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On a model reduction method for computing forced response using non-linear normal modes

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Abstract

This paper presents a method for constructing reduced order models using non-linear normal modes (NNM) in the context of non-linear vibrations. Starting from a discretised version of the non-linear problem, the non-linear normal modes (NNM) of the structure are computed using the Harmonic Balance Method (HBM). A two parameters (amplitude and phase) parametrization of the NNM is introduced and they are then used for the forced response construction, assuming that the solution involves only a single (non-linear) resonant mode. The solution is then eventually corrected by linear terms which helps decreasing the error outside resonance, in particular around anti-resonances. The procedure results in two algebraic equations containing only two variables, one controlling the amplitude of vibration and the other controlling the phase, thus leading to a drastic reduction in the number of degrees of freedom. The procedure is illustrated on a simple, but representative, example. It is shown that a single mode approximation is sufficient for computing a good approximation of the forced response around a particular mode.

Introduction

This work deals with the vibration study of non-linear structures with conservative non-linearities, such as geometric nonlinearity. In order to compute the response of nonlinear systems, the structure is generally discretized using Finite Element Model (FEM)[26], leading to a set of non-linear differential equations describing the time evolution of the degrees of freedom (dof). The resolution of such non-linear equations relies most of the time on iterative procedures based on the Newton-like algorithms. For systems with large number of dof, the resolution procedure can become very time consuming, and may even not be realized in a reasonable time for non-linear industrial structures. In order to reduce the computation time, it has been proposed to build reduced models which allow

to represent the dynamics of the system with a reduced number of degrees of freedom.

One of the most used reduction method, is the Galerkin projection which consist in constructing a reduction basis, and then to express the movement of the full system as a linear combination of the reduction basis vectors. The motion equations are then projected on a projection basis, which leads to a set of reduced non-linear differential equations giving the evolution of the reduced dof. In general, the projection basis can be different from the reduction basis (Petrov-Galerkin Projection), but, in practice, most of the studies choses a projection basis equals to the reduction basis (Galerkin projection). The construction of the reduction basis can be realized either *a priori* or *a posteriori*. In the first case, the reduction basis is constructed using only the available information on the structural system (eg. modal synthesis [25], modal derivative [24, 10], Krylov subspace [4, 5], ...). In the second case, the reduction basis is extracted from data collected from numerical simulations, or experiments (POD [9], compagnon/dual modes [8],...).

Even if the Galerkin projection method can gives good results, it is not always clear how to select the vectors to be retained in the reduction basis. Indeed, the non-linearities can induce a coupling between several modes of vibration, and thus all those "coupled" modes have to be selected in the reduction basis.

In order to directly take into account the modal coupling induced by the nonlinearities, some authors have proposed to reduce the system using Non-linear Normal Modes (NNM). NNM can be view as an extension of the linear normal modes for nonlinear dynamical systems. Several definitions are available depending on the type of considered solution. ([14, 22, 23]). Using the idea of the center manifold theorems, the Shaw and Pierre definition [23] has been used, for example, to propose reduced order models of nonlinear dynamic systems (see e.g.[2, 21]). Another closely related way of reducing nonlinear system is through the use of the normal form theory, either in the first [13, 7] or second [18] order formalism. More recently, a method based on the use of complex normal mode for constructiong reduced order models of systems with friction has been presented in [15].

In this paper, we propose a new model reduction approach based on the use of NNM computed with HBM for reducing structural systems with conservative non-linearities. The originality of the method rely on considering the spatio-temporal evolution of the non-linear mode, whereas most of the reduction technique presented in the introduction only retains the the spatial distribution.

In a first section we will briefly recall the use of the harmonic balance method for computing solutions of non-linear vibration problems. Next, we will expose the definition of NNM used in this paper and define the parametrization. Finally, the model reduction procedure will be exposed and we will demonstrate its efficiency on numerical examples.

1 Non-linear vibration and Harmonic Balance Method (HBM)

In this paper, the HBM will be used to compute the nonlinear modes and reference solutions. In this section, one briefly recall the main steps of the HBM for computing periodic solutions of non-linear differential equations.

Let's consider a structural system with conservative non-linearities. After discretization (n dof), the motion equations are given by a set of n non-linear differential equations: Forced dynamic equations

$$\mathbf{M} \ddot{\mathbf{u}} + \mathbf{C} \dot{\mathbf{u}} + \mathbf{K} \mathbf{u} + \mathbf{F}^{nl}(\mathbf{u}) = \mathbf{F} \quad (1)$$

where $\mathbf{u}(t)$ is a vector containing the time evolution of the degrees of freedom, \mathbf{M} , \mathbf{C} and \mathbf{K} are respectively the mass, damping and stiffness matrices, $\mathbf{F}(t)$ is the excitation force vector which is assumed to be periodic with period $T = \frac{\omega}{2\pi}$, and finally $\mathbf{F}^{nl}(\mathbf{u})$ is the vector of non-linear forces (supposed to depends only on the position).

The HBM consists in searching for an approximated solution $\hat{\mathbf{u}}$ in the form of a truncated Fourier series up to the H -th harmonic, as follows:

$$\hat{\mathbf{u}}(t) = \mathbf{a}_0 + \sum_{k=1}^H \mathbf{a}_k \cos(k\omega t) + \mathbf{b}_k \sin(k\omega t) \quad (2)$$

where \mathbf{a}_k and \mathbf{b}_k are the vectors of Fourier coefficients associated with harmonic k . Substituting Eq.(2) into Eq.(1) and projecting over the truncated Fourier basis results is a set of non-linear algebraic equations which can be written as:

$$\mathbf{Z}(\omega) \tilde{\mathbf{u}} + \tilde{\mathbf{F}}_{nl}(\tilde{\mathbf{u}}) - \tilde{\mathbf{F}} = 0 \quad (3)$$

where $\tilde{\mathbf{u}} = [\mathbf{a}_0^T, \mathbf{a}_1^T, \mathbf{b}_1^T, \dots, \mathbf{a}_H^T, \mathbf{b}_H^T]^T$ is a vector of unknown of size $n_h = n(2H + 1)$, and \mathbf{Z} is the matrix of dynamic stiffness defined block-diagonally as $\mathbf{Z} = \text{blkdiag}(\mathbf{K}, (\mathbf{Z}_k)_{1 \leq k \leq H})$ where \mathbf{Z}_k is given by:

$$\mathbf{Z}_k = \begin{bmatrix} \mathbf{K} - (k\omega)^2 \mathbf{M} & k\omega \mathbf{C} \\ -k\omega \mathbf{C} & \mathbf{K} - (k\omega)^2 \mathbf{M} \end{bmatrix} \quad (4)$$

$\tilde{\mathbf{F}}$ and $\tilde{\mathbf{F}}_{nl}(\tilde{\mathbf{u}})$ correspond respectively to the excitation force and non-linear force in the frequency domain.

For an imposed frequency ω , the solutions of the algebraic system in Eq.(3) can be computed using Newton like algorithms. For different forcing frequencies, the solution can be computed by applying continuation procedures, see e.g.[12, 17].

2 Non-linear Normal Modes

Here we recall the definition of Non-linear Normal Modes (NNM) that will be used through this paper, and the numerical procedure used to computed them.

2.1 Definition

In this paper, we consider the computation of NNM of structural system with conservative nonlinearities such as Eq.(1). Several definition of NNM are available, here we will only use the Rosenberg's definition of NNM [22], and one can refer to [14, 20] for further detail about NNM definition and computation.

In Rosenberg's definition, the NNM are the "vibration in unisson" of the undamped, unforced system of equation. This definition correspond to the fact that for a motion on a NNM:

- (i) all dof goes to zero (resp. to their extremum) at the same time,
- (ii) all dof vibrate at the same frequency,
- (iii) the displacement of each dof can be parametrized by the displacement of a unique reference dof.

The Rosenberg definition of NNM is very simple and puts some restrictions on the kind of computed NNM. The first restriction of this definition is the fact that NNM are solutions of undamped systems. So that using this definition, one needs to drop all damping terms (frictions, structural damping, ...) before computing NNM. Another restriction of the Rosenberg definition is the fact that all dof should vibrate in unisson, implying that there can be no phase shift between dof [point (i)]. In other words, all dofs have to be perfectly tuned, and solutions such as travelling waves (solutions with a constant space shift between two contiguous dof) do not fall into this definition of NNM. Travelling wave will not be considered in this study. However, the procedure presented in this paper could be extended to travelling waves in a quite straight forward manner (provided one has computed the travelling NNM).

2.2 Computation

NNM will be sought as the solution of the undamped unforced equation:

$$\mathbf{M} \ddot{\mathbf{u}} + \mathbf{K} \mathbf{u} + \mathbf{f}^{\text{nl}}(\mathbf{u}) = \mathbf{0} \quad (5)$$

Here, the HBM will be used to compute the NNM of Eq.(5). In this case the solution is again sought under the form of the truncated Fourier series in Eq.(2), but the angular frequency ω is also an unknown now.

Applying the HBM, leads to an under-determined set of $n_h = (2H + 1)n$ algebraic equations for $n_h + 1$ unknowns. This indetermination is due to the absence of initial conditions and from invariance by time translation of the unforced, undamped equation. That is if $\tilde{\mathbf{u}}$ is a solution of Eq.(5) then the time shifted signal $\tilde{\mathbf{u}}_\tau$ is also a solution of Eq.(5) for all $\tau \in \mathbb{R}$.

For each dof n , the HBM approximation can be re - written as

$$\hat{u}^n(t) = a_0^n + \sum_{k=1}^H a_k^n \cos(k\omega t) + b_k^n \sin(k\omega t) = a_0^n + \sum_{k=1}^H c_k^n \cos(k\omega t + \phi_k^n) \quad (6)$$

where c_k^n and ϕ_k^n are the amplitude and phase depending in the dof n and the harmonic k .

A new equation, termed phase equation [19, 3], is thus added to the n_h algebraic equations in order to set the phase of the solution. Most of the time, the phase equation is constructed such that the velocities of one of the dof is set to 0 at $t = 0$: $\dot{\mathbf{x}}_{i_0}(t = 0) = 0$. In the case of the HBM, this phase condition turns to the following algebraic equation:

$$\sum_{k=1}^H k \mathbf{b}_k^{i_0} = 0 \quad (7)$$

As we are using Rosenberg's definition of NNM, all dof must go to their maximum/zeros at the same time. A simple choice to ensure this property is to set each phase ϕ_k^n to zero for each dof and each harmonics: $\phi_k^n = 0, \forall k, n$. This case corresponds to the the case of the so called monophasic NNM wich was also studied by M.Peeters in [19].

The HBM approximation, for the mono-phase NNM, then reduces to the following:

$$\hat{\mathbf{u}}(t) = \mathbf{a}_0 + \sum_{k=1}^H \mathbf{a}_k \cos(k\omega t) \quad (8)$$

In other word, only the cosine terms will be retained in the computation of the NNM, i.e we search for solutions that are even in time ($\mathbf{u}(-t) = \mathbf{u}(t)$)

Using the approximation in Eq.(8), and applying the HBM projection, one gets a set of $n_{nnm} = (H + 1)n$ algebraic equations for $n_{nnm} + 1$ unknowns \mathbf{a}_k , $k = 0, \dots, H$ and ω , which can be put under the following form:

$$\mathbf{Z}(\omega)\tilde{\mathbf{u}} + \tilde{\mathbf{F}}_{nl}(\tilde{\mathbf{x}}) = 0 \quad (9)$$

with $\tilde{\mathbf{u}} = [\mathbf{a}_0^T, \mathbf{a}_1^T, \dots, \mathbf{a}_H^T]^T$, and $\mathbf{Z} = \text{blkdiag}(\mathbf{K}, (\mathbf{K} - \omega^2\mathbf{M}), \dots, \mathbf{K} - (H\omega)^2\mathbf{M})$, where "blkdiag" is an operator that constructs a block diagonal matrix from its inputs.

Equation (9) is solved using continuation techniques such as the arc-length continuation. At the first iteration, the initial frequency for the Newton-Raphson procedure is taken from the frequency of the linearised system, and the form of the first harmonic \mathbf{a}_1 is initialized with the corresponding linear mode shape (the NNM are tangent to the LNM at low amplitude). Then the arc-length continuation procedure is applied in order to compute the NNM for further amplitude.

For the forced response computation two parameters will be used to "tune" the non-linear mode with respect to the excitation: the amplitude of a control dof and the phase. Here we will simply choose the control parameter as the amplitude $q \in \mathbb{R}$ of the first harmonic of a reference dof m_0 : $q = \mathbf{a}_1^{m_0}$. The NNM will be represented as follows:

$$\hat{\mathbf{u}}_{NNM} = q \left[\frac{\mathbf{a}_0}{q} + \sum_{k=1}^H \frac{\mathbf{a}_k}{q} \cos(k\omega t) \right] = q \left[\Psi_0(q) + \sum_{k=1}^H \Psi_k(q) \cos(k\omega t) \right] \quad (10)$$

Where, by construction, we have $\Psi_1^{m_0}(q) = 1$. Note that other normation condition could be used, such as norming the first harmonic shape w.r.t the mass matrix (see e.g. [15]). The choice of the control dof m_0 is important, in particular its amplitude q should not go to zero when varying the continuation parameter ω .

The parametrisation w.r.t. the phase ϕ is simply re-introduced by considering the effect of a (global) time translation on the HBM solution:

$$\hat{\mathbf{u}}_{NNM}(t + \tau) = \mathbf{a}_0 + \sum_{k=1}^H \mathbf{a}_k \cos(k\omega t + k \underbrace{\omega\tau}_{\phi}) \quad (11)$$

The j -th NNM can therefore be represented as:

$$\Psi_j(q, \phi)(t) = \Psi_0(q) + \sum_{k=1}^H \Psi_k(q) \cos(k\omega t + k\phi) \quad (12)$$

This representation has the advantage that the position on the NNM backbone is controlled by only two parameters q and ϕ . Also, note that not only the spacial shape but also the time evolution of the NNM is kept into account using this parametrisation, this will be a key point of the reduction procedure that follows.

3 Model order reduction using Non-linear Normal Modes

3.1 Single-mode reduction

We now turn to the resolution of the damped and forced equation (1) by means of NNM reduction. A single mode approximations is considered around the main resonance of the j -th NNM, so that an approximated solution $\hat{\mathbf{u}}$ of Eq.(1) is sought under the following form:

$$\hat{\mathbf{u}}(t) = \Psi_j(q, \phi)(t) \quad (13)$$

where $\Psi_j(q, \phi)(t)$ is the resonant NNM. Note that this approximation allows to retain the spatial and temporal feature of the NNM at the same time. The amplitude and phase of the resonant NNM are obtained using a variational method. First the matrix problem in Eq.(1) is transformed into the following (space-time) weak formulation:

$$\int_0^T \delta \mathbf{u}^T [\mathbf{M} \ddot{\mathbf{u}} + \mathbf{C} \dot{\mathbf{u}} + \mathbf{K} \mathbf{u} + \mathbf{F}^{nl}(\mathbf{u}) - \mathbf{F}] dt = 0, \quad \forall \delta \mathbf{u} \quad (14)$$

Then the single NNM approximation and its variation are computed as follows:

$$\hat{\mathbf{u}}(t) = \Psi_j(q, \phi)(t), \quad \delta \hat{\mathbf{u}} = \frac{\partial \Psi_j}{\partial q} \delta q + \frac{\partial \Psi_j}{\partial \phi} \delta \phi \quad (15)$$

and they are substituted into the weak formulation, resulting in two algebraic equations for the amplitude and phase (q, ϕ) of the resonant NNM:

$$\delta q \int_0^T \frac{\partial \Psi_j}{\partial q}{}^T [\mathbf{M} \ddot{\Psi}_j + \mathbf{C} \dot{\Psi}_j + \mathbf{K} \Psi_j + \mathbf{F}^{nl}(\Psi_j) - \mathbf{F}] dt = 0 \quad (16)$$

$$\delta \phi \int_0^T \frac{\partial \Psi_j}{\partial \phi}{}^T [\mathbf{M} \ddot{\Psi}_j + \mathbf{C} \dot{\Psi}_j + \mathbf{K} \Psi_j + \mathbf{F}^{nl}(\Psi_j) - \mathbf{F}] dt = 0 \quad (17)$$

The two equations in (16)-(17) can simply be written as $\mathbf{G}(q, \phi, \omega) = 0$. Their resolution for a fixed excitation frequency ω then gives the values for the variable q and ϕ . Continuation procedure, can be applied in order to extend the solution for further frequencies as described in the next section.

3.2 Numeric considerations

Let's consider the system in Eq.(16),(17) we have to evaluate the derivative of the NNM with relation to phase and amplitude. From the parametrisation of the NNM in Eq.(12), the computation of the derivative with respect to q gives:

$$\frac{\partial \Psi_j}{\partial q} = \Psi_0 + q \frac{\partial \Psi_0}{\partial q} + \sum_{k=1}^{N_h} [\Psi_k + q \frac{\partial \Psi_k}{\partial q}] \cos(k\omega t + k\phi) \quad (18)$$

and the derivative with respect to the phase is given by the following:

$$\frac{\partial \Psi_j}{\partial \phi} = -q \left[\sum_{k=1}^{N_h} k \Psi_k \sin(k\omega t + k\phi) \right] \quad (19)$$

Plugging equations (12),(18) and (19) into the variational equations (16)-(17) and computing the integrals lead to the following form for the algebraic equation of the resonant NNM:

$$\begin{aligned} & [\Psi_0 + q \frac{\partial \Psi_0}{\partial q}]^T [q \mathbf{K} \Psi_0 + \tilde{\mathbf{F}}_{nl}^0 - \tilde{\mathbf{F}}_{ex}^0] \\ & + \sum_k [\Psi_k + q \frac{\partial \Psi_k}{\partial q}]^T [q [\mathbf{K} - (k\omega)^2 \mathbf{M}] \Psi_k \\ & \quad + \tilde{\mathbf{F}}_{nl}^{k,c} \cos(k\phi) - \tilde{\mathbf{F}}_{nl}^{k,s} \sin(k\phi) \\ & \quad - (\tilde{\mathbf{F}}_{ex}^{k,c} \cos(k\phi) - \tilde{\mathbf{F}}_{ex}^{k,s} \sin(k\phi))] = 0 \end{aligned} \quad (20)$$

$$\begin{aligned} & -q \sum_{k=1}^{N_h} k \Psi_k^T [-q k \omega \mathbf{C} \Psi_k \\ & \quad + \tilde{\mathbf{F}}_{nl}^{k,c} \sin(k\phi) - \tilde{\mathbf{F}}_{nl}^{k,s} \cos(k\phi) \\ & \quad - (\tilde{\mathbf{F}}_{ex}^{k,c} \sin(k\phi) - \tilde{\mathbf{F}}_{ex}^{k,s} \cos(k\phi))] = 0 \end{aligned} \quad (21)$$

The two algebraic equations (20),(21) resulting from the variational procedure are defined in a continuous manner over q and ϕ . However, the continuation procedure used to compute the NNM only gives outputs $\Psi_j(q, \phi)$ at a discrete number of point q_s . Therefore, for a given q a linear interpolation of the harmonic coefficients Ψ_k (and their derivatives $\frac{\partial \Psi_k}{\partial q}$) is computed between the two closest points of the (discrete) NNM results.

3.3 Forced response computation

For the forced response computation the two algebraic equations $\mathbf{G}(q, \phi, \omega)$ in Eqs.(16),(17) (or equivalently Eqs.(20),(21)) must be solve for different values of the excitation frequency. In order to account for possible turning points, continuation method are used in order to "follow" the response, for example using arc-length like parameters.

In the case of the proposed method, one can also adopt a different point of view and use the phase of the resonant NNM as the continuation parameter. This is motivated by the fact that the phase is always increasing during the non-linear resonance, and therefore does not undergo turning points (at least for the principal resonance). The phase then appears as a good parameter for the continuation, and a simple sequential continuation algorithm is sufficient to compute the whole resonance curve even in the presence of turning points (w.r.t. frequency) (see Fig.4 and 5).

In this case of simple sequential continuation the phase ϕ will be discretized from 0 to -2π and the amplitude and frequency are considered variable. In other word, we will search for $q(\phi)$, $\omega(\phi)$ at discrete points $\phi = \phi_s$. Note that here only a linear discretization is used for the phase, but grid adaptation could also be considered.

3.4 Linear corrections

In order to improve the solution given by the single resonant NNM approximation, a linear correction can be added as suggested in [15]. The corrected solution can be sought under the following form:

$$\hat{\mathbf{u}}(t) = \Psi_j(q, \phi)(t) + \sum_{i \neq j} \Phi_i \eta_i(t) \quad (22)$$

where the first term is the resonant NNM obtained by the procedure exposed in the previous sections and the second term is the participation of the linear modes. The second term, added a posteriori, is added mostly to take into account for eventual anti-resonances and also allows to decrease the error of the single NNM approximation away from resonance.

In this study, the participations of the (non-resonant) linear modes will be assumed small so that it is reasonable to compute them using only the linear (linearised) system. The time evolution of the non resonant linear modes are then simply obtained from:

$$\ddot{\eta}_i + 2\xi_i \omega_i \dot{\eta}_i + \omega_i^2 \eta_i - \Phi_i^T \mathbf{F} = 0 \quad (23)$$

where ξ_i and ω_i are respectively the modal damping and natural angular frequency of the i -th linear mode.

Note that we have assumed that the amplitude and the phase of the resonant NNM could be computed independently of the amplitudes and phases of the linear modes, which seems reasonable as long as the modes are well separated. Consequently, the response of the linear modes can be computed in parallel (or before, or after) of the computation of the resonant NNM response.

Also, if the structure has a large number of degrees of freedom, not all linear modes need to be retained in the correction, and the sum in Eq.(13) can be truncated as it is usually done in linear modal synthesis (the effect of the discarded modes could also be taken into account through a usual static correction).

3.5 Sum up of the reduction procedure

Finally, the reduction procedure is summarized in the following bullet points

- Compute the j -th NNM with HBM and parametrize it with relation to phase and amplitude to obtain $\Psi_j(q, \phi)$ as in Eq.(12).
- Use the following approximation for the forced solution

$$\hat{\mathbf{u}}(t) = \Psi_j(q, \phi)(t) + \sum_{i \neq j} \Phi_i \eta_i(t)$$

- Non-linear part of the resolution
 - For a given excitation frequency ω find the parameters q and ϕ of the resonant NNM by solving the two algebraic equations in Eq.(20)-(21). Generate the frequency response using arc-length continuation.
 - Or, for a given ϕ find the parameters q and ω by solving the algebraic system in Eq.(20)-(21). Generate the FRF using sequential continuation on $\phi \in]0, -2\pi[$ (no turning points with this parametrization)
- Linear part of the resolution (optional)
 - Compute linear corrections η_i by solving Eq.(23)
 - Optionally, compute a static correction (if not all modes are included in the previous linear correction)

4 Numerical examples

In order to demonstrate the validity and efficiency of the previously exposed method, we propose an application to a simple (but representative) example: a linear beam with non-linear attachment.

4.1 Description of the system

In this section we consider the case of a linear beam clamped at one end and attached to a non-linear spring at the other end. The beam is made out of steel ($E = 210\text{GPa}$, $\rho = 7800\text{ kg.m}^{-3}$), has a length $L = 0.6\text{ m}$ and a square section of width $a = 14\text{ mm}$. The beam is modelled by standard Euler Bernoulli elements with 3 dof per node (axial disp. u , transverse disp. v , section rotation θ). A total number of 20 elements is used, leading to a problem containing $n = 60$ degrees of freedom (the first node is clamped).

The effect of the attachment is modelled by a non-linear spring with the following expression for the restoring force:

$$F^{nl}(v_L) = k_1 v_L + k_3 v_L^3 + k_5 v_L^5 \quad (24)$$

where $v_L = v(L)$ represents the transverse tip displacement, and $k_{1,3,5}$ are some stiffness coefficients. Two sets of coefficient will be studied. First only a cubic restoring force will be considered ($k_5 = 0$). Then a quintic term will be introduced in the restoring force in order to generated more complicated behaviour for validating the proposed reduction method.

The solutions computed with the HBM, in the free and/or forced cases, will be considered as reference solutions for comparison purpose.

4.2 Cubic restoring force

In this first part of the example, only a cubic restoring force will be considered, corresponding to the following set of coefficients for the formula in Eq.(24): $k_1 = 7.3e3$, $k_3 = 1e8$ and $k_5 = 0$.

4.2.1 Non-linear mode computation

We compute the first three NNMs of the system by using the HBM with $H = 5$ harmonics. The backbone curve of the computed modes, in a Frequency-Energy Plot (FEP), are depicted on Fig.1.

In this example, we will study the response of the beam in a frequency range around its first mode of vibration. Therefore, we will only analyse the first NNM (similar results could be obtained for the other modes).

It can be seen on Fig.1. that the non-linear attachment has a stiffening effect on the first mode, and that at $f = 70\text{ Hz}$ an 3:1 internal resonance occurs in the system between mode 1 and 2.

The information contained in the NNM can be analysed by projecting each harmonic of the NNM (i.e. each vector $\Psi_k(q)$ of Eq.(12)) onto the (sub)space generated by (a subset of) the linear mode shapes. The results of such projection onto the first five Linear Mode (LM) shape is given in Fig.2. Note that the abscissa is the point index, which can be seen as a curvilinear abscissa on the backbone curve.

One can see that, at first (for low point index), only LM-1 has a significant contribution onto the first harmonic (i.e. at low amplitude NNM-1 is equivalent

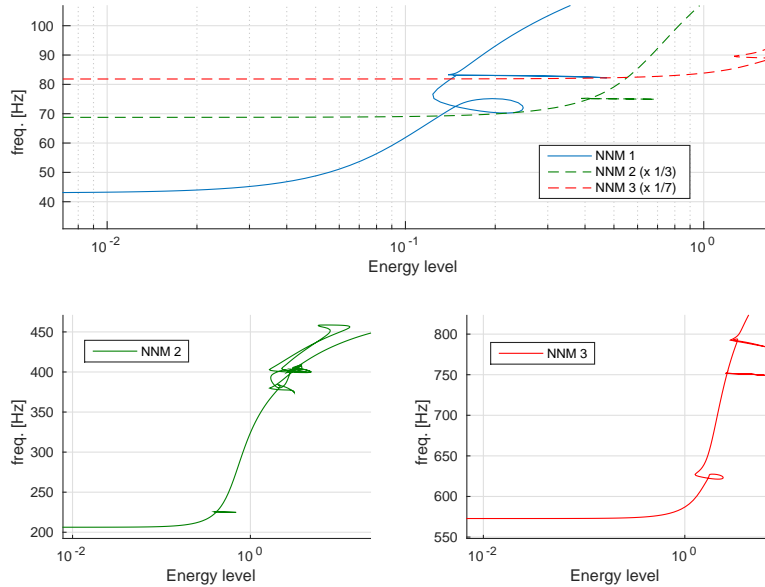


Figure 1: Backbone curve in Frequency-energy Plot for the first three non-linear normal modes of the cantilever beam with cubic non-linear attachment

to LM-1). When the point index increases, the contribution of LM-1 is decreasing and the contribution of mode 2 is increasing (with its highest contribution on the third harmonic): this confirms the presence of a 1:3 internal resonance between mode 1 and 2 (around point index = 125). Following the same lines, one can see that there also exist a 1:7 internal resonance between mode 1 and 3 (around point index = 155).

An illustration of the evolution of NNM-1 is given on Fig.3 where we plotted the evolution in the "phase space" and in the "configuration space" for four different positions on the NNM depicted by small circles on Fig.2. In particular, one can see that when the internal resonance occurs, the configuration line cannot be parametrised only with one coordinates (the graph in the configuration space is not a bijection anymore).

This last remark is a direct consequence of the fact that the 2D invariant manifold intersects itself when experiencing internal resonances.

In order to avoid problems with internal resonance and parametrization, the maximum amplitude for the tip displacement in the numerical simulations will be limited to 0.024m.

As we restrict our study to a single non-linear mode approximation, NNM-1 is considered over a frequency range that does not include internal resonances (ie: [45, 70] Hz, corresponding to an amplitude range [0, 0.024]m) and its control parameter q is chosen to be the amplitude of the first harmonic of the

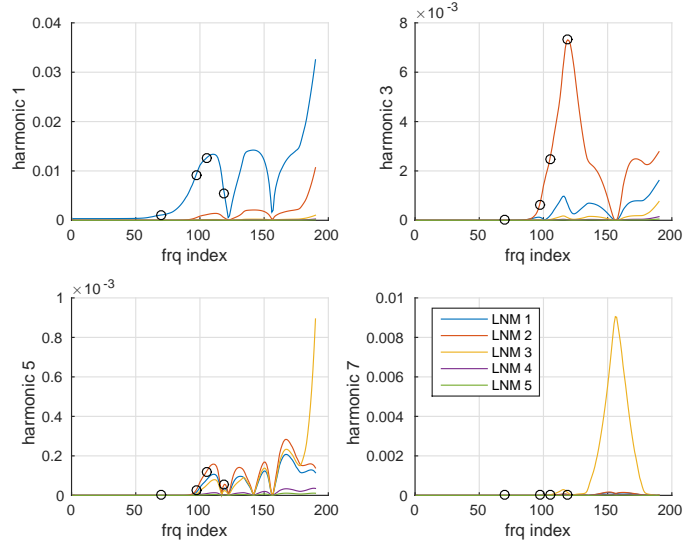


Figure 2: Projection of NNM-1 harmonic's onto the first five linear modes as a function of the point index (equivalent to a curvilinear abscissa on the NNM curve)

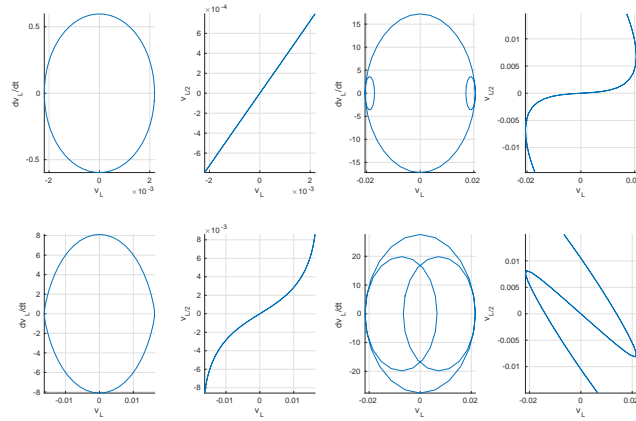


Figure 3: Evolution of NNM-1 along the backbone curve (phase space plot for the tip displacement v_L /velocity $\frac{dv_L}{dt}$, and configuration space plot for the middle $v_{L/2}$ and tip displacement v_L).

tip displacement.

4.2.2 Forced response synthesis

We now use the previously computed NNM to synthesise an approximation for the forced response of the system. Several forcing amplitudes will be considered in order to gradually increase the level of non-linearity in the responses. We recall the HBM solution will be considered as a reference solution for comparison purpose.

Figure 6 shows the FRF of the system (i.e.: the evolution of the transverse amplitude of the tip displacement as a function of the excitation frequency) for three forcing amplitudes. The NNM-synthesised solution are plotted along with the HBM solution (reference).

First one can plot the evolution of the NNM amplitude and phase q and ϕ as a function of the frequency (Fig.4). Those curves exhibit the standard turning points associated with such nonlinear system. One can also take a different point of view and plot the amplitude q and the frequency as a function of the phase ϕ (Fig.5). those curve don't exhibit turning point, making this representation better for continuation purpose, since a simple sequential continuation is sufficient.

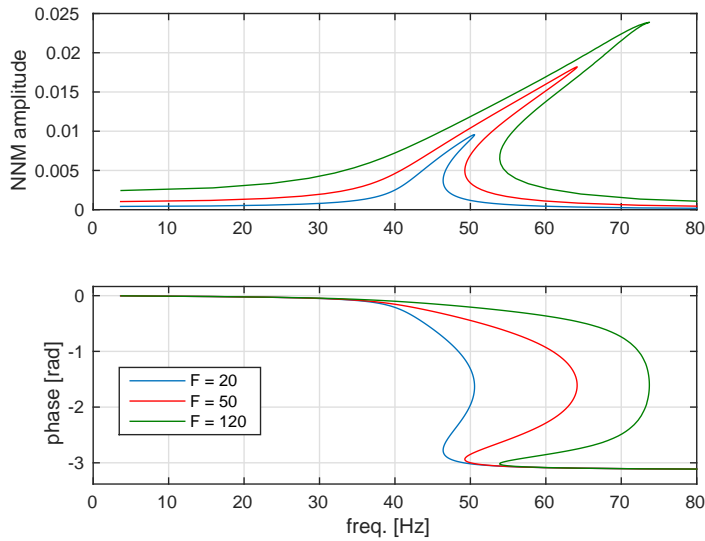


Figure 4: Amplitude q and phase ϕ of the NNM as a function of the excitation frequency for different force level $F = 20, 50, 120$ N

In the three forced cases the reduced model is able to give an excellent representation of the beam displacement (as compared to the reference solution) using only two variables q and ϕ , even for amplitudes close to the internal resonance.

A zoom of the resonance area is given on Fig.7. We can observe that, close to

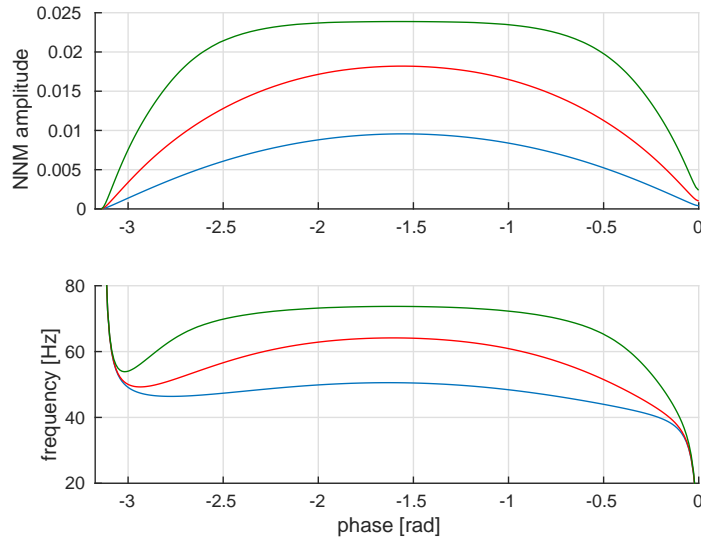


Figure 5: Amplitude q of the NNM and excitation frequency ω as a function of the phase for different force level $F = 20, 50, 120$ N. Note the absence of turning points as compared to Fig.4

the resonance, the NNM-reduced solution is closer to the backbone curve than the HBM solution. However, the difference remains small, and the maximum amplitude (at resonance) is well approximated by the reduction method.

Finally, one can compare the time evolution between the reference solution and the NNM-reduced solution at the resonance for the highest force case. For comparison purpose we can also plot the time evolution obtained by a Galerkin projection keeping only the first linear mode, or the first four linear modes. This comparison is done in Fig.8 where one can see that the time evolution of the solution obtained by two different methods (HBM and red NNM) are very close to each other. This also shows that keeping only the first linear mode is not sufficient at all to accurately represent the time evolution of the system at resonance. To obtain a level of accuracy close to the one obtained with the NNM reduction method, one would need to include at least the first four linear mode shapes in the reduction basis.

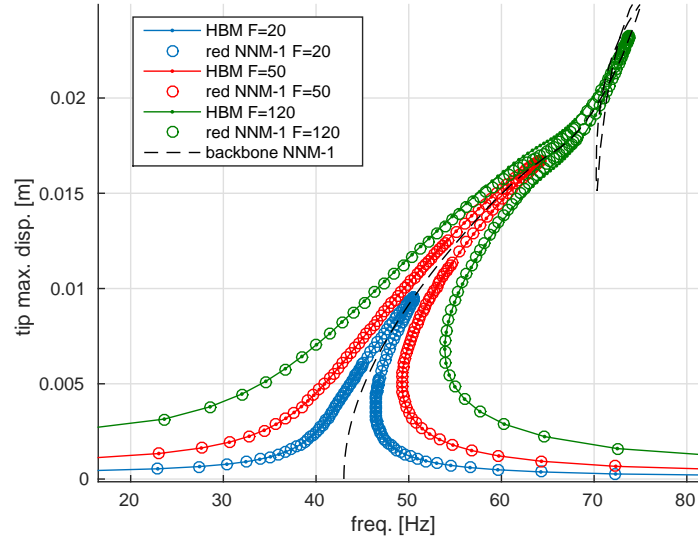


Figure 6: Comparison between the HBM solution (reference) and the NNM-reduced solution for different force level $F = 20, 50, 120$ N for the case of a cubic restoring force

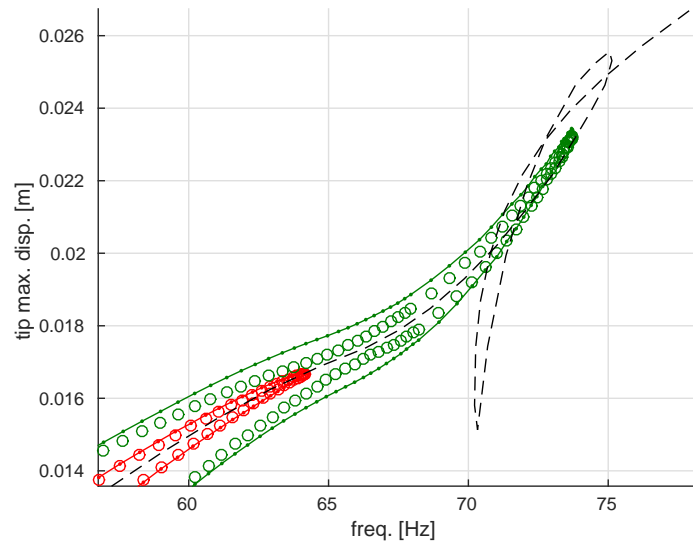


Figure 7: Detail of the previous figure around the resonance peak

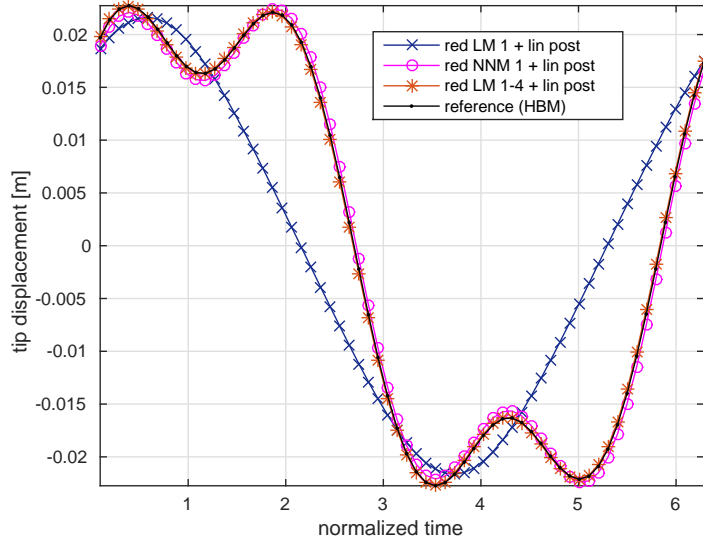


Figure 8: Comparison of the time evolution of the tip displacement for $F = 120$, at resonance ($f = Hz$)

4.3 Quintic restoring force

In the remainder of the example a quintic term is added in the expression of the non-linear forces Eq.(24), leading to the following set of coefficients: $k_1 = 7.3e3$, $k_3 = 1e8$ and $k_5 = -7e10$.

This set of (somehow arbitrary) values has been chosen so that the system have a more complex behaviour compared to the previous case. In particular, the present choice of coefficients allows for the second NNM to exhibit an hardening behaviour followed by a softening one. In the remainder of the example, only this mode (i.e. the second NNM) will be considered. This will show that the proposed reduction method performs well in this non trivial case.

4.3.1 NNM computation

We computed the second NNM by using the HBM with $H = 5$ harmonics. The backbone curve of the computed mode is depicted in a Frequency-Energy Plot (FEP) on Fig.9.

It can be seen that the non-linear attachment has, at first, a stiffening effect on the second mode (up to 215 Hz), followed by a softening effect. Also note that a modal interaction (internal resonance) occurs in the softening region (presence of a loop). In order to avoid (parametrisation) problems for the NNM, the amplitude of the tip displacement will be considered smaller than 0.05 (i.e. below the amplitude corresponding to internal resonance). The control parameter q is

kept the same, i.e. the amplitude of the first harmonic of the tip displacement.

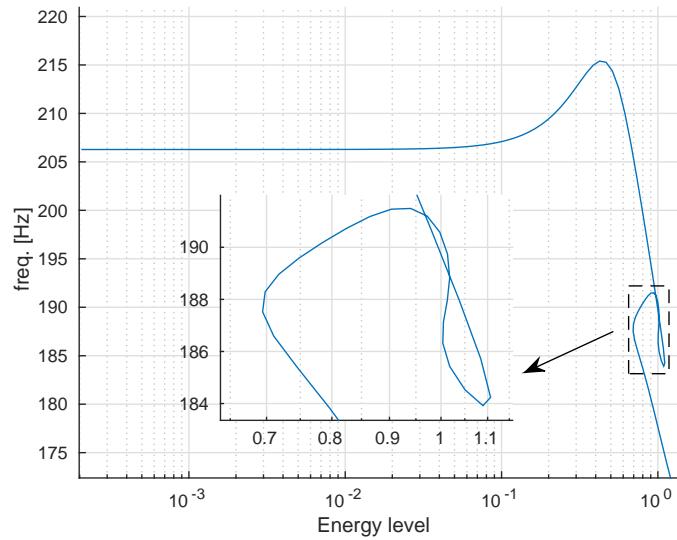


Figure 9: Backbone curve in Frequency-energy Plot for the second non-linear normal mode of the cantilever beam with quintic non-linear attachment

4.3.2 Forced response

We now use the previously computed NNM to synthesise an approximation for the forced response of the system. Again, several forcing amplitudes will be considered in order to gradually increase the level of non-linearity.

Figure 10 shows the FRF of the system for four forcing amplitudes. The NNM-synthesised solutions are plotted along with the HBM solution (reference).

In the four cases there is an excellent agreement between the two methods. This shows that the forced response can be well approximated using only two algebraic variables q and ϕ , even in this non trivial case mixing hardening and softening behaviours.

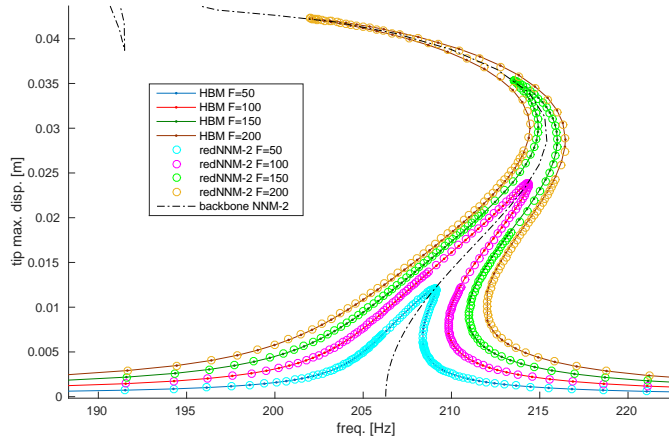


Figure 10: Comparison between the HBM solution (reference) and the NNM-reduced solution for different force level $F = 50, 100, 150, 200$ N for the case of a quintic restoring force

5 Conclusion

In this paper we presented a reduction model procedure based on the use of non-linear normal modes (NNM). First the NNM are computed using the harmonic balance method (HBM) and the Rosenberg definition of NNM. Therefore only a particular case of NNM is considered, namely mono-phase NNM.

The NNM are then used to compute single mode approximation of non-linear damped and forced equation. The solution of the equation is sought under the form of an NNM with an added phase in order to take into account the damping. This leads to only two algebraic unknowns which are the amplitude and the phase of the solution. The two equations controlling the amplitude and phase, are derived using a variational procedure. The procedure can be applied to localized or distributed non-linearity as long as they are conservatives.

The procedure is applied on a numerical example showing that this single mode approximation gives promising results. In the case of large industrial models, the computation of the nonlinear normal modes may be very costly. To reduce this computation cost, it might be judicious to first reduced the mechanical system using linear modes and a stiffness evaluation procedure [8]

Further work should extends the procedure to travelling waves solutions and should also address the problem of multi (non-linear) modes approximation, in particular for the reduction of system with internal resonances.

6 annexe: Coupled version of the variational problem

The variation of the approximation is

$$\hat{\mathbf{u}}(t) = \frac{\partial \Psi_j}{\partial q} \delta q + \frac{\partial \Psi_j}{\partial \phi} \delta \phi + \sum_{i \neq j} \Phi_i \delta \eta_i(t) \quad (25)$$

Substituting into the variational formulation leads to a coupled version of the equations for the resonant NNM and the linear modes. The algebraic equations for the phase and amplitude are given by:

$$\begin{aligned} \int_0^T \frac{\partial \Psi_j}{\partial (q, \phi)}{}^T & [\mathbf{M} \ddot{\Psi}_j + \mathbf{C} \dot{\Psi}_j + \mathbf{K} \Psi_j + \mathbf{F}^{\text{nl}}(\Psi_j + \Phi)] \\ & + \frac{\partial \Psi_j}{\partial (q, \phi)}{}^T [\mathbf{M} \ddot{\Phi} + \mathbf{C} \dot{\Phi} + \mathbf{K} \Phi] \\ & - \frac{\partial \Psi_j}{\partial (q, \phi)}{}^T \mathbf{F} dt = 0 \end{aligned} \quad (26)$$

and the linear modes equations ($n - 1$ differential equations, variable $\eta_i(t)$) are given by:

$$\begin{aligned} \Phi_i^T & [\mathbf{M} \ddot{\Psi}_j + \mathbf{C} \dot{\Psi}_j + \mathbf{K} \Psi_j + \mathbf{F}^{\text{nl}}(\Psi_j + \Phi)] \\ & + m_i \ddot{\eta}_i + c_i \dot{\eta}_i + k_i \eta_i \\ & - \Phi_i^T \mathbf{F} = 0 \end{aligned}$$

In this paper, we made the hypothesis that we can neglect the coupling terms between linear and nonlinear modes. This leads to the set of formulas in Eq.(16) and (17).

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