

Model reduction with NNM

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Overview

- 1 Introduction
- 2 Non-linear modes
- 3 Non-linear synthesis
- 4 Application
- 5 Conclusions

Introduction

- Non-linear vibrations with conservative non-linearities
- Periodic forces, periodic response
- Model reduction: decrease computational cost
- Illustration using two examples
 - Clamped-Clamped beam
 - Linear beam + NL attachment

Linear modal synthesis for nonlinear systems

- Forced Dynamic equations (n dof):

$$\mathbf{M} \ddot{\mathbf{u}}(t) + \mathbf{C} \dot{\mathbf{u}}(t) + \mathbf{K} \mathbf{u}(t) + \mathbf{f}^{nl}(\mathbf{u}(t)) = \mathbf{f}(t)$$
- Approximated solution: retain only linear mode shape, search for time evolutions

- basis Φ fixed, size $n \times r$: $\hat{\mathbf{u}}(t) = \sum_{i=1}^r \Phi_i \eta_i(t)$
- reduced dynamic equations

$$\underbrace{\mathbf{M}_r}_{\Phi^T \mathbf{M} \Phi} \ddot{\eta}(t) + \underbrace{\mathbf{C}_r}_{\Phi^T \mathbf{C} \Phi} \dot{\eta}(t) + \underbrace{\mathbf{K}_r}_{\Phi^T \mathbf{K} \Phi} \eta(t) + \underbrace{\mathbf{f}_r^{nl}(\eta(t))}_{\Phi^T \mathbf{f}^{nl}(\Phi \eta(t))} = \underbrace{\mathbf{f}_r(t)}_{\Phi^T \mathbf{f}(t)}$$

diagonal matrices reduced forces

- Coupled through nonlinearity

Linear modal synthesis for nonlinear systems

- Reduced dynamic equations
$$\mathbf{M}_r \ddot{\eta}(t) + \mathbf{C}_r \dot{\eta}(t) + \mathbf{K}_r \eta(t) + \mathbf{f}_r^{\text{nl}}(\eta(t)) = \mathbf{f}_r(t)$$
- Can be solved with HBM, time integration, ...
- If $r \ll n$, equations can be solved much quicker...
- ... BUT how do we choose the "good" modes to include ?
 - using static/dynamic non linear computations
 - using modal derivatives
 - using non linear modes (indirect way)
- Could we find a direct way of using NNM for reduction ?

$$\hat{\mathbf{u}}(t) = \underbrace{\boldsymbol{\Psi}_j(t)}_{\text{resonant NNM}} + \underbrace{\sum_{i \neq j} \boldsymbol{\Phi}_i \eta_i(t)}_{\text{other linear modes}}$$

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Non-linear Mode

- Free dynamic equations

$$\mathbf{M} \ddot{\mathbf{u}}(t) + \mathbf{K} \mathbf{u}(t) + \mathbf{f}^{nl}(\mathbf{u}(t)) = \mathbf{0} \quad (1)$$

- Computation with HBM

$$\begin{aligned} \mathbf{u}_{mnl}^n(t) &= \mathbf{a}_0^n + \sum_{k=1}^{N_h} \mathbf{a}_k^n \cos(k\omega t) + \mathbf{b}_k^n \sin(k\omega t) \\ &= \mathbf{a}_0^n + \sum_{k=1}^{N_h} \mathbf{c}_k^n \cos(k\omega t + \phi_k^n) \end{aligned}$$

- Monophase NNM, i.e all dof goes to zero at the same time ($\phi_k^n = 0$):

$$\mathbf{u}_{mnl}^n(t) = \mathbf{a}_0^n + \sum_{k=1}^{N_h} \mathbf{c}_k^n \cos(k\omega t)$$

Non-linear Mode

- Time translation

$$\mathbf{u}_{mnl}^n(t + \tau) = \mathbf{a}_0^n + \sum_{k=1}^{N_h} \mathbf{c}_k^n \cos(k\omega t + k\phi)$$

- Parametrisation w.r.t control amplitude and phase

$$\Psi(q, \phi)(t) = q \left[\Psi_0(q) + \sum_{k=1}^{N_h} \Psi_k^c(q) \cos(k\omega t + k\phi) \right]$$

- Nonlinear synthesis: find q_j, ϕ_j ?

Example: Clamped-Clamped Beam

- Clamped-clamped beam with Euler Bernoulli kinematic
- Green Lagrange nonlinear strain \implies Von Karman hypothesis

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \sum_k \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \implies \epsilon_{xx} = u' + \frac{1}{2} (v')^2$$

- Deformation energy

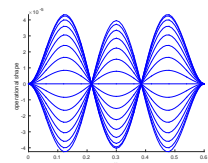
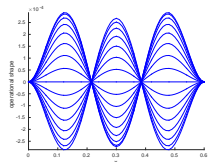
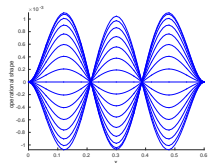
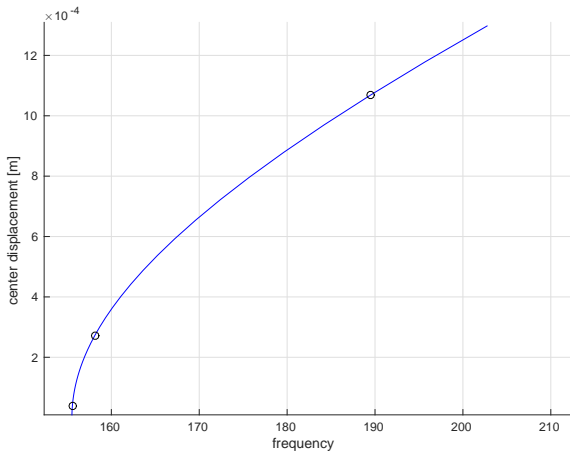
$$U = \frac{1}{2} \int_0^L ES(u')^2 + EI(v'')^2 dx + \frac{1}{2} \int_0^L ES \left[u' + \frac{1}{2} (v')^2 \right] \frac{1}{2} (v')^2 dx$$

- Finite element formulation (3 DOF per node (u_n, v_n, θ_n))

$$\mathbf{M} \ddot{\mathbf{u}} + \mathbf{C} \dot{\mathbf{u}} + \mathbf{K} \mathbf{u} + \underbrace{\mathbf{K}_2(\mathbf{u}) \mathbf{u} + \mathbf{K}_3(\mathbf{u}, \mathbf{u}) \mathbf{u}}_{\mathbf{f}^{\text{nl}}(\mathbf{u})} = \mathbf{f}$$

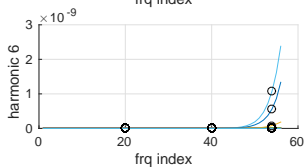
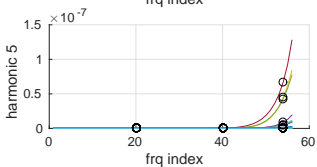
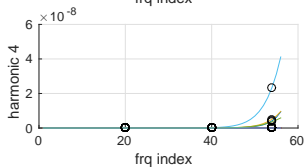
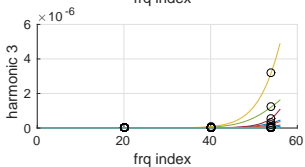
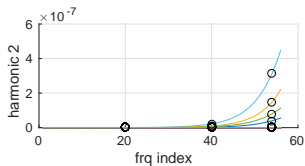
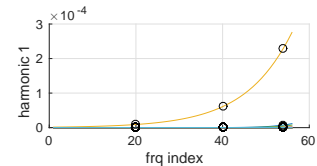
Example: NNM computation

NNM 3. Computed with HBM (constant term + 7 harmonics)



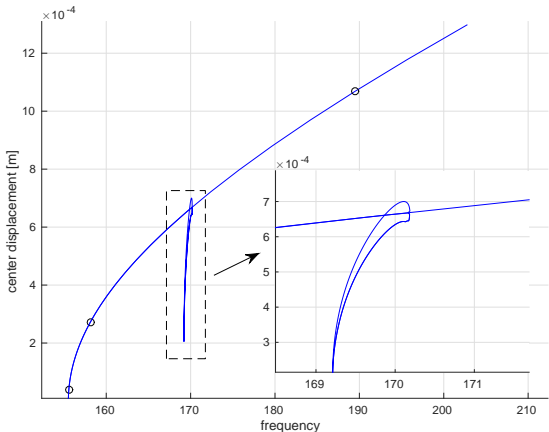
Example: NNM analysis

nnm analysis: what is inside in term of linear mode participation ?



Example: NNM analysis

- Bifurcation/internal resonances not taken into account (parametrisation problem: not always a bijection)



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Non-linear synthesis

- Forced Dynamic equations

$$\mathbf{M} \ddot{\mathbf{u}}(t) + \mathbf{C} \dot{\mathbf{u}}(t) + \mathbf{K} \mathbf{u}(t) + \mathbf{f}^{\text{nl}}(\mathbf{u}(t)) = \mathbf{f}(t) \quad (2)$$

- Forced solution approximation: single NNM - multi LNM
(retain modal shape + time evolution from NNM)

$$\hat{\mathbf{u}}(t) = \underbrace{\boldsymbol{\Psi}_j(q, \phi)(t)}_{\text{resonant NNM}} + \underbrace{\sum_{i \neq j} \boldsymbol{\Phi}_i \eta_i(t)}_{\text{other linear modes}}$$

- Decoupling hypothesis:
 - $\eta_i(t)$ are obtained from a linear resolution
 - the parameters (q, ϕ) are obtained using a variational method

Variational methods

- Forced dynamic equations

$$\mathbf{M} \ddot{\mathbf{u}} + \mathbf{C} \dot{\mathbf{u}} + \mathbf{K} \mathbf{u} + \mathbf{f}^{\text{nl}}(\mathbf{u}) = \mathbf{f}$$

- Variational formulation: \mathbf{u} solution $\iff \forall \delta \mathbf{u}$ we have:

$$\int_0^T \delta \mathbf{u}^T [\mathbf{M} \ddot{\mathbf{u}} + \mathbf{C} \dot{\mathbf{u}} + \mathbf{K} \mathbf{u} + \mathbf{f}^{\text{nl}}(\mathbf{u}) - \mathbf{f}] dt = 0$$

- Single non-linear mode approximation and its variation

$$\hat{\mathbf{u}}(t) = \boldsymbol{\Psi}_j(q, \phi)(t), \quad \delta \hat{\mathbf{u}} = \frac{\partial \boldsymbol{\Psi}_j}{\partial q} \delta q + \frac{\partial \boldsymbol{\Psi}_j}{\partial \phi} \delta \phi$$

- Resonant NNM equations

$$\delta q \int_0^T \frac{\partial \boldsymbol{\Psi}_j}{\partial q}{}^T [\mathbf{M} \ddot{\boldsymbol{\Psi}}_j + \mathbf{C} \dot{\boldsymbol{\Psi}}_j + \mathbf{K} \boldsymbol{\Psi}_j + \mathbf{f}^{\text{nl}}(\boldsymbol{\Psi}_j) - \mathbf{f}] dt = 0$$

$$\delta \phi \int_0^T \frac{\partial \boldsymbol{\Psi}_j}{\partial \phi}{}^T [\mathbf{M} \ddot{\boldsymbol{\Psi}}_j + \mathbf{C} \dot{\boldsymbol{\Psi}}_j + \mathbf{K} \boldsymbol{\Psi}_j + \mathbf{f}^{\text{nl}}(\boldsymbol{\Psi}_j) - \mathbf{f}] dt = 0$$

Numeric considerations

- Two algebraic equations $\mathbf{G}(q, \phi, \omega) = \mathbf{0}$:

$$\int_0^T \frac{\partial \Psi_j}{\partial (q, \phi)}^T [\mathbf{M} \ddot{\Psi}_j + \mathbf{C} \dot{\Psi}_j + \mathbf{K} \Psi_j + \mathbf{f}^{nl}(\Psi_j) - \mathbf{f}] dt = 0$$

- From the HBM, $\Psi_j(q, \phi)$ is computed only at a discrete number of points q_s
- Use linear interpolation between HBM points
- Recall $\Psi_j(q, \phi)(t) = q[\Psi_0(q) + \sum_{k=1}^{N_h} \Psi_k^c(q) \cos(k\omega t + k\phi)]$
 - $\frac{\partial \Psi_j}{\partial q}$ computed with finite difference
 - $\frac{\partial \Psi_j}{\partial \phi}$ computed analytically
- In frequency domain (Fourier basis), no integrals to compute, only matrix vector product

Sum up

- Compute and parametrise $\Psi_j(q, \phi)$ with HBM
- Forced solution approximation

$$\hat{\mathbf{u}}(t) = \underbrace{\Psi_j(q, \phi)(t)}_{\text{resonant NNM}} + \underbrace{\sum_{i \neq j} \Phi_i \eta_i(t)}_{\text{other linear modes}}$$

- parameters q and ϕ obtained by solving two algebraic equations

$$\mathbf{G}(q, \phi, \omega) = \mathbf{0}$$

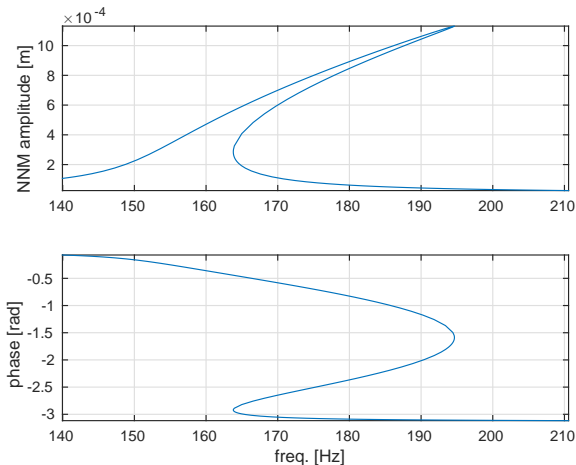
- $\eta_i(t)$ are obtained from a linear resolution (lin. post.)

$$m_i \ddot{\eta}_i + c_i \dot{\eta}_i + k_i \eta_i - \Phi_i^T \mathbf{f} = 0$$

- FRF are generated using continuation procedures

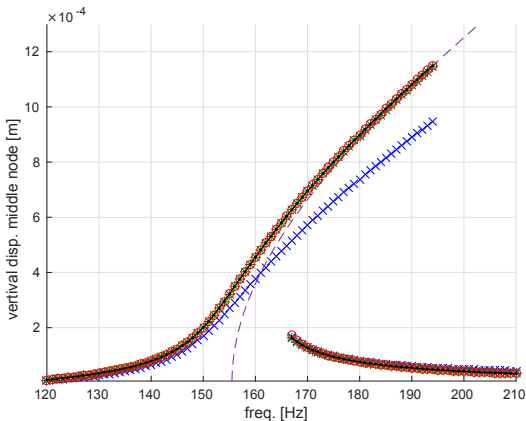
Example: (CC beam) synthesis around third resonance

- Periodic vertical force (at the middle of the beam)
- Amplitude (q) and phase (ϕ) of the NNM vs frequency



Example: (CC beam) comparison with LM reduction

- Reference solution: HBM with 7 harmonics
- Using sequential continuation for comparison purpose
 - HBM, **x** red LM 1, ***** red LM 5T4I, **o** red NNM

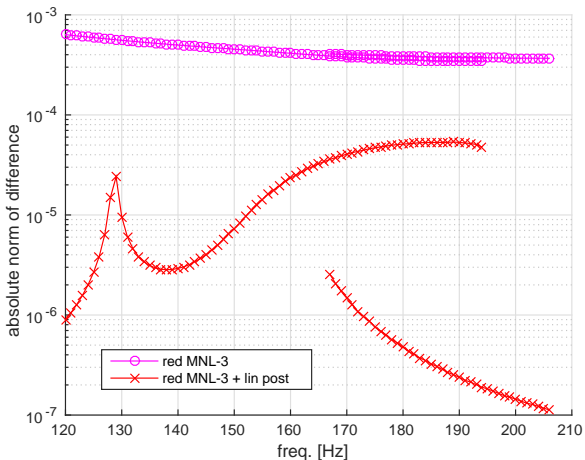


Comparison

- red. NNM 1:
2 alg. eqs
- red. LM 1:
1 Equa. diff. NL
= 15 alg. eqs
- red. LM 5T4I:
9 Equa. diff. NL
= 125 alg. eqs

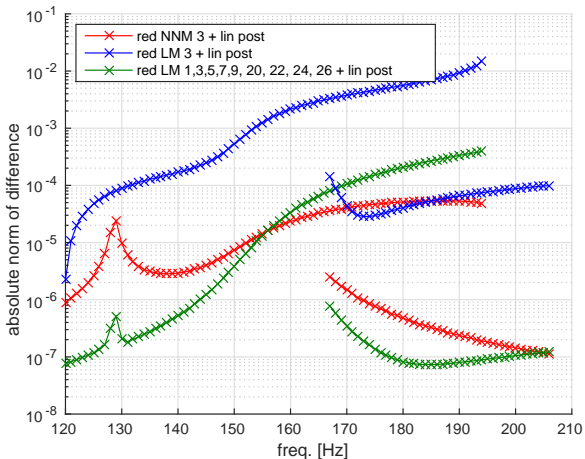
Example: (CC beam) comparison with LM reduction

- absolute error $\epsilon^2 = \int_t \|\mathbf{u} - \mathbf{u}_{ref}\|^2 dt$
- effect of "linear post process"



Example: (CC beam) comparison with LM reduction

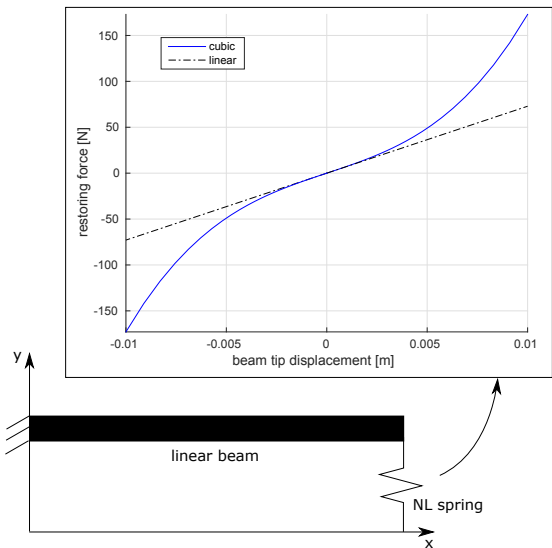
- absolute error $\epsilon^2 = \int_t \|\mathbf{u} - \mathbf{u}_{ref}\|^2 dt$
- comparison with LM reduction



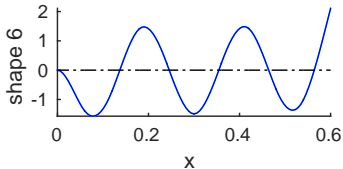
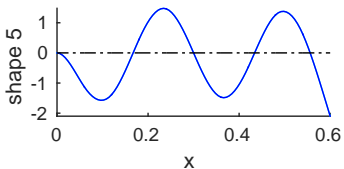
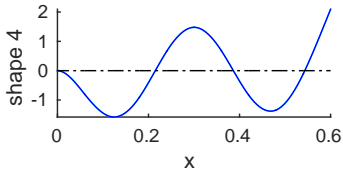
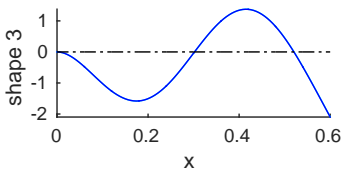
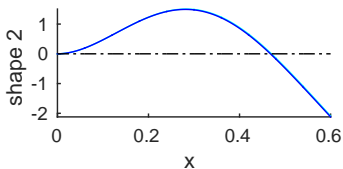
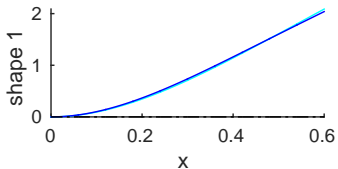
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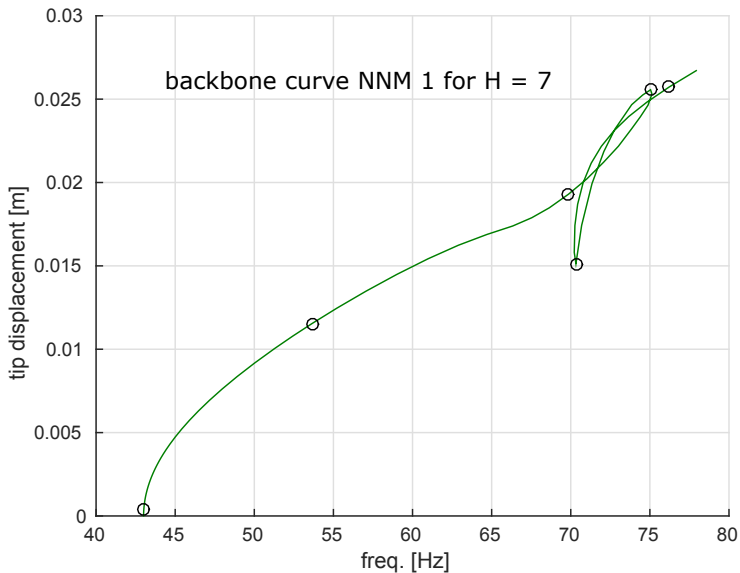
Exemple description



Exemple description

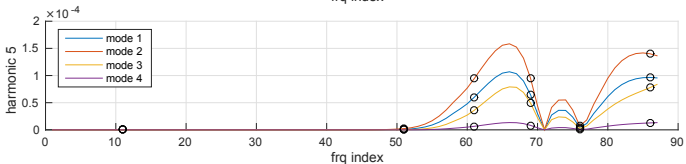
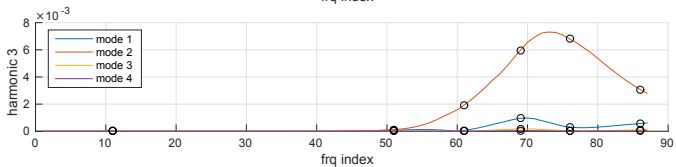
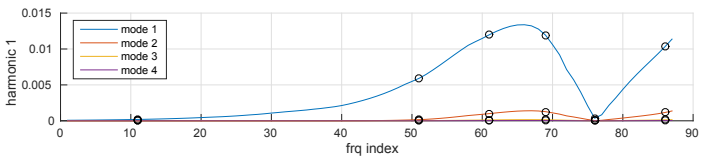


Nonlinear mode computation and analysis



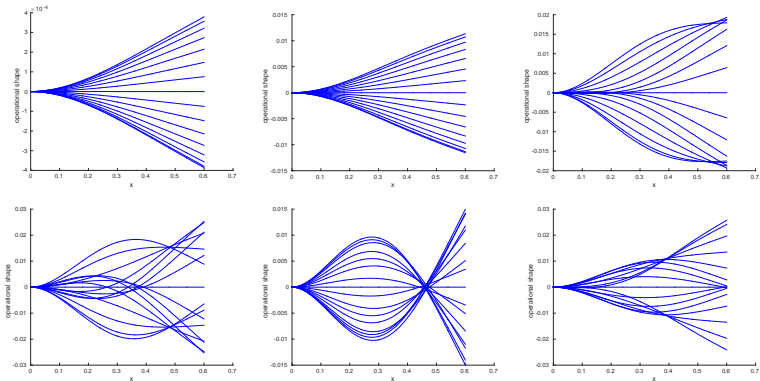
Nonlinear mode computation and analysis

- Projection of the NNM harmonics onto the linear modes
- Internal resonance 1:3 (LM 1- LM 2)



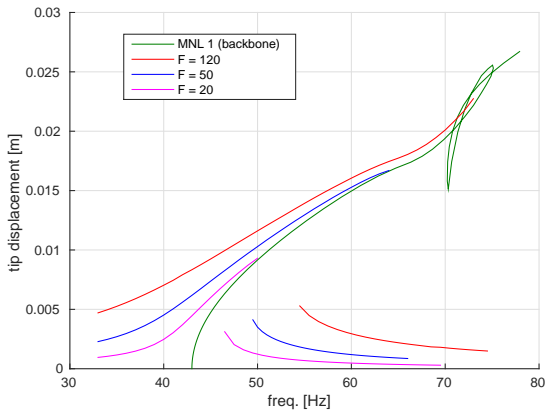
Nonlinear mode computation and analysis

- Mode evolution over one half period (sampling $T/32$)



Non-linear synthesis

- ref. solutions (HBM, 7H) for different force level
(20 NL diff. Eqs. = 300 alg. eqs)

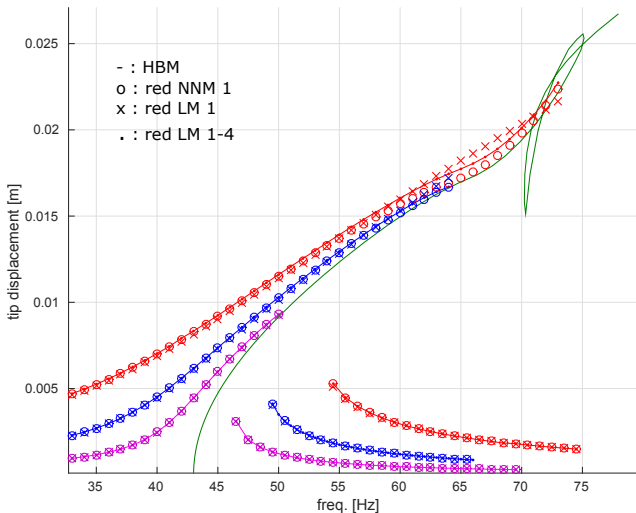


Comparison

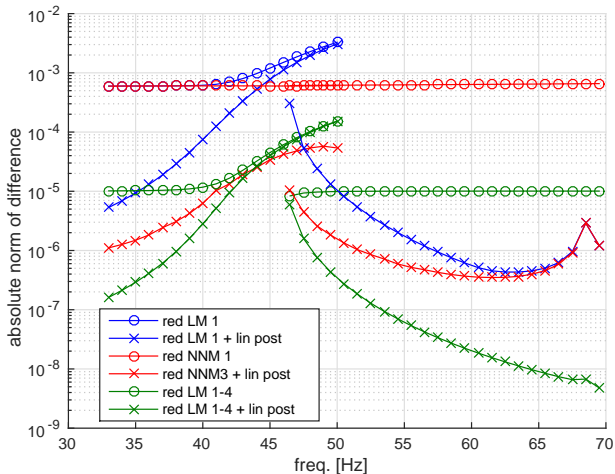
- red. LM 1:
2 alg. eqs
- red. NNM 1:
1 Equa diff NL
= 15 alg. eqs
- red. LM 1 to 4:
4 Equa diff NL
= 60 alg. eqs

Non-linear synthesis

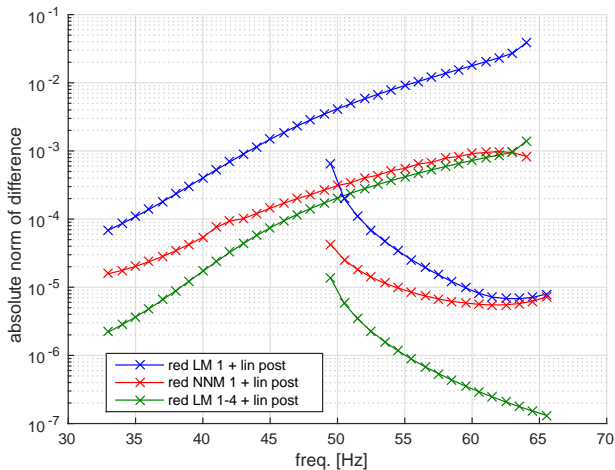
- comparison of FRF (maximum tip displacement vs frequency)



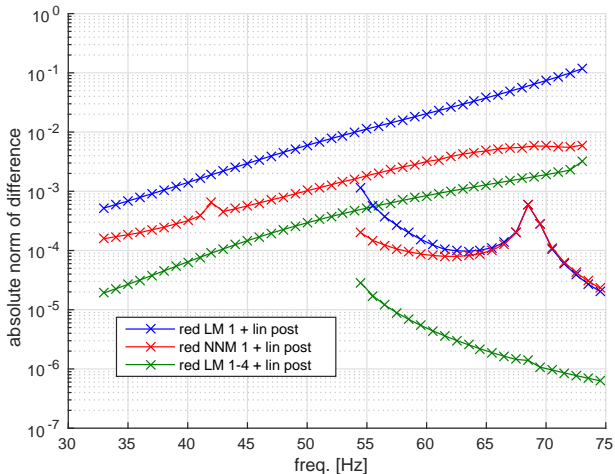
- absolute error $\epsilon^2 = \int_t \|\mathbf{u} - \mathbf{u}_{ref}\|^2 dt$
- low amplitude forcing: $F = 20$, effect of "linear post process"



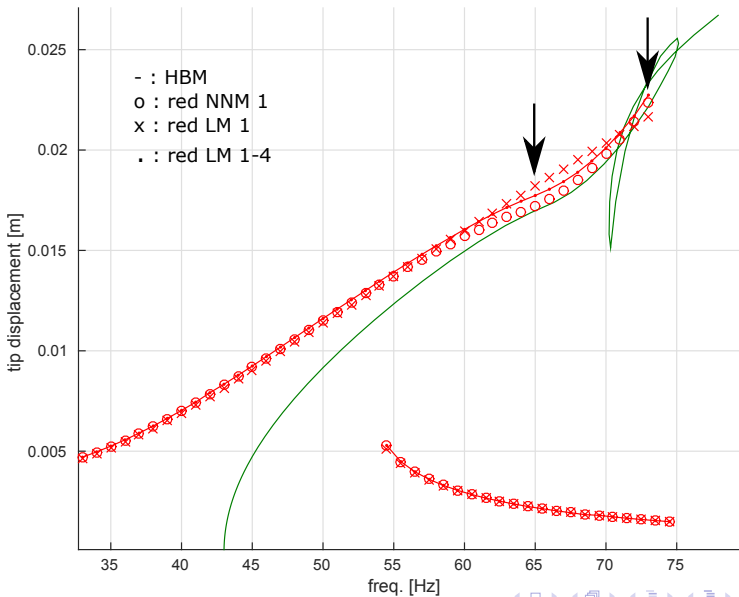
- absolute error $\epsilon^2 = \int_t \|\mathbf{u} - \mathbf{u}_{ref}\|^2 dt$
- mid amplitude forcing: $F = 50$



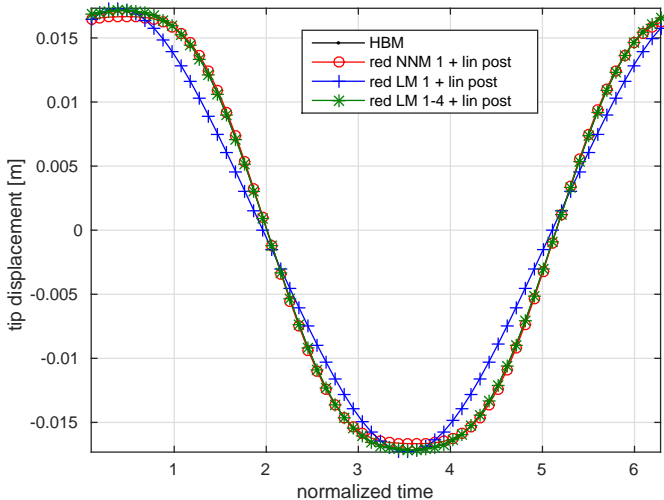
- absolute error $\epsilon^2 = \int_t \|\mathbf{u} - \mathbf{u}_{ref}\|^2 dt$
- high amplitude forcing: $F = 120$



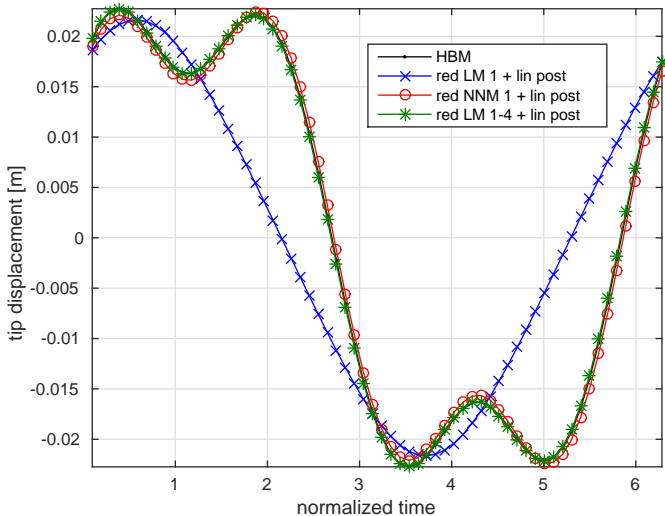
● Comparison of time evolution



- comparison tip displacement before resonance $F = 120$



- comparison tip displacement at resonance $F = 120$



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Conclusion

- Simple non-linear synthesis based on non-linear modes
- Take into account space and time variation (at the same time)
- Only two algebraic equations to solve "on-line"
- Generate very good results as compared to the full model
- BUT... some open questions
- collateral problem: how to evaluate the reduced forces without going to physical coordinates ?
- extend to travelling waves solutions (not unisson)
- ...

Model reduction with NNM

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variational methods: approximation

coupled version of the variational equations resonant NNM equations (2 algebraic equations, variable: (q, ϕ))

$$\int_0^T \frac{\partial \Psi_j}{\partial (q, \phi)}^T [\mathbf{M} \ddot{\Psi}_j + \mathbf{C} \dot{\Psi}_j + \mathbf{K} \Psi_j + \mathbf{f}^{nl}(\Psi_j + \Phi)] + \frac{\partial \Psi_j}{\partial (q, \phi)}^T [\mathbf{M} \ddot{\Phi} + \mathbf{C} \dot{\Phi} + \mathbf{K} \Phi] - \frac{\partial \Psi_j}{\partial (q, \phi)}^T \mathbf{f} dt = 0$$

other modes equations ($n - 1$ differential equations, variable $\eta_i(t)$)

$$\Phi_i^T [\mathbf{M} \ddot{\Psi}_j + \mathbf{C} \dot{\Psi}_j + \mathbf{K} \Psi_j + \mathbf{f}^{nl}(\Psi_j + \Phi)] + m_i \ddot{\eta}_i + c_i \dot{\eta}_i + k_i \eta_i - \Phi_i^T \mathbf{f} = 0$$