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Nina Aguillon, Pierre-Antoine Guiheneuf. Dynamical behavior of a nondiffusive scheme for the advection equation. 2019. hal-02304798v1

HAL Id: hal-02304798

<https://hal.science/hal-02304798v1>

Preprint submitted on 7 Oct 2019 (v1), last revised 13 Jul 2020 (v2)

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DYNAMICAL BEHAVIOR OF A NONDIFFUSIVE SCHEME FOR THE ADVECTION EQUATION

by

Nina Aguillon & Pierre-Antoine Guihéneuf

Abstract. — We study the long time behaviour of a dynamical system strongly linked to the anti-diffusive scheme of Després and Lagoutiere for the 1-dimensional transport equation. This scheme is nondiffusive in the sens that discontinuities are not smoothed out through time. Numerical simulations indicates that the scheme error's is uniformly bounded with time. We prove that this scheme is overcompressive when the Courant–Friedrichs–Levy number is $1/2$: when the initial data is nondecreasing, the approximate solution becomes a Heaviside function. In a special case, we also understand how plateaus are formed in the solution and their stability, a distinctive feature of the Després and Lagoutiere scheme.

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Introduction

The numerical approximation of the solution to the 1-dimensionnal transport equation with a constant velocity $V > 0$ has received a lot of attention for a long time, and still continues to do. One of the reason is that this equation, namely

$$\begin{cases} \partial_t u(t, x) + V \partial_x u(t, x) = 0 & \forall t > 0, \forall x \in \mathbf{R} \\ u(0, x) = u^0(x) & \forall x \in \mathbf{R} \end{cases} \quad (1)$$

is very simple and well understood, and is at the same time a fundamental example in the much larger class of conservation laws. In the multidimensionnal setting with space and time dependent velocity fields, (1) is of crucial importance for practical applications, as it represents the passive advection of the quantity u . For this class of equation it is of crucial importance to have reliable and accurate numerical schemes,

able to capture the exact solution $u(t, x) = u^0(x - Vt)$ of (1), even when u^0 is discontinuous.

One of the simplest schemes to approximate (1) is the so-called upwind scheme. Fixed time step $\Delta t > 0$ and space step $\Delta x > 0$ are given, and the real line \mathbf{R} is separated in intervals of size Δx , with midpoints $x_i = i\Delta x$, and left extremity $x_{i-1/2} = x_i - \frac{\Delta x}{2}$. The scheme is initialized with $u_j^0 = u^0(x_j)$ if u^0 is C^1 -regular or with

$$u_j^0 = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} u^0(x) dx \quad (2)$$

when u^0 only has bounded variations. The approximate solution $(u_j^{n+1})_{j \in \mathbf{Z}}$ at time $(n+1)\Delta t$ is obtained from $(u_j^n)_{j \in \mathbf{Z}}$, the approximate solution at time $n\Delta t$, by

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + V \frac{u_j^n - u_{j-1}^n}{\Delta x} = 0, \quad \forall j \in \mathbf{Z}, \forall n \in \mathbf{N}. \quad (3)$$

An interpretation is the following. At time $n\Delta t$, define a piecewise constant function by

$$u_{\Delta x}^n(x) = u_j^n \quad \text{if } x \in [x_{j-1/2}, x_{j+1/2}).$$

Translate it to the right of a distance $V\Delta t$, i.e. consider v the exact solution of (1) with initial data $u_{\Delta x}^n$ at time Δt . If the solution does not cross more than a cell, namely if $V\Delta t < \Delta x$, then

$$\begin{aligned} \int_{x_{i-1/2}}^{x_{i+1/2}} v(x) dx &= \int_{x_{i-1/2}}^{x_{i+1/2}} u_{\Delta x}^n(x - V\Delta t) dx \\ &= V\Delta t u_{j-1}^n + (\Delta x - V\Delta t) u_j^n = \Delta x u_j^{n+1}. \end{aligned}$$

It is possible to prove that the resulting scheme converges towards the exact solution.

Theorem A. — Suppose that the Courant–Friedrichs–Levy number $\frac{V\Delta t}{\Delta x}$ is fixed in the interval $(0, 1)$.

- If u^0 is C^2 -regular and with the initialization $u_j^0 = u^0(x_j)$, there exists a constant C such that

$$\max_{j \in \mathbf{Z}} |u_j^n - u(n\Delta t, x_j)| \leq Cn\Delta t\Delta x.$$

- If u^0 has total variation and with the initialization (2), there exists a constant C such that

$$\forall n \in \mathbf{N}, \quad \Delta x \sum_{j \in \mathbf{Z}} \left| u_j^n - \frac{1}{\Delta x} \int_{x_{i-\Delta x/2}}^{x_i + \Delta x/2} u(n\Delta t, x) dx \right| \leq C\sqrt{n\Delta t\Delta x}.$$

This theorem means that if the final time $T = n\Delta t$ is fixed and if Δt and Δx both tend to zero by keeping the ratio $\frac{V\Delta t}{\Delta x}$ fixed and smaller than 1, then the approximate solution converges towards the exact solution at rate 1 or 1/2 (depending on the regularity of u^0), for the L^∞ and L^1 norm respectively. The error in time grows as T or \sqrt{T} .

A considerable effort has been made over the last decades to improve the rate of convergence. For linear schemes, estimates like

$$\|u_{\Delta x} - u\|_{L^\infty} \leq C(T)\Delta x^p \quad \text{or} \quad \|u_{\Delta x} - u\|_{L^1} \leq C(T)\Delta x^{\frac{p}{p+1}}$$

have been proven in [Des08a] and [Des08b], for regular and BV initial data respectively; Theorem A is a special case of this result. Nonlinear schemes for (1) are widely used, because it is the only way to obtain methods that are of order larger than 2 and that verify a discrete maximum principle. For a description of the most popular methods for the linear advection equation, see [LeV92].

The antidiffusive scheme of Després and Lagoutière. — Among all the schemes available for (1), the scheme introduced by Després and Lagoutière in [DL01] has the property of having an error that does not grow with time. This property has been verified numerically but is still a conjecture, and this paper is a step toward its proof.

Their main idea is to reverse the average step of the upwind scheme (3), by considering that each value u_j^n comes from an average of a discontinuity joining u_{j-1}^n to u_{j+1}^n located somewhere inside the cell (recall that at time $n\Delta t$, the approximate solution is constant equal to u_j^n on the interval $[x_{i-1/2}, x_{i+1/2})$). This scheme can be decomposed in three steps:

1. In $[x_{i-1/2}, x_{i+1/2})$, replace u_j^n by a piecewise constant map of the form

$$(u_{rec}^n)|_{[x_{i-1/2}, x_{i+1/2})} : x \mapsto \begin{cases} u_{j-1}^n & \text{if } x_{i-1/2} \leq x < x_{i-1/2} + d_j^n \\ u_{j+1}^n & \text{if } x_{i-1/2} + d_j^n \leq x < x_{i+1/2} \end{cases}$$

The discontinuity is placed at a distance $d_j^n \in [0, \Delta x]$ of the left extremity of the cell, in such a way that the total mass inside the cell is preserved, i.e.

$$\Delta x u_j^n = d_j^n u_{j-1}^n + (\Delta x - d_j^n) u_{j+1}^n.$$

If this is not possible, do nothing, i.e. $(u_{rec}^n)|_{[x_{i-1/2}, x_{i+1/2})} = u_j^n$.

2. Compute the exact solution of (1) with initial data u_{rec}^n at time Δt , which is nothing but $x \mapsto u_{rec}^n(x - V\Delta t)$.
3. Define u_j^{n+1} as the average of this exact solution on $[x_{j-1/2}, x_{j+1/2}]$:

$$u_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_{rec}^n(x - V\Delta t) dx.$$

This process is illustrated on Figure 1, where regions of same colors are of equal areas. This interpretation in terms of discontinuous reconstruction is equivalent to the original presentation of [DL01] and is presented in [BCLL08]. The scheme is initialized with (2).

The property that makes this scheme unique is that it is exact for a large class of initial data (the vast majority of schemes are exact only for constant initial data).

Proposition 1 (Després, Lagoutière, Theorem 3 of [DL01])

Suppose that u^0 is piecewise constant, with plateaus of width larger than $3\Delta x$.

Then

$$\forall n \in \mathbf{N}, \forall j \in \mathbf{Z}, \quad u_j^n = \int_{x_{j-1/2}}^{x_{j+1/2}} u^0(x - Vn\Delta t) dx.$$

Proof. — The proof can be found in [DL01]. It boils down to proving that u_{rec}^n is the L^2 -projection on the grid of the exact solution at time $n\Delta t$, which is $x \mapsto u^0(x - Vn\Delta t)$. The computation is a bit tedious but the idea is simple. Let us focus

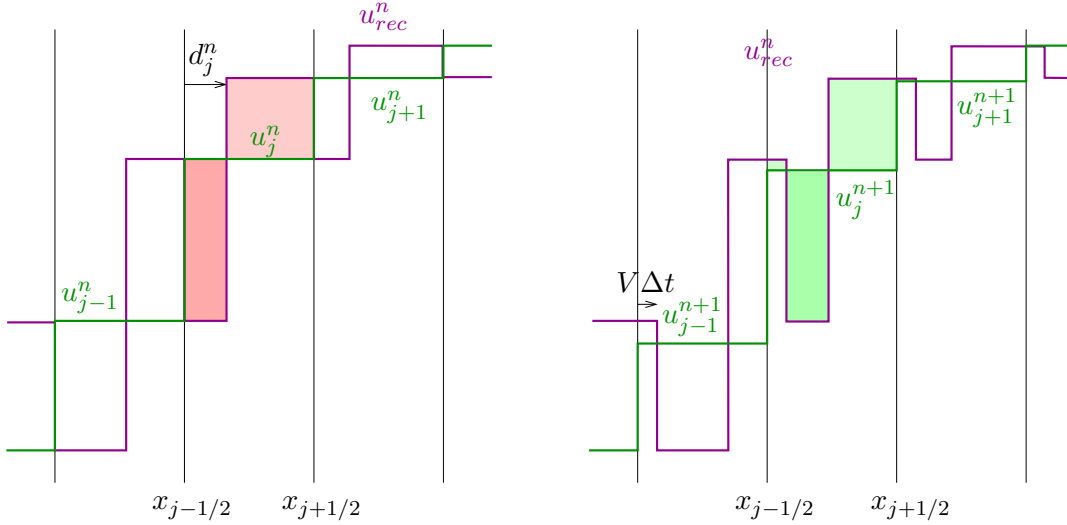


FIGURE 1. Left: initial data $(u_j^n)_j$ (green) and reconstruction u_{rec}^n (purple). Right: the solution at time $n + 1$ (green) is the L^2 -projection of the reconstruction translated of $V\Delta t$ (purple).

on the initialization. As plateaus are wider than $3\Delta x$, each of them contains at least two consecutive cells which have the same value after the initialization (2). It ensures that the first step of the reconstruction scheme is successful only on cells containing a discontinuity of the initial data u^0 , and thus that $u^0 = u_{rec}^0$. \square

Numerically it is observed that this class of initial data behaves as an attractor. Plateaus are created in the first time steps and are then advected exactly. We have the following conjecture, that looks very much like Theorem A but with a time independent constant.

Conjecture 2. — *Let u^0 be a function with total bounded variations to which we associate the initialization $u_j^0 = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} u^0(x) dx$. Suppose that the ratio $\lambda = \frac{\Delta t}{V\Delta x}$ is kept fixed and belongs to $(0, 1] \setminus \{1/2\}$. Then there exists a constant $C > 0$, depending only on u^0 and λ such that*

$$\forall n \in \mathbf{N}, \quad \Delta x \sum_{j \in \mathbf{Z}} \left| u_j^n - \frac{1}{\Delta x} \int_{x_i - \Delta x/2}^{x_i + \Delta x/2} u(n\Delta t, x) dx \right| \leq C\sqrt{\Delta x}.$$

This conjecture states that there is a global attractor A for bounded increasing configurations, made of solutions whose reconstructions have plateaus of width bigger than 3 (see Figure 2). In other words, we expect that for any bounded increasing initial data (u_j^0) and for any $\varepsilon > 0$, there is a solution $(\tilde{u}_j^0) \in A$ and an integer $N \in \mathbf{N}$ such that for any $n \geq N$, one has $\|(u_j^n)_j - (\tilde{u}_j^n)_j\|_\infty \leq \varepsilon$. Note that the elements of A are almost periodic. For more details about the concept of attractor, see [Mil85a, Mil85b], [Mil06].

The main result of our work is to prove that this conjecture does not hold if $\lambda = 1/2$. We roughly speaking prove the following result (see Theorem 11 for a precise statement).

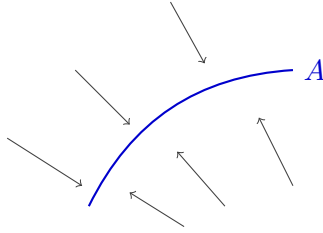


FIGURE 2. Picture of an attractor A : any solution eventually approaches a solution inside the set A of solutions whose reconstructions have plateaus of width bigger than 3

Theorem B. — Suppose that the initial sequence $(u_j^0)_{j \in \mathbf{Z}}$ strictly increases from a constant value α to another constant value β . Then for all $n \in \mathbf{N}$ large enough, there exists an integer $j_\infty^n \in \mathbf{Z}$ such that $(u_j^n)_{j \in \mathbf{Z}}$ is a discrete Heaviside function, that is,

$$u_j^n = \begin{cases} \alpha & \text{if } j < j_\infty^n \\ \beta & \text{if } j > j_\infty^n \end{cases}$$

We also get a similar statement for initial conditions that are “half infinite staircases” in Proposition 12.

In a last part of the article, we prove a special case of Conjecture 2 in the case where the grid is alternatively shifted to the left and to the right of a parameter $\lambda \neq 1/2$ (see Proposition 13 for a precise statement).

Theorem C. — For any $\lambda \in (0, 1) \setminus \{1/2\}$, there exists a nonempty open set of initial data for which $(u_i^n)_i$ converges uniformly exponentially fast towards a limit configuration.

In Section 1, we simplify the problem and present important lemmas. Some numerical illustrations of Conjecture 2 and Proposition 1 are given in Section 2. Theorem B is proven in Section 3. Eventually in Section 4, a particular case illustrating the exponential convergence toward solutions with plateaus is studied. This result is far from being a complete proof of Conjecture 2 which remains open at the moment.

1. A related shifted grids dynamical process

In order to simplify the analysis while retaining the most important aspects, we do the following modifications. First, we set $V = 1$, $\Delta x = 1$ and $x_j = j$. More importantly, instead of moving the reconstructed solution of λ to the right, we shift the grid of λ to the left for odd iteration in time and of λ to the right for even iteration in time. The advantage of shifting the grid alternatively to the left and to the right is obviously that we end up with the same grid after two iterations. If $\lambda = 1/2$, there is no difference with the case where the grid is always shifted to the left. For any real number a we denote by \mathcal{C}_a the interval centered around a of size 1: $\mathcal{C}_a = (a - 1/2, a + 1/2)$.

The structure of the scheme follows the same guidelines than the Després and Lagoutière scheme presented in the previous section. The process is initialized with

the sequence $(u_j^0)_{j \in \mathbf{Z}}$ given by (2). For odd iteration in time the process is centered on integer points j at the beginning on the time step and the grid is shifted of λ to the left. The three steps are the following

1. *Reconstruction step.* Compute the distance d_j^{2n} from the *right* interface $j + 1/2$ such that

$$(1 - d_j^{2n})u_{j-1}^{2n} + d_j^{2n}u_{j+1}^{2n} = u_j^{2n}.$$

One gets

$$d_j^{2n} = \frac{u_j^{2n} - u_{j-1}^{2n}}{u_{j+1}^{2n} - u_{j-1}^{2n}}, \quad (4)$$

and we set arbitrarily $d_j^{2n} = -1$ if it is not defined. Then, define

$$u_{j,L}^{2n} = \begin{cases} u_{j-1}^{2n} & \text{if } 0 < d_j^{2n} < 1, \\ u_j^{2n} & \text{otherwise,} \end{cases} \quad u_{j,R}^{2n} = \begin{cases} u_{j+1}^{2n} & \text{if } 0 < d_j^{2n} < 1, \\ u_j^{2n} & \text{otherwise.} \end{cases}$$

The reconstructed solution at iteration $2n$ is obtained as

$$u_{rec}^{2n}(x) = \sum_{j \in \mathbf{Z}} \left(u_{j,L}^{2n} \mathbf{1}_{d_j^{2n} < (j+1/2) - x < 1} + u_{j,R}^{2n} \mathbf{1}_{0 < (j+1/2) - x < d_j^{2n}} \right) \mathbf{1}_{x \in \mathcal{C}_j}.$$

2. *Shifting.* Shift the grid of λ to the left and define

$$u_{j-\lambda}^{2n+1} = \int_{\mathcal{C}_{j-\lambda}} u_{rec}^{2n}(x) dx$$

$$\text{and } u^{2n+1}(x) = \sum_{j \in \mathbf{Z}} u_{j-\lambda}^{2n} \mathbf{1}_{x \in \mathcal{C}_{j-\lambda}}.$$

At the beginning of an even iteration in time, the cells are centered around the points $(j - \lambda)_{j \in \mathbf{Z}}$, and we follow the same process but move the grid to the right:

1. *Reconstruction step.* Compute the distance $d_{j-\lambda}^{2n+1}$ from the *left* interface such that

$$d_{j-\lambda}^{2n+1}u_{j-1-\lambda}^{2n+1} + (1 - d_{j-\lambda}^{2n+1})u_{j+1-\lambda}^{2n+1} = u_{j-\lambda}^{2n+1}.$$

If does not exists, set $d_{j-\lambda}^{2n+1} = -1$. Then define

$$u_{j-\lambda,L}^{2n+1} = \begin{cases} u_{j-1-\lambda}^{2n+1} & \text{if } 0 < d_{j-\lambda}^{2n+1} < 1, \\ u_{j-\lambda}^{2n+1} & \text{otherwise,} \end{cases} \quad u_{j-\lambda,R}^{2n+1} = \begin{cases} u_{j+1-\lambda}^{2n+1} & \text{if } 0 < d_{j-\lambda}^{2n+1} < 1, \\ u_{j-\lambda}^{2n+1} & \text{otherwise.} \end{cases}$$

The reconstructed solution at iteration $2n + 1$ is

$$u_{rec}^{2n+1}(x) = \sum_{j \in \mathbf{Z}} \left(u_{j,L}^{2n+1} \mathbf{1}_{0 < x - (j - \lambda - 1/2) < d_{j-\lambda}^{2n+1}} + u_{j,R}^{2n+1} \mathbf{1}_{d_{j-\lambda}^{2n+1} < x - (j - \lambda - 1/2) < 1} \right) \mathbf{1}_{x \in \mathcal{C}_{j-\lambda}}.$$

2. *Shifting.* Shift the grid of λ to the right and define

$$u_j^{2n+2} = \int_{\mathcal{C}_j} u_{rec}^{2n+1}(x) dx$$

$$\text{and } u^{2n+2}(x) = \sum_{j \in \mathbf{Z}} u_j^{2n+2} \mathbf{1}_{x \in \mathcal{C}_j}.$$

The notations are gathered on Figure 3. In our analysis we restrict our attention to the case of nondecreasing initial data. If the initial data is increasing, this property is inherited at each time step.

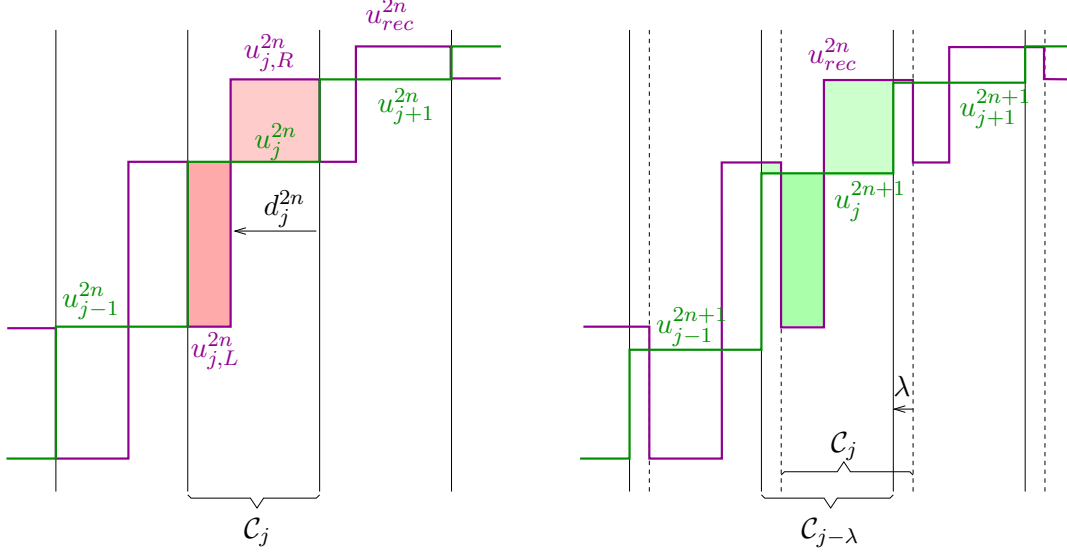


FIGURE 3. The process where the grid is shifted to the left (odd iterations in time).

Lemma 3. — Suppose that $x \mapsto u^n(x)$ is non-decreasing. Then $x \mapsto u^{n+1}(x)$ is also non-decreasing.

Proof. — Suppose that n is even. Let us denote by u_{rec}^n the reconstruction map, which is increasing. Then the mean value function

$$r \mapsto \frac{1}{r} \int_{j-1/2}^{j-1/2+r} u_{rec}^n(x) dx$$

increases on $(0, 1)$ from u_{j-1}^n for $r = 0$ to u_j^n for $r = 1$. One equally proves that the mean of u_{rec}^n on $[j-1/2-r, j-1/2]$, $r \in (0, 1)$ belongs to $[u_{j-1}^n, u_j^n]$. Thus, the mean of u_{rec}^n on $C_{j-\lambda}$ belongs to $[u_{j-1}^n, u_j^n]$, in other words $u_{j-\lambda}^{n+1} \in [u_{j-1}^n, u_j^n]$ which is smaller than $u_{j+1-\lambda}^{n+1} \in [u_j^n, u_{j+1}^n]$. When n is odd, we similarly prove that u_j^{n+1} belongs to $[u_{j-\lambda}^n, u_{j+1-\lambda}^n]$. \square

The general case follows from the study of the non decreasing one, as explained by the following lemma.

Lemma 4. — Suppose that $(u_j^0)_{j \in \mathbf{Z}}$ is given. Without loss of generality, we suppose that $u_0^0 = 0$. Let us define $(v_j^0)_{j \in \mathbf{Z}}$, $(w_j^0)_{j \in \mathbf{Z}}$ by $v_0^0 = w_0^0 = 0$ and

$$v_{j+1}^0 = \begin{cases} v_j^0 + (u_{j+1}^0 - u_j^0) & \text{if } u_{j+1}^0 > u_j^0 \\ v_j^0 & \text{if } u_{j+1}^0 \leq u_j^0 \end{cases}$$

$$w_{j+1}^0 = \begin{cases} w_j^0 & \text{if } u_{j+1}^0 > u_j^0 \\ w_j^0 + (u_{j+1}^0 - u_j^0) & \text{if } u_{j+1}^0 \leq u_j^0 \end{cases}$$

Then for all $n \geq 0$,

$$(u_{j-\lambda(n \bmod 2)}^n)_{j \in \mathbf{Z}} = (v_{j-\lambda(n \bmod 2)}^n)_{j \in \mathbf{Z}} + (w_{j-\lambda(n \bmod 2)}^n)_{j \in \mathbf{Z}}.$$

Proof. — The proof boils down to show that for all n , $u_{rec}^n = v_{rec}^n + w_{rec}^n$. Let us prove it for $n = 0$. We distinguish cases depending on the relative positions of u_{j-1}^0 , u_j^0 and u_{j+1}^0 .

- If $u_{j-1}^0 = u_j^0 = u_{j+1}^0$, the reconstruction on cell \mathcal{C}_j is constant equal to u_j^0 . It is clear that $v_{j-1}^0 = v_j^0 = v_{j+1}^0$ and $w_{j-1}^0 = w_j^0 = w_{j+1}^0$, and thus

$$(v_{rec}^0)_{|\mathcal{C}_j} + (w_{rec}^0)_{|\mathcal{C}_j} = v_j^0 + w_j^0 = u_j^0 = (u_{rec}^0)_{|\mathcal{C}_j}.$$

- If $u_{j-1}^0 \leq u_j^0 \leq u_{j+1}^0$ with one strict inequality, a discontinuity is reconstructed in cell \mathcal{C}_j at a distance $d_j^0 = \frac{u_j^0 - u_{j-1}^0}{u_{j+1}^0 - u_{j-1}^0}$ of the right interface and

$$(u_{rec}^0)_{|\mathcal{C}_j}(x) = u_{j-1}^0 \mathbf{1}_{d_j^0 < j+1/2-x < 1} + u_{j+1}^0 \mathbf{1}_{0 < j+1/2-x < d_j^0}.$$

In this case we have

$$(v_{j-1}^0, v_j^0, v_{j+1}^0) = v_j^0 - u_j^0 + (u_{j-1}^0, u_j^0, u_{j+1}^0)$$

and thus $(v_{rec}^0)_{|\mathcal{C}_j} = v_j^0 - u_j^0 + (u_{rec}^0)_{|\mathcal{C}_j}$. Moreover $w_{j-1}^0 = w_j^0 = w_{j+1}^0$, thus $(w_{rec}^0)_{|\mathcal{C}_j} = w_j^0$ and the results follows.

- If $u_{j-1}^0 \geq u_j^0$ and $u_j^0 \leq u_{j+1}^0$ with one strict inequality, then d_j^0 does not belong to $(0, 1)$ and thus $(u_{rec}^0)_{|\mathcal{C}_j} = u_j^0$. On the other hand $v_j^0 = v_{j+1}^0$ and $w_{j-1}^0 = w_j^0$ which yields $(v_{rec}^0)_{|\mathcal{C}_j} = v_j^0$ and $(w_{rec}^0)_{|\mathcal{C}_j} = w_j^0$.
- The other cases are treated similarly by exchanging the roles of v and w .

□

2. Numerical simulations

We now give some numerical illustrations of the long time behavior of the scheme and the influence on the parameter λ .

2.1. Illustration of Theorem A and Conjecture 2. — To begin with, we consider the smooth 1-periodic initial data defined by

$$\forall x \in [0, 1], \quad u^0(x) = \cos(2\pi x) \sin(10\pi x) \tag{5}$$

and we compare three classical schemes for the transport equation (1):

- the upwind scheme (3), which is linear and first order;
- the Lax–Wendroff scheme

$$u_j^{n+1} = u_j^n - \frac{V\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) + \frac{V^2\Delta t^2}{2\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n),$$

which is linear and first order;

- the Després and Lagoutière scheme explained in the introduction, which is first order and nonlinear;

We set $V = 1$ and a final time of $T = 10$. The space interval $[0, 1]$ is discretized with M cells and the time step is related to the space step $\Delta x = 1/M$ by $\Delta t = \frac{0.4\Delta x}{V}$,

which ensures stability and convergence of the schemes (this implies that $\lambda = 0.4$). We are interested in the evolution of the L^∞ -error

$$Err(n) = \max_{j \in \{1, \dots, M\}} |u_j^n - u^0(x_j - Vn\Delta t)|.$$

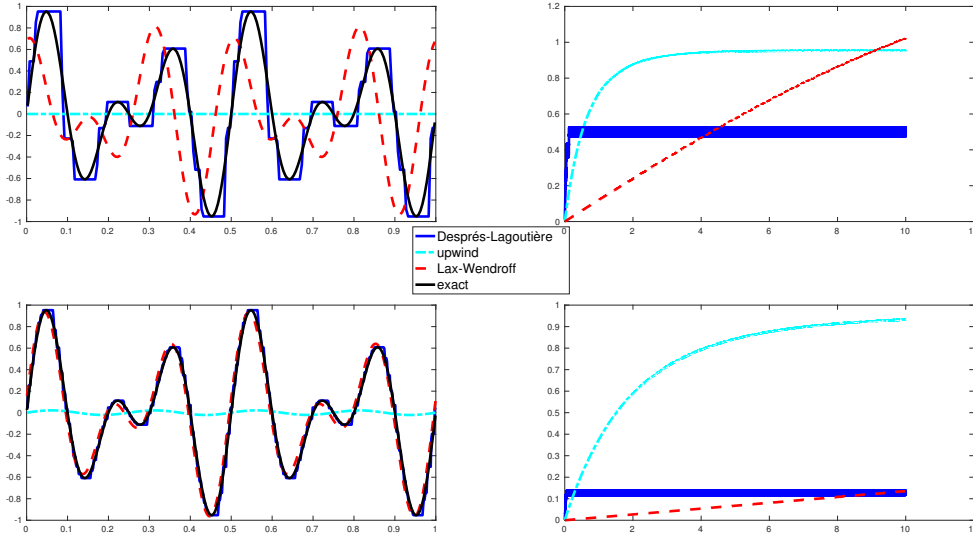


FIGURE 4. Left: approximate solution after 10 periods for the initial data (5) for different schemes with $M = 100$ (top, 5000 time steps) and $M = 600$ (bottom, 15000 time steps). Right: evolution of the L^∞ -error with time. The blue line looks thicker because it has high frequency oscillations.

The results for $M = 200$ and $M = 600$ are given on Figure 2.1. We see that the upwind scheme is so diffusive that all oscillations are flattened and the approximate solution is almost constant. The Lax-Wendroff scheme is second order and thus much less diffusive, however the approximate solution is not acceptable for $M = 100$. With the reconstruction scheme, stairs appear in the first iterations in time and are then advected exactly. It is the only scheme for which the maximum value does not decrease with time. As expected, the results are better for $M = 600$. With finer and finer meshes, we could illustrate the validity of Theorem A on the time interval $[0, 15]$. However, whatever the value of M we can reproduce Figure 2.1 by increasing the final time T .

2.2. Influence of λ . — The initial data is now 1.5-periodic with

$$\forall x \in [-0.3, 1.2], u^0(x) = \begin{cases} -1 & \text{if } -0.3 \leq x \leq 0 \\ \sin(\pi x - \pi/2) & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 \leq x \leq 1.2 \end{cases} \quad (6)$$

It contains a discontinuity at $x = 1.2$ and a smooth part in the interval $[0, 1]$.

On Figure 5, we plot the result at time 22.5 (which corresponds to 15 periods) for the original scheme of Després and Lagoutière described in the introduction and

for CFL numbers $\lambda = \frac{V\Delta t}{\Delta x}$ of 0.47, 0.48, 0.49 and 0.5 the critical value. We took $M = 100$ and $V = 1$.

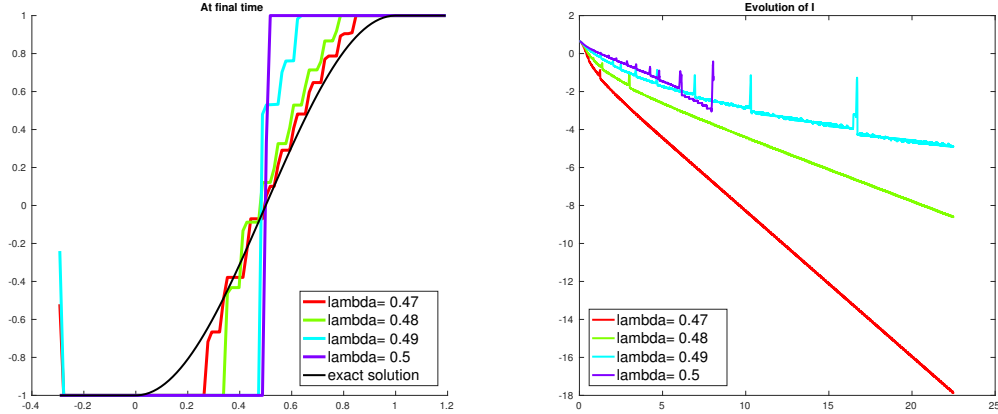


FIGURE 5. Approximate solution after 15 periods for the initial data (6) and for different CFL number, when the grid is fixed. Left: solution at the final time, right: evolution of the quantity I in logarithmic scale.

The final time is large enough to observe the long time behavior of Theorem B when $\lambda = 1/2$: the approximate solution is an Heaviside function. The closest λ is close to $1/2$, the fewer steps there is. On the right of this Figure, we plot the quantity

$$I(n) = \sum_{j=1}^M \min(|u_{j-1}^n - u_j^n|, |u_j^n - u_{j+1}^n|, |u_{j+1}^n - u_{j+2}^n|)$$

(with periodic boundary conditions $u_0^n = u_M^n$, $u_{M+1}^n = u_0^n$ and $u_{M+2}^n = u_1^n$). This quantity is null if (u_j^n) is piecewise constant with plateaus of width larger than 3 cells. Intermediate values between two plateaus are allowed. It somehow illustrate Proposition 1.

The results of the same simulation for the related scheme of Section 1, where the grid is shifted alternatively to the left and to the right, are given on Figure 6. We see that the results are more symmetrical and that the convergence is faster.

Remark 5. — The long time behavior observed here for $\lambda = 1/2$ is not in contradiction with Theorem A. Indeed for this final time, when the number of cells M is large enough, stairs do not have time to completely merge together and the scheme converges, in the sense that the approximate solution *at time* T is closer and closer to the exact one.

3. The symmetric case of a half cell shift

In this section we study the long time behavior of the scheme when $\lambda = 1/2$ with two types of increasing initial data.

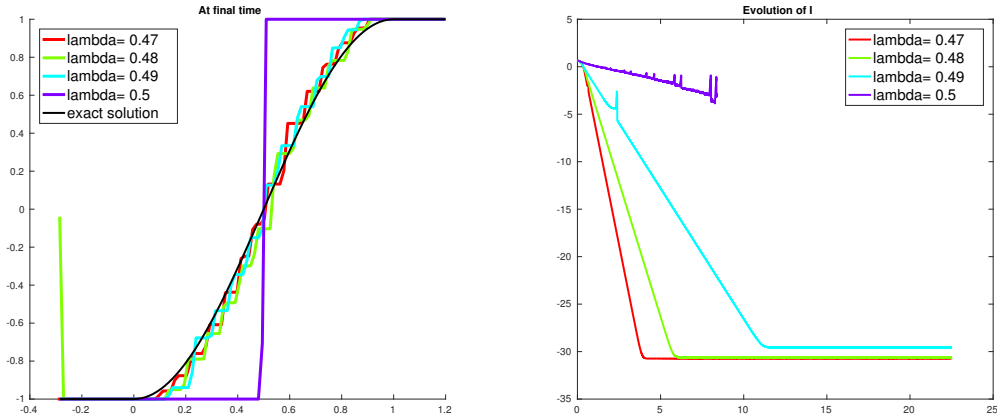


FIGURE 6. Approximate solution after 15 periods for the initial data (6) for different CFL number, when the grid is fixed. Left: solution at the final time, right: evolution of the quantity I in logarithmic scale.

3.1. Overcompressivity for nonnegative and compactly supported jumps.

— First we suppose that the initial data consists in a finite succession of strictly positive jumps. We prove that after a finite number of iterations, the numerical solution contains only one intermediate value. Following the notation of Section 1, we denote by $(u_{j+1/2}^1)_{j \in \mathbf{Z}}$ the solution after one step of the scheme, and by $(u_j^2)_{j \in \mathbf{Z}}$ the solution after the second time step.

Definition 6. — The jumps associated to the solution (u) are defined by the sequence $(S_j^n = u_{j+1/2}^n - u_{j-1/2}^n)_{j \in \mathbf{Z}}$ for odd steps n , and by sequence $(S_{j+1/2}^n = u_{j+1}^n - u_j^n)_{j \in \mathbf{Z}}$ for even steps n .

We are interested in the following class of sequences.

Definition 7. — Let α be a nonnegative real number and $M \geq 1$ be an integer. The set of M -configurations with inner jumps larger than α is the set of jumps

$$H_\alpha^M = \left\{ (v_j)_{j \in \mathbf{Z}} \in \mathbf{R}^{\mathbf{N}} \text{ such that } \exists j_0 \in \mathbf{Z} : \begin{array}{l} \bullet v_j = 0 \text{ if } j \leq j_0 \\ \bullet v_j = 1 \text{ if } j \geq j_0 + M \\ \bullet v_{j_0+1} - v_{j_0} > 0 \text{ and } v_{j_0+M} - v_{j_0+M-1} > 0 \\ \bullet v_{j+1} - v_j > \alpha \quad \forall j \in \{j_0 + 1, \dots, j_0 + M - 2\} \end{array} \right\} \quad (H_\alpha^M)$$

and the set of configurations with inner jumps larger than α is $H_\alpha = \bigcup_{M \in \mathbf{N}} H_\alpha^M$.

From now on we suppose that the initial data $(u_j^0)_{j \in \mathbf{Z}}$ belongs to H_α^M for some $\alpha > 0$ and some integer M . The long time behavior follows from the following points.

- If the initial data is in H_α , then all iterations of it also belong to H_α . The case $\alpha = 0$ is easy (Lemma 9), the case $\alpha > 0$ requires a finer analysis of the first and last jumps (Lemma 10).
- The number M of positive jumps essentially decreases with time.

For the sake of simplicity, intermediate results are stated at iteration $n = 0$. We start with a useful but simple lemma.

Lemma 8. — Consider three adjacent cells u_{j-1}^0 , u_j^0 and u_{j+1}^0 , with $u_{j-1}^0 \leq u_j^0 \leq u_{j+1}^0$. Denote by $x \mapsto u_{rec}^0(x)$ the associated reconstruction.

– If $S_{j-1/2}^0 \geq S_{j+1/2}^0$, then

$$\int_{j-1/2}^j u_{rec}^0(x) dx = u_j^0 - \frac{u_{j+1}^0}{2} \quad \text{and} \quad \int_j^{j+1/2} u_{rec}^0(x) dx = \frac{u_{j+1}^0}{2}. \quad (7)$$

– If $S_{j-1/2}^0 \leq S_{j+1/2}^0$, then

$$\int_{j-1/2}^j u_{rec}^0(x) dx = \frac{u_{j-1}^0}{2} \quad \text{and} \quad \int_j^{j+1/2} u_{rec}^0(x) dx = u_j^0 - \frac{u_{j-1}^0}{2}. \quad (8)$$

In any case we have

$$\frac{u_{j-1}^0}{2} \leq \int_{j-1/2}^j u_{rec}^0(x) dx \leq \frac{u_j^0}{2} \quad \text{and} \quad \frac{u_j^0}{2} \leq \int_j^{j+1/2} u_{rec}^0(x) dx \leq \frac{u_{j+1}^0}{2}. \quad (9)$$

As a consequence, the scheme is monotonicity preserving: if the initial sequence $(u_j^0)_{j \in \mathbf{Z}}$ is nondecreasing, so is $(u_{j+1/2}^1)_{j \in \mathbf{Z}}$.

Proof. — The proof is illustrated on Figure 7. For readability we denote by $a = u_{j-1}^0$, $b = u_j^0$ and $c = u_{j+1}^0$.

If $S_{j-1/2}^0 \geq S_{j+1/2}^0$, i.e. if $c - b \leq b - a$, the reconstructed discontinuity lies in the left half cell and its integral is $\frac{b}{2} - \frac{c-b}{2} \in [\frac{a}{2}, \frac{b}{2}]$. On the right half cell the reconstruction is constant equal to c .

If $S_{j-1/2}^0 \leq S_{j+1/2}^0$, i.e. if $c - b \geq b - a$, the discontinuity falls in the right half cell and $\int_j^{j+1/2} u_{rec}^0(x) dx = \frac{b}{2} + \frac{b-a}{2} \in [\frac{b}{2}, \frac{c}{2}]$. The reconstruction is equal to a on the left half cell.

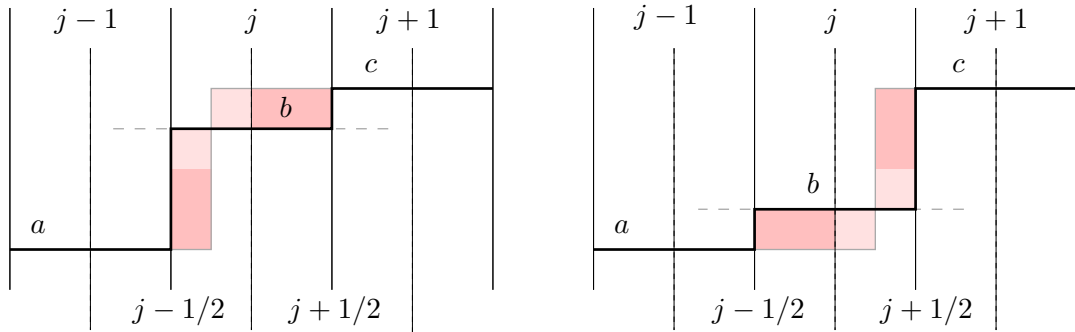


FIGURE 7. Big jump / small jump (left) or small jump / big jump (right)? Either way, the areas per half cells are easy to compute. The rectangles of the same color have the same area.

□

We now prove that the class H_α of configurations with inner jumps larger than α is preserved by the scheme.

Lemma 9. — *If $(u_j^0)_{j \in \mathbf{Z}}$ belongs to H_0 , so does $(u_{j+1/2}^1)_{j \in \mathbf{Z}}$. More precisely, if the jumps are nonnegative, then*

$$\forall j \in \mathbf{Z}, S_j^1 \geq \min(S_{j-1/2}^0, S_{j+1/2}^0).$$

Proof. — We denote by $u_{j-1}^0 = a$, $u_j^0 = b$ and $u_{j+1}^0 = c$, so that $S_{j-1/2}^0 = b - a$ and $S_{j+1/2}^0 = c - b$.

We begin with the case $S_{j-1/2}^n \geq S_{j+1/2}^n$ (Figure 7, left). Lemma 8 gives

$$\begin{aligned} S_j^1 &= u_{j+1/2}^1 - u_{j-1/2}^1 \\ &= \int_j^{j+1/2} u_{rec}^0(x) dx + \int_{j+1/2}^{j+1} u_{rec}^0(x) dx \\ &\quad - \int_{j-1}^{j-1/2} u_0^{rec}(x) dx - \int_{j-1/2}^j u_0^{rec}(x) dx \\ &\geq \frac{c}{2} + \frac{b}{2} - \frac{b}{2} - \left(b - \frac{c}{2}\right) = c - b = \min(S_{j-1/2}^0, S_{j+1/2}^0) \end{aligned}$$

A similar computation gives the result in the other case. One can also easily see that $S_j^1 = 0$ for any $j \leq 0$ or $j \geq M$. \square

This proof does not work immediately for $\alpha > 0$ because at the left extremity, it only yields to $S_{j_0}^1 \geq \min(S_{j_0-1/2}^0, S_{j_0+1/2}^0)$ which may be non zero and $S_{j_0+1}^1 \geq \min(S_{j_0+1/2}^0, S_{j_0+3/2}^0)$, which may be smaller than α , as an element of H_α does not have any constraint on the first jump $S_{j_0+1/2}^0$.

Lemma 10. — *Suppose that the initial data $(u_j^0)_{j \in \mathbf{Z}}$ belongs to H_α for some $\alpha > 0$. Then $(u_{j-1/2}^1)_{j \in \mathbf{Z}}$ also belongs to H_α . More precisely,*

- if $0 < S_{j_0+1/2}^0 \leq S_{j_0+3/2}^0$, i.e. if the first jump is smaller than the second one, then $S_{j_0}^1 = 0$;
- if $0 < S_{j_0+3/2}^0 < S_{j_0+1/2}^0$, i.e. if the first jump is larger than the second one, then $0 < S_{j_0}^1 \leq S_{j_0+1}^1$. Moreover, if $S_{j_0+5/2}^0 \geq \alpha$, then $S_{j_0+1/2}^2 \leq S_{j_0+1/2}^0 - \frac{\alpha}{4}$.

Proof. — First of all, from the previous lemma and Hypothesis (H_α^M) we have

$$\forall j \in \{j_0 + 1, \dots, j_0 + M - 1\}, S_j^1 \geq \alpha. \quad (10)$$

We easily see that $u_{rec}^0 = 0$ on $(-\infty, j_0 + 1/2)$ and $u_{rec}^0 = 1$ on $(j_0 + M - 1/2, +\infty)$. It yields

$$\forall j \leq j_0 - 1 \text{ and } \forall j \geq j_0 + M + 1, S_j^1 = 0.$$

We now focus on the two jumps near the left extremities, the results extending trivially to the right extremity. First, if $0 < S_{j_0+1/2}^0 \leq S_{j_0+3/2}^0$, then the reconstruction in \mathcal{C}_{j_0+1} lies in the right half of the cell $[j_0 + 1, j_0 + \frac{3}{2}]$. The reconstruction u_{rec}^0 is null on $[j_0 + \frac{1}{2}, j_0 + 1]$, so $u_{j_0+1/2}^1 = 0$ and $S_{j_0}^1 = 0$. The first inner jump is $S_{j_0+2}^1$ and is larger than α by (10).

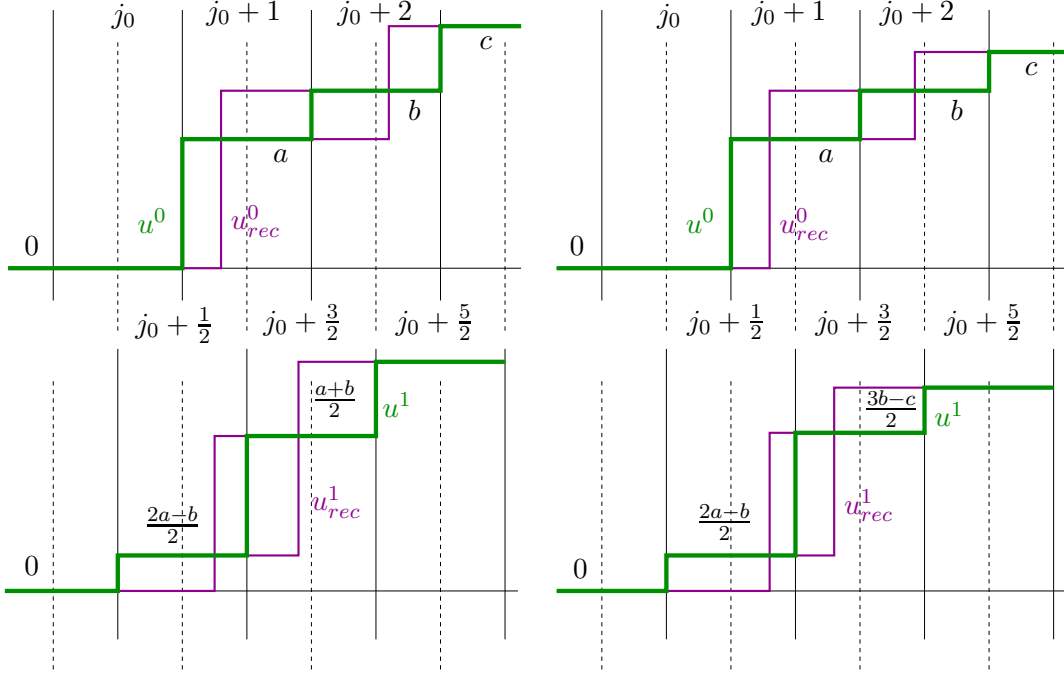


FIGURE 8. Behavior at the left extremity when the first jump is larger than the second one, depending on the relative sizes of the second and third jumps.

We now focus on the second case $S_{j_0+1/2}^0 > S_{j_0+3/2}^0 \geq \alpha$. In this case, one has on the one hand $S_{j_0}^1 > 0$ and on the other hand, by Lemma 9, $S_{j_0+1}^1 \geq \min(S_{j_0+1/2}^0, S_{j_0+3/2}^0) \geq \alpha$. Hence, $(u_{j-1/2}^1)_j \in H_\alpha$, and it remains to prove the bound about the second iteration. We will do it by a case disjunction, in each case it will follow from a simple computation.

We denote as usual $u_{j_0+1}^0 = a$, $u_{j_0+2}^0 = b$ and $u_{j_0+3}^0 = c$. The elements of proof are illustrated on Figure 8. Suppose first that the second jump $S_{j_0+3/2}^0$ is smaller than the third jump $S_{j_0+5/2}^0$ (Figure 8, left). Using Lemma 8 we obtain

$$u_{j_0-1/2}^1 = 0, \quad u_{j_0+1/2}^1 = \frac{2a-b}{2} \quad \text{and} \quad u_{j_0+3/2}^1 = \frac{a+b}{2}.$$

It follows that

$$S_{j_0}^1 = \frac{2a-b}{2} < \frac{a}{2} \quad \text{and} \quad S_{j_0+1}^1 = \frac{a+b}{2} - \frac{2a-b}{2} = \frac{2b-a}{2} > \frac{a}{2}.$$

We thus have $S_{j_0}^1 < S_{j_0+1}^1$, which implies that $u_{j_0}^2 = 0$ and $S_{j_0-1/2}^2 = 0$. We can bound the jump in $j_0 + 1/2$, using once again Lemma 8:

$$\begin{aligned} S_{j_0+1/2}^2 &= u_{j_0+1}^2 - u_{j_0}^2 = u_{j_0+1}^2 \\ &= \int_{j_0+1/2}^{j_0+1} u_{rec}^1(x) dx + \int_{j_0+1}^{j_0+3/2} u_{rec}^1(x) dx \\ &\leq \frac{2u_{j_0+1/2}^1 - u_{j_0-1/2}^1}{2} + \frac{u_{j_0+3/2}^1}{2} \quad \text{with (8) and (9)} \\ &= a - \frac{b-a}{4} \leq S_{j_0+1/2}^0 - \frac{\alpha}{4}. \end{aligned}$$

To conclude, we treat the case where the third jump $S_{j_0+5/2}^0$ is smaller than the second jump $S_{j_0+3/2}^0$ (Figure 8, right). We still have $u_{j_0+1/2}^1 = \frac{2a-b}{2}$, and using Lemma 8, first case both on $[j_0+1, j_0+\frac{3}{2}]$ and $[j_0+\frac{3}{2}, j_0+2]$ we obtain $u_{j_0+3/2}^1 = \frac{3b-c}{2}$. It follows that

$$S_{j_0}^1 = \frac{2a-b}{2} \quad \text{and} \quad S_{j_0+1}^1 = \frac{4b-2a-c}{2}.$$

The second jump is larger than the first jump:

$$S_{3/2}^0 \geq S_{5/2}^0 \implies 4(b-a) > (c-b) \implies \frac{4b-2a-c}{2} \geq \frac{2a-b}{2}.$$

Thus the first point of Lemma 10 gives $u_{j_0}^2 = 0$ and $S_{j_0-1/2}^2 = 0$. Eventually we compute $S_{j_0+1/2}^2 = u_{j_0+1}^2$. Using once again (8) and (9) we obtain

$$\int_{j_0+\frac{1}{2}}^{j_0+1} u_{rec}^1 = u_{j_0+1/2}^1 = \frac{2a-b}{2} \quad \text{and} \quad \int_{j_0+1}^{j_0+\frac{3}{2}} u_{rec}^1 \leq \frac{u_{j_0+3/2}^1}{2} = \frac{3b-c}{4}.$$

We end up with

$$S_{j_0+1/2}^2 \leq \frac{4a+b-c}{4} = a - \frac{c-b}{4} \leq S_{j_0+1/2}^0 - \frac{\alpha}{4},$$

which concludes the proof. □

We are now in position to prove the following theorem. It states that after a finite number of iterations, the process described in Section 1 is 2-periodic and at each time step, the sequence contains at most one intermediate value.

Theorem 11. — *Suppose that the initial data $(u_j^0)_{j \in \mathbf{Z}}$ belongs to H_α^M for some $\alpha > 0$ and some integer $M > 0$. Then there exists an integer $p = p(M, \alpha)$ such that for all $n \geq p$, $(u_j^n)_{j \in \mathbf{Z}}$ is in H_α^2 or H_α^1 , and the solution is 2-periodic: for all $n \geq p$, $u_j^{n+2} = u_j^n$.*

Proof. — All along the proof, we make a slight abuse of notation and drop distinction between odd and even iteration in time, always denoting $(u_j^n)_{j \in \mathbf{Z}}$. We do not wish to systematically distinguish between the two cases, which would only make the notation heavier.

We know by Lemma 10 that for all iteration in time n , there exists an integer M^n such that $(u_j^n)_{j \in \mathbf{Z}}$ belongs to $H_\alpha^{M^n}$. Note that if $M^n = 1$ or $M^n = 2$, α plays no role in the definition of H_α^M . We prove that the number M^n of strictly positive jumps at iteration n reaches 2 in a finite number of iterations. If $M^n \leq 2$, then $(u_j^n)_{j \in \mathbf{Z}}$ has only one intermediate value, thus u_{rec} contains a single discontinuity and it is easily seen that this form is 2-periodic (this is a special case of [DL01], Theorem 3).

We now estimate the number M^n of non zero jumps. Following Definition 7, we denote by j_0^n the last cell where u_j^n is null and recall that the first cell where u_j^n is 1 is $j_0^n + M^n$. Depending on whether the first jump is larger or smaller than the second one, Lemma 10 gives the relation between j_0^{n+1} and j_0^n . With a similar argument at the right extremity we deduce that M^{n+1} is equal to $M^n + 1$, M^n or $M^n - 1$.

There are four cases that can occur at the extremities. In what follows, L stands for “large jump” and S stands for “small jump” (relatively to each other). The elements of proof are gathered on Picture 9, where the points represents the jumps. The inner jumps (in grey) are all larger than α .

- **Case LS/SL, Fig. 9, top:** in that case

$$S_{j_0^n+1/2}^n > S_{j_0^n+3/2}^n \quad \text{and} \quad S_{j_0^n+M^n-3/2}^n < S_{j_0^n+M^n-1/2}^n,$$

hence (Lemma 10, second case)

$$0 < S_{j_0^n}^{n+1} \leq S_{j_0^n+1}^{n+1} \quad \text{and} \quad 0 < S_{j_0^n+M^n+1}^{n+1} \leq S_{j_0^n+M^n}^{n+1}.$$

Thus the number of non zero jumps increases by 1: $M^{n+1} = M^n + 1$ and the solution at time $n + 1$ is in configuration SL/LS.

- **Case SL/LS, Fig. 9, middle:** in that case

$$S_{j_0^n+1/2}^n \leq S_{j_0^n+3/2}^n \quad \text{and} \quad S_{j_0^n+M^n-3/2}^n \geq S_{j_0^n+M^n-1/2}^n.$$

By Lemma 10, first case,

$$S_{j_0^n}^{n+1} = 0 \quad \text{and} \quad S_{j_0^n+M^n+1}^{n+1} = 0.$$

The number of non zero jumps decreases by 1: $M^{n+1} = M^n - 1$. We do not have any information on the relative positions of the first and second jump (neither on last and second last) at the iteration $n + 1$, and actually the four configurations LS/SL, LS/SL, LS/LS and SL/SL are possible.

- **Case SL/SL, Fig. 9, bottom right:** in that case

$$S_{j_0^n+1/2}^n \leq S_{j_0^n+3/2}^n \quad \text{and} \quad S_{j_0^n+M^n-3/2}^n \leq S_{j_0^n+M^n-1/2}^n$$

and

$$0 = S_{j_0^n}^{n+1} \quad \text{and} \quad 0 < S_{j_0^n+M^n+1}^{n+1} \leq S_{j_0^n+M^n}^{n+1}.$$

It follows that $M^{n+1} = M^n$. At the next iteration in time, we only have information on the last two jumps, so it is possible to end up in situations LS/LS or SL/LS.

- **Case LS/LS, Fig. 9, bottom left:** in that case

$$S_{j_0^n+1/2}^n > S_{j_0^n+3/2}^n \quad \text{and} \quad S_{j_0^n+M^n-3/2}^n \geq S_{j_0^n+M^n-1/2}^n.$$

and

$$0 < S_{j_0^n}^{n+1} \leq S_{j_0^n+1}^{n+1} \quad \text{and} \quad 0 = S_{j_0^n+M^n+1}^{n+1}.$$

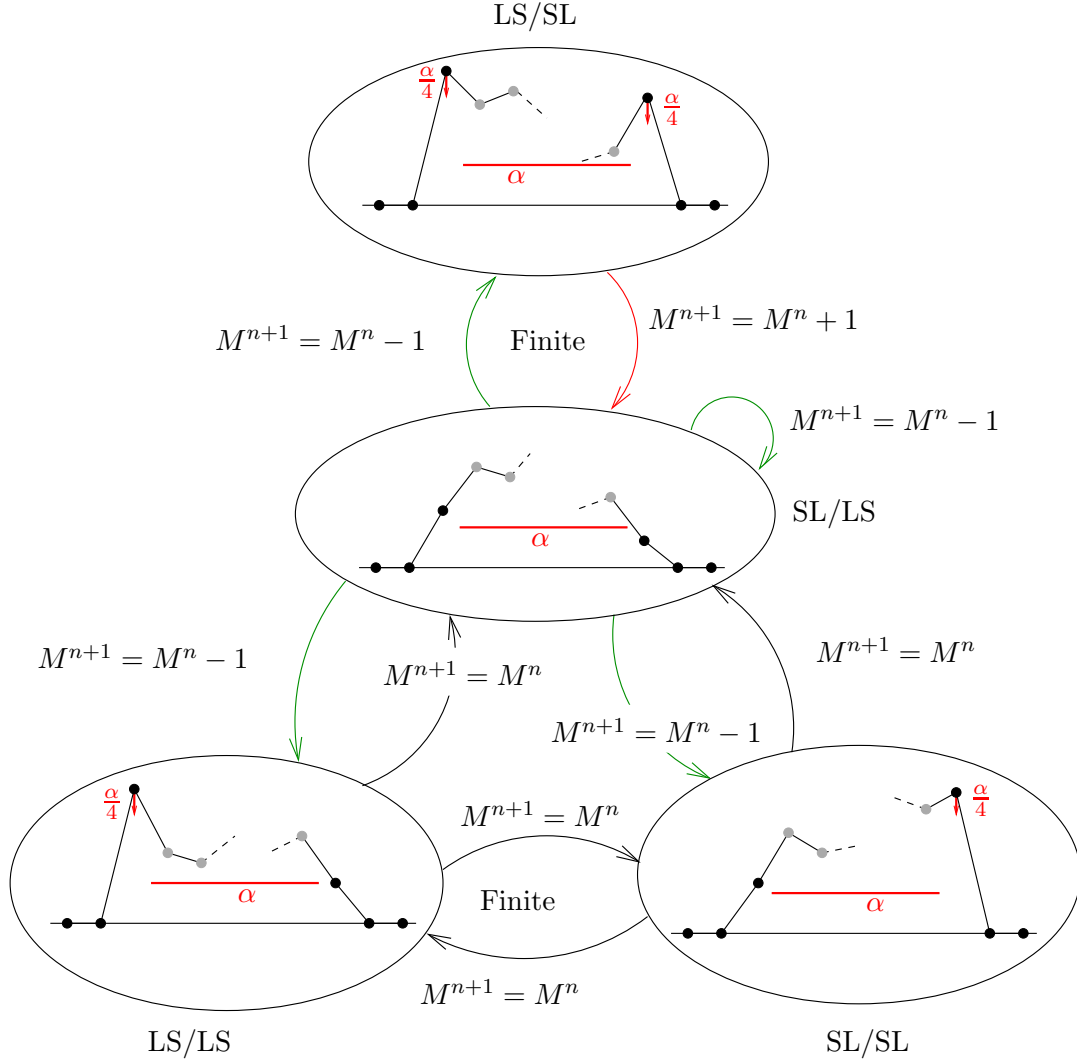


FIGURE 9. Transitions between the four possible configurations at the extremities and evolution of the number M^n of non null jumps.

The number of non zero jump remains unchanged $M^{n+1} = M^n$. At time $n + 1$, it is possible to be in cases SL/SL and SL/LS.

Looking at the transitions between the four possible situations on Figure 9, we see that the number of jumps decreases of 1 each time Case SL/LS is left. It remains to prove that this is the most frequent case, i.e. that it is not possible to cycle indefinitely from case SL/LS to case LS/SL or from case SL/SL to case LS/LS. It follows from the fact that if $S_{j_0+1/2}^n > S_{j_0+3/2}^n$, then $S_{j_0+1/2}^{n+2} \leq S_{j_0+1/2}^n - \frac{\alpha}{4}$, see Lemma 10. Thus, the left extreme value decreases of $\frac{\alpha}{4}$ after one cycle, and in particular will be smaller than α after a finite number of cycles (smaller than $2/\alpha$). At this stage, it is necessary smaller than the first inner jump in $S_{j_0+3/2}^n$ which is larger than α , thus the solution exits the cycle and falls in situation SL/LS. This concludes the proof. \square

3.2. Half infinite staircase with steps of equal heights. — We now consider staircase-like initial data. It once again illustrates the importance of the behavior at extremities. We say that $(u_j^n)_j$ satisfies Hypothesis (H') if, up to a horizontal translation,

1. $(u_j^n)_j$ is constant on $\{j \leq 0\}$, i.e. for any $j \leq 0$, $S_{j-1/2}^n = 0$;
2. for any $j \geq 3$, we have $S_{j-1/2}^n = 1$;
3. $S_{3/2}^n \geq 1$;
4. $S_{1/2}^n \geq 0$.

As usual, we here considered an even time n , the odd case being identical (up to a shift of $1/2$ in the notations).

The following proposition expresses that if the initial condition satisfies Hypothesis (H'), then the solution will satisfy Hypothesis (H') at all time, and the total height of the two first steps will tend to infinity as the time goes to infinity.

Proposition 12. — *If $(u_j^n)_j$ satisfies Hypothesis (H'), then so does $(u_{j-1/2}^{n+1})_j$. Moreover, one has, $S_1^n + S_2^n \rightarrow \infty$.*

This proposition is illustrated on Figure 10: the first step falls down at each iteration, until it disappears.

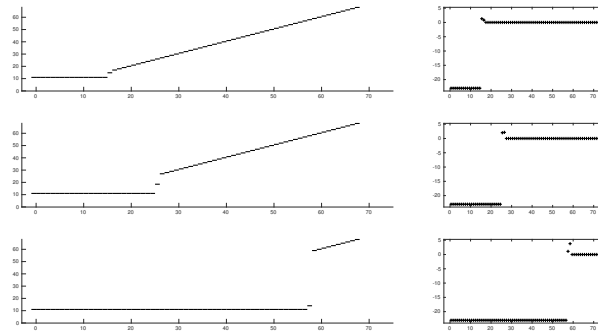


FIGURE 10. Numerical solution after 100 iterations (light gray) to 800 iterations (dark gray) when the initial data is a half infinite staircase

Proof. — We first prove the first part of the proposition.

Of course, Hypothesis (1) of (H') is still satisfied at time $n + 1$. As $S_{5/2}^n = 1$, Equation (4) gives:

$$d_1^n = \frac{S_{3/2}^n}{S_{1/2}^n + S_{3/2}^n} \quad \text{and} \quad d_2^n = \frac{1}{1 + S_{3/2}^n}$$

By Hypothesis (3) of (H'), we have $d_2^n \leq 1/2$. Then the expression of the reconstruction depends on the sign of $d_1^n - 1/2$. We have two cases:

- (i) $d_1^n \leq 1/2 \iff S_{1/2}^n \geq S_{3/2}^n$. In this case, a computation leads to

$$u_{1/2}^{n+1} = \frac{S_{1/2}^n - S_{3/2}^n}{2}, \quad u_{3/2}^{n+1} = u_2^n - \frac{1}{2},$$

and for $j \geq 2$, $u_{j+1/2}^{n+1} = \frac{u_j^n + u_{j+1}^n}{2}$. So Hypothesis (2) of (H') is satisfied at step $n + 1$. Moreover

$$S_0^{n+1} = \frac{S_{1/2}^n - S_{3/2}^n}{2} \quad \text{and} \quad S_1^{n+1} = \frac{3S_{3/2}^n + S_{1/2}^n - 1}{2}$$

and using that $S_{1/2}^n \geq S_{3/2}^n \geq 1$, we get that $S_1^{n+1} \geq 3/2$, so Hypothesis (3) is satisfied at step $n + 1$.

Remark that in this case, we have

$$S_0^{n+1} - S_1^{n+1} = \frac{1 - 4S_{3/2}^n}{2} \leq -\frac{3}{2}, \quad (11)$$

so $S_1^{n+1} < S_2^{n+1}$. In other words if case (i) occurs at time n , then it occurs case (ii) at time $n + 1$.

(ii) $d_1^n \geq 1/2 \iff S_{1/2}^n \leq S_{3/2}^n$. In this case,

$$u_{1/2}^{n+1} = 0, \quad u_{3/2}^{n+1} = \frac{2u_1^n + u_2^n - 1}{2},$$

and for $j \geq 2$, $u_{j+1/2}^{n+1} = \frac{u_j^n + u_{j+1}^n}{2}$. Hypothesis (2) is immediately satisfied. Thus we have

$$S_1^{n+1} = \frac{3S_{1/2}^n + S_{3/2}^n - 1}{2} \quad \text{and} \quad S_2^{n+1} = \frac{S_2^n - S_1^n}{2} + 1 \geq 1$$

so Hypothesis (3) is satisfied. Further computations yields $S_1^{n+1} + S_2^{n+1} = S_1^n + S_2^n + \frac{1}{2}$.

Let us now prove the second part of the proposition. A simple computation gives the sum of the two first jumps

$$\begin{cases} S_0^{n+1} + S_1^{n+1} = S_{1/2}^n + S_{3/2}^n - \frac{1}{2} & \text{in case (i)} \\ S_1^{n+1} + S_2^{n+1} = S_{1/2}^n + S_{3/2}^n + \frac{1}{2} & \text{in case (ii)} \end{cases}$$

By (11), if at time n we are in case (i), then at time $n + 1$ we have to be in case (ii). So the sequence $(S_1^{2n} + S_2^{2n})_n$ is increasing. To prove that it tends to $+\infty$, we only have to prove that there are infinitely times $m \in \mathbf{N}$ such that at both times m and $m + 1$ we are in case (ii).

If at time n we are in case (i), then at time $n + 1$ we have to be in case (ii), and a simple computation leads to

$$S_1^{n+2} = S_1^n - \frac{3}{4} \quad \text{and} \quad S_2^{n+2} = S_2^n + \frac{3}{4},$$

in particular

$$S_1^{n+2} - S_2^{n+2} = S_1^n - S_2^n - \frac{3}{2}.$$

Thus, by a straightforward induction, for any integer $k \leq k_0$, where

$$k_0 = \left\lfloor \frac{2}{3}(S_1^n - S_2^n) \right\rfloor,$$

we are in case (i) in time $n + 2k$ and in case (ii) in time $n + 2k + 1$, while we are again in case (ii) in time $n + 2k + 2$. In other words, for any $n \in \mathbf{N}$, we have found a time $m > n$ such that at both times m and $m + 1$ we are in case (ii). \square

4. The non symmetric case $\lambda \neq 1/2$

Contrary to the last section, we apply the scheme with parameter $\lambda \in (0, 1/2)$. The grid is shifted alternatively to the left and to the right as explained in Section 1. We will prove that there is an open set of initial conditions which are 5-configurations (five jumps, four intermediate values, see Definition 7) on which the solutions converge exponentially to a 5-configuration having a size 2 plateau.

We first need some notations. We take $j_0 = 0$, denote $\varepsilon^n = u_3^n - u_2^n$, and

$$\begin{aligned} u_1^\infty &= u_1^0 - \frac{2\lambda - \lambda^2}{1 - 4\lambda^2} \varepsilon^0 \\ u_2^\infty = u_3^\infty &= \frac{1 + \lambda}{1 + 2\lambda} u_2^0 + \frac{\lambda}{1 + 2\lambda} u_3^0 \\ u_4^\infty &= u_4^0 + \frac{1 - \lambda^2}{1 - 4\lambda^2} \varepsilon^0 \end{aligned}$$

Proposition 13. — *If (u_j^0) is in a 5-configuration (meaning that $(S_n^0) \in M_0^5$), and if*

- (a) $u_2^0 - u_1^0 \geq 2\varepsilon^0$;
- (b) $u_1^\infty \geq \lambda u_2^\infty$;
- (c) $\lambda(u_4^0 - u_3^0) \geq (1 - \lambda)\varepsilon^0$;
- (d) $u_4^0 - u_3^0 \geq \lambda(1 - u_3^0)$;
- (e) $1 - u_4^\infty \geq (1 - \lambda^2)\varepsilon^0$;

then $(u_i^n)_i$ is in a 5-configuration for any $n \geq 0$. Moreover, $(u_i^n)_i$ converges uniformly exponentially fast towards the configuration $(u_i^\infty)_i$.

This convergence is illustrated on Figure 11

Remark 14. — The set of 5-configurations satisfying (a) to (e) contains a nonempty open set. Indeed, consider any number $v_2 = v_3 \in (0, 1)$, and two numbers v_1 and v_4 such that

$$\frac{v_1 - 0}{v_2 - 0}, \frac{v_4 - v_2}{1 - v_2} \in (\lambda, 1)$$

(see Figure 12). In other words, consider the intervals $(0, v_2)$ and $(v_2, 1)$ and divide each of them into two intervals of relative sizes λ and $1 - \lambda$; the numbers v_1 and v_4 have to be in the upper respective subintervals. One easily checks that any 5-configuration sufficiently close to the configuration (v_i) satisfies (a) to (e).

Remark 15. — With a small computation one can see that condition (b) cannot hold when $\lambda = 1/2$ unless ε^0 is null. Thus this result is not in contradiction with Section 3.

Proof. — We prove by induction the following properties:

- (u_j^{2n}) is in a 5-configuration;

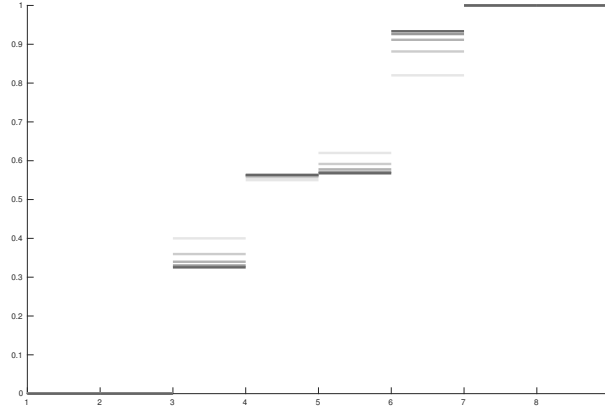


FIGURE 11. First iterations of the scheme for a 5-configuration verifying the hypothesis of Proposition 13. Most recent iterations are of darker color.

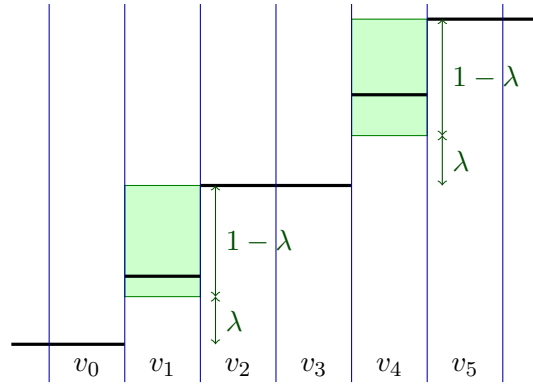


FIGURE 12. One sets $v_0 = 0$, $v_5 = 1$ and chooses $v_2 = v_3 \in (0, 1)$. This defines “good” intervals (in green) which are the upper parts of the intervals $(0, v_2)$ and $(v_3, 1)$ of relative lengths $1 - \lambda$; the numbers v_1 and v_4 can be chosen anywhere in these intervals.

- $\varepsilon^{2n} = (4\lambda^2)^n \varepsilon^0$;
- and the following bounds on the intermediate values:

$$\begin{aligned}
 u_1^0 - (2\lambda - \lambda^2) \frac{1 - (4\lambda^2)^{n+1}}{1 - 4\lambda^2} \varepsilon^0 &\leq u_1^{2n} \leq u_1^0 \\
 u_2^0 &\leq u_2^{2n} \leq u_2^0 + (\lambda - 2\lambda^2) \frac{1 - (4\lambda^2)^{n+1}}{1 - 4\lambda^2} \varepsilon^0 \\
 u_3^0 - (1 - \lambda - 2\lambda^2) \frac{1 - (4\lambda^2)^{n1}}{1 - 4\lambda^2} \varepsilon^0 &\leq u_3^{2n} \leq u_3^0 \\
 u_4^0 &\leq u_4^{2n} \leq u_4^0 + (1 - \lambda^2) \frac{1 - (4\lambda^2)^{n+1}}{1 - 4\lambda^2} \varepsilon^0.
 \end{aligned}$$

Configurations satisfying these properties are said to satisfy property (P). In particular, this will prove that the configurations u_j^{2n} are as in Figure 13, since these conditions imply that

$$\begin{aligned} u_1^\infty &\leq u_1^{2n} \leq u_1^0 \\ u_2^0 &\leq u_2^{2n} \leq u_2^\infty \\ u_3^\infty &\leq u_3^{2n} \leq u_3^0 \\ u_4^0 &\leq u_4^{2n} \leq u_4^\infty, \end{aligned}$$

Suppose that a configuration u_j^{2n} satisfies property (P). We want to prove that the configuration u_j^{2n+2} still satisfies property (P).

For odd iteration in time, the grid is shifted to the left and the distance $d_j^{2n} = \frac{u_j^{2n} - u_{j-1}^{2n}}{u_{j+1}^{2n} - u_{j-1}^{2n}}$ satisfies

$$d_j^{2n} \geq \lambda \iff (1 - \lambda)(u_j^{2n} - u_{j-1}^{2n}) \geq \lambda(u_{j+1}^{2n} - u_j^{2n}).$$

Let us prove that $d_1^{2n} \geq \lambda$, $d_2^{2n} \geq \lambda$, $d_3^{2n} \leq \lambda$ and $d_4^{2n} \geq \lambda$. By the hypotheses made on the initial configuration, we have respectively

- $(1 - \lambda)(u_1^{2n} - 0) \geq \lambda(u_2^{2n} - u_1^{2n})$ because by Condition (b),
 $(1 - \lambda)u_1^{2n} \geq (1 - \lambda)u_1^\infty \geq \lambda(u_2^\infty - u_1^\infty) \geq \lambda(u_2^{2n} - u_1^{2n})$.
- $(1 - \lambda)(u_2^{2n} - u_1^{2n}) \geq \lambda\varepsilon^{2n}$ which is true by Condition (a):
 $(1 - \lambda)(u_2^{2n} - u_1^{2n}) \geq (1 - \lambda)(u_2^0 - u_1^0) \geq \lambda\varepsilon^0 \geq \lambda\varepsilon^{2n}$.
- $(1 - \lambda)\varepsilon^{2n} \leq \lambda(u_2^{2n} - u_3^{2n})$ which is true by Condition (c).
- $(1 - \lambda)(u_4^{2n} - u_3^{2n}) \geq \lambda(1 - u_4^{2n})$ which is true by Condition (d).

In these cases, one can compute $u_{i-\lambda}^{2n+1}$ (using in particular (4)). It is a 5-configuration, with

$$\begin{aligned} u_{1-\lambda}^{2n+1} &= u_1^{2n} - \lambda u_2^{2n} \\ u_{2-\lambda}^{2n+1} &= u_2^{2n} - \lambda\varepsilon^{2n} \\ u_{3-\lambda}^{2n+1} &= u_2^{2n} + \lambda\varepsilon^{2n} \\ u_{4-\lambda}^{2n+1} &= u_4^{2n} - \lambda(1 - u_2^{2n}) + \varepsilon^{2n} \end{aligned}$$

and in particular,

$$\varepsilon^{2n+1} = 2\lambda\varepsilon^{2n}$$

For the next iteration in time, the grid is shifted to the right, the distance $d_{j-\lambda}^{2n+1} = \frac{u_{j+1-\lambda}^{2n+1} - u_{j-\lambda}^{2n+1}}{u_{j+1-\lambda}^{2n+1} - u_{j-1-\lambda}^{2n+1}}$ is counted from the left interface and

$$d_{j-\lambda}^{2n+1} \geq \lambda \iff (1 - \lambda)(u_{j+1-\lambda}^{2n+1} - u_{j-\lambda}^{2n+1}) \geq \lambda(u_{j-\lambda}^{2n+1} - u_{j-1-\lambda}^{2n+1}).$$

Now, we have $d_{1-\lambda}^{2n+1} \geq \lambda$, $d_{2-\lambda}^{2n+1} \leq \lambda$, $d_{3-\lambda}^{2n+1} \geq \lambda$ and $d_{4-\lambda}^{2n+1} \geq \lambda$, because by Hypothesis (P), we have respectively

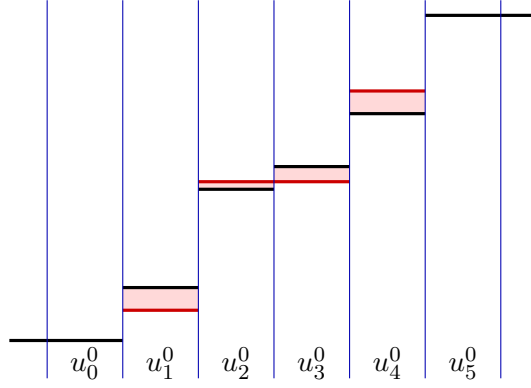


FIGURE 13. The solutions u_i^{2n} all lie in the red domains delimited by the initial configuration u_i^0 (black) and the limit configuration u_i^∞ (red) having a 2 plateau

- $\lambda(u_{1-\lambda}^{2n+1} - 0) \leq (1 - \lambda)(u_{2-\lambda}^{2n+1} - u_{1-\lambda}^{2n+1}) \iff u_2^{2n} - u_1^{2n} \geq (\lambda - \lambda^2)\varepsilon^{2n}$ which is true by condition (a);
- $\lambda(u_{2-\lambda}^{2n+1} - u_{1-\lambda}^{2n+1}) \geq (1 - \lambda)\varepsilon^{2n+1} \iff u_2^{2n} - u_1^{2n} + \lambda u_2^{2n} \geq (2 - \lambda)\varepsilon^{2n}$ which is true by condition (a);
- $\lambda\varepsilon^{2n+1} \leq (1 - \lambda)(u_{4-\lambda}^{2n+1} - u_{3-\lambda}^{2n+1}) \iff \lambda(1 - \lambda)(1 - u_4^{2n}) \leq (2 - 4\lambda)\varepsilon^{2n} + (1 - \lambda)^2(u_4^{2n} - u_3^{2n})$ which is true by condition (d);
- $\lambda(u_{4-\lambda}^{2n+1} - u_{3-\lambda}^{2n+1}) \leq (1 - \lambda)(1 - u_{4-\lambda}^{2n+1}) \iff (1 - \lambda^2)\varepsilon^{2n} \leq 1 - u_4^{2n}$ which is true by condition (e).

These conditions allow to compute the sequence u_j^{n+2} :

$$\begin{aligned} u_1^{2n+2} &= u_1^{2n+1} + \lambda u_2^{2n+1} - (1 - \lambda)\varepsilon^{2n+1} \\ u_2^{2n+2} &= u_3^{2n+1} - \lambda\varepsilon^{2n+1} \\ u_3^{2n+2} &= u_3^{2n+1} + \lambda\varepsilon^{2n+1} \\ u_4^{2n+2} &= u_4^{2n+1} + \lambda(1 - u_3^{2n+1}) \end{aligned}$$

thus

$$\begin{aligned} u_1^{2n+2} &= u_1^{2n} - (2\lambda - \lambda^2)\varepsilon^{2n} \\ u_2^{2n+2} &= u_2^{2n} + (\lambda - 2\lambda^2)\varepsilon^{2n} \\ u_3^{2n+2} &= u_3^{2n} - (1 - \lambda - 2\lambda^2)\varepsilon^{2n} \\ u_4^{2n+2} &= u_4^{2n} + (1 - \lambda^2)\varepsilon^{2n} \end{aligned}$$

and in particular

$$\varepsilon^{2n+2} = 4\lambda^2\varepsilon^{2n}.$$

□

As a conclusion, let us mention that on general initial data, the sequence $(u_j^n)_{j \in \mathbf{R}}$ quickly goes from its initial state to some “stairshaped” organization. Then, we observe an exponential convergence of the smaller jumps, as illustrated on Figure 14.

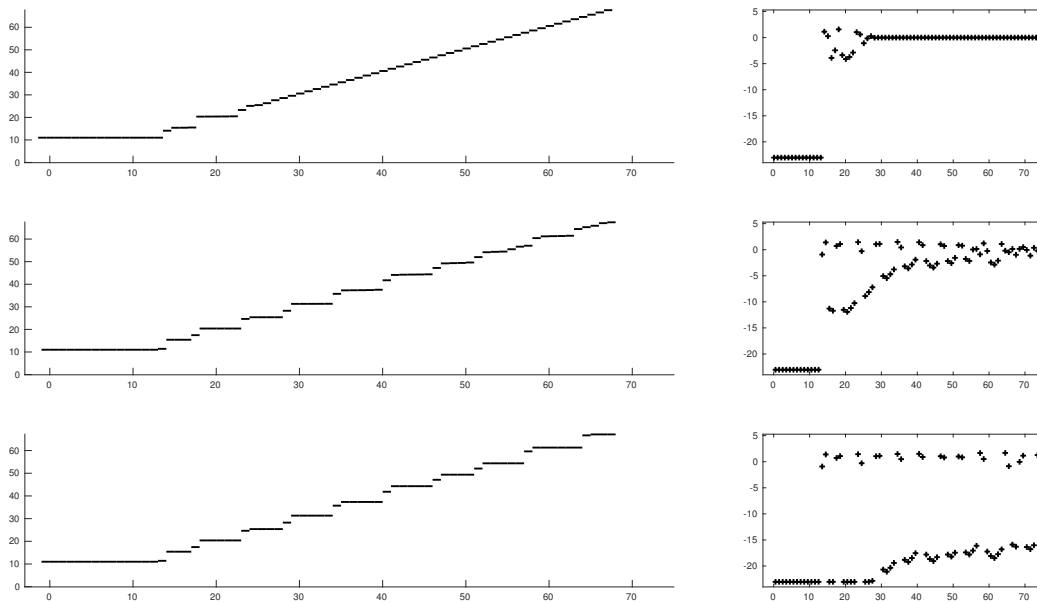


FIGURE 14. Left: apparition of large plateaus when the initial data is a half infinite staircase and $\lambda = 0.4$. Right: height of the jumps in log-scale; we observe an exponential convergence as in Proposition 13

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