \|u_\lambda\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \leq \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \end{aligned} \quad (3.21)$$

Since the scaled dynamics (3.20) satisfies assumption (A'), we know from Proposition 4 that the scaled solution u_λ satisfies the A Priori Estimates (Equation (3.18)):

$$\|u_\lambda\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \leq C c_0^{\frac{\beta-\gamma n}{\alpha}} [\|g_\lambda\|_{C_{b,d}^{\alpha+\beta}} + \|f_\lambda\|_{L^\infty(C_{b,d}^\beta)}] + C(c_0^{\frac{\beta-\gamma n}{\alpha}} \|\mathbf{F}_\lambda\|_H + c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}) \|u_\lambda\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \quad (3.22)$$

for some constant c_0 in $(0, 1]$ to be chosen later.

We would like now to exploit a circular argument in order to bring to the left-hand side of (3.22) the term involving u_λ on the right-hand one. To do that, we need to choose properly λ and c_0 in order to have

$$C(c_0^{\frac{\beta-\gamma n}{\alpha}} \|\mathbf{F}_\lambda\|_H + c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}) < 1.$$

This is true if for example we choose firstly c_0 such that

$$C c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}} = \frac{1}{4}$$

and fixed c_0 , we choose λ so that

$$C c_0^{\frac{\beta-\gamma n}{\alpha}} \lambda^{\beta/\alpha} \|\mathbf{F}\|_H = C c_0^{\frac{\beta-\gamma n}{\alpha}} \|\mathbf{F}_\lambda\|_H = \frac{1}{4}.$$

With this choice, it thus follows from (3.22) that

$$\|u_\lambda\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \leq 2C c_0^{\frac{\beta-\gamma n}{\alpha}} [\|g_\lambda\|_{C_{b,d}^{\alpha+\beta}} + \|f_\lambda\|_{L^\infty(C_{b,d}^\beta)}].$$

We can finally conclude using Lemma 4 to go back to the original dynamics and write that

$$\|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \leq \lambda^{-\frac{\alpha+\beta}{\alpha}} \|u_\lambda\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \leq \overline{C} [\|g\|_{C_{b,d}^{\alpha+\beta}} + \|f\|_{L^\infty(C_{b,d}^\beta)}]$$

for some constant $\overline{C} > 0$ defined by

$$\overline{C} := 2\lambda^{-\frac{\alpha+\beta}{\alpha}} C c_0^{\frac{\beta-\gamma n}{\alpha}}.$$

3.3.2 Schauder Estimates for General Time

Up to this point, we have assumed to be in a small enough final time horizon (i.e. $T \leq 1$) to let our procedure work. We are going now to extend the Schauder estimates (Equation (2.19)) to an arbitrary but fixed final time $T_0 > 0$. Our proof will consist essentially in a backward iterative procedure through a chain of identical differential dynamics on different, small enough, time intervals. We recall indeed that the Schauder estimates precisely provide a stability result in the chosen functional space.

Proposition 5. *Under (A), let $T_0 > T$ and u a mild solution in $L^\infty(0, T_0, C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ of the IPDE (1.1) on $[0, T_0]$ that satisfies the Schauder Estimates (Equation (2.19)) on $[0, T]$. Then, there exists a constant $C_0 := C_0(T_0)$ such that*

$$\|u\|_{L^\infty(0, T_0; C_{b,d}^{\alpha+\beta})} \leq C_0 \left[\|f\|_{L^\infty(0, T_0; C_{b,d}^\beta)} + \|g\|_{C_{b,d}^{\alpha+\beta}} \right].$$

Proof. Fixed $N = \lceil \frac{T_0}{T} \rceil$, we are going to consider a system of N Cauchy problems:

$$\begin{cases} \partial_t u_k(t, \mathbf{x}) + \langle A\mathbf{x} + \mathbf{F}(t, \mathbf{x}), D_{\mathbf{x}} u_k(t, \mathbf{x}) \rangle + L_{\alpha} u_k(t, \mathbf{x}) = -f(t, \mathbf{x}), & \text{on } ((1 - \frac{k}{N})T_0, (1 - \frac{k-1}{N})T_0) \times \mathbb{R}^{nd} \\ u_k((1 - \frac{k-1}{N})T_0, \mathbf{x}) = u_{k-1}((1 - \frac{k-1}{N})T_0, \mathbf{x}) & \text{on } \mathbb{R}^{nd}. \end{cases}$$

for $k = 1, \dots, N$ with the notation that $u_0(T_0, \mathbf{x}) = g(\mathbf{x})$. Reasoning iteratively, we find that any mild solution of the IPDE (1.1) on $[0, T_0]$ is also a mild solution of any of the equations of the system. Moreover, since any solution u_k is defined on $[(1 - \frac{k}{N})T_0, (1 - \frac{k-1}{N})T_0]$ and

$$(1 - \frac{k-1}{N})T_0 - (1 - \frac{k}{N})T_0 = \frac{k}{N}T_0 - \frac{k-1}{N}T_0 = \frac{1}{N}T_0 \leq T,$$

the Schauder Estimates (Equation (1)) hold for any u_k with terminal condition $u_{k-1}((1 - \frac{k-1}{N})T_0, \cdot)$. In particular,

$$\begin{aligned} \|u_k\|_{L^\infty((1 - \frac{k}{N})T_0, (1 - \frac{k-1}{N})T_0; C_{b,d}^{\alpha+\beta})} &\leq C \left[\|f\|_{L^\infty((1 - \frac{k}{N})T_0, (1 - \frac{k-1}{N})T_0; C_{b,d}^\beta)} + \|u_{k-1}((1 - \frac{k-1}{N})T_0, \cdot)\|_{C_{b,d}^{\alpha+\beta}} \right] \\ &\leq C^2 \left[\|f\|_{L^\infty((1 - \frac{k}{N})T_0, (1 - \frac{k-1}{N})T_0; C_{b,d}^\beta)} + \|f\|_{L^\infty((1 - \frac{k-1}{N})T_0, (1 - \frac{k-2}{N})T_0; C_{b,d}^\beta)} + \|u_{k-2}((1 - \frac{k-2}{N})T_0, \cdot)\|_{C_{b,d}^{\alpha+\beta}} \right] \\ &\leq C^2 \left[\|f\|_{L^\infty((1 - \frac{k}{N})T_0, (1 - \frac{k-2}{N})T_0; C_{b,d}^\beta)} + \|u_{k-2}((1 - \frac{k-2}{N})T_0, \cdot)\|_{C_{b,d}^{\alpha+\beta}} \right] \end{aligned}$$

since u_{k-1} satisfies the Schauder Estimates with terminal condition $u_{k-2}((1 - \frac{k-2}{N})T_0, \cdot)$. Applying the same procedure recursively, we finally find that

$$\|u_k\|_{L^\infty((1 - \frac{k}{N})T_0, (1 - \frac{k-1}{N})T_0; C_{b,d}^{\alpha+\beta})} \leq C^k \left[\|f\|_{L^\infty((1 - \frac{k}{N})T_0, T_0; C_{b,d}^\beta)} + \|g\|_{C_{b,d}^{\alpha+\beta}} \right].$$

Hence,

$$\|u\|_{L^\infty(0, T_0; C_{b,d}^{\alpha+\beta})} \leq C^N \left[\|f\|_{L^\infty(0, T_0; C_{b,d}^\beta)} + \|g\|_{C_{b,d}^{\alpha+\beta}} \right]$$

and we have concluded. \square

4 Schauder Estimates for the Proxy

The aim of this section is to show how to properly control a solution $\tilde{u}^{\tau, \xi}$ of the "frozen" IPDE (3.2) in order to prove the Schauder estimates (Proposition 1) for the proxy. We recall the definition of $\tilde{u}^{\tau, \xi}$ through the Duhamel representation (3.10). Namely, for any freezing couple (τ, ξ) in $[0, T] \times \mathbb{R}^{nd}$, it holds that

$$\tilde{u}^{\tau, \xi}(t, \mathbf{x}) = \tilde{P}_{t,T}^{\tau, \xi} g(\mathbf{x}) + \tilde{G}_{t,T}^{\tau, \xi} f(t, \mathbf{x}) \quad (4.1)$$

where we have denoted for simplicity with $(\tilde{G}_{v,r}^{\tau, \xi})_{t>v \geq 0}$ the family of Green kernels associated with the frozen density $\tilde{p}^{\tau, \xi}$. Namely, for any $v < r$ in $[0, T]$,

$$\tilde{G}_{v,r}^{\tau, \xi} f(t, x) := \int_v^r \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) f(s, \mathbf{y}) d\mathbf{y} ds. \quad (4.2)$$

We can then differentiate the above equation with respect to \mathbf{x}_1 so that to obtain an analogous Duhamel type representation for the derivative $D_{\mathbf{x}_1} \tilde{u}^{\tau, \xi}$:

$$D_{\mathbf{x}_1} \tilde{u}^{\tau, \xi}(t, \mathbf{x}) = D_{\mathbf{x}_1} \tilde{P}^{\tau, \xi}_{t, T} g(\mathbf{x}) + D_{\mathbf{x}_1} \tilde{G}^{\tau, \xi}_{t, T} f(t, \mathbf{x}) \quad (4.3)$$

It is then clear that in order to control $\tilde{u}^{\tau, \xi}(t, \mathbf{x})$ in the norm $\|\cdot\|_{L^\infty(C_{b, d}^{\alpha+\beta})}$, we can analyze separately the contributions appearing from the frozen semigroup $\tilde{P}^{\tau, \xi}_{t, T} g(\mathbf{x})$ and those from the frozen Green kernel $\tilde{G}^{\tau, \xi}_{t, T} f(t, \mathbf{x})$.

4.1 First Besov Control

We focus for the moment on the contribution in the Duhamel representation (4.1) associated with the source g that, as it will be seen, is the more delicate to treat. In the non-degenerate setting (i.e. with respect to \mathbf{x}_1), it precisely write:

$$D_{\mathbf{x}_1} \tilde{P}^{\tau, \xi} g(\mathbf{x}) = \int_{\mathbb{R}^{nd}} D_{\mathbf{x}_1} \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}, \mathbf{y}) g(\mathbf{y}) d\mathbf{y}.$$

Looking at the particular structure of $\tilde{p}^{\tau, \xi}$ (cf. Equation (3.3)), it can be seen from Lemma 1 that

Lemma 5. *Let i in $\llbracket 1, n \rrbracket$. Then, there exist constants $\{C_j\}_{j \in \llbracket i, n \rrbracket}$ such that*

$$D_{\mathbf{x}_i} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) = \sum_{j=i}^n C_j (s-t)^{j-i} D_{\mathbf{y}_j} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) \quad (4.4)$$

for any $t < s$ in $[0, T]$, any \mathbf{x}, \mathbf{y} in \mathbb{R}^{nd} and any freezing couple (τ, ξ) in $[0, T] \times \mathbb{R}^{nd}$.

We can now use equation (4.4) to rewrite $D_{\mathbf{x}_1} \tilde{P}^{\tau, \xi} g(\mathbf{x})$ as

$$|D_{\mathbf{x}_1} \tilde{P}^{\tau, \xi} g(\mathbf{x})| = \left| \int_{\mathbb{R}^{nd}} D_{\mathbf{x}_1} \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}, \mathbf{y}) g(\mathbf{y}) d\mathbf{y} \right| \leq C \sum_{j=1}^n (s-t)^{j-1} \left| \int_{\mathbb{R}^{nd}} D_{\mathbf{y}_j} \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}, \mathbf{y}) g(\mathbf{y}) d\mathbf{y} \right|. \quad (4.5)$$

Remembering that g is in $C_{b, d}^{\alpha+\beta}(\mathbb{R}^{nd})$ for $\alpha + \beta > 1$ by hypothesis, we know that it is differentiable with respect to the first (non-degenerate) variable \mathbf{x}_1 . Then, the above expression can be controlled easily for $j = 1$ as

$$\left| \int_{\mathbb{R}^{nd}} D_{\mathbf{y}_1} \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}, \mathbf{y}) g(\mathbf{y}) d\mathbf{y} \right| = \left| \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}, \mathbf{y}) D_{\mathbf{y}_1} g(\mathbf{y}) d\mathbf{y} \right| \leq \|D_{\mathbf{y}_1} g\|_{L^\infty} \leq \|g\|_{C_{b, d}^{\alpha+\beta}}$$

using integration by parts formula. We can then focus on the degenerate components in (4.5), i.e.

$$\left| \int_{\mathbb{R}^{nd}} D_{\mathbf{y}_j} \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}, \mathbf{y}) g(\mathbf{y}) d\mathbf{y} \right| \quad (4.6)$$

for some $j > 1$. Since g is not differentiable with respect to \mathbf{y}_j if $j > 1$, we cannot apply the same reasoning above but we will need a more subtle control. Our main idea will be to use the duality in Besov spaces to derive bounds for expression (4.6). Namely, we introduce for a given \mathbf{y} in \mathbb{R}^d ,

$$\mathbf{y}_{\setminus j} := (\mathbf{y}_1, \dots, \mathbf{y}_{j-1}, \mathbf{y}_{j+1}, \dots, \mathbf{y}_n) \in \mathbb{R}^{(n-1)d}.$$

With this definition at hand, we then denote for any function ϕ on \mathbb{R}^{nd} , the function $\phi(\mathbf{y}_{\setminus j}, \cdot)$ on \mathbb{R}^d with a slight abuse of notation as

$$\phi(\mathbf{y}_{\setminus j}, z) := \phi(\mathbf{y}_1, \dots, \mathbf{y}_{j-1}, z, \mathbf{y}_{j+1}, \dots, \mathbf{y}_n). \quad (4.7)$$

The key point now is to control the Hölder modulus of $g(\mathbf{y}_{\setminus j}, \cdot)$ on \mathbb{R}^d , uniformly in $\mathbf{y}_{\setminus j} \in \mathbb{R}^{(n-1)d}$. To do so, we will need the identification $C_b^{\alpha_j + \beta_j}(\mathbb{R}^d) = B_{\infty, \infty}^{\alpha_j + \beta_j}(\mathbb{R}^d)$ with the usual notations for the Besov spaces.

We recall now some useful definitions/characterizations about Besov spaces $B_{p, q}^{\tilde{\gamma}}(\mathbb{R}^d)$. For a more detailed analysis of this argument, we suggest the reader to see Section 2.6.4 of Triebel [Tri83]. For $\tilde{\gamma}$ in $(0, 1)$, q, p in $(0, +\infty]$, we define the Besov space of indexes $(\tilde{\gamma}, p, q)$ on \mathbb{R}^d as:

$$B_{p, q}^{\tilde{\gamma}}(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\mathcal{H}_{p, q}^{\tilde{\gamma}}} < +\infty\}$$

where $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class on \mathbb{R}^d and

$$\|f\|_{\mathcal{H}_{p,q}^{\tilde{\gamma}}} := \|(\phi_0 \hat{f})^\vee\|_{L^p} + \left(\int_0^1 v^{-\frac{\tilde{\gamma}}{\alpha}} \|\partial_v p_h(v, \cdot) * f\|_{L^p}^q dv \right)^{\frac{1}{q}} \quad (4.8)$$

with ϕ_0 a function in $C_0^\infty(\mathbb{R}^d)$ such that $\phi_0(0) \neq 0$ and p_h the isotropic α -stable heat kernel on \mathbb{R}^d , i.e. the stable density on \mathbb{R}^d whose Lévy symbol is equivalent to $|\lambda|^\alpha$.

We point out that the quantities in (4.8) are well-defined for any $q \neq +\infty$. The modifications for $q = +\infty$ are obvious and can be written passing to the limit. The previous definition of $B_{p,q}^{\tilde{\gamma}}(\mathbb{R}^d)$ is known as the stable thermic characterization of Besov spaces and it is particularly adapted to our framework. By a little abuse of notation, we will write $\|f\|_{B_{p,q}^{\tilde{\gamma}}} := \|f\|_{\mathcal{H}_{p,q}^{\tilde{\gamma}}}$ when this quantity is finite.

For the heat-kernel p_h , it is possible to show an improvement of the smoothing effect (cf. equation (2.8)), due essentially to its better decay at infinity. Namely, we are no more bounded to the condition $\gamma < \alpha$ but we can integrate up to an order γ strictly smaller than $1 + \alpha$.

Lemma 6 (Smoothing Effect of the Isotropic Stable Heat-Kernel). *Let l be in $\{1, 2\}$ and γ in $[0, 1 + \alpha)$. Then, there exists a positive constant $C := C(\gamma)$ such that*

$$\int_{\mathbb{R}^d} |y|^\gamma |\partial_v D_y^l p_h(v, y)| dy \leq C t^{\frac{\gamma-l}{\alpha}-1}. \quad (4.9)$$

A proof of the above result can be derived using the estimates of Kolokoltsov [Kol00] (see also [BJ07]).

As already indicated before, it can be seen from the α -thermic characterization (4.8) that

$$C_b^{\tilde{\gamma}}(\mathbb{R}^d) = B_{\infty,\infty}^{\tilde{\gamma}}(\mathbb{R}^d). \quad (4.10)$$

Moreover, it is well known (see for example Proposition 3.6 in [LR02]) that $B_{\infty,\infty}^{\tilde{\gamma}}(\mathbb{R}^d)$ and $B_{1,1}^{-\tilde{\gamma}}(\mathbb{R}^d)$ are in duality. Namely, it holds

$$\left| \int_{\mathbb{R}^d} f g dx \right| \leq C \|f\|_{B_{\infty,\infty}^{\tilde{\gamma}}} \|g\|_{B_{1,1}^{-\tilde{\gamma}}}. \quad (4.11)$$

for any f in $B_{\infty,\infty}^{\tilde{\gamma}}(\mathbb{R}^d)$ and any function g in $B_{1,1}^{-\tilde{\gamma}}(\mathbb{R}^d)$.

With these definitions and properties at hand, we can now go back at expression (4.6) to write that

$$\begin{aligned} \left| \int_{\mathbb{R}^{nd}} D_{\mathbf{y}_j} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}, \mathbf{y}) g(\mathbf{y}) d\mathbf{y} \right| &\leq \int_{\mathbb{R}^{(n-1)d}} \left| D_{\mathbf{y}_j} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}, \mathbf{y}) g(\mathbf{y}) d\mathbf{y}_j \right| d\mathbf{y}_{\setminus j} \\ &\leq \int_{\mathbb{R}^{(n-1)d}} \left\| D_{\mathbf{y}_j} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}, \mathbf{y}_{\setminus j}, \cdot) \right\|_{B_{1,1}^{-(\alpha_j + \beta_j)}} \left\| g(\mathbf{y}_{\setminus j}, \cdot) \right\|_{B_{\infty,\infty}^{\alpha_j + \beta_j}} d\mathbf{y}_{\setminus j} \\ &\leq \|g\|_{C_{b,d}^{\alpha+\beta}} \int_{\mathbb{R}^{(n-1)d}} \left\| D_{\mathbf{y}_j} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}, \mathbf{y}_{\setminus j}, \cdot) \right\|_{B_{1,1}^{-(\alpha_j + \beta_j)}} d\mathbf{y}_{\setminus j}. \end{aligned}$$

In order to control the above quantities, we will then need a control on the integral of the Besov norms of the derivatives of the proxy. Since however an additional derivative with respect to \mathbf{x}_1 will often appear, for example in Equation (4.24) below, we state the following result in a more general way.

Lemma 7 (First Besov Control). *Let j be in $\llbracket 2, n \rrbracket$ and $l \in \{0, 1\}$. Under (A), there exists a constant $C := C(j, l)$ such that*

$$\int_{\mathbb{R}^{(n-1)d}} \left\| D_{\mathbf{y}_j} D_{\mathbf{x}_1}^l \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \cdot) \right\|_{B_{1,1}^{-(\alpha_j + \beta_j)}} d\mathbf{y}_{\setminus j} \leq C(s-t)^{\frac{\alpha+\beta}{\alpha} - \frac{1}{\alpha_j} - \frac{l}{\alpha}}$$

for any $t < s$ in $[0, T]$, any \mathbf{x} in \mathbb{R}^{nd} and any frozen couple (τ, ξ) in $[0, T] \times \mathbb{R}^{nd}$.

Proof. To control the Besov norm in $B_{1,1}^{-(\alpha_j + \beta_j)}(\mathbb{R}^d)$, we are going to use the stable thermic characterization (4.8) with $\tilde{\gamma} = -(\alpha_j + \beta_j)$. We start considering the second term in the characterization, i.e.

$$\int_0^1 v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \partial_v p_h(v, z - \mathbf{y}_j) D_{\mathbf{y}_j} D_{\mathbf{x}_1}^l \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y}) d\mathbf{y}_j \right| dz dv.$$

Fixed a constant $\delta_j \geq 1$ to be chosen later, we split the integral with respect to v in two components:

$$\begin{aligned} & \|D_{\mathbf{y}_j} D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \cdot)\|_{B_{1,1}^{-(\alpha_j + \beta_j)}} \\ &= \int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \partial_v p_h(v, z - \mathbf{y}_j) D_{\mathbf{y}_j} D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) d\mathbf{y}_j \right| dz dv \\ &+ \int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \partial_v p_h(v, z - \mathbf{y}_j) D_{\mathbf{y}_j} D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) d\mathbf{y}_j \right| dz dv =: (I_1 + I_2)(\mathbf{y}_{\setminus j}). \end{aligned}$$

The second component I_2 has no time-singularity and can be easily controlled by

$$I_2(\mathbf{y}_{\setminus j}) = \int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} D_z \partial_v p_h(v, z - \mathbf{y}_j) \otimes D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) d\mathbf{y}_j \right| dz dv$$

using integration by parts formula and noticing that $D_{\mathbf{y}_j} p_h(v, z - \mathbf{y}_j) = -D_z p_h(v, z - \mathbf{y}_j)$. Then,

$$I_2(\mathbf{y}_{\setminus j}) \leq \int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |D_z \partial_v p_h(v, z - \mathbf{y}_j)| |D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| d\mathbf{y}_j dz dv.$$

We can then use Fubini theorem to separate the integrals and apply the smoothing effect of the heat-kernel p_h (Lemma 6) to show that

$$\begin{aligned} I_2(\mathbf{y}_{\setminus j}) &\leq \int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |D_z \partial_v p_h(v, z - \mathbf{y}_j)| dz \right) |D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| d\mathbf{y}_j dv \\ &\leq C \left(\int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j + \beta_j - 1}{\alpha} - 1} dv \right) \left(\int_{\mathbb{R}^d} |D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| d\mathbf{y}_j \right) \\ &\leq C(s-t)^{\frac{\delta_j(\alpha_j + \beta_j - 1)}{\alpha}} \int_{\mathbb{R}^d} |D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| d\mathbf{y}_j. \end{aligned}$$

Using the smoothing effect (Equation (3.6)) of the frozen density $\tilde{p}^{\tau, \xi}$, we have thus found that

$$\int_{\mathbb{R}^{(n-1)d}} I_2(\mathbf{y}_{\setminus j}) d\mathbf{y}_{\setminus j} \leq (s-t)^{\frac{\delta_j(\alpha_j + \beta_j - 1)}{\alpha}} \int_{\mathbb{R}^{nd}} |D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| d\mathbf{y} \leq C(s-t)^{\frac{\delta_j(\alpha_j + \beta_j - 1) - l}{\alpha}}. \quad (4.12)$$

On the other hand, the term I_1 needs a more delicate treatment in order to avoid time-integrability problems. We start using a cancellation argument with respect to the derivative $\partial_v p_h$ of the heat-kernel to rewrite I_1 as

$$\begin{aligned} I_1(\mathbf{y}_{\setminus j}) &= \int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \partial_v p_h(v, z - \mathbf{y}_j) [D_{\mathbf{y}_j} D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) - D_{\mathbf{y}_j} D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, z)] d\mathbf{y}_j \right| dz dv \\ &= \int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} D_z \partial_v p_h(v, z - \mathbf{y}_j) \otimes [D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) - D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, z)] d\mathbf{y}_j \right| dz dv \end{aligned}$$

where in the second passage we used again integration by parts formula to move the derivative to p_h and the equality $D_{\mathbf{y}_j} p_h(v, z - \mathbf{y}_j) = -D_z p_h(v, z - \mathbf{y}_j)$. We can then apply a Taylor expansion with respect to variable \mathbf{y}_j to write that

$$\begin{aligned} I_1(\mathbf{y}_{\setminus j}) &= \int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} D_z \partial_v p_h(v, z - \mathbf{y}_j) \int_0^1 D_{\mathbf{y}_j} D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \mathbf{y}_j + \lambda(z - \mathbf{y}_j)) \cdot (z - \mathbf{y}_j) d\mu d\mathbf{y}_j \right| dz dv \\ &\leq \int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 |D_z \partial_v p_h(v, z - \mathbf{y}_j)| |D_{\mathbf{y}_j} D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \mathbf{y}_j + \lambda(z - \mathbf{y}_j))| |z - \mathbf{y}_j| d\lambda d\mathbf{y}_j dz dv \end{aligned}$$

We can then use Fubini theorem and changes of variables $\tilde{z} = z - \mathbf{y}_j$ (fixed \mathbf{y}_j) and $\tilde{\mathbf{y}}_j = \mathbf{y}_j + \lambda \tilde{z}$ (considering \tilde{z} and λ fixed) to separate the integrals so that

$$I_1(\mathbf{y}_{\setminus j}) \leq \int_0^{(s-t)^{\delta_i}} v^{\frac{\alpha_j + \beta_j}{\alpha}} \left(\int_{\mathbb{R}^d} |D_z \partial_v p_h(v, \tilde{z})| |\tilde{z}| dz \right) \left(\int_{\mathbb{R}^d} |D_{\mathbf{y}_j} D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \tilde{\mathbf{y}}_j)| d\mathbf{y}_j \right) dv.$$

The smoothing effect of the heat-kernel p_h (Lemma 6) allows now to control the first term:

$$\begin{aligned} I_1(\mathbf{y}_{\setminus j}) &\leq C \left(\int_0^{(s-t)^{\delta_i}} v^{\frac{\alpha_j + \beta_j - 1}{\alpha}} dv \right) \left(\int_{\mathbb{R}^d} |D_{\mathbf{y}_j} D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, z + \lambda(\mathbf{y}_j - z))| d\mathbf{y}_j \right) \\ &\leq C(s-t)^{\delta_j \frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} |D_{\mathbf{y}_j} D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, z + \lambda(\mathbf{y}_j - z))| d\mathbf{y}_j. \end{aligned}$$

It then follows using the smoothing effect of the frozen semigroup (Lemma 3) that

$$\begin{aligned} \int_{\mathbb{R}^{(n-1)d}} I_1(\mathbf{y}_{\setminus j}) d\mathbf{y}_{\setminus j} &\leq C(s-t)^{\delta_j \frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^{nd}} |D_{\mathbf{y}_j} D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, z + \lambda(\mathbf{y}_j - z))| d\mathbf{y} \\ &\leq C(s-t)^{\delta_j \frac{\alpha_j + \beta_j}{\alpha} - \frac{1}{\alpha} - \frac{1}{\alpha_j}}. \quad (4.13) \end{aligned}$$

Going back to equations (4.12) and (4.13), we notice that we need δ_j to be such that

$$\delta_j \left[\frac{\alpha_j + \beta_j}{\alpha} \right] = \frac{\alpha + \beta}{\alpha} \quad \text{and} \quad \delta_j \left[\frac{\alpha_j + \beta_j - 1}{\alpha} \right] = \frac{\alpha + \beta}{\alpha} - \frac{1}{\alpha_j}.$$

Recalling Equation (2.20) for the relative definitions, we can thus conclude choosing $\delta_j = (\alpha + \beta)/(\alpha_j + \beta_j) = 1 + \alpha(j-1)$.

Reproducing the previous computations, we can also write for a test function in ϕ_0 in $C_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned} \int_{\mathbb{R}^{(n-1)d}} \left\| \left(\phi_0(D_{\mathbf{y}_j} D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \cdot))^{\wedge} \right)^{\vee} \right\|_{L^1} d\mathbf{y}_{\setminus j} \\ = \int_{\mathbb{R}^{(n-1)d}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} D_{\mathbf{y}_j} \hat{\phi}_0(z - \mathbf{y}_j) \cdot D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) d\mathbf{y} \right| dz d\mathbf{y}_{\setminus j} \\ \leq C \int_{\mathbb{R}^{nd}} |D_{\mathbf{x}_1}^l \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| d\mathbf{y} \leq C(s-t)^{-\frac{1}{\alpha}}. \end{aligned}$$

□

4.2 Proof of Proposition 1

Thanks to the First Besov Control (Lemma 7), we are now ready to prove the Schauder Estimates for the proxy (Proposition 1). Such a proof will be divided in three parts: the estimates for the supremum norms of the solution and its non-degenerate gradient are stated in Lemma 8 while the controls of the Hölder moduli of the solution and its gradient with respect to the non-degenerate variable are given in Lemmas 9 and 10, respectively.

Lemma 8. (Controls on Supremum Norm) Under (A), there exists a constant $C := C(T) \geq 1$ such that for any freezing couple (τ, ξ) in $[0, T] \times \mathbb{R}^{nd}$, any t in $[0, T]$ and any \mathbf{x} in \mathbb{R}^{nd} ,

$$|\tilde{u}^{\tau, \xi}(t, \mathbf{x})| + |D_{\mathbf{x}_1} \tilde{u}^{\tau, \xi}(t, \mathbf{x})| \leq C \left[\|g\|_{C_{b,d}^{\alpha+\beta}} + \|f\|_{L^\infty(C_{b,d}^\beta)} \right].$$

Proof. We start noticing that $\tilde{P}_{t,T}^{\tau, \xi} g(\mathbf{x})$ and $\tilde{G}_{t,T}^{\tau, \xi} f(t, \mathbf{x})$ can be easily bounded using the supremum norm of f and g , respectively.

Moreover, we can use the controls on the frozen semigroup (Equation (3.7)) to control $D_{\mathbf{x}_1} \tilde{G}_{t,T}^{\tau, \xi} f(t, \mathbf{x})$. Indeed,

$$|D_{\mathbf{x}_1} \tilde{G}_{t,T}^{\tau, \xi} f(t, \mathbf{x})| \leq \int_t^T |D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau, \xi} f(s, \mathbf{x})| ds \leq C(T-t)^{\frac{\alpha+\beta-1}{\alpha}} \|f\|_{L^\infty(C^\beta)} \leq CT^{\frac{\alpha+\beta-1}{\alpha}} \|f\|_{L^\infty(C^\beta)}$$

remembering in the last inequality that $\alpha + \beta - 1 > 0$ by hypothesis **(P)**.

It remains to control $D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau,\xi} g(\mathbf{x})$. As shown the previous Sub-section 4.1, we start using the scaling lemma 5 to write that

$$\begin{aligned} |D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau,\xi} g(\mathbf{x})| &= \left| \int_{\mathbb{R}^{nd}} D_{\mathbf{x}_1} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}, \mathbf{y}) g(\mathbf{y}) d\mathbf{y} \right| \\ &\leq C \sum_{j=1}^n (T-t)^{j-1} \left| \int_{\mathbb{R}^{nd}} D_{\mathbf{y}_j} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}, \mathbf{y}) g(\mathbf{y}) d\mathbf{y} \right| =: C \sum_{j=1}^n (T-t)^{j-1} J_j. \end{aligned}$$

Since g is differentiable in the first, non-degenerate variable \mathbf{x}_1 , the contribution J_1 can be easily bounded using integration by parts formula:

$$J_1 = \left| \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}, \mathbf{y}) D_{\mathbf{y}_1} g(\mathbf{y}) d\mathbf{y} \right| \leq \|D_{\mathbf{y}_1} g\|_{L^\infty} \leq \|g\|_{C_{b,d}^{\alpha+\beta}}. \quad (4.14)$$

To control the other terms J_j for $j > 1$, we use instead the duality in Besov spaces (4.11) and the identification (4.10), so that

$$J_j \leq C \|g\|_{C_{b,d}^{\alpha+\beta}} \int_{\mathbb{R}^{(n-1)d}} \|D_{\mathbf{y}_j} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}, \mathbf{y}_{\setminus j}, \cdot)\|_{B_{1,1}^{-(\alpha_j+\beta_j)}} d\mathbf{y}_{\setminus j} \leq C \|g\|_{C_{b,d}^{\alpha+\beta}} (T-t)^{\frac{\alpha+\beta}{\alpha} - \frac{1}{\alpha_j}} \quad (4.15)$$

where in the last inequality we applied the first Besov Control (Lemma 7).

Looking back at equations (4.14) and (4.15), it finally holds that

$$|D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau,\xi} g(\mathbf{x})| \leq C \|g\|_{C_{b,d}^{\alpha+\beta}} \left(1 + \sum_{j=2}^n (T-t)^{j-1} (T-t)^{\frac{\alpha+\beta}{\alpha} - \frac{1}{\alpha_j}} \right) \leq C (1 + T^{\frac{\alpha+\beta-1}{\alpha}}) \|g\|_{C_{b,d}^{\alpha+\beta}}$$

where in the last passage we used again that $\alpha + \beta - 1 > 0$ by hypothesis **(P)**. \square

Before starting with the calculations on the Hölder modulus. For fixed $(t, \mathbf{x}, \mathbf{x}')$ in $[0, T] \times \mathbb{R}^{2nd}$, we will need to distinguish two cases. We will say that the *off-diagonal regime* holds if $T-t \leq c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$ for a constant c_0 to be specified but meant to be smaller than 1. This means in particular that the spatial distance is larger than the characteristic time-scale up to the prescribed constant c_0 which will be useful further on in the computations for a circular argument.

On the other hand, we will say that the global diagonal regime is in force when $T-t \geq c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$ and the spatial points are instead closer than the typical time-scale magnitude. In particular, when a time integration is involved (for example in the control of the frozen Green kernel), the same two regime appears even if in a local base. Considering a variable s in $[t, T]$, there are again a local off-diagonal regime if $s-t \leq c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$ and a local diagonal regime when $s-t \geq c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$. In particular we will denote with t_0 the critical time at which a change of regime occurs in the globally diagonal regime. Namely,

$$t_0 := (t + c_0 d^\alpha(\mathbf{x}, \mathbf{x}')) \wedge T. \quad (4.16)$$

We highlight however that this approach was already used in [CdRHM18] to obtain Schauder estimates for degenerate Kolmogorov equations and can be adapted in the current setting.

Moreover, it is important to notice that the norm $\|\cdot\|_{C_d^{\alpha+\beta}}$ is essentially defined as the sum of the norms $\|\cdot\|_{C^{\frac{\alpha+\beta}{1+\alpha(i-1)}}}$ with respect to the i -th variable and uniformly on the other components. Thus, there is a big difference between the case $i = 1$ where $\alpha + \beta$ is in $(1, 2)$ and we have to deal with a proper derivative and the other situations ($i > 1$) where instead $(\alpha + \beta)/(1 + \alpha(i - 1)) < 1$ and the norm is calculated directly on the function. For this reason, we are going to analyze the two cases separately. Lemma 9 will work on the non-degenerate setting ($i = 1$) while Lemma 10 will concern the degenerate one ($i > 1$).

Lemma 9 (Controls on Hölder Moduli: Non-Degenerate). *Let \mathbf{x}, \mathbf{x}' be in \mathbb{R}^{nd} such that $\mathbf{x}_j = \mathbf{x}'_j$ for any $j \neq 1$. Under **(A)**, there exists a constant $C \geq 1$ such that for any t in $[0, T]$ and any freezing couple (τ, ξ) in $[0, T] \times \mathbb{R}^{nd}$, it holds that*

$$|D_{\mathbf{x}_1} \tilde{u}^{\tau,\xi}(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{u}^{\tau,\xi}(t, \mathbf{x}')| \leq C c_0^{\frac{\alpha+\beta-2}{\alpha}} (\|g\|_{C_{b,d}^{\alpha+\beta}} + \|f\|_{L^\infty(C_{b,d}^\beta)}) d^{\alpha+\beta-1}(\mathbf{x}, \mathbf{x}').$$

Before proving the above result, we point out the control on the Hölder modulus of $\tilde{u}^{\tau, \xi}$ with respect to the degenerate variables ($i > 1$):

Lemma 10 (Controls on Hölder Moduli: Degenerate). *Let i be in $\llbracket 2, n \rrbracket$ and \mathbf{x}, \mathbf{x}' in \mathbb{R}^{nd} such that $\mathbf{x}_j = \mathbf{x}'_j$ for any $j \neq i$. Under (A), there exists a constant $C := C(i)$ such that for any t in $[0, T]$ and any freezing couple (τ, ξ) in $[0, T] \times \mathbb{R}^{nd}$, it holds that*

$$|\tilde{u}^{\tau, \xi}(t, \mathbf{x}) - \tilde{u}^{\tau, \xi}(t, \mathbf{x}')| \leq C c_0^{\frac{\beta - \gamma_i}{\alpha}} (\|g\|_{C_{b,d}^{\alpha+\beta}} + \|f\|_{L^\infty(C_{b,d}^\beta)}) d^{\alpha+\beta}(\mathbf{x}, \mathbf{x}').$$

Proof of Lemma 9 *Controls on frozen semigroup.* Let us consider firstly the off-diagonal regime, i.e. the case $T - t \leq c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$. Using the scaling lemma 5, it holds that

$$D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau, \xi} g(\mathbf{x}) = \int_{\mathbb{R}^{nd}} D_{\mathbf{x}_1} \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}, \mathbf{y}) g(\mathbf{y}) d\mathbf{y} = \sum_{j=1}^n C_j (T-t)^{j-1} \int_{\mathbb{R}^{nd}} D_{\mathbf{y}_j} \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}, \mathbf{y}) g(\mathbf{y}) d\mathbf{y}.$$

It then follows that

$$\begin{aligned} |D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau, \xi} g(\mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau, \xi} g(\mathbf{x}')| &\leq C \sum_{j=1}^n (T-t)^{j-1} \left| \int_{\mathbb{R}^{nd}} [D_{\mathbf{y}_j} \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}, \mathbf{y}) - D_{\mathbf{y}_j} \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}', \mathbf{y})] g(\mathbf{y}) d\mathbf{y} \right| \\ &=: C \sum_{j=1}^n (T-t)^{j-1} I_j^{od}. \end{aligned} \quad (4.17)$$

We are going to treat separately the cases $j = 1$ and $j > 1$ for the *off-diagonal* contributions $(I_j^{od})_{j \in \llbracket 1, n \rrbracket}$. Indeed, the function g is differentiable only with respect to the first component \mathbf{y}_1 . In this first case, we can apply integration by parts formula to move the derivative on g , so that

$$I_1^{od} = \left| \int_{\mathbb{R}^{nd}} [\tilde{p}^{\tau, \xi}(t, T, \mathbf{x}, \mathbf{y}) - \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}', \mathbf{y})] D_{\mathbf{y}_1} g(\mathbf{y}) d\mathbf{y} \right|.$$

Noticing that $D_{\mathbf{y}_1} g$ is in $C_{b,d}^{\alpha+\beta-1}(\mathbb{R}^{nd})$ thanks to the reverse Taylor expansion (Lemma 23), the last expression can be then rewritten as

$$\begin{aligned} I_1^{od} &\leq \left| \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}, \mathbf{y}) [D_{\mathbf{y}_1} g(\mathbf{y}) \pm D_{\mathbf{y}_1} g(\tilde{\mathbf{m}}_{t,T}^{\tau, \xi}(\mathbf{x}))] - \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}', \mathbf{y}) [D_{\mathbf{y}_1} g(\mathbf{y}) \pm D_{\mathbf{y}_1} g(\tilde{\mathbf{m}}_{t,T}^{\tau, \xi}(\mathbf{x}'))] d\mathbf{y} \right| \\ &\leq C \|g\|_{C_{b,d}^{\alpha+\beta}} \left\{ \int_{\mathbb{R}^{nd}} [\tilde{p}^{\tau, \xi}(t, T, \mathbf{x}, \mathbf{y}) d^{\alpha+\beta-1}(\mathbf{y}, \tilde{\mathbf{m}}_{t,T}^{\tau, \xi}(\mathbf{x})) + \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}', \mathbf{y}) d^{\alpha+\beta-1}(\mathbf{y}, \tilde{\mathbf{m}}_{t,T}^{\tau, \xi}(\mathbf{x}'))] d\mathbf{y} \right. \\ &\quad \left. + d^{\alpha+\beta-1}(\tilde{\mathbf{m}}_{t,T}^{\tau, \xi}(\mathbf{x}), \tilde{\mathbf{m}}_{t,T}^{\tau, \xi}(\mathbf{x}')) \right\} \end{aligned} \quad (4.18)$$

Now, we use the smoothing effect of $\tilde{p}^{\tau, \xi}$ (Equation (3.6)) to control the two integrals in the last expression, so that

$$I_1^{od} \leq C \|g\|_{C_{b,d}^{\alpha+\beta}} [(T-t)^{\frac{\alpha+\beta-1}{\alpha}} + d^{\alpha+\beta-1}(\tilde{\mathbf{m}}_{t,T}^{\tau, \xi}(\mathbf{x}), \tilde{\mathbf{m}}_{t,T}^{\tau, \xi}(\mathbf{x}'))].$$

We can then conclude the case $j = 1$ recalling that the mapping $\mathbf{x} \rightarrow \tilde{\mathbf{m}}_{t,T}^{\tau, \xi}(\mathbf{x})$ is affine (see Equation (3.4) for definition of $\tilde{\mathbf{m}}_{t,T}^{\tau, \xi}(\mathbf{x})$) in order to show that

$$I_1^{od} \leq C \|g\|_{C_{b,d}^{\alpha+\beta}} [(T-t)^{\frac{\alpha+\beta-1}{\alpha}} + d^{\alpha+\beta-1}(\mathbf{x}, \mathbf{x}')]. \quad (4.19)$$

Let us consider now the case $j > 1$. Using the duality in Besov spaces (Equation (4.11)) and the identification (4.10), we can write from Equation (4.17) that

$$\begin{aligned} I_j^{od} &\leq C \|g\|_{C_{b,d}^{\alpha+\beta}} \int_{\mathbb{R}^{(n-1)d}} \|D_{\mathbf{y}_j} \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}, \mathbf{y}_{\setminus j}, \cdot) - D_{\mathbf{y}_j} \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}', \mathbf{y}_{\setminus j}, \cdot)\|_{B_{1,1}^{-(\alpha_j+\beta_j)}} d\mathbf{y}_{\setminus j} \\ &\leq C \|g\|_{C_{b,d}^{\alpha+\beta}} \int_{\mathbb{R}^{(n-1)d}} \|D_{\mathbf{y}_j} \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}, \mathbf{y}_{\setminus j}, \cdot)\|_{B_{1,1}^{-(\alpha_j+\beta_j)}} + \|D_{\mathbf{y}_j} \tilde{p}^{\tau, \xi}(t, T, \mathbf{x}', \mathbf{y}_{\setminus j}, \cdot)\|_{B_{1,1}^{-(\alpha_j+\beta_j)}} d\mathbf{y}_{\setminus j} \\ &\leq C \|g\|_{C_{b,d}^{\alpha+\beta}} (T-t)^{\frac{\alpha+\beta}{\alpha} - \frac{1}{\alpha_j}} \end{aligned} \quad (4.20)$$

where in the last inequality we applied the first Besov Control (Lemma 7). Going back at equations (4.19) and (4.20), we finally conclude that

$$\begin{aligned} |D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau,\xi} g(\mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau,\xi} g(\mathbf{x}')| &\leq C \|g\|_{C_{b,d}^{\alpha+\beta}} \left[(T-t)^{\frac{\alpha+\beta-1}{\alpha}} + d^{\alpha+\beta-1}(\mathbf{x}, \mathbf{x}') + \sum_{j=2}^n (T-t)^{j-1} (T-t)^{\frac{\alpha+\beta}{\alpha} - \frac{1}{\alpha_j}} \right] \\ &\leq C \|g\|_{C_{b,d}^{\alpha+\beta}} \left[(T-t)^{\frac{\alpha+\beta-1}{\alpha}} + d^{\alpha+\beta-1}(\mathbf{x}, \mathbf{x}') \right] \leq C \|g\|_{C_{b,d}^{\alpha+\beta}} d^{\alpha+\beta-1}(\mathbf{x}, \mathbf{x}') \quad (4.21) \end{aligned}$$

where in the last passage we used that $T-t \leq c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$ for some $c_0 \leq 1$.

We focus now on the diagonal regime, i.e. when $T-t > c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$. Remembering that we assumed that $\mathbf{x}_j = \mathbf{x}'_j$ for any $j \in \llbracket 2, n \rrbracket$, we start using a Taylor expansion on the density $\tilde{p}^{\tau,\xi}$ with respect to the first, non-degenerate variable \mathbf{x}_1 . Namely,

$$\begin{aligned} D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau,\xi} g(\mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau,\xi} g(\mathbf{x}') &= \int_{\mathbb{R}^{nd}} [D_{\mathbf{x}_1} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}, \mathbf{y}) - D_{\mathbf{x}_1} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}', \mathbf{y})] g(\mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^{nd}} \int_0^1 D_{\mathbf{x}_1}^2 \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}' + \lambda(\mathbf{x} - \mathbf{x}'), \mathbf{y}) (\mathbf{x} - \mathbf{x}')_1 g(\mathbf{y}) d\lambda d\mathbf{y}. \end{aligned}$$

Moreover, from the Scaling Lemma 5, it holds that

$$D_{\mathbf{x}_1}^2 \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}' + \lambda(\mathbf{x} - \mathbf{x}'), \mathbf{y}) = \sum_{j=1}^n C_j (T-t)^{j-1} D_{\mathbf{y}_j} D_{\mathbf{x}_1} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}' + \lambda(\mathbf{x} - \mathbf{x}'), \mathbf{y})$$

and we can use it to write

$$\begin{aligned} |D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau,\xi} g(\mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau,\xi} g(\mathbf{x}')| &\leq C |(\mathbf{x} - \mathbf{x}')_1| \sum_{j=1}^n (T-t)^{j-1} \left| \int_0^1 \int_{\mathbb{R}^{nd}} D_{\mathbf{y}_j} D_{\mathbf{x}_1} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}' + \lambda(\mathbf{x} - \mathbf{x}'), \mathbf{y}) g(\mathbf{y}) d\mathbf{y} d\lambda \right| \\ &=: C |(\mathbf{x} - \mathbf{x}')_1| \sum_{j=1}^n (T-t)^{j-1} I_j^d. \quad (4.22) \end{aligned}$$

Similarly to the off-diagonal regime, we are going to treat separately the cases $j = 1$ and $j > 1$ for the *diagonal* contributions $(I_j^d)_{j \in \llbracket 1, n \rrbracket}$. In the first case, we can apply integration by parts formula to show that

$$I_1^d = \left| \int_0^1 \int_{\mathbb{R}^{nd}} D_{\mathbf{x}_1} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}' + \lambda(\mathbf{x} - \mathbf{x}'), \mathbf{y}) \otimes D_{\mathbf{y}_1} g(\mathbf{y}) d\mathbf{y} d\lambda \right|.$$

A cancellation argument with respect to $D_{\mathbf{x}_1} \tilde{p}^{\tau,\xi}$ then leads to

$$\begin{aligned} I_1^d &= \left| \int_0^1 \int_{\mathbb{R}^{nd}} D_{\mathbf{x}_1} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}' + \lambda(\mathbf{x} - \mathbf{x}'), \mathbf{y}) \otimes [D_{\mathbf{y}_1} g(\mathbf{y}) - D_{\mathbf{y}_1} g(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}' + \lambda(\mathbf{x} - \mathbf{x}')))] d\mathbf{y} d\lambda \right| \\ &\leq C \|g\|_{C_{b,d}^{\alpha+\beta}} \int_0^1 \int_{\mathbb{R}^{nd}} |D_{\mathbf{x}_1} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}' + \lambda(\mathbf{x} - \mathbf{x}'), \mathbf{y})| d^{\alpha+\beta-1}(\mathbf{y}, \tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}' + \lambda(\mathbf{x} - \mathbf{x}'))) d\mathbf{y} d\lambda. \end{aligned}$$

Since $\alpha + \beta - 1 < \alpha$ by hypothesis **(P)**, we can conclude using the smoothing effect of $\tilde{p}^{\tau,\xi}$ (Lemma 3) to show that

$$I_1^d \leq C \|g\|_{C_{b,d}^{\alpha+\beta}} (T-t)^{\frac{\alpha+\beta-2}{\alpha}}. \quad (4.23)$$

For the case $j > 1$, we use instead the duality in Besov spaces (4.11) and the identification (4.10) to write

$$\begin{aligned} I_j^d &\leq \int_0^1 \int_{\mathbb{R}^{(n-1)d}} \|D_{\mathbf{y}_j} D_{\mathbf{x}_1} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}' + \lambda(\mathbf{x} - \mathbf{x}'), \mathbf{y}_{\setminus j}, \cdot)\|_{B_{1,1}^{-(\alpha_j+\beta_j)}} d\mathbf{y}_{\setminus j} d\lambda \\ &\leq C \|g\|_{C_{b,d}^{\alpha+\beta}} (T-t)^{\frac{\alpha+\beta}{\alpha} - \frac{1}{\alpha_j} - \frac{1}{\alpha}} \quad (4.24) \end{aligned}$$

where in the last passage we applied the First Besov control (Lemma 7). From equations (4.22), (4.23) and (4.24), it is possible to conclude that

$$\begin{aligned} |D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\xi} g(\mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\xi} g(\mathbf{x}')| &\leq C \|g\|_{C_{b,d}^{\alpha+\beta}} |(\mathbf{x} - \mathbf{x}')_1| \sum_{j=1}^n (T-t)^{j-1} (T-t)^{\frac{\alpha+\beta}{\alpha} - \frac{1}{\alpha_j} - \frac{1}{\alpha}} \\ &\leq C \|g\|_{C_{b,d}^{\alpha+\beta}} |(\mathbf{x} - \mathbf{x}')_1| (T-t)^{\frac{\alpha+\beta-2}{\alpha}} \leq C c_0^{\frac{\alpha+\beta-2}{\alpha}} \|g\|_{C_{b,d}^{\alpha+\beta}} d^{\alpha+\beta-1}(\mathbf{x}, \mathbf{x}') \end{aligned}$$

where in the last passage we used that $|(\mathbf{x} - \mathbf{x}')_1| = d(\mathbf{x}, \mathbf{x}')$ and, since $\frac{\alpha+\beta-2}{\alpha} < 0$, that

$$|(\mathbf{x} - \mathbf{x}')_1| (T-t)^{\frac{\alpha+\beta-2}{\alpha}} \leq c_0^{\frac{\alpha+\beta-2}{\alpha}} d^{\alpha+\beta-1}(\mathbf{x}, \mathbf{x}').$$

Remembering that c_0 is considered fixed and bigger then zero, the searched control follows immediately.

Controls on frozen Green kernel. In order to preserve the previous terminology of off-diagonal/diagonal regime for the frozen semigroup, we have introduced the transition time t_0 , defined in (4.16). Then, while integrating in s from t to T , we will say that the "local" off-diagonal regime holds for $\tilde{G}^{\tau, \xi}$ if s is in $[t, t_0]$ and that the "local" diagonal regime holds if s is in $[t_0, T]$. With the notations of (4.2), it seems quite natural now to decompose the derivative of the frozen Green kernel with respect to t_0 , i.e.

$$D_{\mathbf{x}_1} \tilde{G}_{t,T}^{\tau, \xi} f(t, \mathbf{x}) = D_{\mathbf{x}_1} \tilde{G}_{t,t_0}^{\tau, \xi} f(t, \mathbf{x}) + D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \xi} f(t, \mathbf{x}).$$

We remark however that the globally off-diagonal regime is considered in the above decomposition, too. Indeed, when $T-t \leq c_0 d^{\alpha}(\mathbf{x}, \mathbf{x}')$, t_0 coincides with T and the second term on the right-hand side vanishes. We start considering the off-diagonal regime represented by $|D_{\mathbf{x}_1} \tilde{G}_{t,t_0}^{\tau, \xi} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t,t_0}^{\tau, \xi} f(t, \mathbf{x}')|$. It holds that

$$|D_{\mathbf{x}_1} \tilde{G}_{t,t_0}^{\tau, \xi} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t,t_0}^{\tau, \xi} f(t, \mathbf{x}')| \leq \int_t^{t_0} \left[|D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau, \xi} f(s, \mathbf{x})| + |D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau, \xi} f(s, \mathbf{x}')| \right] ds$$

We then use the control on the frozen semigroup (Equation (3.7)) to find that

$$|D_{\mathbf{x}_1} \tilde{G}_{t,t_0}^{\tau, \xi} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t,t_0}^{\tau, \xi} f(t, \mathbf{x}')| \leq C \|f\|_{L^{\infty}(C_{b,d}^{\beta})} \int_t^{t_0} (s-t)^{\frac{\beta-1}{\alpha}} ds \leq C \|f\|_{L^{\infty}(C_{b,d}^{\beta})} (t_0 - t)^{\frac{\beta+\alpha-1}{\alpha}}.$$

Our choice of t_0 (cf. Equation (4.16)) allows then to conclude that

$$|D_{\mathbf{x}_1} \tilde{G}_{t,t_0}^{\tau, \xi} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t,t_0}^{\tau, \xi} f(t, \mathbf{x}')| \leq C \|f\|_{L^{\infty}(C_{b,d}^{\beta})} d^{\beta+\alpha-1}(\mathbf{x}, \mathbf{x}')$$

remembering that, by assumption, $c_0 \leq 1$.

We can focus now on the diagonal regime represented by $|D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \xi} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \xi} f(t, \mathbf{x}')|$.

We start applying a Taylor expansion on the derivative of the semigroup $\tilde{P}^{\tau, \xi} f(t, \mathbf{x})$ so that

$$\begin{aligned} |D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \xi} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \xi} f(t, \mathbf{x}')| &= \left| \int_{t_0}^T \left[D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau, \xi} f(s, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau, \xi} f(s, \mathbf{x}') \right] ds \right| \\ &= \left| \int_{t_0}^T \int_0^1 D_{\mathbf{x}_1}^2 \tilde{P}_{t,s}^{\tau, \xi} f(s, \mathbf{x} + \lambda(\mathbf{x}' - \mathbf{x})) (\mathbf{x}' - \mathbf{x})_1 d\lambda ds \right|. \end{aligned}$$

Then, the Fubini theorem and the control on the frozen semigroup (Equation (3.7)) allow us to write that

$$\begin{aligned} |D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \xi} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \xi} f(t, \mathbf{x}')| &\leq C \|f\|_{L^{\infty}(C_{b,d}^{\beta})} |(\mathbf{x} - \mathbf{x}')_1| \int_{t_0}^T (s-t)^{\frac{\beta-2}{\alpha}} ds \\ &\leq C \|f\|_{L^{\infty}(C_{b,d}^{\beta})} |(\mathbf{x} - \mathbf{x}')_1| \left[(s-t)^{\frac{\alpha+\beta-2}{\alpha}} \right]_{t_0}^T. \end{aligned}$$

Since by hypothesis **(P)** in assumption **(A)**, it holds that $\alpha + \beta - 2 < 0$, it follows that

$$|D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \xi} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \xi} f(t, \mathbf{x}')| \leq C \|f\|_{L^{\infty}(C_{b,d}^{\beta})} |(\mathbf{x} - \mathbf{x}')_1| (t_0 - t)^{\frac{\alpha+\beta-2}{\alpha}}.$$

Using that $|(\mathbf{x} - \mathbf{x}')_1| = d(\mathbf{x}, \mathbf{x}')$ and remembering our choice of t_0 in (4.16), we can then conclude that

$$|D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \xi} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \xi} f(t, \mathbf{x}')| \leq C c_0^{\frac{\alpha+\beta-2}{\alpha}} \|f\|_{L^{\infty}(C_{b,d}^{\beta})} d^{\alpha+\beta-1}(\mathbf{x}, \mathbf{x}').$$

Proof of Lemma 10 *Controls on frozen semigroup.* Using the change of variables $\mathbf{z} = \tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}) - \mathbf{y}$, we can rewrite $\tilde{P}_{t,T}^{\tau,\xi}g(\mathbf{x})$ as

$$\begin{aligned}\tilde{P}_{t,T}^{\tau,\xi}g(\mathbf{x}) &= \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t, T, \mathbf{x}, \mathbf{y})g(\mathbf{y}) d\mathbf{y} = \int_{\mathbb{R}^{nd}} \frac{1}{\det(\mathbb{M}_{T-t})} p_S(T-t, \mathbb{M}_{T-t}^{-1}(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}) - \mathbf{y}))g(\mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^{nd}} \frac{1}{\det(\mathbb{M}_{T-t})} p_S(T-t, \mathbb{M}_{T-t}^{-1}\mathbf{z})g(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}) - \mathbf{z}) d\mathbf{z}.\end{aligned}$$

It then follows that

$$|\tilde{P}_{t,T}^{\tau,\xi}g(\mathbf{x}) - \tilde{P}_{t,T}^{\tau,\xi}g(\mathbf{x}')| = \left| \int_{\mathbb{R}^{nd}} \frac{1}{\det(\mathbb{M}_{T-t})} p_S(T-t, \mathbb{M}_{T-t}^{-1}\mathbf{z}) [g(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}) - \mathbf{z}) - g(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}') - \mathbf{z})] d\mathbf{z} \right|.$$

We observe now that the function $\mathbf{x} \rightarrow \tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x})$ is affine (cf. Equation (3.4)) and thus, that

$$(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}) - \mathbf{z})_1 = (\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}') - \mathbf{z})_1$$

since $\mathbf{x}_1 = \mathbf{x}'_1$. It then holds that

$$|g(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}) - \mathbf{z}) - g(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}') - \mathbf{z})| \leq C\|g\|_{C_{b,d}^{\alpha+\beta}} d^{\alpha+\beta}(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}), \tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}')) \leq C\|g\|_{C_{b,d}^{\alpha+\beta}} d^{\alpha+\beta}(\mathbf{x}, \mathbf{x}').$$

Hence, we can conclude using it to write

$$\begin{aligned}|\tilde{P}_{t,T}^{\tau,\xi}g(\mathbf{x}) - \tilde{P}_{t,T}^{\tau,\xi}g(\mathbf{x}')| &\leq C\|g\|_{C_{b,d}^{\alpha+\beta}} d^{\alpha+\beta}(\mathbf{x}, \mathbf{x}') \int_{\mathbb{R}^{nd}} \frac{1}{\det(\mathbb{M}_{T-t})} p_S(T-t, \mathbb{M}_{T-t}^{-1}\mathbf{z}) d\mathbf{z} \\ &\leq C\|g\|_{C_{b,d}^{\alpha+\beta}} d^{\alpha+\beta}(\mathbf{x}, \mathbf{x}').\end{aligned}$$

Controls on frozen Green kernel. We will assume the same notations appeared in the previous lemma for the frozen Green Kernel. In particular, we decompose the frozen Green Kernel as

$$\tilde{G}_{t,T}^{\tau,\xi}f(t, \mathbf{x}) = \tilde{G}_{t,t_0}^{\tau,\xi}f(t, \mathbf{x}) + \tilde{G}_{t_0,T}^{\tau,\xi}f(t, \mathbf{x})$$

with t_0 defined in (4.16).

We start rewriting the off-diagonal regime contribution as

$$\begin{aligned}&|\tilde{G}_{t,t_0}^{\tau,\xi}f(t, \mathbf{x}) - \tilde{G}_{t,t_0}^{\tau,\xi}f(t, \mathbf{x}')| \\ &= \left| \int_t^{t_0} \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y}) [f(s, \mathbf{y}) \pm f(s, \tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x}))] - \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}', \mathbf{y}) [f(s, \mathbf{y}) \pm f(s, \tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x}'))] dy ds \right| \\ &\leq \left| \int_t^{t_0} \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y}) [f(s, \mathbf{y}) - f(s, \tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x}))] dy ds - \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}', \mathbf{y}) [f(s, \mathbf{y}) - f(s, \tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x}'))] dy ds \right| \\ &\quad + \left| \int_t^{t_0} f(s, \tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x})) - f(s, \tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x}')) ds \right|.\end{aligned}$$

We can then use the smoothing effect for $\tilde{p}^{\tau,\xi}$ (Equation (3.6) in Lemma 3) to show that

$$|\tilde{G}_{t,t_0}^{\tau,\xi}f(t, \mathbf{x}) - \tilde{G}_{t,t_0}^{\tau,\xi}f(t, \mathbf{x}')| \leq C\|f\|_{L^\infty(C_{b,d}^\beta)} \int_t^{t_0} [(s-t)^{\beta/\alpha} + d^\beta(\tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x}), \tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x}'))] ds. \quad (4.25)$$

Recalling from Equation (3.4) that $\mathbf{x} \rightarrow \tilde{\mathbf{m}}_{t,s}^{\tau,\xi}(\mathbf{x})$ is affine, it follows that

$$\begin{aligned}|\tilde{G}_{t,t_0}^{\tau,\xi}f(t, \mathbf{x}) - \tilde{G}_{t,t_0}^{\tau,\xi}f(t, \mathbf{x}')| &\leq C\|f\|_{L^\infty(C_{b,d}^\beta)} \int_t^{t_0} [(s-t)^{\beta/\alpha} + d^\beta(\mathbf{x}, \mathbf{x}')] ds \\ &\leq C\|f\|_{L^\infty(C_{b,d}^\beta)} [(t_0-t)d^\beta(\mathbf{x}, \mathbf{x}') + (t_0-t)^{\frac{\beta+\alpha}{\alpha}}].\end{aligned}$$

Using that $t_0 - t \leq c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$ for some $c_0 \leq 1$, we can finally conclude that

$$|\tilde{G}_{t,t_0}^{\tau,\xi} f(t, \mathbf{x}) - \tilde{G}_{t,t_0}^{\tau,\xi} f(t, \mathbf{x}')| \leq C \|f\|_{L^\infty(C_{b,d}^\beta)} d^{\beta+\alpha}(\mathbf{x}, \mathbf{x}').$$

Now, we can focus our analysis to the diagonal regime contribution, i.e. $|\tilde{G}_{t_0,T}^{\tau,\xi} f(t, \mathbf{x}) - \tilde{G}_{t_0,T}^{\tau,\xi} f(t, \mathbf{x}')|$.

We start applying a Taylor expansion on the frozen semigroup $\tilde{P}_{t,s}^{\tau,\xi} f$ with respect to the i -th variable \mathbf{x}_i , which is, by hypothesis, the only one for which the entries of \mathbf{x} and \mathbf{x}' differ. Namely,

$$\begin{aligned} |\tilde{G}_{t_0,T}^{\tau,\xi} f(t, \mathbf{x}) - \tilde{G}_{t_0,T}^{\tau,\xi} f(t, \mathbf{x}')| &= \left| \int_{t_0}^T \tilde{P}_{t,s}^{\tau,\xi} f(s, \mathbf{x}) - \tilde{P}_{t,s}^{\tau,\xi} f(s, \mathbf{x}') ds \right| \\ &= \left| \int_{t_0}^T \int_0^1 D_{\mathbf{x}_i} \tilde{P}_{t,s}^{\tau,\xi} f(s, \mathbf{x} + \lambda(\mathbf{x}' - \mathbf{x})) \cdot (\mathbf{x}' - \mathbf{x})_i d\lambda ds \right|. \end{aligned}$$

The control on the frozen semigroup (Equation (3.7)) then implies that

$$|\tilde{G}_{t_0,T}^{\tau,\xi} f(t, \mathbf{x}) - \tilde{G}_{t_0,T}^{\tau,\xi} f(t, \mathbf{x}')| \leq C \|f\|_{L^\infty(C_{b,d}^\beta)} |(\mathbf{x} - \mathbf{x}')_i| \int_{t_0}^T (s-t)^{\frac{\beta}{\alpha} - \frac{1}{\alpha_i}} ds. \quad (4.26)$$

Noticing from assumption **(P)** that $\beta + \alpha - 1 - \alpha(i-1) < 0$ for $i \geq 2$, it holds that

$$\int_{t_0}^T (s-t)^{\frac{\beta}{\alpha} - \frac{1}{\alpha_i}} ds = \int_{t_0}^T (s-t)^{\frac{\beta - [1 + \alpha(i-1)]}{\alpha}} ds \leq C \left[-(s-t)^{\frac{\beta + \alpha - 1 - \alpha(i-1)}{\alpha}} \right]_{t_0}^T \leq C(t_0 - t)^{\frac{\beta - 1 - \alpha(i-2)}{\alpha}}.$$

Using that $|(\mathbf{x} - \mathbf{x}')_i| = d^{1+\alpha(i-1)}(\mathbf{x}, \mathbf{x}')$ and our choice of t_0 (cf. Equation (4.16)), we can then conclude from (4.26) that

$$|\tilde{G}_{t_0,T}^{\tau,\xi} f(t, \mathbf{x}) - \tilde{G}_{t_0,T}^{\tau,\xi} f(t, \mathbf{x}')| \leq C c_0^{\frac{\beta - 1 - \alpha(i-2)}{\alpha}} \|f\|_{L^\infty(C_{b,d}^\beta)} d^{\alpha+\beta}(\mathbf{x}, \mathbf{x}') \leq C c_0^{\frac{\beta - \gamma_i}{\alpha}} \|f\|_{L^\infty(C_{b,d}^\beta)} d^{\alpha+\beta}(\mathbf{x}, \mathbf{x}'),$$

remembering the definition of γ_i in (2.14).

5 A Priori Estimates

Since the aim of this section is to prove Proposition 4, we will assume tacitly from this point further that assumption **(A')** holds. Moreover, we recall here that we are throughout this section considering the regularized framework of Section 3.2.

WARNING: For notational simplicity, we drop here the sub-scripts and the superscripts in m associated with the regularization. For any fixed (τ, ξ) in $[0, T] \times \mathbb{R}^{nd}$, we rewrite, with some abuse in notations, the Duhamel expansion (Equation (3.16)) as:

$$u(t, \mathbf{x}) = \tilde{u}^{\tau,\xi}(t, \mathbf{x}) + \int_t^T \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s, \mathbf{x}) ds, \quad (5.27)$$

where $\tilde{u}^{\tau,\xi}$ is defined through the Duhamel representation (3.10) and

$$R^{\tau,\xi}(t, \mathbf{x}) = \langle \mathbf{F}(t, \mathbf{x}) - \mathbf{F}(t, \boldsymbol{\theta}_{\tau,t}(\xi)), D_{\mathbf{x}} u(t, \mathbf{x}) \rangle, \quad (t, \mathbf{x}) \in (0, T) \times \mathbb{R}^{nd}.$$

It is however important to keep in mind that f, g, \mathbf{F} are now smooth and bounded functions so that all the terms above are clearly defined. We recall however that we aim at obtaining controls in the $L^\infty(C_{b,d}^{\alpha+\beta})$ -norm, uniformly with respect to the regularization parameter.

From the expansion above, we know moreover that for any (t, ξ) in $[0, T] \times \mathbb{R}^{nd}$, it holds that

$$D_{\mathbf{x}_1} u(t, \mathbf{x}) = D_{\mathbf{x}_1} \tilde{u}^{\tau,\xi}(t, \mathbf{x}) + \int_t^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s, \mathbf{x}) ds. \quad (5.28)$$

As seen in the previous section, these decompositions will allow us to control u in norm $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ analyzing separately the contributions from the Duhamel representation $\tilde{u}^{\tau,\xi}$ and those from the expansion error $R^{\tau,\xi}(t, \mathbf{x})$ for suitable choices of freezing parameters (τ, ξ) .

5.1 Second Besov Control

This sub-section focuses on the contribution associated with the remainder term $R^{m,\tau,\xi}$ appearing in the Duhamel-type expansion (5.27). We recall that we aim at controlling it with the $L^\infty(C_{b,d}^{\alpha+\beta})$ -norm of the coefficients, uniformly in the regularization parameter. Let us start decomposing it through

$$\left| \int_t^T \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s, \mathbf{x}) ds \right| = \left| \sum_{j=1}^n \int_t^T \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y}) [\mathbf{F}_j(s, \mathbf{y}) - \mathbf{F}_j(s, \boldsymbol{\theta}_{\tau,s}(\xi))] \cdot D_{\mathbf{y}_j} u(s, \mathbf{y}) d\mathbf{y} ds \right|.$$

We then notice that the non-degenerate contribution in the sum (corresponding to the index $j = 1$) can be treated easily remembering that u is differentiable with respect to the first component with a bounded derivative. Indeed, using the smoothing effect for the frozen density $\tilde{p}^{\tau,\xi}$ (Equation (3.6)), it holds that

$$\begin{aligned} & \left| \int_t^T \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y}) [\mathbf{F}_1(s, \mathbf{y}) - \mathbf{F}_1(s, \boldsymbol{\theta}_{\tau,s}(\xi))] \cdot D_{\mathbf{y}_1} u(s, \mathbf{y}) d\mathbf{y} ds \right| \\ & \leq C \|D_{\mathbf{y}_1} u(s, \mathbf{y})\|_{l^\infty(L^\infty)} \|\mathbf{F}\|_H \int_t^T \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y}) d^{\alpha+\beta}(\mathbf{y}, \boldsymbol{\theta}_{\tau,s}(\xi)) d\mathbf{y} ds \\ & \leq C \|D_{\mathbf{y}_1} u(s, \mathbf{y})\|_{l^\infty(L^\infty)} \|\mathbf{F}\|_H \int_t^T (s-t)^{\frac{\beta}{\alpha}} ds \leq C \|D_{\mathbf{y}_1} u(s, \mathbf{y})\|_{l^\infty(L^\infty)} \|\mathbf{F}\|_H (T-t)^{\frac{\alpha+\beta}{\alpha}}. \end{aligned}$$

In order to deal with the degenerate indexes, we will use, similarly to the previous subsection, a reasoning in Besov spaces. Since u is not differentiable with respect to \mathbf{y}_j if $j > 1$, we move the derivative to the other terms using integration by parts formula:

$$\left| \int_t^T \int_{\mathbb{R}^{nd}} D_{\mathbf{y}_j} \cdot \left\{ \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y}) [\mathbf{F}_j(s, \mathbf{y}) - \mathbf{F}_j(s, \boldsymbol{\theta}_{\tau,s}(\xi))] \right\} u(s, \mathbf{y}) d\mathbf{y} ds \right|.$$

In order to rely again on the duality in Besov spaces (4.11), we rewrite the above expression as

$$\begin{aligned} & \left| \int_t^T \int_{\mathbb{R}^{nd}} D_{\mathbf{y}_j} \cdot \left\{ \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y}) [\mathbf{F}_j(s, \mathbf{y}) - \mathbf{F}_j(s, \boldsymbol{\theta}_{\tau,s}(\xi))] \right\} u(s, \mathbf{y}) d\mathbf{y} ds \right| \leq \\ & \int_t^T \int_{\mathbb{R}^{(n-1)d}} \left\| D_{\mathbf{y}_j} \cdot \left\{ \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \cdot) [\mathbf{F}_j(s, \mathbf{y}_{\setminus j}, \cdot) - \mathbf{F}_j(s, \boldsymbol{\theta}_{\tau,s}(\xi))] \right\} \right\|_{B_{1,1}^{-(\alpha_j+\beta_j)}} \|u(s, \mathbf{y}_{\setminus j}, \cdot)\|_{B_{\infty,\infty}^{\alpha_j+\beta_j}} d\mathbf{y}_{\setminus j} ds. \end{aligned}$$

Remembering identification (4.10), it holds now that

$$\begin{aligned} & \left| \int_t^T \int_{\mathbb{R}^{nd}} D_{\mathbf{y}_j} \cdot \left\{ \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y}) [\mathbf{F}_j(s, \mathbf{y}) - \mathbf{F}_j(s, \boldsymbol{\theta}_{\tau,s}(\xi))] \right\} u(s, \mathbf{y}) d\mathbf{y} ds \right| \\ & \leq \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \int_t^T \int_{\mathbb{R}^{(n-1)d}} \left\| D_{\mathbf{y}_j} \cdot \left\{ \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \cdot) [\mathbf{F}_j(s, \mathbf{y}_{\setminus j}, \cdot) - \mathbf{F}_j(s, \boldsymbol{\theta}_{\tau,s}(\xi))] \right\} \right\|_{B_{1,1}^{-(\alpha_j+\beta_j)}} d\mathbf{y}_{\setminus j} ds. \end{aligned}$$

It then remains to control the integral of the Besov norm above. To do that, we will need a refinement of the smoothing effect (3.6) that involves only partial differences of variables. For a fixed i in $\llbracket 2, n \rrbracket$, we start denoting by $d_{i:n}(\cdot, \cdot)$ the part of the anisotropic distance considering only the last $n - (i - 1)$ variables. Namely,

$$d_{i:n}(\mathbf{x}, \mathbf{x}') := \sum_{j=i}^n |(\mathbf{x} - \mathbf{x}')_j|^{\frac{1}{1+\alpha(j-1)}}.$$

Lemma 11 (Partial Smoothing Effect). *Let i be in $\llbracket 2, n \rrbracket$, γ in $(0, 1 \wedge \alpha(1 + \alpha(i - 1)))$ and ϑ, ϱ two n -multi-indexes such that $|\vartheta + \varrho| \leq 3$. Then, there exists a constant $C := C(\vartheta, \varrho, \gamma)$ such that for any $t < s$ in $[0, T]$, any \mathbf{x} in \mathbb{R}^{nd} ,*

$$\int_{\mathbb{R}^{nd}} |D_{\mathbf{y}}^\varrho D_{\mathbf{x}}^\vartheta \tilde{p}^{\tau,\xi}(t, s, \mathbf{x}, \mathbf{y})| d_{i:n}^\gamma(\mathbf{y}, \boldsymbol{\theta}_{\tau,s}(\xi)) d\mathbf{y} \leq C(s-t)^{\frac{\gamma}{\alpha} - \sum_{k=i}^n \frac{\vartheta_k + \varrho_k}{\alpha_k}} \quad (5.29)$$

taking $(\tau, \xi) = (t, \mathbf{x})$.

The above assumption on γ should not appear to much strange. Indeed, in the partial distance $d_{i:n}^\gamma(\mathbf{x}, \mathbf{x}')$, the stronger term to be integrated is at level i with intensity of order $\gamma/(1 + \alpha(i - 1))$. Since by the smoothing effect (Equation (3.6)) of the frozen density, we know we can integrate against $\tilde{p}^{\tau, \xi}$ contributions of order up to α , so it appears the condition $\gamma < \alpha(1 + \alpha(i - 1))$.

A proof of this result can be obtained mimicking with slightly modifications the proof in Lemma 3.

As done above for the first Besov control, we will however state the result considering a possibly additional derivative with respect to \mathbf{x}_1 . Namely, we would like to control the following:

$$D_{\mathbf{y}_j} \cdot \left\{ D_{\mathbf{x}}^\vartheta \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \cdot) \otimes [F_j(s, \mathbf{y}_{\setminus j}, \cdot) - F_j(s, \boldsymbol{\theta}_{\tau, s}(\xi))] \right\}$$

where we have denoted as in (4.7), $F_j(s, \mathbf{y}_{\setminus j}, \cdot) := F_j(s, \mathbf{y}_1, \dots, \mathbf{y}_{j-1}, \cdot, \mathbf{y}_{j+1}, \dots, \mathbf{y}_n)$ and, with a slightly abuse of notation, by $D_{\mathbf{y}_j} \cdot$ an extended form of the divergence over the j -th variable. In other words, this "enhanced" divergence form decreases by one the order of the input tensor.

Lemma 12 (Second Besov Control). *Let j be in $\llbracket 2, n \rrbracket$ and ϑ a multi-index in \mathbb{N}^n such that $|\vartheta| \leq 2$. Under (\mathbf{A}') , there exists a constant $C := C(j, \vartheta)$ such that for any \mathbf{x} in \mathbb{R}^{nd} and any $t < s$ in $[0, T]$*

$$\begin{aligned} \int_{\mathbb{R}^{(n-1)d}} \left\| D_{\mathbf{y}_j} \cdot \left\{ D_{\mathbf{x}}^\vartheta \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \cdot) \otimes [F_j(s, \mathbf{y}_{\setminus j}, \cdot) - F_j(s, \boldsymbol{\theta}_{\tau, s}(\xi))] \right\} \right\|_{B_{1,1}^{-(\alpha_j + \beta_j)}} d\mathbf{y}_{\setminus j} \\ \leq C \|F\|_H(s - t)^{\frac{\beta}{\alpha} - \sum_{k=1}^n \frac{\vartheta_k}{\alpha_k}} \end{aligned}$$

taking $(\tau, \xi) = (t, \mathbf{x})$.

Proof. To control the Besov norm in $B_{1,1}^{-(\alpha_j + \beta_j)}(\mathbb{R}^d)$, we are going to use the stable thermic characterization (4.8) with $\tilde{\gamma} = -(\alpha_j + \beta_j)$. Since the first term can be controlled as in the First Besov Control (Lemma 7), we will focus on the second one, i.e.

$$\int_0^1 v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \partial_v p_h(v, z - \mathbf{y}_j) D_{\mathbf{y}_j} \cdot \left\{ D_{\mathbf{x}}^\vartheta \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) \otimes [F_j(s, \mathbf{y}) - F_j(s, \boldsymbol{\theta}_{\tau, s}(\xi))] \right\} d\mathbf{y}_j \right| dz dv.$$

We start applying integration by parts formula noticing that $D_{\mathbf{y}_j} p_h(v, z - \mathbf{y}_j) = -D_z p_h(v, z - \mathbf{y}_j)$, to write that

$$\int_0^1 v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} D_z \partial_v p_h(v, z - \mathbf{y}_j) \cdot \left\{ D_{\mathbf{x}}^\vartheta \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) \otimes [F_j(s, \mathbf{y}) - F_j(s, \boldsymbol{\theta}_{\tau, s}(\xi))] \right\} d\mathbf{y}_j \right| dz dv.$$

Fixed a constant $\delta_j \geq 1$ to be chosen later, we then split the above integral with respect to v into two components

$$\begin{aligned} \int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} D_z \partial_v p_h(v, z - \mathbf{y}_j) \cdot \left\{ D_{\mathbf{x}}^\vartheta \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) [F_j(s, \mathbf{y}) - F_j(s, \boldsymbol{\theta}_{\tau, s}(\xi))] \right\} d\mathbf{y}_j \right| dz dv \\ + \int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} D_z \partial_v p_h(v, z - \mathbf{y}_j) \cdot \left\{ D_{\mathbf{x}}^\vartheta \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) [F_j(s, \mathbf{y}) - F_j(s, \boldsymbol{\theta}_{\tau, s}(\xi))] \right\} d\mathbf{y}_j \right| dz dv \\ =: (I_1 + I_2)(\mathbf{y}_{\setminus j}). \end{aligned}$$

The second component I_2 has no time-singularity and it can be easily controlled using Fubini theorem

$$\begin{aligned} I_2(\mathbf{y}_{\setminus j}) &\leq \\ C \|F\|_H \int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |D_z \partial_v p_h(v, z - \mathbf{y}_j)| dz \right) |D_{\mathbf{x}}^\vartheta \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| d_{j:n}^{1+\alpha(j-2)+\beta}(\mathbf{y}, \boldsymbol{\theta}_{\tau, s}(\xi)) d\mathbf{y}_j dv, \end{aligned}$$

remembering that $F_j(t, \cdot)$ depends only on the last $(n - j)$ variables and it is in $C_{b,d}^{1+\alpha(j-2)+\beta}(\mathbb{R}^{nd})$ by assumption (\mathbf{R}) . We can then use the smoothing effect of the heat-kernel p_h (Equation (4.9)) and the Fubini

theorem again to write that

$$\begin{aligned} I_2(\mathbf{y}_{\setminus j}) &\leq C\|\mathbf{F}\|_H \int_{(s-t)^{\delta_j}}^1 \frac{v^{\frac{\alpha_j+\beta_j-1}{\alpha}}}{v} \int_{\mathbb{R}^d} |D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y})| d_{j:n}^{1+\alpha(j-2)+\beta}(\mathbf{y}, \boldsymbol{\theta}_{\tau, s}(\boldsymbol{\xi})) d\mathbf{y} dv \\ &\leq C\|\mathbf{F}\|_H \left(\int_{(s-t)^{\delta_j}}^1 \frac{v^{\frac{\alpha_j+\beta_j-1}{\alpha}}}{v} dv \right) \left(\int_{\mathbb{R}^d} |D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y})| d_{j:n}^{1+\alpha(j-2)+\beta}(\mathbf{y}, \boldsymbol{\theta}_{\tau, s}(\boldsymbol{\xi})) d\mathbf{y} \right). \end{aligned}$$

Noticing from (2.20) that $\alpha_j + \beta_j - 1 < 0$, it holds now that

$$I_2(\mathbf{y}_{\setminus j}) \leq C\|\mathbf{F}\|_H (s-t)^{\delta_j \frac{\alpha_j+\beta_j-1}{\alpha}} \int_{\mathbb{R}^d} |D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y})| d_{j:n}^{1+\alpha(j-2)+\beta}(\mathbf{y}, \boldsymbol{\theta}_{\tau, s}(\boldsymbol{\xi})) d\mathbf{y}.$$

We can finally add the integral with respect to the other components $\mathbf{y}_{\setminus j}$. In order to use now the partial smoothing effect (Equation (5.29)), we take $\tau = t$ and $\boldsymbol{\xi} = \mathbf{x}$ and notice that by assumption **(P)**,

$$1 + \alpha(j-2) + \beta = 1 + \alpha(j-1) - (\alpha - \beta) < 1 + \alpha(j-1) - (1 - \alpha)(1 + \alpha(j-1)) = \alpha(1 + \alpha(j-1)). \quad (5.30)$$

It then holds that

$$\begin{aligned} \int_{\mathbb{R}^{(n-1)d}} I_2(\mathbf{y}_{\setminus j}) d\mathbf{y}_{\setminus j} &\leq C\|\mathbf{F}\|_H (s-t)^{\delta_j \frac{\alpha_j+\beta_j-1}{\alpha}} \int_{\mathbb{R}^{nd}} |D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y})| d_{j:n}^{1+\alpha(j-2)+\beta}(\mathbf{y}, \boldsymbol{\theta}_{\tau, s}(\boldsymbol{\xi})) d\mathbf{y} \\ &\leq C\|\mathbf{F}\|_H (s-t)^{\delta_j \frac{\alpha_j+\beta_j-1}{\alpha} + \frac{1+\alpha(j-2)+\beta}{\alpha} - \sum_{k=1}^n \frac{\vartheta_k}{\alpha_k}}. \quad (5.31) \end{aligned}$$

To control the other term I_1 , we focus at first on the inner integral with respect to \mathbf{y}_j :

$$\int_{\mathbb{R}^d} D_z \partial_v p_h(v, z - \mathbf{y}_j) \cdot \left\{ D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y}) \otimes [\mathbf{F}_j(s, \mathbf{y}) - \mathbf{F}_j(s, \boldsymbol{\theta}_{\tau, s}(\boldsymbol{\xi}))] \right\} d\mathbf{y}_j.$$

We start using a cancellation argument with respect to the density p_h to divide it in

$$\begin{aligned} &\int_{\mathbb{R}^d} D_z \partial_v p_h(v, z - \mathbf{y}_j) \cdot \left\{ D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y}) \otimes [\mathbf{F}_j(s, \mathbf{y}) - \mathbf{F}_j(s, \boldsymbol{\theta}_{\tau, s}(\boldsymbol{\xi}))] \right\} \\ &\quad - D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, z) \otimes [\mathbf{F}_j(s, \mathbf{y}_{\setminus j}, z) - \mathbf{F}_j(s, \boldsymbol{\theta}_{\tau, s}(\boldsymbol{\xi}))] \Big\} d\mathbf{y}_j \\ &= \int_{\mathbb{R}^d} D_z \partial_v p_h(v, z - \mathbf{y}_j) \cdot \left\{ D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y}) \otimes [\mathbf{F}_j(s, \mathbf{y}) - \mathbf{F}_j(s, \mathbf{y}_{\setminus j}, z)] \right\} d\mathbf{y}_j \\ &+ \int_{\mathbb{R}^d} D_z \partial_v p_h(v, z - \mathbf{y}_j) \cdot \left\{ [D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y}) - D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, z)] \otimes [\mathbf{F}_j(s, \mathbf{y}_{\setminus j}, z) - \mathbf{F}_j(s, \boldsymbol{\theta}_{\tau, s}(\boldsymbol{\xi}))] \right\} d\mathbf{y}_j \\ &=: (J_1 + J_2)(v, \mathbf{y}_{\setminus j}, z). \end{aligned}$$

Remembering notation (4.7) for $\mathbf{F}_j(s, \mathbf{y}_{\setminus j}, z)$ and that \mathbf{F}_j is $\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}$ -Hölder continuous with respect to its j -th variable by assumption **(R)**, the first component J_1 can be easily controlled using the Fubini theorem by

$$\begin{aligned} &\int_{\mathbb{R}^d} |J_1(v, \mathbf{y}_{\setminus j}, z)| dz \\ &\leq C\|\mathbf{F}\|_H \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |z - \mathbf{y}_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} |D_z \partial_v p_h(v, z - \mathbf{y}_j)| dz \right) |D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y})| d\mathbf{y}_j \\ &\leq C\|\mathbf{F}\|_H v^{\frac{1}{\alpha} \frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)} - \frac{1}{\alpha} - 1} \int_{\mathbb{R}^d} |D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y})| d\mathbf{y}_j \end{aligned}$$

where in the last passage we used the smoothing effect of the heat-kernel p_h (Equation (4.9)), noticing that

$$\frac{1 + \alpha(j-2) + \beta}{1 + \alpha(j-1)} = 1 + \frac{\beta - \alpha}{1 + \alpha(j-1)} < 1 + \alpha,$$

since $\alpha > \beta$ by assumption **(P)**. Using now the identity

$$\frac{\alpha_j + \beta_j}{\alpha} + \frac{1}{\alpha} \left(\frac{1 + \alpha(j-2) + \beta}{1 + \alpha(j-1)} - 1 \right) = \frac{2\beta_j}{\alpha}, \quad (5.32)$$

we add the integral with respect to v and write that

$$\begin{aligned} \int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} |J_1(v, \mathbf{y}_{\setminus j}, z)| dz dv &\leq C \|\mathbf{F}\|_H \int_0^{(s-t)^{\delta_j}} \frac{v^{\frac{2\beta_j}{\alpha}}}{v} \int_{\mathbb{R}^d} |D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| d\mathbf{y} dv \\ &\leq C \|\mathbf{F}\|_H (s-t)^{\delta_j \frac{2\beta_j}{\alpha}} \int_{\mathbb{R}^d} |D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| d\mathbf{y}. \end{aligned}$$

Adding the integral with respect to the other components $\mathbf{y}_{\setminus j}$, we can finally conclude that

$$\begin{aligned} \int_{\mathbb{R}^{(n-1)d}} \int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} |J_1(v, \mathbf{y}_{\setminus j}, z)| dz dv d\mathbf{y}_{\setminus j} &\leq C \|\mathbf{F}\|_H (s-t)^{\delta_j \frac{2\beta_j}{\alpha}} \int_{\mathbb{R}^{nd}} |D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| d\mathbf{y} \\ &\leq C \|\mathbf{F}\|_H (s-t)^{\delta_j \frac{2\beta_j}{\alpha} - \sum_{k=1}^n \frac{\vartheta_k}{\alpha_k}} \quad (5.33) \end{aligned}$$

To control the second component J_2 , we start applying a Taylor expansion on $\tilde{p}^{\tau, \xi}$ with respect to \mathbf{y}_j :

$$\begin{aligned} J_2(v, \mathbf{y}_{\setminus j}, z) &= \int_{\mathbb{R}^d} D_z \partial_v p_h(v, z - \mathbf{y}_j) \cdot \left\{ [F_j(s, \mathbf{y}_{\setminus j}, z) - F_j(s, \boldsymbol{\theta}_{\tau, s}(\xi))] \right. \\ &\quad \left. \otimes \int_0^1 D_{\mathbf{y}_j} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \mathbf{y}_j + \lambda(z - \mathbf{y}_j)) \cdot (z) \right\} d\lambda d\mathbf{y}_j. \quad (5.34) \end{aligned}$$

We then notice that for any fixed λ in $[0, 1]$, it holds that

$$\begin{aligned} |F_j(s, \mathbf{y}_{\setminus j}, z) - F_j(s, \boldsymbol{\theta}_{\tau, s}(\xi))| &\leq C \|\mathbf{F}\|_H \left\{ |(z - \boldsymbol{\theta}_{\tau, s}(\xi))_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + \sum_{k=j+1}^n |(\mathbf{y} - \boldsymbol{\theta}_{\tau, s}(\xi))_k|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(k-1)}} \right\} \\ &\leq C \|\mathbf{F}\|_H \left\{ |\lambda(\mathbf{y}_j - z)|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + |(\mathbf{y}_j + \lambda(z - \mathbf{y}_j) - \boldsymbol{\theta}_{\tau, s}(\xi))_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + \sum_{k=j+1}^n |(\mathbf{y} - \boldsymbol{\theta}_{\tau, s}(\xi))_k|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(k-1)}} \right\} \\ &\leq C \|\mathbf{F}\|_H \left\{ |z - \mathbf{y}_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + d_{j:n}^{1+\alpha(j-2)+\beta}((\mathbf{y}_{\setminus j}, \mathbf{y}_j + \lambda(z - \mathbf{y}_j)), \boldsymbol{\theta}_{\tau, s}(\xi)) \right\} \end{aligned}$$

where as in (4.7), we denoted $(\mathbf{y}_{\setminus j}, \mathbf{y}_j + \lambda(z - \mathbf{y}_j)) := (\mathbf{y}_1, \dots, \mathbf{y}_{j-1}, \mathbf{y}_1, \dots, \mathbf{y}_{j-1}, \mathbf{y}_j + \lambda(z - \mathbf{y}_j), \mathbf{y}_{j+1}, \dots, \mathbf{y}_n)$. We can thus split J_2 as

$$\begin{aligned} |J_2(v, \mathbf{y}_{\setminus j}, z)| &\leq \\ &C \|\mathbf{F}\|_H \int_0^1 \left\{ \int_{\mathbb{R}^d} |z - \mathbf{y}_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}+1} |D_z \partial_v p_h(v, z - \mathbf{y}_j)| |D_{\mathbf{y}_j} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \mathbf{y}_j + \lambda(z - \mathbf{y}_j))| d\mathbf{y}_j \right. \\ &\quad \left. + \int_{\mathbb{R}^d} |z - \mathbf{y}_j| |D_z \partial_v p_h(v, z - \mathbf{y}_j)| |D_{\mathbf{y}_j} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \mathbf{y}_j + \lambda(z - \mathbf{y}_j))| \right. \\ &\quad \left. \times d_{j:n}^{1+\alpha(j-2)+\beta}((\mathbf{y}_{\setminus j}, \mathbf{y}_j + \lambda(z - \mathbf{y}_j)), \boldsymbol{\theta}_{\tau, s}(\xi)) d\mathbf{y}_j \right\} d\lambda =: C \|\mathbf{F}\|_H \int_0^1 (J_{2,1} + J_{2,2})(v, \mathbf{y}_{\setminus j}, z, \lambda) d\lambda \quad (5.35) \end{aligned}$$

Adding now the integral with respect to z , the first term $J_{2,1}$ can be rewritten as

$$\begin{aligned} \int_{\mathbb{R}^d} J_{2,1}(v, \mathbf{y}_{\setminus j}, z, \lambda) dz &\leq \\ &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |z - \mathbf{y}_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}+1} |D_z \partial_v p_h(v, z - \mathbf{y}_j)| |D_{\mathbf{y}_j} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \mathbf{y}_j + \lambda(z - \mathbf{y}_j))| d\mathbf{y}_j dz. \end{aligned}$$

The Fubini theorem and the change of variables $\tilde{z} = z - \mathbf{y}_j$ and $\tilde{\mathbf{y}}_j = \mathbf{y}_j + \lambda\tilde{z}$ allow then to divide the integrals:

$$\int_{\mathbb{R}^d} J_{2,1}(v, \mathbf{y}_{\setminus j}, z, \lambda) dz \leq \left(\int_{\mathbb{R}^d} |\tilde{z}|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}+1} |D_{\tilde{z}} \partial_v p_h(v, \tilde{z})| d\tilde{z} \right) \left(\int_{\mathbb{R}^d} |D_{\tilde{\mathbf{y}}_j} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \tilde{\mathbf{y}}_j)| d\tilde{\mathbf{y}}_j \right)$$

Noticing now from assumption **(P)** that

$$\frac{1 + \alpha(j-2) + \beta}{1 + \alpha(j-1)} + 1 = 1 - \frac{\beta - \alpha}{1 + \alpha(j-1)} + 1 < 2 - (1 - \alpha) = 1 + \alpha,$$

we can use the smoothing effect of the heat-kernel p_h (Equation (4.9)) to show that

$$\int_{\mathbb{R}^d} J_{2,1}(v, \mathbf{y}_{\setminus j}, z, \lambda) dz \leq \frac{v^{\frac{1+\alpha(j-2)+\beta}{\alpha(1+\alpha(j-1))}}}{v} \int_{\mathbb{R}^d} |D_{\tilde{\mathbf{y}}_j} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \tilde{\mathbf{y}}_j)| d\tilde{\mathbf{y}}_j.$$

Remembering equation (5.32), we can add the in integral with respect to v and show that

$$\int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} J_{2,1}(v, \mathbf{y}_{\setminus j}, z, \lambda) dz dv \leq (s-t)^{\delta_j \frac{2\beta_j + 1}{\alpha}} \int_{\mathbb{R}^d} |D_{\tilde{\mathbf{y}}_j} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \tilde{\mathbf{y}}_j)| d\tilde{\mathbf{y}}_j.$$

Adding the integral with respect to $\mathbf{y}_{\setminus j}$, we can conclude with $J_{2,1}$ that

$$\begin{aligned} & \int_{\mathbb{R}^{(n-1)d}} \int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} J_{2,1}(v, \mathbf{y}_{\setminus j}, z, \lambda) dz dv d\mathbf{y}_{\setminus j} \\ & \leq C(s-t)^{\delta_j \frac{2\beta_j + 1}{\alpha}} \int_{\mathbb{R}^{nd}} |D_{\mathbf{y}_j} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y})| dy \leq C(s-t)^{\delta_j \frac{2\beta_j + 1}{\alpha} - \frac{1}{\alpha_j} - \sum_{k=1}^n \frac{\vartheta_k}{\alpha_k}} \end{aligned} \quad (5.36)$$

where, for simplicity, we have changed back the variable $\tilde{\mathbf{y}}_j$ with \mathbf{y}_j .

To control instead the term $J_{2,2}$ (cf. Equation (5.35)), we can use again the Fubini theorem and the changes of variables $\tilde{z} = z - \mathbf{y}_j$, $\tilde{\mathbf{y}}_j = \mathbf{y}_j + \lambda\tilde{z}$ to divide the integrals and show that

$$\begin{aligned} & \int_{\mathbb{R}^d} J_{2,2}(v, \mathbf{y}_{\setminus j}, z, \lambda) dz \\ & \leq \left(\int_{\mathbb{R}^d} |\tilde{z}| |D_{\tilde{z}} \partial_v p_h(v, \tilde{z})| d\tilde{z} \right) \left(\int_{\mathbb{R}^d} |D_{\tilde{\mathbf{y}}_j} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \tilde{\mathbf{y}}_j)| d_{j:n}^{1+\alpha(j-2)+\beta}((\mathbf{y}_{\setminus j}, \tilde{\mathbf{y}}_j), \boldsymbol{\theta}_{\tau, s}(\boldsymbol{\xi})) d\tilde{\mathbf{y}}_j \right) \\ & \leq \frac{1}{v} \int_{\mathbb{R}^d} |D_{\mathbf{y}_j} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y})| d_{j:n}^{1+\alpha(j-2)+\beta}(\mathbf{y}, \boldsymbol{\theta}_{\tau, s}(\boldsymbol{\xi})) d\mathbf{y}_j dv \end{aligned}$$

where in the second inequality we used the smoothing effect of the heat-kernel p_h (Equation (4.9)) and changed back the variable $\tilde{\mathbf{y}}_j$ with \mathbf{y}_j for simplicity. It then follows that

$$\begin{aligned} & \int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} J_{2,2}(v, z, \mathbf{y}_{\setminus j}) dz dv \\ & \leq (s-t)^{\delta_j \frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} |D_{\mathbf{y}_j} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y})| d_{j:n}^{1+\alpha(j-2)+\beta}(\mathbf{y}, \boldsymbol{\theta}_{\tau, s}(\boldsymbol{\xi})) d\mathbf{y}_j. \end{aligned}$$

Taking now $\tau = t$ and $\boldsymbol{\xi} = \mathbf{x}$, we conclude with $J_{2,2}$ applying the partial smoothing effect (5.29) of $\tilde{p}^{\tau, \boldsymbol{\xi}}$ under the hypothesis $1 + \alpha(j-2) + \beta \leq \alpha(1 + \alpha(j-1))$ (see Equation (5.30)) to write that

$$\begin{aligned} & \int_{\mathbb{R}^{(n-1)d}} \int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^d} J_{2,2}(v, z, \mathbf{y}_{\setminus j}) dz dv d\mathbf{y}_{\setminus j} \\ & \leq (s-t)^{\delta_j \frac{\alpha_j + \beta_j}{\alpha}} \int_{\mathbb{R}^{nd}} |D_{\mathbf{y}_j} D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \boldsymbol{\xi}}(t, s, \mathbf{x}, \mathbf{y})| d_{j:n}^{1+\alpha(j-2)+\beta}(\mathbf{y}, \boldsymbol{\theta}_{\tau, s}(\boldsymbol{\xi})) d\mathbf{y} \\ & \leq C(s-t)^{\delta_j \frac{\alpha_j + \beta_j}{\alpha} + \frac{1+\alpha(j-2)+\beta}{\alpha} - \frac{1}{\alpha_j} - \sum_{k=1}^n \frac{\vartheta_k}{\alpha_k}}. \end{aligned} \quad (5.37)$$

Looking back to Equations (5.31), (5.33), (5.36) and (5.37), we can finally choose the right δ_j . Since $s - t \leq T - t < 1$ by hypothesis (ST), it is enough to take δ_j such that the following quantities

$$\delta_j \frac{\alpha_j + \beta_j - 1}{\alpha} + \frac{1 + \alpha(j-2) + \beta}{\alpha}, \quad \delta_j \frac{2\beta_j}{\alpha}, \quad \delta_j \frac{2\beta_j + 1}{\alpha} - \frac{1}{\alpha_j} \quad \text{and} \quad \delta_j \frac{\alpha_j + \beta_j}{\alpha} + \frac{1 + \alpha(j-2) + \beta}{\alpha} - \frac{1}{\alpha_j}$$

are bigger than β/α . This is true if for example we choose

$$\delta_j = \frac{[1 + \alpha(j-2)][1 + \alpha(j-1)]}{1 + \alpha(j-2) - \beta}.$$

□

5.2 Proof of Proposition 4

We have now all the tools necessary to prove the A Priori estimates (Proposition 4). In Lemma 13 below, we will state the estimates for the supremum norms of the solution and its non-degenerate gradient while the controls of the Hölder moduli of the solution and its gradient with respect to the non-degenerate variable are given in Lemmas 17 and 18, respectively.

The Schauder estimates (Theorem 1) for a solution u in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ of equation (1.1) will then follows immediately.

Lemma 13 (Supremum Estimates). *Let u be as in Equation (5.27). Then, there exists a constant $C \geq 1$ such that for any t in $[0, T]$ and any \mathbf{x} in \mathbb{R}^{nd} ,*

$$|u(t, \mathbf{x})| + |D_{\mathbf{x}_1} u(t, \mathbf{x})| \leq C \left[\|g\|_{C_{b,d}^{\alpha+\beta}} + \|f\|_{L^\infty(C_{b,d}^\beta)} + \|\mathbf{F}\|_H \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \right].$$

Proof. As indicated above, we can control the supremum norm of u and its gradient with respect to \mathbf{x}_1 analyzing separately the contributions from the proxy $\tilde{u}^{\tau, \xi}$, that have already been handled in Lemma 8, and those from the perturbative term $R^{\tau, \xi}(s, \mathbf{x})$. To control the contribution $\int_t^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau, \xi} R^{\tau, \xi}(s, \mathbf{x}) ds$, we start splitting it up in the following way

$$\begin{aligned} \int_t^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau, \xi} R^{\tau, \xi}(s, \mathbf{x}) ds &= \sum_{j=1}^n \int_t^T \int_{\mathbb{R}^{nd}} D_{\mathbf{x}_1} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) [\mathbf{F}_j(s, \mathbf{y}) - \mathbf{F}_j(s, \boldsymbol{\theta}_{\tau, s}(\xi))] \cdot D_{\mathbf{y}_j} u(s, \mathbf{y}) d\mathbf{y} ds \\ &=: \sum_{j=1}^n I_j(t, \mathbf{x}). \end{aligned} \quad (5.38)$$

Since by hypothesis u has a proper derivative with respect to the first variable \mathbf{x}_1 , it is possible to bound I_1 through

$$|I_1(t, \mathbf{x})| \leq C \|\mathbf{F}\|_H \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \int_t^T \int_{\mathbb{R}^{nd}} |D_{\mathbf{x}_1} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y})| d^\beta(\mathbf{y}, \boldsymbol{\theta}_{\tau, s}(\xi)) d\mathbf{y} ds.$$

We take now $(\tau, \xi) = (t, \mathbf{x})$ so that $\boldsymbol{\theta}_{\tau, s}(\xi) = \tilde{\mathbf{m}}_{t,s}^{\tau, \xi}$ (cf. Equation (3.5) in Lemma 2) and we then use the smoothing effect for the frozen density $\tilde{p}^{\tau, \xi}$ (Equation (3.6)) to show that

$$|I_1(t, \mathbf{x})| \leq C \|\mathbf{F}\|_H \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} (T - t)^{\frac{\beta + \alpha - 1}{\alpha}}. \quad (5.39)$$

Hence, it holds that $|I_1(t, \mathbf{x})| \leq C \|\mathbf{F}\|_H \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})}$, since $T \leq 1$ and $\alpha + \beta > 1$ by assumptions (ST) and (P).

The control for the terms I_j with $j > 1$ can be obtained easily from the second Besov control (Lemma 12). For this reason, we start applying integration by parts formula to show that

$$|I_j(t, \mathbf{x})| = \left| \int_t^T \int_{\mathbb{R}^{nd}} D_{\mathbf{y}_j} \cdot \left\{ D_{\mathbf{x}_1} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}) [\mathbf{F}_j(s, \mathbf{y}) - \mathbf{F}_j(s, \boldsymbol{\theta}_{\tau, s}(\xi))] \right\} u(s, \mathbf{y}) d\mathbf{y} ds \right|$$

We can then use identification (4.10) and duality in Besov spaces (4.11) to write that

$$\begin{aligned} |I_j(t, \mathbf{x})| &\leq \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \int_{\mathbb{R}^{(n-1)d}} \left\| D_{\mathbf{y}_j} \cdot \left\{ D_{\mathbf{x}_1} \tilde{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \cdot) [\mathbf{F}_j(s, \mathbf{y}_{\setminus j}, \cdot) - \mathbf{F}_j(s, \boldsymbol{\theta}_{\tau, s}(\xi))] \right\} \right\|_{B_{1,1}^{-(\alpha_j + \beta_j)}} d\mathbf{y}_{\setminus j}. \end{aligned}$$

Taking now $(\tau, \xi) = (t, \mathbf{x})$, the second Besov control (Lemma 12) can be applied to show that

$$|I_j(t, \mathbf{x})| \leq C \|\mathbf{F}\|_H \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \int_t^T (s-t)^{\frac{\beta-1}{\alpha}} ds \leq C \|\mathbf{F}\|_H \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} (T-t)^{\frac{\beta+\alpha-1}{\alpha}}. \quad (5.40)$$

Since by assumption **(ST)** $T \leq 1$, we can conclude that $|I_j(t, \mathbf{x})| \leq C \|\mathbf{F}\|_H \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})}$.

The control on the perturbative term

$$\int_t^T \tilde{P}_{t,s}^{\tau, \xi} R^{\tau, \xi}(s, \mathbf{x}) ds$$

can be obtained in a similar way. Namely, the inequalities (5.39) and (5.40) hold again with $(T-t)^{\frac{\beta+\alpha-1}{\alpha}}$ replaced by $(T-t)^{\frac{\beta+\alpha}{\alpha}}$. \square

As already specified in the previous sub-section, there is a big difference between the non-degenerate case $i = 1$ where $\alpha + \beta$ is in $(1, 2)$ and we have to deal with a proper derivative and the other degenerate situations ($i > 1$) where instead $(\alpha + \beta)/(1 + \alpha(i - 1)) < 1$ and the norm is calculated directly on the function. Again, we are going to analyze the two cases separately. Lemma 9 will work on the non-degenerate setting ($i = 1$) while lemma 10 will concerns the degenerate one ($i > 1$).

Moreover, we will need to divide the proofs in two cases, depending on which regime we are considering. Since the global off-diagonal regime, i.e. when $T - t \leq c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$, will work essentially as the already shown Schauder estimates (Proposition 1) for the proxy, the proof will be quite shorter.

Instead, in the global diagonal case, such that $T - t \geq c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$, when a time integration is involved (for example in the control of the frozen Green kernel or the perturbative term), two different situations appear. There are again a local off-diagonal regime if $s - t \leq c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$ and a local diagonal regime when $s - t \geq c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$. In order to handle these terms properly, the key tool is to be able to change the freezing points depending on which regime we are. It seems reasonable that, when the spatial points are in a local diagonal regime, the auxiliary frozen densities are considered for the same freezing parameter and conversely that in the local off-diagonal regime, the densities are frozen along their own spatial argument. For this reason, we have postponed the relative proofs in two specific sub-sections.

Before presenting the main results of this section, we are going to state three auxiliary estimates associated with our proxy we will need below. We refer to the Section A.2 for a precise proof of these results.

The first one concerns the sensitivity of the Hölder flow $\theta_{t,s}$ with respect to the initial point. Indeed,

Lemma 14 (Controls on the Flows). *Let $t < s$ be two points in $[0, T]$ and \mathbf{x}, \mathbf{x}' two points in \mathbb{R}^{nd} . Then, there exists a constant $C \geq 1$ such that*

$$d(\theta_{t,s}(\mathbf{x}), \theta_{t,s}(\mathbf{x}')) \leq C \|\mathbf{F}\|_H [d(\mathbf{x}, \mathbf{x}') + (s-t)^{1/\alpha}].$$

The second result is the following:

Lemma 15. *Let $t < s$ be two points in $[0, T]$ and \mathbf{x}, \mathbf{x}' two points in \mathbb{R}^{nd} and \mathbf{y}, \mathbf{y}' two points in \mathbb{R}^{nd} such that $\mathbf{y}_1 = \mathbf{y}'_1$. Then, there exists a constant $C \geq 1$ such that*

$$|(\tilde{\mathbf{m}}_{t,s}^{t,\mathbf{x}}(\mathbf{y}) - \tilde{\mathbf{m}}_{t,s}^{t,\mathbf{x}'}(\mathbf{y}'))_1| \leq C \|\mathbf{F}\|_H [(s-t)d^\beta(\mathbf{x}, \mathbf{x}') + (s-t)^{\frac{\alpha+\beta}{\alpha}}].$$

Finally, the impact of the freezing point in the linearization procedure is the argument of this last Lemma. Namely,

Lemma 16. *Let t be in $[0, T]$ and \mathbf{x}, \mathbf{x}' two points in \mathbb{R}^{nd} . Then, there exists a constant $C \geq 1$ such that*

$$d(\tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}}(\mathbf{x}'), \tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}'}(\mathbf{x}')) \leq C c_0^{\frac{1}{1+\alpha(n-1)}} \|\mathbf{F}\|_H d(\mathbf{x}, \mathbf{x}')$$

where t_0 is the change of regime time defined in (4.16).

Thanks to the above controls, we will eventually prove the following results:

Lemma 17 (Controls on Hölder Moduli: Non-Degenerate). *Let \mathbf{x}, \mathbf{x}' be in \mathbb{R}^{nd} such that $\mathbf{x}_j = \mathbf{x}'_j$ for any $j \neq 1$ and u as in Equation (5.27). Then, there exists a constant $C \geq 1$ such that for any t in $[0, T]$,*

$$|D_{\mathbf{x}_1} u(t, \mathbf{x}) - D_{\mathbf{x}_1} u(t, \mathbf{x}')| \leq C \left\{ c_0^{\frac{\alpha+\beta-2}{\alpha}} (\|g\|_{C^{\alpha+\beta}} + \|f\|_{L^\infty(C^\beta)}) + (c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}} + c_0^{\frac{\alpha+\beta-2}{\alpha}} \|\mathbf{F}\|_H) \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \right\} d^{\alpha+\beta-1}(\mathbf{x}, \mathbf{x}').$$

We can point out now the analogous result in the degenerate setting.

Lemma 18 (Controls on Hölder Moduli: Degenerate). *Let i be in $\llbracket 1, n \rrbracket$ and \mathbf{x}, \mathbf{x}' in \mathbb{R}^{nd} such that $x_j = x'_j$ for any $j \neq i$ and u as in Equation (5.27). Then, there exists a constant $C \geq 1$ such that for any t in $[0, T]$,*

$$|u(t, \mathbf{x}) - u(t, \mathbf{x}')| \leq C \left\{ c_0^{\frac{\beta-\gamma_i}{\alpha}} (\|g\|_{C^{\alpha+\beta}} + \|f\|_{L^\infty(C^\beta)}) + (c_0^{\frac{\alpha+\beta}{1+\alpha(n-1)}} + c_0^{\frac{\beta-\gamma_i}{\alpha}} \|\mathbf{F}\|_H) \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \right\} d^{\alpha+\beta}(\mathbf{x}, \mathbf{x}').$$

5.2.1 Off-Diagonal Regime

We focus here on the proof of the Controls on the Hölder Moduli either in the non-degenerate setting (Proposition 17) and in the degenerate one (Proposition 18), when a off-diagonal regime is assumed. For this reason, all the statements presented in this sub-section will tacitly assume that $T - t \leq c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$ for some given $(t, \mathbf{x}, \mathbf{x}')$ in $[0, T] \times \mathbb{R}^{2nd}$.

To show these two controls, we will need to adapt the auxiliary estimates above to the off-diagonal regime case we consider here. Namely,

$$d(\tilde{\mathbf{m}}_{t,T}^{t,\mathbf{x}}(\mathbf{x}), \tilde{\mathbf{m}}_{t,T}^{t,\mathbf{x}'}(\mathbf{x}')) = d(\boldsymbol{\theta}_{t,T}(\mathbf{x}), \boldsymbol{\theta}_{t,T}(\mathbf{x}')) \leq C \|\mathbf{F}\|_H d(\mathbf{x}, \mathbf{x}'); \quad (5.41)$$

$$\text{if } \mathbf{x}_1 = \mathbf{x}'_1, \quad |(\tilde{\mathbf{m}}_{t,T}^{t,\mathbf{x}}(\mathbf{x}) - \tilde{\mathbf{m}}_{t,T}^{t,\mathbf{x}'}(\mathbf{x}'))_1| \leq C \|\mathbf{F}\|_H d^{\alpha+\beta}(\mathbf{x}, \mathbf{x}') \quad (5.42)$$

They can be obtained easily from Equation (3.5) in Lemma 2 and the sensitivity controls (Lemmas 14 and 15, respectively), taking $s = T$ and $(\mathbf{y}, \mathbf{y}') = (\mathbf{x}, \mathbf{x}')$.

Proof of Proposition 17 in the Off-Diagonal Regime. From the Duhamel-type expansion (5.28), we can represent a mild solution u of Equation for any fixed $(\tau, \boldsymbol{\xi}), (\tau', \boldsymbol{\xi}')$ in $[0, T] \times \mathbb{R}^{nd}$ as

$$|D_{\mathbf{x}_1} u(t, \mathbf{x}) - D_{\mathbf{x}_1} u(t, \mathbf{x}')| \leq |D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau,\boldsymbol{\xi}} g(\mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau',\boldsymbol{\xi}'} g(\mathbf{x}')| + |D_{\mathbf{x}_1} \tilde{G}_{t,T}^{\tau,\boldsymbol{\xi}} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t,T}^{\tau',\boldsymbol{\xi}'} f(t, \mathbf{x}')| + \left| \int_t^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\boldsymbol{\xi}} R^{\tau,\boldsymbol{\xi}}(s, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau',\boldsymbol{\xi}'} R^{\tau',\boldsymbol{\xi}'}(s, \mathbf{x}') ds \right|.$$

After possible differentiations, we will choose $\tau = \tau' = t$, $\boldsymbol{\xi} = \mathbf{x}$ and $\boldsymbol{\xi}' = \mathbf{x}'$ in order to exploit the sensitivity controls (5.42) and (5.41).

Control on the frozen semigroup. It can be handled following the analogous part in the proof of the Hölder control for the proxy (Lemma 9). The only difference is that we cannot control

$$d(\tilde{\mathbf{m}}_{t,T}^{\tau,\boldsymbol{\xi}}(\mathbf{x}), \tilde{\mathbf{m}}_{t,T}^{\tau',\boldsymbol{\xi}'}(\mathbf{x}'))$$

in Equation (4.18) using the affinity of the mapping $\mathbf{x} \rightarrow \tilde{\mathbf{m}}_{t,T}^{\tau,\boldsymbol{\xi}}(\mathbf{x})$, since the two freezing point are now different. Instead, we can take $\tau = \tau' = t$, $\boldsymbol{\xi} = \mathbf{x}$ and $\boldsymbol{\xi}' = \mathbf{x}'$ and apply the sensitivity control (5.41) to write that

$$d(\tilde{\mathbf{m}}_{t,T}^{\tau,\boldsymbol{\xi}}(\mathbf{x}), \tilde{\mathbf{m}}_{t,T}^{\tau',\boldsymbol{\xi}'}(\mathbf{x}')) = d(\boldsymbol{\theta}_{t,T}(\mathbf{x}), \boldsymbol{\theta}_{t,T}(\mathbf{x}')) \leq C \|\mathbf{F}\|_H d(\mathbf{x}, \mathbf{x}').$$

Control on the Green kernel. It follows immediately from the proof of the Hölder control (Lemma 9) for the proxy, noticing that $t_0 = T$, since we are in the off-diagonal regime.

Control on the perturbative error. Since we do not exploit the difference of the spatial points $(\mathbf{x}, \mathbf{x}')$ in the off-diagonal regime but instead we control the two contributions separately, we can rely on the controls on

the supremum norms we have already shown in Lemma 13. Namely, we start writing that

$$\begin{aligned} \left| \int_t^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau',\xi'} R^{\tau',\xi'}(s, \mathbf{x}') ds \right| \\ \leq \left| \int_t^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s, \mathbf{x}) ds \right| + \left| \int_t^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau',\xi'} R^{\tau',\xi'}(s, \mathbf{x}') ds \right| \end{aligned} \quad (5.43)$$

Then, we can follow the same reasonings of Lemma 13 concerning the remainder term (cf. Equations (5.38), (5.39) and (5.40)) to show that

$$\left| \int_t^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s, \mathbf{x}) ds \right| \leq C \|F\|_H \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} (T-t)^{\frac{\alpha+\beta-1}{\alpha}}. \quad (5.44)$$

Using it in the above Equation (5.43), we can finally conclude that

$$\left| \int_t^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau',\xi'} R^{\tau',\xi'}(s, \mathbf{x}') ds \right| \leq C \|F\|_H \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} d^{\alpha+\beta-1}(\mathbf{x}, \mathbf{x}') \quad (5.45)$$

remembering that we assumed to be in the off-diagonal regime, i.e. $T-t \leq c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$ for some $c_0 \leq 1$.

Proof of Proposition 18 in the Off-Diagonal Regime. As done before, we are going to analyze separately the single terms appearing from the Duhamel-type representation (5.27) of a solution u :

$$\begin{aligned} |u(t, \mathbf{x}) - u(t, \mathbf{x}')| \leq & \left| \tilde{P}_{t,T}^{\tau,\xi} g(\mathbf{x}) - \tilde{P}_{t,T}^{\tau',\xi'} g(\mathbf{x}') \right| + \left| \tilde{G}_{t,T}^{\tau,\xi} f(t, \mathbf{x}) - \tilde{G}_{t,T}^{\tau',\xi'} f(t, \mathbf{x}') \right| \\ & + \left| \int_t^T \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s, \mathbf{x}) - \tilde{P}_{t,s}^{\tau',\xi'} R^{\tau',\xi'}(s, \mathbf{x}') ds \right| \end{aligned}$$

for some $(\tau, \xi), (\tau', \xi')$ in $[0, T] \times \mathbb{R}^{nd}$ fixed but to be chosen later as $\tau = \tau' = t$, $\xi = \mathbf{x}$ and $\xi' = \mathbf{x}'$.

Control on the frozen semigroup. We can essentially follow the proof of the Hölder control (Lemma 10) for the proxy. However, this time we cannot exploit the affinity of the mapping $\mathbf{x} \rightarrow \tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x})$ to control the difference

$$|g(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}) - z) - g(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}') - z)|.$$

Instead, we notice now that we can bound it as

$$|g(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}) - z) - g(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}') - z)| \leq C \|g\|_{C_{b,d}^{\alpha+\beta}} \left[d^{\alpha+\beta}(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}), \tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}')) + |(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}) - \tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}'))_1| \right]$$

since g is differentiable and thus Lipschitz continuous, in the first non-degenerate variable.

Taking now $\tau = \tau' = t$, $\xi = \mathbf{x}$ and $\xi' = \mathbf{x}'$, we can use the sensitivity controls (5.41) and (5.42) (noticing that by assumption, $\mathbf{x}_1 = \mathbf{x}'_1$) to write that

$$|g(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}) - z) - g(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}') - z)| \leq C \|F\|_H \|g\|_{C_{b,d}^{\alpha+\beta}} d^{\alpha+\beta}(\mathbf{x}, \mathbf{x}')$$

Control on the Green kernel. It can be obtained following the analogous part in the proof of the Hölder control (Lemma 10) for the proxy. Similarly to the paragraph "Control on the frozen semigroup" in the previous proof, we need to take $(\tau, \xi) = (t, \mathbf{x})$, $(\tau, \xi') = (t, \mathbf{x})$ and apply the sensitivity control (5.41) to control the term

$$d(\tilde{\mathbf{m}}_{t,T}^{\tau,\xi}(\mathbf{x}), \tilde{\mathbf{m}}_{t,T}^{\tau',\xi'}(\mathbf{x}'))$$

appearing in Equation (4.25).

Control on the perturbative error. The proof of this estimate essentially matches the previous, analogous one in the non-degenerate setting. Namely, Equations (5.43), (5.44) and (5.45) hold again with $(T-t)^{\frac{\beta+\alpha}{\alpha}}$ instead of $(T-t)^{\frac{\beta+\alpha-1}{\alpha}}$.

5.2.2 Diagonal Regime

Since the aim of this section is to prove Lemmas 17 and 18 when a diagonal regime is assumed, we will assume from this point further that $T - t \geq c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$ for some given $(t, \mathbf{x}, \mathbf{x}')$ in $[0, T] \times \mathbb{R}^{2nd}$.

As preannounced in the introduction of this section, we need here a modification of the Duhamel-type representation (3.16) that allows to change the freezing points along the time integration variable. Remembering the previous notations for $\tilde{G}_{r,v}^{\tau,\xi}$ and $R^{\tau,\xi}$ in (4.2) and (3.15) respectively, it holds that

Lemma 19 (Change of Frozen Point). *Let (τ, ξ) be a freezing couple in $[0, T] \times \mathbb{R}^{nd}$ and $\tilde{\xi}$ another freezing point in \mathbb{R}^{nd} . Then, any classical solution u in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ of Equation (1.1) can be represented for any (t, \mathbf{x}) in $[0, T] \times \mathbb{R}^{nd}$ as*

$$\begin{aligned} u(t, \mathbf{x}) = & \tilde{P}_{t,T}^{\tau,\tilde{\xi}} g(\mathbf{x}) + \tilde{G}_{t,t_0}^{\tau,\xi} f(t, \mathbf{x}) + \tilde{G}_{t_0,T}^{\tau,\tilde{\xi}} f(t, \mathbf{x}) \\ & + \int_t^{t_0} \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s, \mathbf{x}) ds + \int_{t_0}^T \tilde{P}_{t,s}^{\tau,\tilde{\xi}} R^{\tau,\tilde{\xi}}(s, \mathbf{x}) ds + \tilde{P}_{t,t_0}^{\tau,\xi} u(t_0, \mathbf{x}) - \tilde{P}_{t,t_0}^{\tau,\tilde{\xi}} u(t_0, \mathbf{x}) \end{aligned} \quad (5.46)$$

where t_0 is the change of regime time defined in (4.16).

Proof. Fixed t in $(0, T)$, we start considering another point r in (t, T) . On $(0, r)$, it is clear that u is again a mild solution of equation (1.1) but with terminal condition $u(r, \mathbf{x})$. Then, Duhamel expansion (3.16) can be applied with respect to the frozen couple (τ, ξ) , allowing us to write that

$$u(t, \mathbf{x}) = \tilde{P}_{t,r}^{\tau,\xi} g(\mathbf{x}) + \int_t^r \tilde{P}_{t,s}^{\tau,\xi} f(s, \mathbf{x}) ds + \int_t^r \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi} u(s, \mathbf{x}) ds.$$

Noticing that u is independent from r , it is possible now to differentiate the above equality with respect to r in (t, T) to show that

$$0 = \partial_r [\tilde{P}_{t,r}^{\tau,\xi} u(r, \mathbf{x})] + \tilde{P}_{t,r}^{\tau,\xi} f(r, \mathbf{x}) + \tilde{P}_{t,r}^{\tau,\xi} R^{\tau,\xi}(r, \mathbf{x}). \quad (5.47)$$

We highlight now that the above expression holds for any chosen frozen couple (τ, ξ) and any fixed time r . Thus, it is possible to integrate it with respect to r for a fixed ξ between t and t_0 and for another frozen point $\tilde{\xi}$ between t_0 and T , leading to

$$\begin{aligned} 0 = & \tilde{P}_{t,t_0}^{\tau,\xi} u(t_0, \mathbf{x}) - \tilde{P}_{t,t}^{\tau,\xi} u(t, \mathbf{x}) + \int_t^{t_0} \tilde{P}_{t,r}^{\tau,\xi} f(r, \mathbf{x}) dr + \int_t^{t_0} \tilde{P}_{t,r}^{\tau,\xi} R^{\tau,\xi}(r, \mathbf{x}) dr \\ & + \tilde{P}_{t,T}^{\tau,\tilde{\xi}} u(T, \mathbf{x}) - \tilde{P}_{t,t_0}^{\tau,\tilde{\xi}} u(t_0, \mathbf{x}) + \int_{t_0}^T \tilde{P}_{t,r}^{\tau,\tilde{\xi}} f(r, \mathbf{x}) dr + \int_{t_0}^T \tilde{P}_{t,r}^{\tau,\tilde{\xi}} R^{\tau,\tilde{\xi}}(r, \mathbf{x}) dr. \end{aligned}$$

With our previous notations, the above expression can be finally rewritten as

$$\begin{aligned} 0 = & \tilde{P}_{t,t_0}^{\tau,\xi} u(t_0, \mathbf{x}) - u(t, \mathbf{x}) + \tilde{G}_{t,t_0}^{\tau,\xi} f(t, \mathbf{x}) + \int_t^{t_0} \tilde{P}_{t,r}^{\tau,\xi} R^{\tau,\xi}(r, \mathbf{x}) dr \\ & + \tilde{P}_{t,T}^{\tau,\tilde{\xi}} g(\mathbf{x}) - \tilde{P}_{t,t_0}^{\tau,\tilde{\xi}} u(t_0, \mathbf{x}) + \tilde{G}_{t_0,T}^{\tau,\tilde{\xi}} f(t, \mathbf{x}) + \int_{t_0}^T \tilde{P}_{t,r}^{\tau,\tilde{\xi}} R^{\tau,\tilde{\xi}}(r, \mathbf{x}) dr \end{aligned}$$

and we have concluded. \square

Similarly to the off-diagonal case, we are going to apply the auxiliary estimates associated with the proxy (Lemmas 15 and 16) in the current diagonal regime. Namely, taking $s = t_0$ and $(\mathbf{y}, \mathbf{y}') = (\mathbf{x}, \mathbf{x})$ in Lemma 15, we know that there exists a constant $C \geq 1$ such that for any t in $[0, T]$ and any \mathbf{x}, \mathbf{x}' in \mathbb{R}^{nd} ,

$$\text{if } \mathbf{x}_1 = \mathbf{x}'_1, \quad |(\tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}}(\mathbf{x}) - \tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}'}(\mathbf{x}))_1| \leq C \|\mathbf{F}\|_H d^{\alpha+\beta}(\mathbf{x}, \mathbf{x}'). \quad (5.48)$$

Moreover, in order to control the perturbative term when a local diagonal regime appears, i.e. when the time integration variable s is in $[t_0, T]$, we will quite often use a Taylor expansion on the frozen density. To be

able to exploit the already proven controls, such that the smoothing effect for the frozen density (Equation (3.6)) or the Besov control (Lemma 12), we will need the following:

$$\text{if } s - t \geq c_0 d^\alpha(\mathbf{x}, \mathbf{x}'), \quad |D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi'}(t, s, \mathbf{x} + \lambda(\mathbf{x}' - \mathbf{x}), \mathbf{y})| \leq C |D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi'}(t, s, \mathbf{x}, \mathbf{y})| \quad (5.49)$$

for any multi-index ϑ in \mathbb{N}^d such that $|\vartheta| \leq 2$ and any λ in $[0, 1]$. The proof of these results can be found in Section A.2.

We are now ready to prove Propositions 17 and 18 when a global diagonal regime is considered.

Proof of Proposition 17 in the Diagonal Regime. We start recalling that in Lemma 17 we assumed fixed a time t in $[0, T]$ and two spatial points \mathbf{x}, \mathbf{x}' in \mathbb{R}^{nd} such that $\mathbf{x}_j = \mathbf{x}'_j$ if $j \neq 1$. From the above representation (5.46) and the Duhamel-type formula (3.16), we know that

$$\begin{aligned} D_{\mathbf{x}_1} u(t, \mathbf{x}) - D_{\mathbf{x}_1} u(t, \mathbf{x}') &= \left(D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau, \tilde{\xi}} g(\mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau', \xi'} g(\mathbf{x}') \right) \\ &\quad + \left(D_{\mathbf{x}_1} \tilde{G}_{t,t_0}^{\tau, \tilde{\xi}} f(t, \mathbf{x}) + D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \tilde{\xi}} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t,T}^{\tau', \xi'} f(t, \mathbf{x}') \right) \\ &\quad + \left(\int_t^{t_0} D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau, \tilde{\xi}} R^{\tau, \tilde{\xi}}(s, \mathbf{x}) ds + \int_{t_0}^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau, \tilde{\xi}} R^{\tau, \tilde{\xi}}(s, \mathbf{x}) ds - \int_t^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau', \xi'} R^{\tau', \xi'}(s, \mathbf{x}') ds \right) \\ &\quad + \left(D_{\mathbf{x}_1} \tilde{P}_{t,t_0}^{\tau, \tilde{\xi}} u(t_0, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,t_0}^{\tau', \xi'} u(t_0, \mathbf{x}') \right) \end{aligned}$$

for some freezing couples $(\tau, \tilde{\xi}), (\tau, \tilde{\xi}'), (\tau', \xi')$ in $[0, T] \times \mathbb{R}^{nd}$ fixed but to be chosen later. To help the readability of the following, we assume from this point further $\tau = \tau'$ and $\tilde{\xi} = \xi'$.

Control on frozen semigroup. We start focusing on the control of the frozen semigroup, i.e.

$$|D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau, \tilde{\xi}} g(\mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,T}^{\tau, \tilde{\xi}} g(\mathbf{x}')|.$$

Since the freezing couples coincide, the control on the frozen semigroup can be obtained following the proof of the Hölder control (Lemma 9) for the proxy.

Control on the Green kernel. As done before, we split the analysis with respect to the change of regime time t_0 . Namely, we write

$$\begin{aligned} &|D_{\mathbf{x}_1} \tilde{G}_{t,t_0}^{\tau, \tilde{\xi}} f(t, \mathbf{x}) + D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \tilde{\xi}} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t,T}^{\tau, \tilde{\xi}} f(t, \mathbf{x}')| \\ &\leq |D_{\mathbf{x}_1} \tilde{G}_{t,t_0}^{\tau, \tilde{\xi}} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t,t_0}^{\tau, \tilde{\xi}} f(t, \mathbf{x}')| + |D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \tilde{\xi}} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \tilde{\xi}} f(t, \mathbf{x}')|. \end{aligned}$$

While in the local off-diagonal regime, the first term in the r.h.s. of the above expression can be handled as in the global off-diagonal regime, the local diagonal regime contribution represented by

$$|D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \tilde{\xi}} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \tilde{\xi}} f(t, \mathbf{x}')| = |D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \tilde{\xi}} f(t, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{G}_{t_0,T}^{\tau, \tilde{\xi}} f(t, \mathbf{x}')|$$

since $\tilde{\xi} = \xi'$, can be controlled following again the proof of the Hölder control (Lemma 9) for the proxy.

Control on the discontinuity term. We can now focus on the contribution

$$|D_{\mathbf{x}_1} \tilde{P}_{t,t_0}^{\tau, \tilde{\xi}} u(t_0, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,t_0}^{\tau, \tilde{\xi}} u(t_0, \mathbf{x}')|,$$

arising from the change of freezing point in the representation (5.46).

Since at fixed time t_0 , the function u shows the same spatial regularity of g , this control can be handled following the paragraph in the proof of the Hölder control for the proxy (Lemma 9) concerning the frozen semigroup in the off-diagonal regime. The only main difference is in Equation (4.18) where, this time, we need to take $(\tau, \tilde{\xi}, \xi') = (t, \mathbf{x}, \mathbf{x}')$ and exploit the sensitivity estimate (Lemma 16) to control the quantity

$$d(\tilde{\mathbf{m}}_{t,t_0}^{\tau, \tilde{\xi}}(\mathbf{x}), \tilde{\mathbf{m}}_{t,t_0}^{\tau, \tilde{\xi}}(\mathbf{x}')).$$

In the end, it is possible to show again (cf. Equation (4.21)) that

$$\left| D_{\mathbf{x}_1} \tilde{P}_{t,t_0}^{\tau,\xi} u(t_0, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,t_0}^{\tau,\tilde{\xi}} u(t_0, \mathbf{x}) \right| \leq C \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} c_0^{\frac{\alpha+\beta-1}{\alpha}} d^{\alpha+\beta-1}(\mathbf{x}, \mathbf{x}').$$

Control on the perturbative term. We start splitting the analysis into two cases with respect to the critical time t_0 giving the change of regime. Namely, we write

$$\begin{aligned} & \left| \int_t^{t_0} D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s, \mathbf{x}) ds + \int_{t_0}^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}) ds - \int_t^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}') ds \right| \\ & \leq \left| \int_t^{t_0} D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}') ds \right| + \left| \int_{t_0}^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}') ds \right|. \end{aligned}$$

We then notice that the local off-diagonal regime represented by

$$\left| \int_t^{t_0} D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}') ds \right|$$

can be handled following the proof in the global off-diagonal regime of Lemma 17.

We can then focus our attention on the local diagonal regime, i.e.

$$\left| \int_{t_0}^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}') ds \right|.$$

Since the freezing couples coincide, we can use a Taylor expansion with respect to the first variable \mathbf{x}_1 and write that

$$\begin{aligned} & \left| \int_{t_0}^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}') ds \right| \\ & = \left| \int_{t_0}^T \int_{\mathbb{R}^{nd}} \int_0^1 D_{\mathbf{x}_1}^2 \tilde{p}^{\tau,\xi'}(t, s, \mathbf{x} + \lambda(\mathbf{x}' - \mathbf{x}), \mathbf{y})(\mathbf{x}' - \mathbf{x})_1 R^{\tau,\xi'}(s, \mathbf{y}) d\mathbf{y} ds d\lambda \right|. \end{aligned}$$

Noticing that we are integrating from t_0 to T , equation (5.49) can be rewritten as

$$\begin{aligned} & \left| \int_{t_0}^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}') ds \right| \leq |(\mathbf{x}' - \mathbf{x})_1| \sum_{j=1}^n \int_0^1 \int_{t_0}^T \left| \int_{\mathbb{R}^{nd}} D_{\mathbf{x}_1}^2 \tilde{p}^{\tau,\xi'}(t, s, \mathbf{x}, \mathbf{y}) \right. \\ & \quad \left. \left\{ [\mathbf{F}_j(s, \mathbf{y}) - \mathbf{F}_j(s, \boldsymbol{\theta}_{t,s}(\xi'))] \cdot D_{\mathbf{y}_j} u(s, \mathbf{y}) \right\} d\mathbf{y} \right| ds d\lambda =: |(\mathbf{x} - \mathbf{x}')_1| \sum_{j=1}^n \int_{t_0}^T I_j^d(s) ds \quad (5.50) \end{aligned}$$

As done before, we are going to treat separately the cases $j = 1$ and $j > 1$. In the first case, the term I_1^d can be easily controlled by

$$\begin{aligned} I_1^d(s) & \leq \|D_{\mathbf{y}_1} u\|_{L^\infty(L^\infty)} \int_{\mathbb{R}^{nd}} \left| D_{\mathbf{x}_1}^2 \tilde{p}^{\tau,\xi'}(t, s, \mathbf{x}, \mathbf{y}) \right| |\mathbf{F}_1(s, \mathbf{y}) - \mathbf{F}_1(s, \boldsymbol{\theta}_{t,s}(\xi'))| d\mathbf{y} \\ & \leq C \|\mathbf{F}\|_H \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} (s - t)^{\frac{\beta-2}{\alpha}} \quad (5.51) \end{aligned}$$

where in the last passage we used the smoothing effect for the frozen density $\tilde{p}^{\tau,\xi}$ (Equation (3.6)).

On the other side, the case $j > 1$ can be exploited using the second Besov control (Lemma 12). For this reason, we start using integration by parts formula to show that

$$I_j^d(s) = \left| \int_{\mathbb{R}^{nd}} D_{\mathbf{y}_j} \cdot \left\{ D_{\mathbf{x}_1}^2 \tilde{p}^{\tau,\xi'}(t, s, \mathbf{x}, \mathbf{y}) \otimes [\mathbf{F}_j(s, \mathbf{y}) - \mathbf{F}_j(s, \boldsymbol{\theta}_{t,s}(\xi'))] \right\} u(s, \mathbf{y}) d\mathbf{y} \right|.$$

Through the duality in Besov spaces (4.11) and the identification (4.10), we then write that

$$\begin{aligned} I_j^d(s) & \leq \\ & C \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \int_{\mathbb{R}^{(n-1)d}} \|D_{\mathbf{y}_j} \cdot \left\{ D_{\mathbf{x}_1}^2 \tilde{p}^{\tau,\xi'}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \cdot) \otimes [\mathbf{F}_j(s, \mathbf{y}_{\setminus j}, \cdot) - \mathbf{F}_j(s, \boldsymbol{\theta}_{t,s}(\xi'))] \right\}\|_{B_{1,1}^{-(\alpha_j+\beta_j)}} d\mathbf{y}_{\setminus j}. \end{aligned}$$

We can now apply the Second Besov control (Lemma 12) to show that

$$I_j^d(s) \leq C \|\mathbf{F}\|_H \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} (s-t)^{\frac{\beta-2}{\alpha}}. \quad (5.52)$$

Going back at Equations (5.50) (5.51) and (5.52), we can write that

$$\begin{aligned} \left| \int_{t_0}^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}') ds \right| &\leq C \|\mathbf{F}\|_H \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} |(\mathbf{x} - \mathbf{x}')_1| \int_{t_0}^T (s-t)^{\frac{\beta-2}{\alpha}} ds \\ &\leq C \|\mathbf{F}\|_H \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} |(\mathbf{x} - \mathbf{x}')_1| (t_0 - t)^{\frac{\alpha+\beta-2}{\alpha}} \end{aligned} \quad (5.53)$$

where in the last passage we used that $\frac{\alpha+\beta-2}{\alpha} < 0$ to pick the starting point t_0 in the integral. Using that $t_0 - t = c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$, we can conclude that

$$\left| \int_{t_0}^T D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}) - D_{\mathbf{x}_1} \tilde{P}_{t,s}^{\tau,\xi'} R^{\tau,\xi'}(s, \mathbf{x}') ds \right| \leq C c_0^{\frac{\alpha+\beta-2}{\alpha}} \|\mathbf{F}\|_H \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} d^{\alpha+\beta-1}(\mathbf{x}, \mathbf{x}').$$

We can conclude this section showing the Hölder control in the degenerate setting when a diagonal regime is assumed.

Proof of Proposition 18 in Diagonal Regime. We start recalling that in proposition (18) we assumed fixed a time t in $[0, T]$ and two spatial points \mathbf{x}, \mathbf{x}' in \mathbb{R}^{nd} such that $\mathbf{x}_j = \mathbf{x}'_j$ if $j \neq i$ for some i in $\llbracket 2, n \rrbracket$.

Representation (5.46) and Duhamel-type expansion (3.16) allows to control the Hölder modulus of a solution u analyzing separately the different terms:

$$\begin{aligned} u(t, \mathbf{x}) - u(t, \mathbf{x}') &= \left(\tilde{P}_{t,T}^{\tau,\tilde{\xi}} g(\mathbf{x}) - \tilde{P}_{t,T}^{\tau',\tilde{\xi}'} g(\mathbf{x}') \right) + \left(\tilde{G}_{t,t_0}^{\tau,\tilde{\xi}} f(t, \mathbf{x}) + \tilde{G}_{t_0,T}^{\tau,\tilde{\xi}} f(t, \mathbf{x}) - \tilde{G}_{t,T}^{\tau',\tilde{\xi}'} f(t, \mathbf{x}') \right) \\ &+ \left(\int_t^{t_0} \tilde{P}_{t,s}^{\tau,\tilde{\xi}} R^{\tau,\tilde{\xi}}(s, \mathbf{x}) ds + \int_{t_0}^T \tilde{P}_{t,s}^{\tau,\tilde{\xi}} R^{\tau,\tilde{\xi}}(s, \mathbf{x}) ds - \int_t^T \tilde{P}_{t,s}^{\tau',\tilde{\xi}'} R^{\tau',\tilde{\xi}'}(s, \mathbf{x}') ds \right) + \left(\tilde{P}_{t,t_0}^{\tau,\tilde{\xi}} u(t_0, \mathbf{x}) - \tilde{P}_{t,t_0}^{\tau',\tilde{\xi}'} u(t_0, \mathbf{x}') \right) \end{aligned}$$

for some freezing couples $(\tau, \tilde{\xi}), (\tau, \tilde{\xi}'), (\tau, \tilde{\xi}')$ fixed but to be chosen later. As done before, we assume however from this point further that $\tau = \tau'$ and $\tilde{\xi} = \tilde{\xi}'$.

Control on the frozen semigroup. Noticing that we have taken the same freezing couples since $\tilde{\xi} = \tilde{\xi}'$, the control on the frozen semigroup $|\tilde{P}_{t,T}^{\tau,\tilde{\xi}} g(\mathbf{x}) - \tilde{P}_{t,T}^{\tau,\tilde{\xi}} g(\mathbf{x}')|$ can be obtained exploiting the same argument used in the proof of the Hölder control (Lemma 10) for the proxy.

Control on the Green kernel. The proof of this estimate essentially matches the previous, analogous one in the non-degenerate setting. Namely, we follow the proof in the global off-diagonal regime of Proposition 18 to control the local off-diagonal regime contribution $|\tilde{G}_{t,t_0}^{\tau,\tilde{\xi}} f(t, \mathbf{x}) - \tilde{G}_{t,t_0}^{\tau,\tilde{\xi}} f(t, \mathbf{x}')|$ while in the locally diagonal regime term

$$|\tilde{G}_{t_0,T}^{\tau,\tilde{\xi}} f(t, \mathbf{x}) - \tilde{G}_{t_0,T}^{\tau,\tilde{\xi}} f(t, \mathbf{x}')|,$$

the freezing couples coincide and we can thus exploit the same argument used in the proof of the Hölder control (Lemma 10) for the proxy.

Control on the discontinuity term. The proof of this result will follow essentially the one about the off-diagonal regime of the frozen semigroup with respect to the degenerate variables. It holds that

$$\begin{aligned} \tilde{P}_{t,t_0}^{\tau,\tilde{\xi}} u(t_0, \mathbf{x}) &= \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\tilde{\xi}}(t, t_0, \mathbf{x}, \mathbf{y}) u(t_0, \mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^{nd}} \frac{1}{\det(\mathbb{M}_{t_0-t})} p_S(t_0 - t, \mathbb{M}_{t_0-t}^{-1}(\tilde{\mathbf{m}}_{t,t_0}^{\tau,\tilde{\xi}}(\mathbf{x}) - \mathbf{y})) u(t_0, \mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^{nd}} \frac{1}{\det(\mathbb{M}_{t_0-t})} p_S(t_0 - t, \mathbb{M}_{t_0-t}^{-1} \mathbf{z}) u(t_0, \tilde{\mathbf{m}}_{t,t_0}^{\tau,\tilde{\xi}}(\mathbf{x}) - \mathbf{z}) d\mathbf{z} \end{aligned}$$

where in the last passage we used the change of variable $\mathbf{z} = \tilde{\mathbf{m}}_{t,t_0}^{\tau,\xi}(\mathbf{x}) - \mathbf{y}$. Since a similar argument works also for $\tilde{P}_{t,t_0}^{\xi'} u(t_0, \mathbf{x})$, it then follows that

$$\begin{aligned} & |\tilde{P}_{t,t_0}^{\tau,\xi} u(t_0, \mathbf{x}) - \tilde{P}_{t,t_0}^{\tau,\xi'} u(t_0, \mathbf{x})| \\ &= \left| \int_{\mathbb{R}^{nd}} \frac{1}{\det(\mathbb{M}_{t_0-t})} p_S(t_0 - t, \mathbb{M}_{t_0-t}^{-1} \mathbf{z}) [u(t_0, \tilde{\mathbf{m}}_{t,t_0}^{\tau,\xi}(\mathbf{x}) - \mathbf{z}) - u(t_0, \tilde{\mathbf{m}}_{t,t_0}^{\tau,\xi'}(\mathbf{x}) - \mathbf{z})] d\mathbf{z} \right| \end{aligned}$$

Remembering that $u(t_0, \cdot)$ is Lipschitz with respect to the first non-degenerate variable, we can write now that

$$\begin{aligned} & |\tilde{P}_{t,t_0}^{\tau,\xi} u(t_0, \mathbf{x}) - \tilde{P}_{t,t_0}^{\tau,\xi'} u(t_0, \mathbf{x})| \\ &\leq C \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} [d^{\alpha+\beta}(\tilde{\mathbf{m}}_{t,t_0}^{\tau,\xi}(\mathbf{x}), \tilde{\mathbf{m}}_{t,t_0}^{\tau,\xi'}(\mathbf{x})) + |(\tilde{\mathbf{m}}_{t,t_0}^{\tau,\xi}(\mathbf{x}) - \tilde{\mathbf{m}}_{t,t_0}^{\tau,\xi'}(\mathbf{x}))_1|] \int_{\mathbb{R}^{nd}} p_S(t_0 - t, \mathbb{M}_{t_0-t}^{-1} \mathbf{z}) \frac{d\mathbf{z}}{\det(\mathbb{M}_{t_0-t})} \\ &\leq C \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} [d^{\alpha+\beta}(\tilde{\mathbf{m}}_{t,t_0}^{\tau,\xi}(\mathbf{x}), \tilde{\mathbf{m}}_{t,t_0}^{\tau,\xi'}(\mathbf{x})) + |(\tilde{\mathbf{m}}_{t,t_0}^{\tau,\xi}(\mathbf{x}) - \tilde{\mathbf{m}}_{t,t_0}^{\tau,\xi'}(\mathbf{x}))_1|]. \end{aligned}$$

Taking $(\xi = \xi' = \mathbf{x})$, we can then use the sensitivity controls (Lemma 16 and Equation (5.48)) to show that

$$|\tilde{P}_{t,t_0}^{\tau,\xi} u(t_0, \mathbf{x}) - \tilde{P}_{t,t_0}^{\tau,\xi'} u(t_0, \mathbf{x})| \leq C \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \|\mathbf{F}\|_{H^C} c_0^{\frac{\alpha+\beta}{1+\alpha(n-1)}} d^{\alpha+\beta}(\mathbf{x}, \mathbf{x}).$$

Control on the perturbative term. The proof of this Estimate essentially matches the previous, analogous one in the non-degenerate setting. Namely, Inequalities (5.51), (5.52) and (5.53) hold again with $(s - t)^{\frac{\beta-2}{\alpha}}$ replaced by $(s - t)^{\frac{\beta}{\alpha} - \frac{1}{\alpha_i}}$.

5.2.3 Mollifying Procedure

We now make the mollifying parameter m appear again using the notations introduced in Section 3.2 (see Equation (3.16)). Then, Lemmas 13, 17 and 18 rewrite together in the following way. There exists a constant $C > 0$ such that for any m in \mathbb{N} ,

$$\|u_m\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \leq C c_0^{\frac{\beta-\gamma_m}{\alpha}} [\|g_m\|_{C_{b,d}^{\alpha+\beta}} + \|f_m\|_{L^\infty(C_{b,d}^\beta)}] + C (c_0^{\frac{\beta-\gamma_m}{\alpha}} \|\mathbf{F}_m\|_H + c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}) \|u_m\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \quad (5.1)$$

where c_0 is assumed to be fixed but chosen later. Importantly, c_0 and C does not depends on the regularizing parameter m . Thus, letting m go to ∞ and remembering the definition 1 of mild solution u , the above expression immediately implies the A priori estimates (Proposition 4).

6 Existence Result

The aim of this section is to show the well-posedness in a mild sense of the original IPDE (1.1). Recalling Definition 1 for a mild solution of the IPDE (1.1), let us consider three sequences $\{f_m\}_{m \in \mathbb{N}}$, $\{g_m\}_{m \in \mathbb{N}}$ and $\{\mathbf{F}_m\}_{m \in \mathbb{N}}$ of "regularized" coefficients such that

- $\{f_m\}_{m \in \mathbb{N}}$ is in $C_b^\infty((0, T) \times \mathbb{R}^{nd})$ and f_m converges to f in $L^\infty(0, T; C_{b,d}^\beta(\mathbb{R}^{nd}))$;
- $\{g_m\}_{m \in \mathbb{N}}$ is in $C_b^\infty(\mathbb{R}^{nd})$ and g_m converges to g in $C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd})$;
- $\{\mathbf{F}_m\}_{m \in \mathbb{N}}$ is in $C_b^\infty((0, T) \times \mathbb{R}^{nd}; \mathbb{R}^{nd})$ and $\|\mathbf{F}_m - \mathbf{F}\|_H$ converges to 0.

It can be derived through stochastic flows techniques (see e.g. [Kun04]) that there exists a solution u_m in $C_b^\infty((0, T) \times \mathbb{R}^{nd})$ of the "regularized" IPDE:

$$\begin{cases} \partial_t u_m(t, \mathbf{x}) + L_\alpha u_m(t, \mathbf{x}) + \langle A\mathbf{x} + \mathbf{F}_m(t, \mathbf{x}), D\mathbf{x} u_m(t, \mathbf{x}) \rangle = -f_m(t, \mathbf{x}) & \text{on } (0, T) \times \mathbb{R}^{nd}, \\ u_m(T, \mathbf{x}) = g_m(\mathbf{x}) & \text{on } \mathbb{R}^{nd}. \end{cases}$$

In order to pass to the limit in m , we notice now the arguments above for the proof of the Schauder estimates (Equation (2.19)) can be applied to the above dynamics, too. Namely, there exists a constant $C > 0$ such that

$$\|u_m\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \leq C[\|f_m\|_{L^\infty(C_{b,d}^\beta)} + \|g_m\|_{C_{b,d}^{\alpha+\beta}}] \leq C[\|f\|_{L^\infty(C_{b,d}^\beta)} + \|g\|_{C_{b,d}^{\alpha+\beta}}].$$

Importantly, the above estimates is uniformly in m and thus, the sequence $\{u_m\}_{m \in \mathbb{N}}$ is bounded in the space $L^\infty(C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$. From Arzelà-Ascoli Theorem, we deduce now that there exists u in $L^\infty(C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ and a sequence $\{u_{m_k}\}_{k \in \mathbb{N}}$ of smooth and bounded functions converging to u in $L^\infty(C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ and such that u_{m_k} is solution of the "regularized" IPDE (2.18). It is then clear that u is a mild solution of the original IPDE (1.1).

From Mild to Weak Solutions We conclude showing that any mild solution u of the IPDE (1.1) is indeed a weak solution. The proof of this result will be essentially an application of the arguments presented before, especially the Second Besov Control (Lemma 12). Let u be a mild solution of the IPDE (1.1) in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$. Recalling the definition of weak solution in (2.16), we start fixing a test function ϕ in $C_0^\infty((0, T] \times \mathbb{R}^{nd})$ and passing to the "regularized" setting (see Definition 1), we then notice that it holds that

$$\int_0^T \int_{\mathbb{R}^{nd}} \phi(t, \mathbf{y}) (\partial_t + \mathcal{L}_{m,\alpha}) u_m(t, \mathbf{y}) d\mathbf{y} = - \int_0^T \int_{\mathbb{R}^{nd}} \phi(t, \mathbf{y}) f_m(t, \mathbf{y}) d\mathbf{y}$$

where $\mathcal{L}_{m,\alpha}$ is the "complete" operator defined in (2.15) but with respect to the regularized coefficients. An integration by parts allows now to move the operators to the test function. Indeed, remembering that $u_m(T, \cdot) = g_m(\cdot)$, it holds that

$$\int_0^T \int_{\mathbb{R}^{nd}} (-\partial_t + \mathcal{L}_{m,\alpha}^*) \phi(t, \mathbf{y}) u_m(t, \mathbf{y}) d\mathbf{y} dt + \int_{\mathbb{R}^{nd}} \phi(T, \mathbf{y}) g_m(\mathbf{y}) d\mathbf{y} = - \int_0^T \int_{\mathbb{R}^{nd}} \phi(t, \mathbf{y}) f_m(t, \mathbf{y}) d\mathbf{y} dt \quad (6.2)$$

where $\mathcal{L}_{m,\alpha}^*$ denotes the formal adjoint of $\mathcal{L}_{m,\alpha}$. We would like now to go back to the solution u , letting m go to ∞ . We start rewriting the right-hand side term in the following way:

$$\int_0^T \int_{\mathbb{R}^{nd}} \phi(t, \mathbf{y}) f_m(t, \mathbf{y}) d\mathbf{y} dt = \int_0^T \int_{\mathbb{R}^{nd}} \phi(t, \mathbf{y}) f(t, \mathbf{y}) d\mathbf{y} dt + \int_0^T \int_{\mathbb{R}^{nd}} \phi(t, \mathbf{y}) [f_m - f](t, \mathbf{y}) d\mathbf{y} dt.$$

Exploiting that by assumption, f_m converges to f in $L^\infty(0, T; C_{b,d}^\beta(\mathbb{R}^{nd}))$, it is easy to see that the second contribution above goes to 0 if we let m go to ∞ . A similar argument can be used to show that

$$\int_{\mathbb{R}^{nd}} \phi(T, \mathbf{y}) g_m(\mathbf{y}) d\mathbf{y} \xrightarrow{m} \int_{\mathbb{R}^{nd}} \phi(T, \mathbf{y}) g(\mathbf{y}) d\mathbf{y}.$$

On the other hand, we can decompose the first term in the left-hand side of Equation (6.2) as

$$\int_0^T \int_{\mathbb{R}^{nd}} (-\partial_t + \mathcal{L}_{m,\alpha}^*) \phi(t, \mathbf{y}) u_m(t, \mathbf{y}) d\mathbf{y} dt = \int_0^T \int_{\mathbb{R}^{nd}} (-\partial_t + \mathcal{L}_\alpha^*) \phi(t, \mathbf{y}) u(t, \mathbf{y}) d\mathbf{y} dt + R_m^1 + R_m^2 \quad (6.3)$$

where above we have denoted

$$\begin{aligned} R_m^1 &= \int_0^T \int_{\mathbb{R}^{nd}} [\mathcal{L}_\alpha^* - \mathcal{L}_{m,\alpha}^*] \phi(t, \mathbf{y}) u_m(t, \mathbf{y}) d\mathbf{y} dt \\ R_m^2 &= \int_0^T \int_{\mathbb{R}^{nd}} (-\partial_t + \mathcal{L}_\alpha^*) \phi(t, \mathbf{y}) [u_m(t, \mathbf{y}) - u(t, \mathbf{y})] d\mathbf{y} dt \end{aligned}$$

with \mathcal{L}_α^* as the formal adjoint of the complete operator \mathcal{L}_α . Noticing that

$$[\mathcal{L}_\alpha^* - \mathcal{L}_{m,\alpha}^*] \phi(t, \mathbf{y}) = D_{\mathbf{y}} \cdot \{ \phi(t, \mathbf{y}) [\mathbf{F}(t, \mathbf{y}) - \mathbf{F}_m(t, \mathbf{y})] \},$$

it is clear that the remainder contribution R_m^1 can be essentially handled as in the introduction of Section 5.1, exploiting that $\|\mathbf{F} - \mathbf{F}_m\|_H \rightarrow 0$.

To control instead the second contribution R_m^2 , we start decomposing it as

$$\begin{aligned} R_m^2 &= - \int_0^T \int_{\mathbb{R}^{nd}} \partial_t \phi(t, \mathbf{y}) [u_m - u](t, \mathbf{y}) d\mathbf{y} dt + \sum_{j=1}^n \int_0^T \int_{\mathbb{R}^{nd}} D_{\mathbf{y}_j} [\phi \mathbf{F}_j](t, \mathbf{y}) [u_m - u](t, \mathbf{y}) d\mathbf{y} dt \\ &=: R_{0,m}^2 + \sum_{j=1}^n R_{j,m}^2. \end{aligned}$$

We firstly observe that $|R_{0,m}^2|$ goes to 0 if we let m go to ∞ , since $\|u - u_m\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \xrightarrow{m} 0$. On the other hand, an integration by parts allows to show that

$$|R_{1,m}^2| = \left| \int_0^T \int_{\mathbb{R}^{nd}} [\phi \mathbf{F}](t, \mathbf{y}) D_{\mathbf{y}_1} [u_m - u](t, \mathbf{y}) d\mathbf{y} dt \right|$$

which again tends to 0 when m goes to ∞ . To control instead the contributions $R_{j,m}^2$ for $j > 1$, the point is to use the Besov duality argument again. Namely, from Equations (4.11), (4.10) and with the notations in (4.7), it holds that

$$\begin{aligned} |R_{j,m}^2| &\leq \int_0^T \int_{\mathbb{R}^{d(n-1)}} \|D_{\mathbf{y}_j} [\phi \mathbf{F}](t, \mathbf{y}_{\setminus j}, \cdot)\|_{B_{1,1}^{-(\alpha_j+\beta_j)}} \| [u_m - u](t, \mathbf{y}_{\setminus j}, \cdot) \|_{B_{\infty,\infty}^{\alpha_j+\beta_j}} d\mathbf{y}_{\setminus j} dt \\ &\leq \int_0^T \int_{\mathbb{R}^{d(n-1)}} \|D_{\mathbf{y}_j} [\phi \mathbf{F}](t, \mathbf{y}_{\setminus j}, \cdot)\|_{B_{1,1}^{-(\alpha_j+\beta_j)}} \| [u_m - u](t, \mathbf{y}_{\setminus j}, \cdot) \|_{C_b^{\alpha_j+\beta_j}} d\mathbf{y}_{\setminus j} dt. \end{aligned}$$

Following the same arguments in the Proof of the Second Besov Control (Lemma 12), we know that there exists a constant C such that $\|D_{\mathbf{y}_j} [\phi \mathbf{F}](t, \mathbf{y}_{\setminus j}, \cdot)\|_{B_{1,1}^{\alpha_j+\beta_j}} \leq C \psi_j(t, \mathbf{y}_{\setminus j})$ where ψ_j has compact support on $\mathbb{R}^{d(n-1)}$.

Since moreover $\|u_m - u\|$ goes to zero with m , we easily deduce that $R_{m,j}^2 \xrightarrow{m} 0$ for any j in $\llbracket 2, n \rrbracket$. From the above controls, we can deduce now that $R_m^1 + R_m^2 \xrightarrow{m} 0$. From Equation (6.3), it then follows that

$$\int_0^T \int_{\mathbb{R}^{nd}} \left(-\partial_t + \mathcal{L}_{m,\alpha}^* \right) \phi(t, \mathbf{y}) u_m(t, \mathbf{y}) d\mathbf{y} dt \xrightarrow{m} \int_0^T \int_{\mathbb{R}^{nd}} \left(-\partial_t + \mathcal{L}_\alpha^* \right) \phi(t, \mathbf{y}) u(t, \mathbf{y}) d\mathbf{y} dt$$

and we have concluded.

7 Extensions

As already said in the introduction, our assumption of (global) Hölder regularity on the drift $\bar{\mathbf{F}}$, as well as the choice of considering a perturbed Ornstein-Uhlenbeck operator instead of a more general non-linear dynamics, was done to preserve, as possible, the clarity and understandability of the article. In this conclusive section, we would like to explain briefly how it possible to naturally extend it.

7.1 General Drift

We start illustrating how the perturbative method explained above can be easily adapted to work in a more general setting. In particular, the same results (Schauder-type estimates and well-posedness of the IPDE (1.1)) can be shown also for an equation of the form:

$$\begin{cases} \partial_t u(t, \mathbf{x}) + L_\alpha u(t, \mathbf{x}) + \langle \bar{\mathbf{F}}(t, \mathbf{x}), D_{\mathbf{x}} u(t, \mathbf{x}) \rangle = -f(t, \mathbf{x}), & \text{on } (0, T) \times \mathbb{R}^{nd} \\ u(T, \mathbf{x}) = g(\mathbf{x}) & \text{on } \mathbb{R}^{nd}. \end{cases} \quad (7.1)$$

where $\bar{\mathbf{F}}(t, \mathbf{x}) = (\bar{\mathbf{F}}_1(t, \mathbf{x}), \dots, \bar{\mathbf{F}}_n(t, \mathbf{x}))$ has the following structure

$$\bar{\mathbf{F}}_i(t, \mathbf{x}_{(i-1) \vee 1}, \dots, \mathbf{x}_n).$$

We remark in particular that if for any i in $\llbracket 2, n \rrbracket$, $\bar{\mathbf{F}}_i$ is linear with respect to \mathbf{x}_{i-1} and independent from time, the previous analysis works since we can rewrite $\bar{\mathbf{F}}(t, \mathbf{x}) = A\mathbf{x} + \mathbf{F}(t, \mathbf{x})$.

In order to deal with this more general dynamics addressed in the diffusive setting in [CdRHM18], we will need however to add some additional constraints and to modify slightly the ones presented in assumption **(A)**. First of all, the non-degeneracy assumption **(H)** does not make sense in this new framework and it will be replaced by the following condition:

(H') the matrix $D_{\mathbf{x}_{i-1}} \bar{\mathbf{F}}_i(t, \mathbf{x})$ has full rank d for any i in $\llbracket 2, n \rrbracket$ and any (t, \mathbf{x}) in $[0, T] \times \mathbb{R}^{nd}$.

In particular, we will say that assumption **(A)** is in force when

(S') assumption **(ND)** and **(H')** are satisfied and the drift $\bar{\mathbf{F}} = (\bar{\mathbf{F}}_1, \dots, \bar{\mathbf{F}}_n)$ is such that for any i in $\llbracket 2, n \rrbracket$, $\bar{\mathbf{F}}_i$ depends only on time and on the last $n - (i - 2) \vee 0$ components, i.e. $\bar{\mathbf{F}}_i(t, \mathbf{x}_{i-1}, \dots, \mathbf{x}_n)$;

(P') α is a number in $(0, 2)$, β is in $(0, 1)$ and it holds that

$$\beta < \alpha, \quad \alpha + \beta \in (1, 2) \quad \text{and} \quad \beta < (\alpha - 1)(1 + \alpha(n - 1));$$

(R') The source f is in $L^\infty(0, T; C_{b,d}^\beta(\mathbb{R}^{nd}))$, the terminal condition g is in $C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd})$ and for any i in $\llbracket 1, n \rrbracket$, $\bar{\mathbf{F}}_i$ belongs $L^\infty(0, T; C_d^{\gamma_i+\beta}(\mathbb{R}^{nd}))$ where γ_i was defined in (2.14).

To prove Schauder-type estimates for a solution of equation (7.1), our idea is to adapt the perturbative approach to this new dynamics. In particular, we can exploit the differentiability of $\bar{\mathbf{F}}_i$ with respect to \mathbf{x}_{i-1} to "linearize" it along a flow that takes into account the perturbation (cf. Section 3.1). Namely, we are interested in:

$$\begin{cases} \partial_t \bar{u}^{\tau, \xi}(t, \mathbf{x}) + L_\alpha \bar{u}^{\tau, \xi}(t, \mathbf{x}) + \langle \bar{A}_t^{\tau, \xi}(\mathbf{x} - \bar{\boldsymbol{\theta}}_{\tau, t}(\xi)) + \bar{\mathbf{F}}(t, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi)), D_{\mathbf{x}} \bar{u}^{\tau, \xi}(t, \mathbf{x}) \rangle & = -f(t, \mathbf{x}), \\ \bar{u}^{\tau, \xi}(T, \mathbf{x}) & = g(\mathbf{x}) \end{cases} \quad (7.2)$$

where the time-dependent matrix $\bar{A}_t^{\tau, \xi}$ is defined through

$$[\bar{A}_t^{\tau, \xi}]_{i,j} = \begin{cases} D_{\mathbf{x}_{i-1}} \bar{\mathbf{F}}_i(t, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi)), & \text{if } j = i - 1 \\ 0_{d \times d}, & \text{otherwise} \end{cases}$$

and $\bar{\boldsymbol{\theta}}_{\tau, t}(\xi)$ is a fixed flow satisfying the dynamics

$$\bar{\boldsymbol{\theta}}_{\tau, t}(\xi) = \xi + \int_\tau^t \bar{\mathbf{F}}(v, \bar{\boldsymbol{\theta}}_{\tau, v}(\xi)) dv. \quad (7.3)$$

A first significant difference with respect to the previous approach consists in handling a time-dependent matrix $\bar{A}_t^{\tau, \xi}$. Indeed, it is possible to modify slightly the presentation in [PZ09] (allowing time-dependency on A) in order to show that under assumption **(S')**, the two parameters semigroup $(\bar{P}_{t,s}^{\tau, \xi})_{t \leq s}$ associated with the proxy operator

$$L_\alpha + \langle \bar{A}_t^{\tau, \xi}(\mathbf{x} - \bar{\boldsymbol{\theta}}_{\tau, t}(\xi)) + \bar{\mathbf{F}}(t, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi)), D_{\mathbf{x}} \rangle$$

admits a density $\bar{p}^{\tau, \xi}$ and that it can be rewritten as

$$\bar{p}^{\tau, \xi}(t, s, x, y) = \frac{1}{\det(\mathbb{M}_{s-t})} p_S(s - t, \mathbb{M}_{s-t}^{-1}(\mathbf{y} - \bar{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x}))).$$

Here, the notations for p_S and \mathbb{M}_t remain the same of above while this time the shift $\bar{\mathbf{m}}_{t,s}^{\tau, \xi}$ is defined through

$$\bar{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x}) = \mathcal{R}_{t,s}^{\tau, \xi} \mathbf{x} + \int_t^s \mathcal{R}_{v,s}^{\tau, \xi} [\bar{\mathbf{F}}(v, \bar{\boldsymbol{\theta}}_{\tau, v}(\xi)) - \bar{A}_v^{\tau, \xi} \bar{\boldsymbol{\theta}}_{\tau, v}(\xi)] dv$$

where $\mathcal{R}_{t,s}^{\tau, \xi}$ is the time-ordered resolvent of $\bar{A}_s^{\tau, \xi}$ starting at time t , i.e.

$$\begin{cases} d\mathcal{R}_{t,s}^{\tau, \xi} = \bar{A}_s^{\tau, \xi} \mathcal{R}_{t,s}^{\tau, \xi} ds, & \text{on } [t, T] \\ \mathcal{R}_{t,t}^{\tau, \xi} = I. \end{cases}$$

We can as well refer to [HM16] for related issues (see Proposition 3.2 and Section C about the linearization, therein).

Following the same reasonings of Propositions 2 and 3, it is then possible to state a Duhamel type formula suitable for the IPDE 7.1:

$$u(t, \mathbf{x}) = \bar{P}_{t,T}^{\tau, \xi} g(\mathbf{x}) + \int_t^T \bar{P}_{t,s}^{\tau, \xi} [f(s, \cdot) + \bar{R}^{\tau, \xi}(s, \cdot)](\mathbf{x}) ds \quad (7.4)$$

where the remainder term is given now by

$$\bar{R}^{\tau, \xi}(t, \mathbf{x}) = \langle \mathbf{F}(t, \mathbf{x}) - \mathbf{F}(t, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi)) - \bar{A}_t^{\tau, \xi}(\mathbf{x} - \bar{\boldsymbol{\theta}}_{\tau, t}(\xi)), D_{\mathbf{x}} u(t, \mathbf{x}) \rangle.$$

Looking back at the first part of the article, it is important to notice that the main steps of proof (cf. Equation (3.6), Propositions 1, 4 and Section 3.3) does not rely on the explicit formulas for $\bar{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x})$ and $\bar{R}^{\tau, \xi}$ but instead, they exploit only the Besov controls for the remainder $\bar{R}^{\tau, \xi}$ (cf. Section 5.1) and the controls on the shift $\bar{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x})$ (Section A.2). Hence, once we have proven the suitable controls, the proofs of the analogous results for the new dynamics (7.1) can be obtained easily modifying slightly the notations and following the same reasonings above.

For example, exploiting that

$$\bar{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x}) = \mathbf{x} + \int_t^s \mathcal{R}_{t,v}^{\tau, \xi} \left(\bar{\mathbf{m}}_{t,v}^{\tau, \xi}(\mathbf{x}) - \boldsymbol{\theta}_{\tau, v}(\xi) \right) + \mathbf{F}(v, \boldsymbol{\theta}_{\tau, v}(\xi)) dv,$$

we can follow the same method of proof in the above lemma 2 to show again that

$$\bar{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x}) = \bar{\boldsymbol{\theta}}_{\tau, s}(\xi)$$

taking $\tau = t$ and $\xi = \mathbf{x}$.

Letting the interested reader look in the appendix for the suggestions on how to extend the controls on the shift $\bar{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x})$ in this more general setting, we will focus now on proving the Besov controls. First of all, we notice immediately that the proof of the first Besov control 7 relies essentially only on the smoothing effect (3.6) and thus, it can be obtained following the same reasoning above. The proof of the second Besov control (Lemma 12) in this framework is a bit more involved and we are going to explain it below more in details. We start noticing that the second Besov Lemma 12 can be reformulated for the new dynamics in the following way

$$\int_{\mathbb{R}^{(n-1)d}} \left\| D_{\mathbf{y}_j} \cdot \left\{ D_{\mathbf{x}}^{\vartheta} \bar{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, \cdot) \otimes \bar{\Delta}_j^{\tau, \xi}(s, \mathbf{y}_{\setminus j}, \cdot) \right\} \right\|_{B_{1,1}^{-(\alpha_j + \beta_j)}} d\mathbf{y}_{\setminus j} \leq C \|\bar{\mathbf{F}}\|_H (s-t)^{\frac{\beta}{\alpha} - \sum_{k=1}^n \frac{\vartheta_k}{\alpha_k}}$$

taking $(\tau, \xi) = (t, \mathbf{x})$, where we have denoted for simplicity

$$\bar{\Delta}_j^{\tau, \xi}(s, \mathbf{y}) := \bar{\mathbf{F}}_j(s, \mathbf{y}) - \bar{\mathbf{F}}_j(s, \boldsymbol{\theta}_{\tau, s}(\xi)) - D_{\mathbf{x}_{j-1}} \bar{\mathbf{F}}_j(s, \boldsymbol{\theta}_{\tau, s}(\xi)) (\mathbf{y} - \boldsymbol{\theta}_{\tau, s}(\xi))_{j-1}$$

for any j in $\llbracket 2, n \rrbracket$. The above control can be obtained mimicking the proof in the second Besov control (Lemma 12), exploiting this time that

$$|\bar{\Delta}_j^{\tau, \xi}(s, \mathbf{y})| \leq C \|\bar{\mathbf{F}}\|_H d_{j-1:n}^{1+\alpha(j-2)+\beta}(\mathbf{y}, \bar{\boldsymbol{\theta}}_{\tau, s}(\xi))$$

and the additional assumption (\mathbf{P}') in order to make the partial smoothing effect (Equation (5.29)) work in this framework too.

The main difference in the proof is related to the control of the component $J_2(v, \mathbf{y}_{\setminus j}, z)$ appearing in Equation (5.34). Namely,

$$\int_{\mathbb{R}^d} D_z \partial_v p_h(v, z - \mathbf{y}_j) \cdot \left\{ \bar{\Delta}_j^{\tau, \xi}(s, \mathbf{y}_{\setminus j}, z) \otimes \int_0^1 D_{\mathbf{y}_j} D_{\mathbf{x}}^{\vartheta} \bar{p}^{\tau, \xi}(t, s, \mathbf{x}, \mathbf{y}_{\setminus j}, z + \lambda(\mathbf{y}_j - z)) \cdot (\mathbf{y}_j - z) \right\} d\lambda d\mathbf{y}_j$$

with our new notations. Indeed, the dependence of $\bar{\mathbf{F}}$ on \mathbf{x}_{j-1} pushes us to add a new term in the difference

$|\bar{F}_j(s, \mathbf{y}_{\setminus j}, z) - \bar{F}_j(s, \boldsymbol{\theta}_{\tau,s}(\boldsymbol{\xi}))|$ (now, $|\bar{\Delta}_j^{\tau,\boldsymbol{\xi}}(s, \mathbf{y}_{\setminus j}, z)|$) before splitting it up. In particular,

$$\begin{aligned}
& |\bar{\Delta}_j^{\tau,\boldsymbol{\xi}}(s, \mathbf{y}_{\setminus j}, z)| \\
&= |\bar{F}_j(s, \mathbf{y}_{\setminus j}, z) - \bar{F}_j(s, \boldsymbol{\theta}_{\tau,s}(\boldsymbol{\xi})) - D_{\mathbf{x}_{j-1}} \bar{F}_j(s, \bar{\boldsymbol{\theta}}_{\tau,s}(\boldsymbol{\xi}))(\mathbf{y} - \bar{\boldsymbol{\theta}}_{\tau,s}(\boldsymbol{\xi}))_{j-1} \pm \bar{F}_j(s, \mathbf{y}_{1:j-1}, (\bar{\boldsymbol{\theta}}_{\tau,s}(\boldsymbol{\xi}))_{j:n})| \\
&\leq C \|\bar{\mathbf{F}}\|_H (|z - (\bar{\boldsymbol{\theta}}_{\tau,s}(\boldsymbol{\xi}))_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + \sum_{k=j+1}^n |(\mathbf{y} - \bar{\boldsymbol{\theta}}_{\tau,s}(\boldsymbol{\xi}))_k|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(k-1)}} + |(\mathbf{y} - \bar{\boldsymbol{\theta}}_{\tau,s}(\boldsymbol{\xi}))_{j-1}|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-2)}}) \\
&\leq C \|\bar{\mathbf{F}}\|_H (|\lambda(z - \mathbf{y}_j)|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + |z + \lambda(\mathbf{y}_j - z) - \boldsymbol{\theta}_{\tau,s}(\boldsymbol{\xi})_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + \sum_{k=j+1}^n |\mathbf{y} - \bar{\boldsymbol{\theta}}_{\tau,s}(\boldsymbol{\xi})_k|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(k-1)}} \\
&+ |(\mathbf{y} - \bar{\boldsymbol{\theta}}_{\tau,s}(\boldsymbol{\xi}))_{j-1}|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-2)}}) \leq C \|\bar{\mathbf{F}}\|_H (|z - \mathbf{y}_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + d_{j+1:n}^{1+\alpha(j-2)+\beta}((\mathbf{y}_{\setminus j}, z + \lambda(\mathbf{y}_j - z)), \bar{\boldsymbol{\theta}}_{\tau,s}(\boldsymbol{\xi}))).
\end{aligned}$$

The remaining part of the proof exactly matches the original method in Lemma 12.

Even in this more general framework, it is thus possible to obtain the following:

Theorem 3 (Well-posedness). *Under $(\bar{\mathbf{A}})$, there exists a unique mild solution u of (7.1) such that*

$$\|u\|_{L^\infty(C_d^{\alpha+\beta})} \leq C[\|f\|_{L^\infty(C_{b,d}^\beta)} + \|g\|_{C_{b,d}^{\alpha+\beta}}].$$

7.2 Locally Hölder Drift

This part is designed to give a brief explanation on how it is possible to deal with the general IPDE (7.1) when the drift $\bar{\mathbf{F}}$ is only locally Hölder continuous in space. Namely, we assume with the notations in (2.14) that

(LR') there exists a constant $K_0 > 0$ such that for any i in $\llbracket 1, n \rrbracket$

$$d(\bar{\mathbf{F}}(t, \mathbf{x}), \bar{\mathbf{F}}(t, \mathbf{x}')) \leq K_0 d^{\beta+\gamma_i}(\mathbf{x}, \mathbf{x}'), \quad t \in [0, T], \mathbf{x}, \mathbf{x}' \in \mathbb{R}^{nd} \text{ s.t. } d(\mathbf{x}, \mathbf{x}') < 1.$$

In other words, it is required that $\bar{\mathbf{F}}_i$ is in $L^\infty(0, T; C^{\beta+\gamma_i}(B(x_0, 1/2)))$, uniformly in $x_0 \in \mathbb{R}^{nd}$.

Under assumption $(\bar{\mathbf{A}})$ (with condition **(R')** replaced by **(LR')**), it is possible to recover the Schauder-type estimates (Theorem 1), following the approach developed successfully in [CdRMP19] for the non-degenerate, super-critical stable setting. Roughly speaking, in order to handle the local assumption, as well as the potentially unboundedness of the drift $\bar{\mathbf{F}}$, we need to introduce a "localized" version of the Duhamel formulation (cf. Equation (3.16)). The key point here is to multiply a solution u by a suitable bump function $\bar{\eta}^{\tau,\boldsymbol{\xi}}$ that "localizes" in space along the deterministic flow $\bar{\boldsymbol{\theta}}_{\tau,t}(\boldsymbol{\xi})$ that characterizes the proxy. Namely, we fix a smooth function ρ that is equal to 1 on $B(0, 1/2)$ and vanishes outside $B(0, 1)$ and then define for any $(\tau, \boldsymbol{\xi})$ in $[0, T] \times \mathbb{R}^{nd}$,

$$\bar{\eta}^{\tau,\boldsymbol{\xi}}(t, \mathbf{x}) := \rho(\mathbf{x} - \bar{\boldsymbol{\theta}}_{\tau,t}(\boldsymbol{\xi})).$$

We mention however that in the setting of [CdRMP19], the "localization" with the cut-off function $\bar{\eta}^{\tau,\boldsymbol{\xi}}$ is not simply motivated by the local Hölder continuity condition but it is also needed to give a proper meaning to the Duhamel formulation for a solution (cf. Proposition 3) when $\alpha < 1/2$, because of the low integrability properties of the underlying stable density. Such a problem does not however appear here since condition **(P)** forces us to consider only the case $\alpha > 1/2$.

Given a mild solution u of the IPDE (7.1) assuming $\bar{\mathbf{F}}$ to be only locally Hölder continuous as in **(LR')**, it is possible to show, at least formally, that the function $\bar{v}^{\tau,\boldsymbol{\xi}} := u \bar{\eta}^{\tau,\boldsymbol{\xi}}$ solves the following equation

$$\begin{cases} \partial_t \bar{v}^{\tau,\boldsymbol{\xi}}(t, \mathbf{x}) + \langle \bar{\mathbf{F}}(t, \mathbf{x}), D_{\mathbf{x}} \bar{v}^{\tau,\boldsymbol{\xi}}(t, \mathbf{x}) \rangle + L_{\alpha} \bar{v}^{\tau,\boldsymbol{\xi}}(t, \mathbf{x}) = -[\bar{\eta}^{\tau,\boldsymbol{\xi}} f + \bar{\mathcal{S}}^{\tau,\boldsymbol{\xi}}](t, \mathbf{x}) & \text{on } [0, T] \times \mathbb{R}^{nd}; \\ \bar{v}^{\tau,\boldsymbol{\xi}}(T, \mathbf{x}) = \bar{\eta}^{\tau,\boldsymbol{\xi}}(T, \mathbf{x}) g(\mathbf{x}) & \text{on } \mathbb{R}^{nd}, \end{cases} \quad (7.5)$$

where we have denoted above

$$\begin{aligned}
\bar{\mathcal{S}}^{\tau,\boldsymbol{\xi}}(t, \mathbf{x}) &:= \int_{\mathbb{R}^d} [u(t, \mathbf{x} + B\mathbf{y}) - u(t, \mathbf{x})] [\bar{\eta}^{\tau,\boldsymbol{\xi}}(t, \mathbf{x} + B\mathbf{y}) - \bar{\eta}^{\tau,\boldsymbol{\xi}}(t, \mathbf{x})] \nu_{\alpha}(d\mathbf{y}) \\
&\quad - u(t, \mathbf{x}) \langle \bar{\mathbf{F}}(t, \mathbf{x}) - \bar{\mathbf{F}}(t, \bar{\boldsymbol{\theta}}_{\tau,t}(\boldsymbol{\xi})), D\rho(\mathbf{x} - \bar{\boldsymbol{\theta}}_{\tau,t}(\boldsymbol{\xi})) \rangle.
\end{aligned}$$

The IPDE (7.5) can be seen essentially as a "local" version of the original one (7.1), depending on the freezing parameter (τ, ξ) . In particular, it is important to notice that the difference

$$\bar{F}(t, \mathbf{x}) - \bar{F}(t, \bar{\theta}_{\tau,t}(\xi))$$

appearing in the "localizing" error $\bar{S}^{\tau,\xi}$ can be controlled exactly because it is multiplied by the derivative of the bump function ρ in the right point $\mathbf{x} - \bar{\theta}_{\tau,t}(\xi)$, allowing us to exploit the *local* Hölder regularity. On the other hand, the first integral term in the r.h.s. can be seen as a commutator which involves only the non-degenerate variables and thus, that can be handled with interpolation techniques as in [CdRMP19].

Even with the additional difficulty in controlling the remainder term, the perturbative approach explained in Section 3 can be applied, leading to show Schauder-type estimates as in Theorem 1 and the well-posedness of the IPDE (7.1) when assuming \bar{F} to be only locally Hölder continuous.

Our procedure could be also used in order to establish Schauder-type estimates for the full Ornstein-Uhlenbeck operator as done, for example, in [Lun97] for the diffusive case. Indeed, a general operator of the form $\langle \bar{A}\mathbf{x}, D_{\mathbf{x}} \rangle + L_{\alpha}$ can be treated, decomposing the matrix as $\bar{A} = A + U$ where A is, as before, the sub-diagonal matrix that makes the Ornstein-Uhlenbeck operator invariant by the dilation operator associated with the distance d , while U is an upper triangular matrix that could be seen as an additional *locally* Hölder term.

7.3 Diffusion Coefficient

We conclude the article showing briefly how an additional diffusion coefficient $\sigma: [0, T] \times \mathbb{R}^{nd} \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ can be handled in the IPDE (7.1) with an operator *L_alpha* of the form:

$$L_{\alpha}\phi(t, \mathbf{x}) := \text{p.v.} \int_{\mathbb{R}^d} [\phi(t, \mathbf{x} + B\sigma(t, \mathbf{x})y) - \phi(t, \mathbf{x})] \nu_{\alpha}(dy).$$

In this framework, it is quite standard (cf. [HWZ19] and [ZZ18]) to assume the Lévy measure ν_{α} to be absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^d i.e. $\nu_{\alpha}(dy) = f(y)dy$, for some Lipschitz function $f: \mathbb{R}^d \rightarrow \mathbb{R}$. In particular, since ν_{α} is a symmetric, α -stable Lévy measure, it holds passing to polar coordinates $y = \rho s$ where $(\rho, s) \in [0, \infty) \times \mathbb{S}^{d-1}$ that

$$f(y) = \frac{g(s)}{\rho^{d+\alpha}}$$

for an even, Lipschitz function g on \mathbb{S}^{d-1} (see also Equation (2.1)). Moreover, σ is considered uniformly elliptic and in $L^{\infty}(0, T; C^{\beta}(\mathbb{R}^n, \mathbb{R}))$.

Introducing now the "frozen" operator $\bar{L}_{\alpha}^{\tau,\xi}\phi(t, \mathbf{x}) = \text{p.v.} \int_{\mathbb{R}^d} [\phi(t, \mathbf{x} + B\sigma(t, \bar{\theta}_{\tau,t}(\xi))y) - \phi(t, \mathbf{x})] \nu_{\alpha}(dy)$, this would lead to consider for the IPDE an additional term in the Duhamel formula (cf. Equation (7.4)) that would write:

$$u(t, \mathbf{x}) = \check{P}_{t,T}^{\tau,\xi}g(\mathbf{x}) + \int_t^T \check{P}_{t,s}^{\tau,\xi}f(s, \mathbf{x}) + \check{P}_{t,s}^{\tau,\xi}\bar{R}^{\tau,\xi}(s, \mathbf{x}) + \check{P}_{t,s}^{\tau,\xi}[(L_{\alpha} - \bar{L}_{\alpha}^{\tau,\xi})u(s, \cdot)](\mathbf{x}) ds. \quad (7.6)$$

Here, $(\check{P}_{t,s}^{\tau,\xi})_{t \leq s}$ denotes the two parameter semigroup associated with the proxy operator

$$\bar{L}_{\alpha}^{\tau,\xi} + \langle \bar{A}_t^{\tau,\xi}(\mathbf{x} - \bar{\theta}_{\tau,t}(\xi)) + \bar{F}(t, \bar{\theta}_{\tau,t}(\xi)), D_{\mathbf{x}} \rangle.$$

Let us focus on the last term in the integral of Equation (7.6). Looking back at the proof of the A Priori Estimates (Proposition 4), we notice in particular that we aim to establish the following control:

$$|(L_{\alpha} - \bar{L}_{\alpha}^{\tau,\xi})u(t, \mathbf{x})| \leq C\|\sigma\|_{L^{\infty}(C_{b,d}^{\beta})}\|u\|_{L^{\infty}(C_{b,d}^{\alpha+\beta})}d^{\beta}(\mathbf{x}, \bar{\theta}_{\tau,t}(\xi)) \quad (7.7)$$

in order to apply the same reasoning above in this new framework. To this end, we write that

$$\begin{aligned}
& (L_\alpha - \bar{L}_\alpha^{\tau, \xi})u(t, \mathbf{x}) \\
&= \text{p.v.} \int_{\mathbb{R}^d} \{u(t, \mathbf{x} + B\sigma(t, \mathbf{x})y) - u(t, \mathbf{x})\} \nu_\alpha(dy) - \int_{\mathbb{R}^d} \{u(t, \mathbf{x} + B\sigma(t, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi))y) - u(t, \mathbf{x})\} \nu_\alpha(dy) \\
&= \text{p.v.} \int_{\mathbb{R}^d} \{u(t, \mathbf{x} + Bz) - u(t, \mathbf{x})\} \frac{f(\sigma^{-1}(t, \mathbf{x})z)}{\det \sigma(t, \mathbf{x})} dz - \int_{\mathbb{R}^d} \{u(t, \mathbf{x} + Bz) - u(t, \mathbf{x})\} \frac{f(\sigma^{-1}(t, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi))z)}{\det \sigma(t, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi))} dz \\
&= \text{p.v.} \int_0^\infty \frac{1}{\rho^{1+\alpha}} \int_{\mathbb{S}^{d-1}} \{u(t, \mathbf{x} + B\rho s) - u(t, \mathbf{x})\} \bar{D}^{\tau, \xi}(t, \mathbf{x}, s) ds d\rho
\end{aligned}$$

where we have denoted, for notational convenience

$$\bar{D}^{\tau, \xi}(t, \mathbf{x}, s) := \left\{ \frac{g\left(\frac{\sigma^{-1}(t, \mathbf{x})s}{|\sigma^{-1}(t, \mathbf{x})s|}\right)}{|\sigma^{-1}(t, \mathbf{x})s|^{d+\alpha} \det \sigma(t, \mathbf{x})} - \frac{g\left(\frac{\sigma^{-1}(t, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi))s}{|\sigma^{-1}(t, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi))s|}\right)}{|\sigma^{-1}(t, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi))s|^{d+\alpha} \det \sigma(t, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi))} \right\}.$$

Using now that g is Lipschitz and the assumptions on σ , we can show that

$$|\bar{D}^{\tau, \xi}(t, \mathbf{x}, s)| \leq C|\sigma(t, \mathbf{x}) - \sigma(t, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi))| \leq C\|\sigma\|_{L^\infty(C_{b,d}^\beta)} d^\beta(\mathbf{x}, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi)). \quad (7.8)$$

Finally, Equation (7.7) follows from the previous controls using Taylor expansions and the symmetry condition on ν_α . Namely, considering the case $\alpha \geq 1$, which is the most delicate one for this part and precisely requires the symmetry of g , we write that

$$\begin{aligned}
| (L_\alpha - \bar{L}_\alpha^{\tau, \xi})u(t, \mathbf{x}) | &= \left| \text{p.v.} \int_0^\infty \frac{1}{\rho^{1+\alpha}} \int_{\mathbb{S}^{d-1}} \{u(t, \mathbf{x} + B\rho s) - u(t, \mathbf{x})\} \bar{D}^{\tau, \xi}(t, \mathbf{x}, s) ds d\rho \right| \\
&\leq \left| \text{p.v.} \int_{(0,1)} \frac{1}{\rho^{1+\alpha}} \int_{\mathbb{S}^{d-1}} \{u(t, \mathbf{x} + B\rho s) - u(t, \mathbf{x})\} \bar{D}^{\tau, \xi}(t, \mathbf{x}, s) ds d\rho \right| \\
&\quad + \int_{(1,\infty)} \frac{1}{\rho^{1+\alpha}} \int_{\mathbb{S}^{d-1}} |u(t, \mathbf{x} + B\rho s) - u(t, \mathbf{x})| |\bar{D}^{\tau, \xi}(t, \mathbf{x}, s)| ds d\rho =: [\bar{I}_s^{\tau, \xi} + \bar{I}_l^{\tau, \xi}](t, \mathbf{x}). \quad (7.9)
\end{aligned}$$

The *large jump* contribution $\bar{I}_l^{\tau, \xi}$ is easily handled from Equation (7.8). We get that

$$\bar{I}_l^{\tau, \xi}(t, \mathbf{x}) \leq 2C\|\sigma\|_{L^\infty(C_{b,d}^\beta)} \|u\|_{L^\infty(L^\infty)} d^\beta(\mathbf{x}, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi)) \leq 2C\|\sigma\|_{L^\infty(C_{b,d}^\beta)} \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} d^\beta(\mathbf{x}, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi)). \quad (7.10)$$

On the other hand, from the symmetry assumption on ν_α , which transfers to g , we can control the *small jump* contribution $\bar{I}_s^{\tau, \xi}$ through Taylor expansion and a centering argument. Indeed,

$$\begin{aligned}
\bar{I}_s^{\tau, \xi}(t, \mathbf{x}) &= \left| \text{p.v.} \int_{(0,1)} \frac{1}{\rho^{1+\alpha}} \int_{\mathbb{S}^{d-1}} \int_0^1 [D_{\mathbf{x}_1} u(t, \mathbf{x} + \lambda B\rho s) - D_{\mathbf{x}_1} u(t, \mathbf{x})] \rho s \bar{D}^{\tau, \xi}(t, \mathbf{x}, s) d\lambda ds d\rho \right| \\
&\leq C\|\sigma\|_{L^\infty(C_{b,d}^\beta)} d^\beta(\mathbf{x}, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi)) \int_{(0,1)} \frac{1}{\rho^\alpha} \int_{\mathbb{S}^{d-1}} \int_0^1 |D_{\mathbf{x}_1} u(t, \mathbf{x} + \lambda B\rho s) - D_{\mathbf{x}_1} u(t, \mathbf{x})| d\lambda ds d\rho \\
&\leq C\|\sigma\|_{L^\infty(C_{b,d}^\beta)} \|D_{\mathbf{x}_1} u\|_{L^\infty(C_{b,d}^{\alpha+\beta-1})} d^\beta(\mathbf{x}, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi)) \int_{(0,1)} \frac{1}{\rho^\alpha} \rho^{\alpha+\beta-1} d\rho \\
&\leq C\|\sigma\|_{L^\infty(C_{b,d}^\beta)} \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} d^\beta(\mathbf{x}, \bar{\boldsymbol{\theta}}_{\tau, t}(\xi)). \quad (7.11)
\end{aligned}$$

Using Controls (7.10) and (7.11) in the decomposition (7.9), we obtain the expected bound (Equation (7.7)). We remark that the case $\alpha < 1$ could be handled similarly for the contribution $\bar{I}_l^{\tau, \xi}$ and even more directly for $\bar{I}_s^{\tau, \xi}$. Indeed, in that case, the centering argument is not needed since the Taylor expansion already yields an integrable singularity.

A Appendix

A.1 Smoothing Effects for Ornstein-Uhlenbeck Operator

We state and prove here some of the key properties of the Ornstein-Uhlenbeck operator. Namely, we will prove the representation (2.6) and the associated α -smoothing effect (2.8). We highlight however that these results are only a slight modification to our purpose of those in [HMP19].

The two lemma below presents a deep connection with stochastic analysis and their proofs relies on tools that are more familiar in the probabilistic realm. For this reason, we are going to consider the stochastic counterpart of the Ornstein-Uhlenbeck operator L^{ou} . Namely, for a given starting point \mathbf{x} in \mathbb{R}^{nd} , we are interested in the following dynamics

$$\begin{cases} dX_t = AX_t dt + B dZ_t, & \text{on } [0, T] \\ X_0 = \mathbf{x} \end{cases} \quad (\text{A.1})$$

where $(Z_t)_{t \geq 0}$ is an α -stable, \mathbb{R}^{nd} -dimensional process with Lévy measure ν_α , defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Lemma 20 (Representation). *Under (A), the semigroup $(P_t^{ou})_{t \geq 0}$ generated by the Ornstein-Uhlenbeck operator L_{ou} (defined in (2.4)) admits for any fixed $t > 0$, a density $p^{ou}(t, \cdot)$ which writes for any $t > 0$ and any \mathbf{x}, \mathbf{y} in \mathbb{R}^{nd}*

$$p^{ou}(t, \mathbf{x}, \mathbf{y}) = \frac{1}{\det \mathbb{M}_t} p_S(t, \mathbb{M}_t^{-1}(e^{At} \mathbf{x} - \mathbf{y}))$$

where \mathbb{M}_t is the matrix defined in (2.7) and p_S is the smooth density of an \mathbb{R}^{nd} -valued, symmetric and α -stable process S whose Lévy measure μ_S satisfies the non-degeneracy assumption (ND) on \mathbb{R}^{nd} .

Proof. We start noticing that the above dynamics (A.1) can be explicitly integrated and gives

$$X_t = e^{tA} \mathbf{x} + \int_0^t e^{(t-s)A} B dZ_s.$$

It is then readily derived from [PZ09] that, for any $t > 0$, the random variable X_t has a density $p_X(t, \mathbf{x}, \cdot)$ with respect to the Lebesgue measure on \mathbb{R}^{nd} and it is moreover well known (see for example [Dyn65]) that p_X coincides with the density p^{ou} of the Ornstein-Uhlenbeck operator L_{ou} .

For this reason, we fix $t \geq 0$ and consider, for a given N in \mathbb{N} , a uniform partition $\{t_i\}_{i \in \llbracket 0, N \rrbracket}$ of $[0, t]$. Then, it holds for any \mathbf{p} in \mathbb{R}^{nd} ,

$$\mathbb{E} \left[\exp \left(i \langle \mathbf{p}, \sum_{i=1}^N e^{(t-t_{i-1})A} B (Z_{t_i} - Z_{t_{i-1}}) \rangle \right) \right] = \exp \left(- \frac{1}{N} \sum_{i=1}^N \int_{\mathbb{S}^{d-1}} |\langle B^* e^{(t-t_{i-1})A} \mathbf{p}, s \rangle|^\alpha \mu(ds) \right)$$

where μ is the spherical measure associated with ν_α (see Equation (2.2)). By dominated convergence theorem, we let m goes to infinity and show that

$$\mathbb{E} \left[\exp \left(i \langle \mathbf{p}, \int_0^t e^{(t-s)A} B dZ_s \rangle \right) \right] = \exp \left(- \int_0^t \int_{\mathbb{S}^{d-1}} |\langle e^{uA} \mathbf{p}, Bs \rangle|^\alpha \mu(ds) du \right).$$

Thanks to the above equation, we can rewrite the characteristic function of X_t as:

$$\begin{aligned} \psi_{X_t}(\mathbf{p}) &= \mathbb{E} \left[\exp \left(i \langle \mathbf{p}, e^{tA} \mathbf{x} + \int_0^t e^{(t-s)A} B dZ_s \rangle \right) \right] = \exp \left(i \langle \mathbf{p}, e^{tA} \mathbf{x} \rangle - \int_0^t \int_{\mathbb{S}^{d-1}} |\langle e^{uA} \mathbf{p}, Bs \rangle|^\alpha \mu(ds) du \right) \\ &= \exp \left(i \langle \mathbf{p}, e^{tA} \mathbf{x} \rangle - t \int_0^1 \int_{\mathbb{S}^{d-1}} |\langle e^{vtA} \mathbf{p}, Bs \rangle|^\alpha \mu(ds) dv \right) \end{aligned}$$

where in the last passage we used the change of variables $u = vt$. For the next step, we firstly notice that it holds

$$e^{tA} = \mathbb{M}_t e^A \mathbb{M}_t^{-1},$$

shown using the definition of matrix exponential and the trivial relation $\mathbb{M}_t A \mathbb{M}_t^{-1} = tA$. Exploiting the above identity, we then find that

$$\begin{aligned} \psi_{X_t}(\mathbf{p}) &= \exp \left(i \langle \mathbf{p}, e^{tA} \mathbf{x} \rangle - t \int_0^1 \int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_t \mathbf{p}, e^{vA} \mathbb{M}_t^{-1} Bs \rangle|^\alpha \mu(ds) dv \right) \\ &= \exp \left(i \langle \mathbf{p}, e^{tA} \mathbf{x} \rangle - t \int_0^1 \int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_t \mathbf{p}, e^{vA} Bs \rangle|^\alpha \mu(ds) dv \right) \end{aligned}$$

where in the last passage we used the straightforward identity $\mathbb{M}_t^1 B \mathbf{y} = B \mathbf{y}$. We focus now only on the double integral

$$\int_0^1 \int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_t \mathbf{p}, e^{vA} B s \rangle|^\alpha \mu(ds) dv.$$

If we consider the measure $m_\alpha(dv, ds) := |e^{vA} B s|^\alpha \mu(ds) dv$ on $[0, 1] \times \mathbb{S}^{d-1}$ and the normalized lift function $l: [0, 1] \times \mathbb{S}^{d-1} \rightarrow \mathbb{S}^{nd-1}$ given by

$$l(v, s) := \frac{e^{vA} B s}{|e^{vA} B s|},$$

it then follows that

$$\int_0^1 \int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_t \mathbf{p}, e^{vA} B s \rangle|^\alpha \mu(ds) dv = \int_0^1 \int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_t \mathbf{p}, \frac{e^{vA} B s}{|e^{vA} B s|} \rangle|^\alpha m_\alpha(ds, dv) = \int_{\mathbb{S}^{nd-1}} |\langle \mathbb{M}_t \mathbf{p}, \boldsymbol{\xi} \rangle|^\alpha \mu_S(d\boldsymbol{\xi})$$

where $\mu_S := \text{Sym}(l_*(m_\alpha))$ is the symmetrized version of the measure m_α push-forwarded through l . Noticing that μ_S is the Lévy measure of a symmetric α -stable process $(S_t)_{t \geq 0}$ satisfying assumption **(ND)** on \mathbb{R}^{nd} , we can finally write that

$$\psi_{X_t}(\mathbf{p}) = \exp\left(i\langle \mathbf{p}, e^{tA} \mathbf{x} \rangle - t\Psi_S(\mathbb{M}_t \mathbf{p})\right)$$

where Ψ_S is the Lévy symbol associated with S_t (cf. Equation (2.2)).

From Lemma A.1 in [HMP19], we know that under assumption **(ND)**, the above calculations implies that

$$\int_0^1 \int_{\mathbb{S}^{d-1}} |(\mathbb{M}_t \mathbf{p}) \cdot (e^{Av} B s)|^\alpha \mu_S(ds) dv \geq C |\mathbb{M}_t \mathbf{p}|^\alpha$$

for some constant $C > 0$. It follows in particular that the function $\mathbf{p} \rightarrow \psi_{X_t}(\mathbf{p})$ is in $L^1(\mathbb{R}^{nd})$. Thus, by inverse fourier transform and a change of variables we can prove that

$$\begin{aligned} \mathcal{F}^{-1}[\psi_{X_t}](\mathbf{y}) &= \frac{1}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} e^{-i\langle \mathbf{p}, \mathbf{y} \rangle} \exp\left(i\langle \mathbf{p}, e^{tA} \mathbf{x} \rangle - t\Psi_S(\mathbb{M}_t \mathbf{p})\right) d\mathbf{p} \\ &= \frac{\det(\mathbb{M}_t^{-1})}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} \exp\left(-i\langle \mathbb{M}_t^{-1} \mathbf{p}, \mathbf{y} - e^{tA} \mathbf{x} \rangle\right) e^{-t\Psi(\mathbf{p})} d\mathbf{p} \\ &= \frac{\det(\mathbb{M}_t^{-1})}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} \exp\left(-i\langle \mathbf{p}, \mathbb{M}_t^{-1}(\mathbf{y} - e^{tA} \mathbf{x}) \rangle\right) e^{-t\Psi(\mathbf{p})} d\mathbf{p} = \frac{1}{\det(\mathbb{M}_t)} p_S(t, \mathbb{M}_t^{-1}(\mathbf{y} - e^{tA} \mathbf{x})) \end{aligned}$$

and we have concluded since p_S is symmetric. \square

We can now point out the smoothing effect (Equation (2.8)) associated with the Ornstein-Uhlenbeck density p^{ou} .

Lemma 21 (Smoothing Effect). *Under **(A)**, there exists a family $\{q(t, \cdot) : t \in [0, T]\}$ of densities on \mathbb{R}^{nd} such that*

- *for any l in $\llbracket 0, 3 \rrbracket$, there exists a constant $C := C(l, nd)$ such that $|D_y^l p_S(t, \mathbf{y})| \leq C q(t, \mathbf{y}) t^{-l/\alpha}$ for any t in $[0, T]$ and any \mathbf{y} in \mathbb{R}^{nd} ;*
- *(stable scaling property) $q(t, \mathbf{y}) = t^{-nd/\alpha} q(1, t^{-1/\alpha} \mathbf{y})$ for any t in $[0, T]$ and any \mathbf{y} in \mathbb{R}^{nd} ;*
- *(stable smoothing effect) for any γ in $[0, \alpha)$, there exists a constant $c := c(\gamma, nd)$ such that*

$$\int_{\mathbb{R}^{nd}} q(t, \mathbf{y}) |\mathbf{y}|^\gamma d\mathbf{y} \leq c t^{\gamma/\alpha} \text{ for any } t > 0. \quad (\text{A.2})$$

Proof. Fixed a time $t > 0$, we start applying the Ito-Lévy decomposition to S at the associated characteristic stable time scale, i.e. we choose to truncate at threshold $t^{1/\alpha}$, so that we can write $S_t = M_t + N_t$ for some M_t, N_t independent random variables corresponding to the small jumps part and the large jumps part, respectively. Namely, we denote for any $s > 0$

$$N_s := \int_0^s \int_{|\mathbf{x}| > t^{1/\alpha}} \mathbf{x} P(du, d\mathbf{x}) \quad \text{and} \quad M_s := S_s - N_s$$

where P is the Poisson random measure associated with the process S . We can thus rewrite the density p_S in the following way

$$p_S(t, \mathbf{x}) = \int_{\mathbb{R}^{nd}} p_M(t, \mathbf{x} - \mathbf{y}) P_{N_t}(d\mathbf{y})$$

where $p_M(t, \cdot)$ corresponds to the density of M_t and P_{N_t} is the law of N_t .

It is important now to notice that it is precisely our choice of the cutting threshold $t^{1/\alpha}$ that gives M and N the α -similarity property (for any fixed t)

$$N_t \stackrel{law}{=} t^{1/\alpha} N_1 \quad \text{and} \quad M_t \stackrel{law}{=} t^{1/\alpha} M_1$$

we will need below. Indeed, to show the assertion for N , we can start from the Lévy-Khintchine formula for the characteristic function of N :

$$\mathbb{E}[e^{i\langle p, N_t \rangle}] = \exp \left[t \int_{\mathbb{S}^{nd-1}} \int_{t^{1/\alpha}}^{\infty} (\cos(\langle p, r\xi \rangle) - 1) \frac{dr}{r^{1+\alpha}} \bar{\mu}_S(d\xi) \right]$$

for any p in \mathbb{R}^{nd} . We then use the change of variable $rt^{-1/\alpha} = s$ to get that

$$\mathbb{E}[e^{i\langle p, N_t \rangle}] = \mathbb{E}[e^{i\langle p, t^{1/\alpha} N_1 \rangle}].$$

This implies in particular our assertion on N . In a similar way, it is possible to get the analogous assertion on M .

From lemma A.2 in [HMP19] with $m = 3$, we know that there exist a family $\{p_{\bar{M}}(t, \cdot)\}_{t>0}$ of densities on \mathbb{R}^{nd} and a constant $C := C(m, \alpha)$ such that

$$|D_{\mathbf{y}}^l p_M(t, \mathbf{y})| \leq C p_{\bar{M}}(t, \mathbf{y}) t^{-l/\alpha}$$

for any $t > 0$, any \mathbf{x} in \mathbb{R}^{nd} and any $l \in \{0, 1, 2\}$.

Moreover, denoting \bar{M}_t the random variable with density $p_{\bar{M}}(t, \cdot)$ and independent from N_t , we can easily check from $p_{\bar{M}}(t, \mathbf{y}) = t^{-nd/\alpha} p_{\bar{M}}(1, t^{-1/\alpha} \mathbf{y})$ that \bar{M} is α -selfsimilar

$$\bar{M}_t \stackrel{law}{=} t^{1/\alpha} \bar{M}_1.$$

We can finally define the family $\{q(t, \cdot)\}_{t>0}$ of densities as

$$q(t, \mathbf{x}) := \int_{\mathbb{R}^{nd}} p_{\bar{M}}(t, \mathbf{x} - \mathbf{y}) P_{N_t}(d\mathbf{y})$$

corresponding to the density of the random variable

$$\bar{S}_t := \bar{M}_t + N_t$$

for any fixed $t > 0$. Using Fourier transform and the already proven α -selfsimilarity of \bar{M} and N , we can show now that

$$\bar{S}_t \stackrel{law}{=} t^{1/\alpha} \bar{S}_1$$

or equivalently, that

$$q(t, \mathbf{y}) = t^{-nd/\alpha} q(1, t^{-1/\alpha} \mathbf{y})$$

for any t in $[0, T]$ and any \mathbf{y} in \mathbb{R}^{nd} . Moreover,

$$\mathbb{E}[|\bar{S}_t|^\gamma] = \mathbb{E}[|\bar{M}_t + N_t|^\gamma] = Ct^{\gamma/\alpha} (\mathbb{E}[|\bar{M}_1|^\gamma] + \mathbb{E}[|N_t|^\gamma]) \leq Ct^{\gamma/\alpha}.$$

This shows in particular that equation (A.2) holds. □

We conclude this sub-section showing Control (5.49) appearing in the proof of Proposition 4 for the diagonal regime. First of all, we will need the following lemma:

Lemma 22. *Let t in $[0, T]$, \mathbf{x}, \mathbf{b} in \mathbb{R}^{nd} such that $|\mathbf{b}| \leq ct^{1/\alpha}$ for some constant $c > 0$. Under (A), there exists a constant $C := C(c)$ such that*

$$|D_{\mathbf{x}}^l p_S(t, \mathbf{x} + \mathbf{b})| \leq \tilde{C} |D_{\mathbf{x}}^l p_S(t, \mathbf{x})|$$

Proof. Looking back at the proof of the previous lemma 21, we know that

$$D_{\mathbf{x}}^l p_S(t, \mathbf{x} + \mathbf{b}) = \int_{\mathbb{R}^{nd}} D_{\mathbf{x}}^l p_M(t, \mathbf{x} + \mathbf{b} - \mathbf{y}) P_{N_t}(d\mathbf{y})$$

where $p_M(t, \cdot)$ is the density of M_t and P_{N_t} is the law of N_t , corresponding to the small and big jumps in the Ito-Lévy decomposition.

From lemma A.2 in [HMP19] we know moreover that

$$|D_{\mathbf{x}}^l p_M(t, \mathbf{x} + \mathbf{b} - \mathbf{y})| \leq \frac{C}{t^{\frac{1}{\alpha}}} p_{\overline{M}}(t, \mathbf{x} + \mathbf{b} - \mathbf{y}) \quad \text{where} \quad p_{\overline{M}}(t, \mathbf{z}) = \frac{C}{t^{\frac{nd}{\alpha}}} \frac{1}{\left(1 + \frac{|\mathbf{z}|}{t^{\frac{1}{\alpha}}}\right)^3}.$$

It is then enough to show that

$$\begin{aligned} p_{\overline{M}}(t, \mathbf{z} + \mathbf{b}) &= \frac{C}{t^{\frac{nd}{\alpha}}} \frac{1}{\left(1 + \frac{|\mathbf{z} + \mathbf{b}|}{t^{\frac{1}{\alpha}}}\right)^3} \leq \frac{\tilde{C}}{t^{\frac{nd}{\alpha}}} \frac{1}{\left(1 + c + \frac{|\mathbf{z} + \mathbf{b}|}{t^{\frac{1}{\alpha}}}\right)^3} \\ &\leq \frac{C}{t^{\frac{nd}{\alpha}}} \frac{1}{\left(1 + c \frac{|\mathbf{z}|}{t^{\frac{1}{\alpha}}} - \frac{|\mathbf{b}|}{t^{\frac{1}{\alpha}}}\right)^3} \leq \frac{C}{t^{\frac{nd}{\alpha}}} \frac{1}{\left(1 + \frac{|\mathbf{z}|}{t^{\frac{1}{\alpha}}}\right)^3} \leq C p_{\overline{M}}(t, \mathbf{z}). \end{aligned}$$

to conclude the proof. \square

Proof of Equation (5.49). We start looking back to the proof of Lemma 3 to find that

$$|D_{\mathbf{x}}^{\vartheta} \tilde{p}^{\tau, \xi'}(t, s, \mathbf{x} + \lambda(\mathbf{x}' - \mathbf{x}), \mathbf{y})| = C(s-t)^{-\sum_{k=1}^n \frac{\vartheta_k}{\alpha_k}} \frac{1}{\det(\mathbb{M}_{s-t})} |D_{\mathbf{z}}^{|\vartheta|} p_S(s-t, \cdot)(\mathbb{M}_{s-t}^{-1}(\tilde{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x}) - \mathbf{y}))|$$

Moreover, we notice that

$$\mathbb{M}_{s-t}^{-1}(\tilde{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x} + \lambda(\mathbf{x} - \mathbf{x}')) - \mathbf{y}) = \mathbb{M}_{s-t}^{-1}(\tilde{\mathbf{m}}_{t,s}^{\tau, \xi}(\mathbf{x}) - \mathbf{y}) + \lambda \mathbb{M}_{s-t}^{-1} e^{A(s-t)}(\mathbf{x} - \mathbf{x}').$$

Then, Control (5.49) follows immediately from the previous lemma once we have shown that

$$|\lambda \mathbb{M}_{s-t}^{-1} e^{A(s-t)}(\mathbf{x} - \mathbf{x}')| \leq C(s-t)^{1/\alpha}$$

for some constant $C := C(A)$. Indeed, fixed i in $\llbracket 1, n \rrbracket$, we can exploit the structure of A and \mathbb{M}_{s-t} (cf. Equation (2.13) in Scaling Lemma 1) to write that

$$[\mathbb{M}_{s-t}^{-1} e^{A(s-t)}(\mathbf{x} - \mathbf{x}')]_i = \sum_{j=1}^n \sum_{k=1}^n [\mathbb{M}_{s-t}^{-1}]_{i,k} [e^{A(s-t)}]_{k,j} (\mathbf{x} - \mathbf{x}')_j = \sum_{j=i}^n (s-t)^{-(i-1)} C_j (s-t)^{i-j} (\mathbf{x} - \mathbf{x}')_j.$$

Since moreover we assumed to be in a local diagonal regime, i.e. $d^\alpha(\mathbf{x}, \mathbf{x}') \leq (s-t)^{1/\alpha}$, we can conclude that

$$|[\mathbb{M}_{s-t}^{-1} e^{A(s-t)}(\mathbf{x} - \mathbf{x}')]_i| \leq C \sum_{j=i}^n (s-t)^{-(j-1)} |(\mathbf{x} - \mathbf{x}')_j| \leq C \sum_{j=i}^n (s-t)^{-(j-1)} (s-t)^{\frac{1+\alpha(j-1)}{\alpha}} = C(s-t)^{1/\alpha}. \quad \square$$

A.2 Technical Tools

In this section, we present the proof of some technical results already used in the article, for the sake of completeness.

We recall moreover that the results below can be proven also for the flow $\bar{\boldsymbol{\theta}}_{\tau,s}(\boldsymbol{\xi})$ driven by a more general perturbation \mathbf{F} under assumption $(\bar{\mathbf{A}})$ (cf. Section 7.1), exploiting that $\bar{\mathbf{F}}_i$ is Lipschitz continuous in the \mathbf{x}_{i-1} variable for any i in $\llbracket 2, n \rrbracket$.

We begin proving Lemma 14 about the sensitivity of the Hölder flows, appearing in the proof of the a priori estimates (3.18) of Proposition 4. For this reason, we will assume from this point further to be under assumption (\mathbf{A}') .

Proof of Lemma 14. We start noticing that our result follows immediately using Young inequality, once we have shown that it holds

$$|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x}'))_i| \leq C \left[(s-t)^{\frac{1+\alpha(i-1)}{\alpha}} + d^{1+\alpha(i-1)}(\mathbf{x}, \mathbf{x}') \right] \quad \text{for any } i \text{ in } \llbracket 1, n \rrbracket. \quad (\text{A.3})$$

Our proof will rely essentially in iterative applications of the Grönwall lemma. We notice however that under (A), the perturbation \mathbf{F}_i is only Hölder continuous with respect to its i -th variable. To overcome this problem, we are going to mollify (but only with respect to the variable of interest) the function \mathbf{F} in the following way: fixed a mollifier ρ on \mathbb{R}^d , i.e. a compactly supported, non-negative, smooth function such that $\|\rho\|_{L^1} = 1$ and a family δ_i of positive constants to be chosen later, the mollified version of the perturbation is given by $\mathbf{F}^\delta = (\mathbf{F}_1, \mathbf{F}_2^{\delta_2}, \dots, \mathbf{F}_n^{\delta_n})$ where

$$\mathbf{F}_i^{\delta_i}(t, \mathbf{z}_{i:n}) := \mathbf{F}_i * \rho_{\delta_i}(t, \mathbf{z}_{i:n}) = \int_{\mathbb{R}^d} \mathbf{F}_i(t, \mathbf{z}_i - \omega, \mathbf{z}_{i+1}, \dots, \mathbf{z}_n) \frac{1}{\delta_i^d} \rho\left(\frac{\omega}{\delta_i}\right) d\omega.$$

We remark in particular that we do not need to mollify the first component \mathbf{F}_1 since it is regular enough, say β -Hölder continuous in the first d -dimensional variable \mathbf{x}_1 , by assumption (R).

Then, standard results on mollifier theory and our current assumptions on \mathbf{F} show us that the following controls hold

$$|\mathbf{F}_i(u, \mathbf{z}) - \mathbf{F}_i^\delta(u, \mathbf{z})| \leq \|\mathbf{F}_i\|_{L^\infty(C_d^{\gamma+\beta})} \delta_i^{\frac{\gamma_i+\beta}{1+\alpha(i-1)}}, \quad (\text{A.4})$$

$$|\mathbf{F}_i^\delta(u, \mathbf{z}) - \mathbf{F}_i^\delta(u, \mathbf{z}')| \leq C \|\mathbf{F}_i\|_{L^\infty(C_d^{\gamma+\beta})} [\delta_i^{\frac{\gamma_i+\beta}{1+\alpha(i-1)}-1} |(\mathbf{z} - \mathbf{z}')_i| + \sum_{j=i+1}^n |(\mathbf{z} - \mathbf{z}')_j|^{\frac{\gamma_j+\beta}{1+\alpha(j-1)}}]. \quad (\text{A.5})$$

We choose now δ_i for any i in $\llbracket 2, n \rrbracket$ in order to have any contribution associated with the mollification appearing in (A.4) at a good current scale time. Namely, we would like δ_i to satisfy

$$|((s-t)^{\frac{1}{\alpha}} \mathbb{M}_{s-t})^{-1} (\mathbf{F}(u, \mathbf{z}) - \mathbf{F}^\delta(u, \mathbf{z}))| \leq C(s-t)^{-1}$$

for any u in $[t, s]$ and any \mathbf{z} in \mathbb{R}^{nd} . Using the mollifier controls (A.4), it is enough to ask for

$$\sum_{i=2}^n (s-t)^{-\frac{1}{\alpha_i}} \delta_i^{\frac{\gamma_i+\beta}{1+\alpha(i-1)}} \leq C(s-t)^{-1}.$$

Recalling that $\gamma_i := 1 + \alpha(i-2)$ by assumption (R), this is true if we fix for example,

$$\delta_i = (s-t)^{\frac{\gamma_i}{\alpha} \frac{1+\alpha(i-1)}{\gamma_i+\beta}} \quad \text{for } i \text{ in } \llbracket 2, n \rrbracket. \quad (\text{A.6})$$

After this introductive part, we start controlling the last component of the flow. By construction of $\boldsymbol{\theta}_{t,s}$, we can write that

$$\begin{aligned} |(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x}'))_n| &= \left| (\mathbf{x} - \mathbf{x}')_n + \int_t^s \{ [A(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))]_n + \mathbf{F}_n(v, \boldsymbol{\theta}_{t,v}(\mathbf{x})) - \mathbf{F}_n(v, \boldsymbol{\theta}_{t,v}(\mathbf{x}')) \} dv \right| \\ &\leq |(\mathbf{x} - \mathbf{x}')_n| + \int_t^s \{ A_{n,n-1} |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_{n-1}| + |\mathbf{F}_n(v, \boldsymbol{\theta}_{t,v}(\mathbf{x})) - \mathbf{F}_n(v, \boldsymbol{\theta}_{t,v}(\mathbf{x}'))| \} dv \end{aligned} \quad (\text{A.7})$$

where in the last passage we have exploited the sub-diagonal structure of A (cf. Equation (1.2)). If we focus only on the last term involving the difference of the drifts, It holds now that

$$\begin{aligned} |\mathbf{F}_n(v, \boldsymbol{\theta}_{t,v}(\mathbf{x})) - \mathbf{F}_n(v, \boldsymbol{\theta}_{t,v}(\mathbf{x}'))| &\leq |\mathbf{F}_n(v, \boldsymbol{\theta}_{t,v}(\mathbf{x})) \pm \mathbf{F}_n^\delta(v, \boldsymbol{\theta}_{t,v}(\mathbf{x})) - \mathbf{F}_n(v, \boldsymbol{\theta}_{t,v}(\mathbf{x}')) \pm \mathbf{F}_n^\delta(v, \boldsymbol{\theta}_{t,v}(\mathbf{x}'))| \\ &\leq |\mathbf{F}_n(v, \boldsymbol{\theta}_{t,v}(\mathbf{x})) - \mathbf{F}_n^\delta(v, \boldsymbol{\theta}_{t,v}(\mathbf{x}))| + |\mathbf{F}_n(v, \boldsymbol{\theta}_{t,v}(\mathbf{x}')) - \mathbf{F}_n^\delta(v, \boldsymbol{\theta}_{t,v}(\mathbf{x}'))| + |\mathbf{F}_n^\delta(v, \boldsymbol{\theta}_{t,v}(\mathbf{x})) - \mathbf{F}_n^\delta(v, \boldsymbol{\theta}_{t,v}(\mathbf{x}'))|. \end{aligned}$$

Using the controls (A.4), (A.5) on the mollified drifts, we then write from (A.7) and the previous equation that

$$\begin{aligned} |(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x}'))_n| &\leq \\ &|(\mathbf{x} - \mathbf{x}')_n| + 2(s-t) \delta_n^{\frac{\gamma_n+\beta}{1+\alpha(n-1)}} + C \int_t^s \{ |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_{n-1}| + \delta_n^{\frac{\gamma_n+\beta}{1+\alpha(n-1)}-1} |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_n| \} dv. \end{aligned}$$

We now apply the Grönwall lemma to show that

$$|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x}'))_n| \leq C \left[|(\mathbf{x} - \mathbf{x}')_n| + (s-t) \delta_n^{\frac{\gamma_n+\beta}{1+\alpha(n-1)}} + \int_t^s |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_{n-1}| dv \right].$$

From our previous choice for δ_n (cf. Equation (A.6)), we know that $(s-t)^{-\frac{1}{\alpha_n}} \delta_n^{\frac{\gamma_n+\beta}{1+\alpha(n-1)}} \leq C(s-t)^{-1}$ and thus, we can rewrite the last inequality as

$$|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x}'))_n| \leq C \left[|(\mathbf{x} - \mathbf{x}')_n| + (s-t)^{\frac{1+\alpha(n-1)}{\alpha}} + \int_t^s |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_{n-1}| dv \right]. \quad (\text{A.8})$$

We would like now to obtain a similar control on the $(n-1)$ -th term. As already done at the beginning of the proof, we can write that

$$\begin{aligned} |(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x}'))_{n-1}| &\leq |(\mathbf{x} - \mathbf{x}')_{n-1}| + C \delta_{n-1}^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-2)}} (s-t) + \int_t^s |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_{n-2}| \\ &\quad + \delta_{n-1}^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-2)}-1} |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_{n-1}| + |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} dv \end{aligned}$$

We then apply the Grönwall lemma to find that

$$\begin{aligned} |(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x}'))_{n-1}| &\leq C \left[|(\mathbf{x} - \mathbf{x}')_{n-1}| + \delta_{n-1}^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-2)}} (s-t) \right. \\ &\quad \left. + \int_t^s \left\{ |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_{n-2}| + |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} \right\} dv \right]. \end{aligned}$$

Remembering our previous choice of δ_{n-1} , it holds now that

$$\begin{aligned} |(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x}'))_{n-1}| &\leq C \left[|(\mathbf{x} - \mathbf{x}')_{n-1}| + (s-t)^{\frac{1+\alpha(n-2)}{\alpha}} + \int_t^s |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_{n-2}| \right. \\ &\quad \left. + |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} dv \right]. \quad (\text{A.9}) \end{aligned}$$

We then use equation (A.8) and the Jensen inequality to write

$$\begin{aligned} |(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x}'))_{n-1}| &\leq C \left[|(\mathbf{x} - \mathbf{x}')_{n-1}| + (s-t)^{\frac{1+\alpha(n-2)}{\alpha}} + \int_t^s \left\{ |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_{n-2}| + |(\mathbf{x} - \mathbf{x}')_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} \right. \right. \\ &\quad \left. \left. + (v-t)^{\frac{\gamma_{n-1}+\beta}{\alpha}} + \left(\int_t^v |(\boldsymbol{\theta}_{t,\omega}(\mathbf{x}) - \boldsymbol{\theta}_{t,\omega}(\mathbf{x}'))_{n-1}| d\omega \right)^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} \right\} dv \right]. \quad (\text{A.10}) \end{aligned}$$

The idea now is to use Grönwall lemma again. To do so, we firstly move the exponent from the last integral term involving the $(n-1)$ -th term using the Young inequality:

$$\left(\int_t^v |(\boldsymbol{\theta}_{t,\omega}(\mathbf{x}) - \boldsymbol{\theta}_{t,\omega}(\mathbf{x}'))_{n-1}| d\omega \right)^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} \leq B^{-\frac{1+\alpha(n-1)}{\gamma_{n-1}+\beta}} \int_t^v |(\boldsymbol{\theta}_{t,\omega}(\mathbf{x}) - \boldsymbol{\theta}_{t,\omega}(\mathbf{x}'))_{n-1}| d\omega + B^{\frac{1+\alpha(n-1)}{2\alpha-\beta}}$$

for a quantity B to be fixed later.

Since we need homogeneity with respect to time in equation (A.9), we choose B such that

$$B^{\frac{1+\alpha(n-1)}{2\alpha-\beta}} = (v-t)^{\frac{\gamma_{n-1}+\beta}{\alpha}} \Leftrightarrow B = (v-t)^{\frac{\gamma_{n-1}+\beta}{\alpha} \frac{2\alpha-\beta}{1+\alpha(n-1)}}.$$

Plugging it into the general expression (A.10), we find that

$$\begin{aligned}
|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x}'))_{n-1}| &\leq C \left[|(\mathbf{x} - \mathbf{x}')_{n-1}| + (s-t)^{\frac{1+\alpha(n-2)}{\alpha}} \right. \\
&\quad \left. + \int_t^s \left\{ |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_{n-2}| + |(\mathbf{x} - \mathbf{x}')_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} + (v-t)^{\frac{\gamma_{n-1}+\beta}{\alpha}} \right. \right. \\
&\quad \left. \left. + (v-t)^{\frac{\beta}{\alpha}-2} \int_t^v |(\boldsymbol{\theta}_{t,\omega}(\mathbf{x}) - \boldsymbol{\theta}_{t,\omega}(\mathbf{x}'))_{n-1}| d\omega \right\} dv \right] \\
&\leq C \left[|(\mathbf{x} - \mathbf{x}')_{n-1}| + (s-t)^{\frac{1+\alpha(n-2)}{\alpha}} + (s-t) |(\mathbf{x} - \mathbf{x}')_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} + (s-t)^{\frac{\gamma_{n-1}+\beta+\alpha}{\alpha}} \right. \\
&\quad \left. + \int_t^s \left\{ |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_{n-2}| + (v-t)^{\frac{\beta}{\alpha}-1} \sup_{\omega \in [t,v]} |(\boldsymbol{\theta}_{t,\omega}(\mathbf{x}) - \boldsymbol{\theta}_{t,\omega}(\mathbf{x}'))_{n-1}| \right\} dv \right].
\end{aligned}$$

Since the previous inequality is also true for any \bar{s} in $[t, s]$, it follows that

$$\begin{aligned}
\sup_{\bar{s} \in [0, s]} |(\boldsymbol{\theta}_{t,\bar{s}}(\mathbf{x}) - \boldsymbol{\theta}_{t,\bar{s}}(\mathbf{x}'))_{n-1}| &\leq C \left[|(\mathbf{x} - \mathbf{x}')_{n-1}| + (s-t)^{\frac{1+\alpha(n-2)}{\alpha}} + (s-t) |(\mathbf{x} - \mathbf{x}')_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} \right. \\
&\quad \left. + (s-t)^{\frac{\gamma_{n-1}+\beta+\alpha}{\alpha}} + \int_t^s \left\{ |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_{n-2}| + (v-t)^{\frac{\beta}{\alpha}-1} \sup_{\omega \in [t,v]} |(\boldsymbol{\theta}_{t,\omega}(\mathbf{x}) - \boldsymbol{\theta}_{t,\omega}(\mathbf{x}'))_{n-1}| \right\} dv \right].
\end{aligned}$$

We can finally apply the Grönwall lemma to show that for any s in $[t, T]$, there exists a constant C such that

$$\begin{aligned}
|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x}'))_{n-1}| \\
\leq C \left[|(\mathbf{x} - \mathbf{x}')_{n-1}| + (s-t)^{\frac{1+\alpha(n-2)}{\alpha}} + (s-t) |(\mathbf{x} - \mathbf{x}')_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} + \int_t^s |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_{n-2}| dv \right].
\end{aligned}$$

Moreover, thanks to the Young inequality we know that

$$(s-t) |(\mathbf{x} - \mathbf{x}')_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} \leq C \left\{ (s-t)^{\frac{1+\alpha(n-2)}{\alpha}} + |(\mathbf{x} - \mathbf{x}')_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)} \frac{1+\alpha(n-2)}{1+\alpha(n-3)}} \right\}$$

and remembering that $d(\mathbf{x}, \mathbf{x}') \leq 1$ by hypothesis,

$$|(\mathbf{x} - \mathbf{x}')_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)} \frac{1+\alpha(n-2)}{1+\alpha(n-3)}} \leq |(\mathbf{x} - \mathbf{x}')_n|^{\frac{\gamma_{n-1}+\beta}{\gamma_{n-1}} \frac{1+\alpha(n-2)}{1+\alpha(n-1)}} \leq |(\mathbf{x} - \mathbf{x}')_n|^{\frac{1+\alpha(n-2)}{1+\alpha(n-1)}}.$$

We then use it to write for any v in $[t, T]$,

$$\begin{aligned}
|(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_{n-1}| \\
\leq C \left[|(\mathbf{x} - \mathbf{x}')_{n-1}| + (v-t)^{\frac{1+\alpha(n-2)}{\alpha}} + |(\mathbf{x} - \mathbf{x}')_n|^{\frac{1+\alpha(n-2)}{1+\alpha(n-1)}} + \int_t^v |(\boldsymbol{\theta}_{t,\omega}(\mathbf{x}) - \boldsymbol{\theta}_{t,\omega}(\mathbf{x}'))_{n-2}| d\omega \right].
\end{aligned}$$

Going back to equation (A.8), we plug in the last bound to find that

$$\begin{aligned}
|(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x}'))_n| &\leq C \left[|(\mathbf{x} - \mathbf{x}')_n| + (s-t)^{\frac{1+\alpha(n-1)}{\alpha}} + (s-t) |(\mathbf{x} - \mathbf{x}')_{n-1}| \right. \\
&\quad \left. + (s-t) |(\mathbf{x} - \mathbf{x}')_n|^{\frac{1+\alpha(n-2)}{1+\alpha(n-1)}} + \int_t^s \int_t^v |(\boldsymbol{\theta}_{t,\omega}(\mathbf{x}) - \boldsymbol{\theta}_{t,\omega}(\mathbf{x}'))_{n-2}| d\omega dv \right] \\
&\leq C \left[|(\mathbf{x} - \mathbf{x}')_n| + (s-t)^{\frac{1+\alpha(n-1)}{\alpha}} + |(\mathbf{x} - \mathbf{x}')_{n-1}|^{\frac{1+\alpha(n-1)}{1+\alpha(n-2)}} + \int_t^s \int_t^v |(\boldsymbol{\theta}_{t,\omega}(\mathbf{x}) - \boldsymbol{\theta}_{t,\omega}(\mathbf{x}'))_{n-2}| d\omega dv \right]
\end{aligned}$$

where in the last passage we used again the Young inequality to show that

$$(s-t) |(\mathbf{x} - \mathbf{x}')_{n-1}| \leq C (s-t)^{\frac{1+\alpha(n-1)}{\alpha}} + |(\mathbf{x} - \mathbf{x}')_{n-1}|^{\frac{1+\alpha(n-1)}{1+\alpha(n-2)}}$$

and

$$(s-t) |(\mathbf{x} - \mathbf{x}')_n|^{\frac{1+\alpha(n-2)}{1+\alpha(n-1)}} \leq C (s-t)^{\frac{1+\alpha(n-1)}{\alpha}} + |(\mathbf{x} - \mathbf{x}')_n|.$$

This approach may be naturally iterated up to the first term of the chain, so that

$$\begin{aligned} & |(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x}'))_n| \\ & \leq C \left[\sum_{j=2}^n |(\mathbf{x} - \mathbf{x}')_j|^{\frac{1+\alpha(n-1)}{1+\alpha(j-1)}} + (s-t)^{\frac{1+\alpha(n-1)}{\alpha}} + \int_t^{v_n=s} dv_{n-1} \cdots \int_t^{v=2} dv_1 |(\boldsymbol{\theta}_{t,v_1}(\mathbf{x}) - \boldsymbol{\theta}_{t,v_1}(\mathbf{x}'))_1| \right]. \end{aligned}$$

In a similar manner, we can show for any i in $\llbracket 2, n \rrbracket$,

$$\begin{aligned} & |(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \boldsymbol{\theta}_{t,s}(\mathbf{x}'))_i| \\ & \leq C \left[\sum_{j=2}^n |(\mathbf{x} - \mathbf{x}')_j|^{\frac{1+\alpha(i-1)}{1+\alpha(j-1)}} + (s-t)^{\frac{1+\alpha(i-1)}{\alpha}} + \int_t^{v_i=s} dv_{i-1} \cdots \int_t^{v=2} dv_1 |(\boldsymbol{\theta}_{t,v_1}(\mathbf{x}) - \boldsymbol{\theta}_{t,v_1}(\mathbf{x}'))_1| \right]. \quad (\text{A.11}) \end{aligned}$$

Since all the non-integral terms in (A.11) are compatible with the statement of the lemma, it remains to find the proper bound for the first component of the flow. As before, let us consider \bar{s} in $[t, s]$. We can write

$$|(\boldsymbol{\theta}_{t,\bar{s}}(\mathbf{x}) - \boldsymbol{\theta}_{t,\bar{s}}(\mathbf{x}'))_1| \leq |(\mathbf{x} - \mathbf{x}')_1| + C \sum_{j=1}^n \int_t^{\bar{s}} |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_j|^{\frac{\beta}{1+\alpha(j-1)}} dv$$

or, passing to the supremum on both sides,

$$\begin{aligned} & \sup_{\bar{s} \in [t, s]} |(\boldsymbol{\theta}_{t,\bar{s}}(\mathbf{x}) - \boldsymbol{\theta}_{t,\bar{s}}(\mathbf{x}'))_1| \\ & \leq |(\mathbf{x} - \mathbf{x}')_1| + C \left\{ (s-t) \left(\sup_{v \in [t, s]} |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_1| \right)^\beta + \sum_{j=2}^n \int_t^s |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_j|^{\frac{\beta}{1+\alpha(j-1)}} dv \right\}. \end{aligned}$$

Using equation (A.11), it holds now that

$$\begin{aligned} & \sup_{\bar{s} \in [t, s]} |(\boldsymbol{\theta}_{t,\bar{s}}(\mathbf{x}) - \boldsymbol{\theta}_{t,\bar{s}}(\mathbf{x}'))_1| \leq |(\mathbf{x} - \mathbf{x}')_1| + C \left\{ (s-t) \left(\sup_{v \in [t, s]} |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_1| \right)^\beta \right. \\ & + \sum_{j=2}^n \left[(s-t) \left((s-t)^{\frac{1+\alpha(j-1)}{\alpha}} + \sum_{k=2}^n |(\mathbf{x} - \mathbf{x}')_k|^{\frac{1+\alpha(j-1)}{1+\alpha(k-1)}} + (s-t)^{j-1} \sup_{v \in [t, s]} |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_1|^{\frac{\beta}{1+\alpha(j-1)}} \right) \right] \Big\}. \quad (\text{A.12}) \end{aligned}$$

We then apply the Jensen inequality to show that

$$\begin{aligned} & \sup_{\bar{s} \in [t, s]} |(\boldsymbol{\theta}_{t,\bar{s}}(\mathbf{x}) - \boldsymbol{\theta}_{t,\bar{s}}(\mathbf{x}'))_1| \leq |(\mathbf{x} - \mathbf{x}')_1| + C \left\{ (s-t) \left[\sup_{v \in [t, s]} |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_1| \right]^\beta + \sum_{j=2}^n C(s-t) \left[(s-t)^{\frac{\beta}{\alpha}} \right. \right. \\ & + \sum_{k=2}^n |(\mathbf{x} - \mathbf{x}')_k|^{\frac{\beta}{1+\alpha(k-1)}} + (s-t)^{\frac{(j-1)\beta}{1+\alpha(j-1)}} \sup_{v \in [t, s]} |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_1|^{\frac{\beta}{1+\alpha(j-1)}} \Big] \Big\} \\ & \leq C \left\{ |(\mathbf{x} - \mathbf{x}')_1| + (s-t)^{\frac{\alpha+\beta}{\alpha}} + (s-t) \sum_{k=2}^n |(\mathbf{x} - \mathbf{x}')_k|^{\frac{\beta}{1+\alpha(k-1)}} \right. \\ & \quad \left. + \sum_{j=1}^n (s-t)^{1+\frac{(j-1)\beta}{1+\alpha(j-1)}} \sup_{v \in [t, s]} |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_1|^{\frac{\beta}{1+\alpha(j-1)}} \right\}. \quad (\text{A.13}) \end{aligned}$$

From Young inequality, we can deduce now that

$$(s-t) |(\mathbf{x} - \mathbf{x}')_k|^{\frac{\beta}{1+\alpha(k-1)}} \leq C \left((s-t)^{\frac{1}{1-\beta}} + |(\mathbf{x} - \mathbf{x}')_k|^{\frac{1}{1+\alpha(k-1)}} \right)$$

and

$$(s-t)^{1+\frac{(j-1)\beta}{1+\alpha(j-1)}} \sup_{v \in [t, s]} |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_1|^{\frac{\beta}{1+\alpha(j-1)}} \leq C \left\{ (s-t)^{\frac{1+\alpha+\beta}{1+\alpha(j-1)-\beta}} + \sup_{v \in [t, s]} |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_1| \right\}$$

Plugging these inequalities in the main one (A.13), we find that

$$\begin{aligned}
\sup_{\bar{s} \in [t, s]} |(\boldsymbol{\theta}_{t, \bar{s}}(\mathbf{x}) - \boldsymbol{\theta}_{t, \bar{s}}(\mathbf{x}'))_1| &\leq C \left\{ |(\mathbf{x} - \mathbf{x}')_1| + (s - t)^{\frac{\alpha + \beta}{\alpha}} + \sum_{k=2}^n |(\mathbf{x} - \mathbf{x}')_k|^{\frac{1}{1 + \alpha(k-1)}} \right. \\
&\quad \left. + \sum_{j=1}^n (s - t)^{\frac{1 + (\alpha + \beta)(j-1)}{1 + \alpha(j-1) - \beta}} + \sup_{v \in [t, s]} |(\boldsymbol{\theta}_{t, v}(\mathbf{x}) - \boldsymbol{\theta}_{t, v}(\mathbf{x}'))_1| \right\} \\
&\leq C \left\{ (s - t)^{\frac{\alpha + \beta}{\alpha}} + (s - t)^{\frac{1}{1 - \beta}} + d(\mathbf{x}, \mathbf{x}') + (s - t)^{\frac{1 + (\alpha + \beta)(j-1)}{1 + \alpha(j-1) - \beta}} + \sup_{v \in [t, s]} |(\boldsymbol{\theta}_{t, v}(\mathbf{x}) - \boldsymbol{\theta}_{t, v}(\mathbf{x}'))_1| \right\}
\end{aligned}$$

Remembering that $s - t \leq T - t \leq 1$, it finally holds that

$$|\boldsymbol{\theta}_{t, s}(\mathbf{x}) - \boldsymbol{\theta}_{t, s}(\mathbf{x}')|_1 \leq C((s - t)^{1/\alpha} + d(\mathbf{x}, \mathbf{x}'))$$

since by assumption **(P)**,

$$\frac{\alpha + \beta}{\alpha} > \frac{1}{1 - \beta} > \frac{1}{\alpha}$$

and

$$\frac{1 + (\alpha + \beta)(j - 1)}{1 + \alpha(j - 1) - \beta} = 1 + \frac{\beta j}{1 + \alpha j - (\alpha + \beta)} > 1 + \frac{\beta j}{\alpha j} > 1 + \left(\frac{1 - \alpha}{\alpha} \right) = \frac{1}{\alpha}.$$

Plugging this control in equation (A.11), we then conclude since

$$\begin{aligned}
|(\boldsymbol{\theta}_{t, s}(\mathbf{x}) - \boldsymbol{\theta}_{t, s}(\mathbf{x}'))_i| &\leq C \left(d^{1 + \alpha(i-1)}(\mathbf{x}, \mathbf{x}') + (s - t)^{\frac{1 + \alpha(i-1)}{\alpha}} + (s - t)^{i-1} \sup_{\bar{s} \in [t, s]} |(\boldsymbol{\theta}_{t, \bar{s}}(\mathbf{x}) - \boldsymbol{\theta}_{t, \bar{s}}(\mathbf{x}'))_1| \right) \\
&\leq C \left(d^{1 + \alpha(i-1)}(\mathbf{x}, \mathbf{x}') + (s - t)^{\frac{1 + \alpha(i-1)}{\alpha}} + (s - t)^{i-1} ((s - t)^{1/\alpha} + d(\mathbf{x}, \mathbf{x}')) \right) \\
&\leq C \left((s - t)^{\frac{1 + \alpha(i-1)}{\alpha}} + d^{1 + \alpha(i-1)}(\mathbf{x}, \mathbf{x}') \right),
\end{aligned}$$

using again the Young inequality in the last passage. The proof is complete.

We can now prove the two results (Lemmas 15 and Lemma 16) concerning the sensitivity of the frozen shift $\tilde{\mathbf{m}}_{t, s}^{\tau, \xi}$.

Proof of Lemma 15. From the integral representation of $\tilde{\mathbf{m}}_{t, s}^{t, \mathbf{x}}(\mathbf{y})$ (cf. Equation (3.4)), we can write that

$$\begin{aligned}
|(\tilde{\mathbf{m}}_{t, s}^{t, \mathbf{x}}(\mathbf{y}) - \tilde{\mathbf{m}}_{t, s}^{t, \mathbf{x}'}(\mathbf{y}'))_1| &\leq \int_t^s |\mathbf{F}_1(v, \boldsymbol{\theta}_{t, v}(\mathbf{x})) - \mathbf{F}_1(v, \boldsymbol{\theta}_{t, v}(\mathbf{x}'))| dv \\
&\leq C \|\mathbf{F}\|_H \int_t^s d^\beta(\boldsymbol{\theta}_{t, v}(\mathbf{x}), \boldsymbol{\theta}_{t, v}(\mathbf{x}')) dv
\end{aligned}$$

where in the second passage we used that \mathbf{F}_1 is in $C_{b, d}^\beta(\mathbb{R}^{nd})$. Thanks to the Control on the flows (Lemma 14), it then holds that

$$|(\tilde{\mathbf{m}}_{t, s}^{t, \mathbf{x}}(\mathbf{y}) - \tilde{\mathbf{m}}_{t, s}^{t, \mathbf{x}'}(\mathbf{y}'))_1| \leq C \|\mathbf{F}\|_H (s - t) [d^\beta(\mathbf{x}, \mathbf{x}') + (s - t)^{\frac{\beta}{\alpha}}]$$

and we have concluded.

Proof of Lemma 16. We know from Lemma 2 that $\tilde{\mathbf{m}}_{t, t_0}^{t, \mathbf{x}'}(\mathbf{x}') = \boldsymbol{\theta}_{t, t_0}(\mathbf{x}')$. Fixed i in $\llbracket 1, n \rrbracket$, we can then write that

$$(\tilde{\mathbf{m}}_{t, t_0}^{t, \mathbf{x}}(\mathbf{x}') - \tilde{\mathbf{m}}_{t, t_0}^{t, \mathbf{x}'}(\mathbf{x}'))_i = (\tilde{\mathbf{m}}_{t, t_0}^{t, \mathbf{x}}(\mathbf{x}') - \boldsymbol{\theta}_{t, t_0}(\mathbf{x}'))_i = (\tilde{\mathbf{m}}_{t, t_0}^{t, \mathbf{x}}(\mathbf{x}') - \boldsymbol{\theta}_{t, t_0}(\mathbf{x}))_i + (\boldsymbol{\theta}_{t, t_0}(\mathbf{x}) - \boldsymbol{\theta}_{t, t_0}(\mathbf{x}'))_i.$$

We start focusing on the first term of the above expression. From the integral representation of $\tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}}(\mathbf{x}')$ and $\boldsymbol{\theta}_{t,t_0}(\mathbf{x})$, it holds that

$$\tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}}(\mathbf{x}') - \boldsymbol{\theta}_{t,t_0}(\mathbf{x}) = \mathbf{x}' - \mathbf{x} + \int_t^{t_0} A[\tilde{\mathbf{m}}_{t,v}^{t,\mathbf{x}}(\mathbf{x}') - \boldsymbol{\theta}_{t,v}(\mathbf{x})] dv. \quad (\text{A.14})$$

Remembering from (1.2) that A is sub-diagonal, it follows that

$$(\tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}}(\mathbf{x}') - \boldsymbol{\theta}_{t,t_0}(\mathbf{x}))_i = (\mathbf{x}' - \mathbf{x})_i + A_{i,i-1} \int_t^{t_0} (\tilde{\mathbf{m}}_{t,v}^{t,\mathbf{x}}(\mathbf{x}') - \boldsymbol{\theta}_{t,v}(\mathbf{x}))_{i-1} dv \quad (\text{A.15})$$

for any i in $\llbracket 2, n \rrbracket$ and

$$(\tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}}(\mathbf{x}') - \boldsymbol{\theta}_{t,t_0}(\mathbf{x}))_1 = (\mathbf{x}' - \mathbf{x})_1.$$

Iterating the process, we can find that

$$|(\tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}}(\mathbf{x}') - \boldsymbol{\theta}_{t,t_0}(\mathbf{x}))_i| \leq C \sum_{k=1}^i |(\mathbf{x}' - \mathbf{x})_k| (t_0 - t)^{i-k}.$$

On the other side, the integral representation of $\boldsymbol{\theta}_{\tau,s}(\boldsymbol{\xi})$ (Equation (3.1)) allows us to write that

$$\begin{aligned} & (\boldsymbol{\theta}_{t,t_0}(\mathbf{x}) - \boldsymbol{\theta}_{t,t_0}(\mathbf{x}'))_i \\ &= (\mathbf{x} - \mathbf{x}')_i + A_{i,i-1} \int_t^{t_0} \{(\boldsymbol{\theta}_{t,t_0}(\mathbf{x}) - \boldsymbol{\theta}_{t,t_0}(\mathbf{x}'))_{i-1} + \mathbf{F}_i(v, \boldsymbol{\theta}_{t,v}(\mathbf{x})) - \mathbf{F}_i(v, \boldsymbol{\theta}_{t,v}(\mathbf{x}'))\} dv \end{aligned} \quad (\text{A.16})$$

for any i in $\llbracket 2, n \rrbracket$ and

$$(\boldsymbol{\theta}_{t,t_0}(\mathbf{x}) - \boldsymbol{\theta}_{t,t_0}(\mathbf{x}'))_1 = (\mathbf{x} - \mathbf{x}')_1 + \int_t^{t_0} \{\mathbf{F}_1(v, \boldsymbol{\theta}_{t,v}(\mathbf{x})) - \mathbf{F}_1(v, \boldsymbol{\theta}_{t,v}(\mathbf{x}'))\} dv. \quad (\text{A.17})$$

Fixed i in $\llbracket 2, n \rrbracket$, it then follows from (A.14) and (A.16) that

$$\begin{aligned} |(\tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}}(\mathbf{x}') - \tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}'}(\mathbf{x}'))_i| &\leq C \|\mathbf{F}\|_H \left(\sum_{k=1}^{i-1} |(\mathbf{x}' - \mathbf{x})_k| (t_0 - t)^{i-k} \right. \\ &\quad \left. + \int_t^{t_0} \left\{ |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_{i-1}| + \sum_{j=i}^n |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_j|^{\frac{\gamma_i + \beta}{1 + \alpha(j-1)}} \right\} dv \right). \end{aligned}$$

Also, from (A.15) and (A.17), it holds that

$$|(\tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}}(\mathbf{x}') - \tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}'}(\mathbf{x}'))_1| \leq C \|\mathbf{F}\|_H \int_t^{t_0} \sum_{j=1}^n |(\boldsymbol{\theta}_{t,v}(\mathbf{x}) - \boldsymbol{\theta}_{t,v}(\mathbf{x}'))_j|^{\frac{\beta}{1 + \alpha(j-1)}} dv.$$

Using now Lemma 14, we can show that

$$\begin{aligned} |(\tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}}(\mathbf{x}') - \tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}'}(\mathbf{x}'))_i| &\leq C \|\mathbf{F}\|_H \left(\sum_{k=1}^{i-1} |(\mathbf{x}' - \mathbf{x})_k| (t_0 - t)^{i-k} + (t_0 - t)^{\frac{1 + \alpha(i-2)}{\alpha} + 1} \right. \\ &\quad \left. + (t_0 - t) d^{1 + \alpha(i-2)}(\mathbf{x}, \mathbf{x}') + (t_0 - t)^{\frac{1 + \alpha(i-2) + \beta}{\alpha} + 1} + (t_0 - t) d^{1 + \alpha(i-2) + \beta}(\mathbf{x}, \mathbf{x}') \right) \end{aligned}$$

for any i in $\llbracket 2, n \rrbracket$ and

$$|(\tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}}(\mathbf{x}') - \tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}'}(\mathbf{x}'))_1| \leq C \|\mathbf{F}\|_H (t_0 - t)^{\frac{\beta + \alpha}{\alpha}} + (t_0 - t) d^{\beta}(\mathbf{x}, \mathbf{x}').$$

Since $t_0 - t = c_0 d^\alpha(\mathbf{x}, \mathbf{x}')$ by Equation (4.16), we can conclude that

$$\begin{aligned}
|(\tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}}(\mathbf{x}') - \tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}'}(\mathbf{x}'))_i| &\leq C\|\mathbf{F}\|_H \left(\sum_{k=1}^{i-1} d^{1+\alpha(k-1)}(\mathbf{x}', \mathbf{x}) c_0^{i-k} d^{\alpha(i-k)}(\mathbf{x}, \mathbf{x}') + c_0^{\frac{1+\alpha(i-1)}{\alpha}} d^{1+\alpha(i-1)}(\mathbf{x}, \mathbf{x}') \right. \\
&\quad \left. + c_0 d^{1+\alpha(i-1)}(\mathbf{x}, \mathbf{x}') + c_0^{\frac{1+\alpha(i-2)+\beta}{\alpha}+1} d^{1+\alpha(i-1)+\beta}(\mathbf{x}, \mathbf{x}') + c_0 d^{1+\alpha(i-1)+\beta}(\mathbf{x}, \mathbf{x}') \right) \\
&\leq C\|\mathbf{F}\|_H \left[\left(c_0 + c_0^{\frac{1+\alpha(i-1)}{\alpha}} \right) d^{1+\alpha(i-1)}(\mathbf{x}, \mathbf{x}') + \left(c_0 + c_0^{\frac{1+\alpha(i-1)+\beta}{\alpha}} \right) d^{1+\alpha(i-1)+\beta}(\mathbf{x}, \mathbf{x}') \right] \\
&\leq C c_0 \|\mathbf{F}\|_H d^{1+\alpha(i-1)}(\mathbf{x}, \mathbf{x}')
\end{aligned}$$

for any i in $\llbracket 2, n \rrbracket$ and

$$|(\tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}}(\mathbf{x}') - \tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}'}(\mathbf{x}'))_1| \leq C\|\mathbf{F}\|_H (c_0^{\frac{\beta+\alpha}{\alpha}} + c_0) d^{\alpha+\beta}(\mathbf{x}, \mathbf{x}') \leq C c_0 \|\mathbf{F}\|_H d^{\alpha+\beta}(\mathbf{x}, \mathbf{x}')$$

where in the last passage we used that $c_0 \leq 1$ and $d(\mathbf{x}, \mathbf{x}') \leq 1$. After summing all the terms together at the right scale, we finally show that

$$d(\tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}}(\mathbf{x}'), \tilde{\mathbf{m}}_{t,t_0}^{t,\mathbf{x}'}(\mathbf{x}')) \leq C c_0^{\frac{1}{1+\alpha(n-1)}} \|\mathbf{F}\|_H d(\mathbf{x}, \mathbf{x}')$$

thanks to convexity inequalities and $c_0 \leq 1$.

We conclude this section showing the reverse Taylor formula which was used in the proof of Proposition 17 in the diagonal regime to handle the discontinuity term:

Lemma 23 (Reverse Taylor Expansion). *Let γ be in $(1, 2)$, ϕ a function in $C_{b,d}^\gamma(\mathbb{R}^{nd})$ and \mathbf{x}, \mathbf{x}' two points in \mathbb{R}^{nd} . Then, there exists a constant $C := C(\gamma)$ such that*

$$|D_{\mathbf{x}_1} \phi(\mathbf{x}) - D_{\mathbf{x}_1} \phi(\mathbf{x}')| \leq C \|\phi\|_{C_{b,d}^\gamma} d^{\gamma-1}(\mathbf{x}, \mathbf{x}').$$

Proof. We start rewriting the left-hand side in the following way

$$\begin{aligned}
&D_{\mathbf{x}_1} \phi(\mathbf{x}) - D_{\mathbf{x}_1} \phi(\mathbf{x}') \\
&= \left(\int_0^1 D_{\mathbf{x}_1} \phi(\mathbf{x}) - D_{\mathbf{x}_1} \phi(\mathbf{x}_1 + \lambda d(\mathbf{x}, \mathbf{x}'), (\mathbf{x})_{2:n}) d\lambda \right) - \left(\int_0^1 D_{\mathbf{x}_1} \phi(\mathbf{x}') - D_{\mathbf{x}_1} \phi(\mathbf{x}_1 + \lambda d(\mathbf{x}, \mathbf{x}'), (\mathbf{x}')_{2:n}) d\lambda \right) \\
&\quad - \left(\int_0^1 D_{\mathbf{x}_1} \phi(\mathbf{x}_1 + \lambda d(\mathbf{x}, \mathbf{x}'), (\mathbf{x}')_{2:n}) - D_{\mathbf{x}_1} \phi(\mathbf{x}_1 + \lambda d(\mathbf{x}, \mathbf{x}'), (\mathbf{x})_{2:n}) d\lambda \right) =: I_1 + I_2 + I_3.
\end{aligned}$$

The first two components can be treated directly using that $D_{\mathbf{x}_1} \phi$ is in $C^{\gamma-1}(\mathbb{R}^d)$ with respect to the first non-degenerate variable. Indeed,

$$\begin{aligned}
|I_1| &\leq \int_0^1 |D_{\mathbf{x}_1} \phi(\mathbf{x}) - D_{\mathbf{x}_1} \phi(\mathbf{x}_1 + \lambda d(\mathbf{x}, \mathbf{x}'), (\mathbf{x})_{2:n})| d\lambda \\
&\leq C \|\phi\|_{C^\gamma} \int_0^1 |\lambda d(\mathbf{x}, \mathbf{x}')|^{\gamma-1} d\lambda \leq C \|\phi\|_{C^\gamma} d^{\gamma-1}(\mathbf{x}, \mathbf{x}')
\end{aligned}$$

and

$$\begin{aligned}
|I_2| &\leq \int_0^1 |D_{\mathbf{x}_1} \phi(\mathbf{x}') - D_{\mathbf{x}_1} \phi(\mathbf{x}_1 + \lambda d(\mathbf{x}, \mathbf{x}'), (\mathbf{x}')_{2:n})| d\lambda \\
&\leq C \|\phi\|_{C^\gamma} \int_0^1 |(\mathbf{x}' - \mathbf{x})_1 + \lambda d(\mathbf{x}, \mathbf{x}')|^{\gamma-1} d\lambda \leq C \|\phi\|_{C^\gamma} d^{\gamma-1}(\mathbf{x}, \mathbf{x}')
\end{aligned}$$

where in the last expression we used Young inequality.

To control the last term, we assume for the sake of brevity to be in the scalar case, i.e. $d = 1$. In the general

setting, the proof below can be reproduced component-wise. The idea is to use a reverse Taylor expansion to pass from the derivative to the function itself. Namely,

$$\begin{aligned}
|I_3| &= \frac{1}{d(\mathbf{x}, \mathbf{x}')} \left| \int_0^1 [\partial_\lambda \phi(\mathbf{x}_1 + \lambda d(\mathbf{x}, \mathbf{x}'), (\mathbf{x}')_{2:n}) - \partial_\lambda \phi(\mathbf{x}_1 + \lambda d(\mathbf{x}, \mathbf{x}'), (\mathbf{x})_{2:n})] d\lambda \right| \\
&\leq \frac{1}{d(\mathbf{x}, \mathbf{x}')} |\phi(\mathbf{x}_1 + d(\mathbf{x}, \mathbf{x}'), (\mathbf{x}')_{2:n}) - \phi(\mathbf{x}_1, (\mathbf{x}')_{2:n}) + \phi(\mathbf{x}_1 + d(\mathbf{x}, \mathbf{x}'), (\mathbf{x})_{2:n}) - \phi(\mathbf{x})| \\
&\leq C \|\phi\|_{C^\gamma} d^{\gamma-1}(\mathbf{x}, \mathbf{x}').
\end{aligned}$$

□

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