

# Characterization of hypersurfaces in four dimensional product spaces via two different $\text{Spin}^c$ structures

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## Abstract

The Riemannian product  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$ , where  $\mathbb{M}_i(c_i)$  denotes the 2-dimensional space form of constant sectional curvature  $c_i \in \mathbb{R}$ , has two different  $\text{Spin}^c$  structures carrying each a parallel spinor. The restriction of these two parallel spinor fields to a 3-dimensional hypersurface  $M$  characterizes the isometric immersion of  $M$  into  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$ . As an application, we prove that totally umbilical hypersurfaces of  $\mathbb{M}_1(c_1) \times \mathbb{M}_1(c_1)$  and totally umbilical hypersurfaces of  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  ( $c_1 \neq c_2$ ) having a local structure product, are of constant mean curvature.

**Keywords:**  $\text{Spin}^c$  structures on hypersurfaces, totally umbilical hypersurfaces, parallel  $\text{Spin}^c$  spinors, generalized Killing  $\text{Spin}^c$  spinors, Kähler manifolds.

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## 1 Introduction

Over the past years, the real spinorial ( $\text{Spin}$  geometry) and the complex spinorial ( $\text{Spin}^c$  geometry) approaches have been used fruitfully to characterize ([7, 22, 31, 1, 2, 3, 19, 26, 24] and references therein) submanifolds of some special ambient manifolds. These approaches allowed also to study the geometry and topology of submanifolds and solve naturally some extrinsic problems. For instance, elementary proofs of the Alexandrov theorem in the Euclidean space [13], in the hyperbolic space [11] and in the Minkowski spacetime [11] were obtained (see also [8, 9]). In 2006, O. Hijazi, S. Montiel and F. Urbano [12] constructed on Kähler-Einstein manifolds with positive scalar curvature, a  $\text{Spin}^c$  structure carrying Kählerian Killing spinors. The restriction of these spinors to minimal Lagrangian submanifolds provides topological and geometric restrictions on these submanifolds. The authors [24, 25, 26], and by restricting  $\text{Spin}^c$  spinors, gave an elementary  $\text{Spin}^c$  proof for a Lawson type correspondence between constant mean curvature surfaces of 3-dimensional homogeneous manifolds with 4-dimensional isometry group [4]. Furthermore, they gave necessary and sufficient geometric conditions to immerse a 3-dimensional Sasaki manifold and a complex/Lagrangian surface into the complex projective space of complex dimension 2.

The main idea behind characterizing hypersurfaces of  $\text{Spin}$  or  $\text{Spin}^c$  manifolds is the restriction to the hypersurface of a special spinor field (parallel, real Killing, imaginary Killing, Kählerian Killing...). For example, the restriction  $\phi$  of a parallel spinor field on a Riemannian  $\text{Spin}$  or  $\text{Spin}^c$  manifold to an oriented hypersurface  $M$  is a solution of the generalized Killing equation

$$\nabla_X \phi = -\frac{1}{2} \gamma(IX) \phi, \quad (1)$$

where  $\gamma$  and  $\nabla$  are respectively the Clifford multiplication and the Spin or  $\text{Spin}^c$  connection on  $M$ , the tensor  $II$  is the Weingarten tensor of the immersion and  $X$  any vector field on  $M$ . Conversely and in the two-dimensional case, the existence of a generalized Killing Spin spinor field allows to immerse  $M$  in  $\mathbb{R}^3$  [7]. This characterization has been extended to surfaces of other 3-dimensional (pseudo-) Riemannian manifolds [22, 31, 18]. Moreover, the existence of a generalized Killing  $\text{Spin}^c$  spinor on a surface  $M$  allows to immerse  $M$  in the 3-dimensional homogeneous manifolds with 4-dimensional isometry group [26]. All these previous results are the geometrical invariant versions of previous works on the spinorial Weierstrass representation by R. Kusner and N. Schmidt, B. Konoplechenko, I. Taimanov and many others (see [15, 17, 32]).

In the three dimensional case, having a generalized Killing Spin or  $\text{Spin}^c$  spinor is not sufficient to characterize the immersion of  $M$  in the desired 4-dimensional manifold. The problem is that unlike in the 2-dimensional case, the spinor bundle of a 3-dimensional manifold does not decompose into subbundles of positive and negative half-spinors. In fact, Morel [22] proved that the existence of a Codazzi generalized Killing Spin spinor on a 3-dimensional manifold  $M$  is equivalent to immerse  $M$  in  $\mathbb{R}^4$ . But it was proved in [3, 30] that restricting a Spin structure with a spinor field having non-vanishing positive and negative parts is required to get the integrability condition of an immersion in the desired 4-dimensional target space. Hence, Morel's result has been reformulated for hypersurfaces of  $\mathbb{R}^4$  [19] because  $\mathbb{R}^4$  has a Spin structure with positive and negative parallel spinors. The restriction of both spinors to  $M$  gives two generalized Killing spinors which, conversely, allow to characterize the immersion of  $M$  in  $\mathbb{R}^4$ . This result has been extended to other 4-dimensional space forms and product spaces, that is  $\mathbb{S}^4$ ,  $\mathbb{H}^4$ ,  $\mathbb{S}^3 \times \mathbb{R}$  and  $\mathbb{H}^3 \times \mathbb{R}$  [19]. In the  $\text{Spin}^c$  setting, the existence of a Codazzi generalized Killing  $\text{Spin}^c$  spinor on a 3-dimensional manifold  $M$  is equivalent to immerse  $M$  in the 2-dimensional complex space form  $\mathcal{M}_2(k)$  of constant holomorphic sectional curvature  $4k$  [26]. However here, the condition "Codazzi" cannot be dropped as in the Spin case, because  $\mathcal{M}_2(k)$  has only two different  $\text{Spin}^c$  structures (the canonical and the anti-canonical  $\text{Spin}^c$  structures) carrying each one parallel spinor lying in the positive half-part of the corresponding  $\text{Spin}^c$  bundles.

The aim of the present article is to use  $\text{Spin}^c$  geometry to characterize hypersurfaces of the Riemannian product  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$ , where  $\mathbb{M}_i(c_i)$  denotes the 2-dimensional space form of constant sectional curvature  $c_i \in \mathbb{R}$ . The key starting point is that this product has two different  $\text{Spin}^c$  structures carrying each a non-vanishing parallel spinor. The first structure  $S_1$  is the product of the canonical  $\text{Spin}^c$  structure on  $\mathbb{M}_1(c_1)$  with the canonical  $\text{Spin}^c$  structure on  $\mathbb{M}_2(c_2)$  and it has a non-vanishing parallel spinor lying in the positive half-part of the  $\text{Spin}^c$  bundle. The second structure  $S_2$  is the product of the canonical  $\text{Spin}^c$  structure on  $\mathbb{M}_1(c_1)$  with the anti-canonical  $\text{Spin}^c$  structure on  $\mathbb{M}_2(c_2)$  and it has a non-vanishing parallel spinor lying in the negative half-part of the  $\text{Spin}^c$  bundle. Having said that one could expect that restricting both structures  $S_1$  and  $S_2$ , and hence both parallel spinors, to a hypersurface  $M$  of  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  could allow to characterize the immersion.

We denote by  $\nabla^j$ ,  $\gamma_j$  and  $i\Omega^j$  respectively the Clifford multiplication, the  $\text{Spin}^c$  connection and the curvature of the auxiliary line bundle on the hypersurface  $M$  obtained after restricting the  $\text{Spin}^c$  structure  $S_j$  on  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  (here  $j \in \{1, 2\}$ ). The main theorem of the paper is:

**Theorem 1.** *Let  $(M^3, g = (\cdot, \cdot))$  be a simply connected oriented Riemannian manifold endowed with an almost contact metric structure  $(\mathfrak{X}, \xi, \eta)$ . Let  $E$  be a field of symmetric endomorphisms on  $M$ ,  $h$  a function on  $M$  and  $V$  a vector field on  $M$ . Then, the following statements are equivalent:*

1. *There exists an isometric immersion of  $(M^3, g)$  into  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  with shape operator  $E$  and so that, over  $M$ , the complex structure of  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  is given*

by  $J = \mathfrak{X} + \eta(\cdot)\nu$ , where  $\nu$  is the unit normal vector of the immersion and the product structure is given by  $F = f + (V, \cdot)\nu$  for some endomorphism  $f$  on  $M$ .

2. There exists two  $\text{Spin}^c$  structures on  $M$  carrying each one a non-trivial spinor  $\varphi_1$  and  $\varphi_2$  satisfying

$$\nabla_X^1 \varphi_1 = -\frac{1}{2} \gamma_1(EX) \varphi_1 \quad \text{and} \quad \gamma_1(\xi) \varphi_1 = -i \varphi_1.$$

$$\nabla_X^2 \varphi_2 = \frac{1}{2} \gamma_2(EX) \varphi_2 \quad \text{and} \quad \gamma_2(V) \varphi_2 = -i \gamma_2(\xi) \varphi_2 + h \varphi_2.$$

The curvature 2-form  $i\Omega^j$  of the connection on the auxiliary bundle associated with each  $\text{Spin}^c$  structure is given by ( $j \in \{1, 2\}$ )

$$\begin{cases} \Omega^j(e_1, e_2) = \frac{1}{2}(-1)^{j-1} c_1(h-1) - \frac{1}{2} c_2(h+1), \\ \Omega^j(e_1, \xi) = \frac{1}{2} \left( (-1)^{j-1} c_1 - c_2 \right) (e_1, V), \\ \Omega^j(e_2, \xi) = \frac{1}{2} \left( (-1)^{j-1} c_1 - c_2 \right) (e_2, V), \end{cases}$$

in the basis  $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$ .

Again, these two  $\text{Spin}^c$  structures (resp. two generalized Killing  $\text{Spin}^c$  spinors) on  $M$  comes from the restriction of the two  $\text{Spin}^c$  structures  $S_1$  and  $S_2$  (resp. the two parallel spinors) on  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$ . Needless to say, when  $c_1 = c_2 = 0$ , these two  $\text{Spin}^c$  structures on  $M$  coincide and it is in fact the  $\text{Spin}$  structure coming from the restriction of the unique  $\text{Spin}$  structure on  $\mathbb{R}^4$  having positive and negative parallel spinors. When  $c_1 \neq 0$  or  $c_2 \neq 0$ , the two structures in  $M$  are different because they are the restriction of the two different structures  $S_1$  and  $S_2$ .

As an application of Theorem 1, we prove that totally umbilical hypersurfaces of  $\mathbb{M}_1(c_1) \times \mathbb{M}_1(c_1)$  and totally umbilical hypersurfaces of  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  ( $c_1 \neq c_2$ ) having a local structure product are of constant mean curvature (see Proposition 5.2 and Proposition 5.3).

## 2 Preliminaries

In this section we briefly introduce basic facts about  $\text{Spin}^c$  geometry of hypersurfaces (see [20, 21, 6, 28, 27]). Then we describe hypersurfaces of the Riemannian product  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  [16, 5], where  $\mathbb{M}_i(c_i)$  denotes the 2-dimensional space form of constant sectional curvature  $c_i \in \mathbb{R}$ .

### 2.1 Hypersurfaces and induced $\text{Spin}^c$ structures

**$\text{Spin}^c$  structures on manifolds:** Let  $(N^{n+1}, \bar{g})$  be a Riemannian  $\text{Spin}^c$  manifold of dimension  $n+1 \geq 2$  without boundary. On such a manifold, we have a Hermitian complex vector bundle  $\Sigma N$  endowed with a natural scalar product  $(\cdot, \cdot)$  and with a connection  $\nabla^N$  which parallelizes the metric. This complex vector bundle, called the  $\text{Spin}^c$  bundle, is endowed with a Clifford multiplication denoted by “ $\cdot$ ”,  $\cdot : TN \rightarrow \text{End}_{\mathbb{C}}(\Sigma N)$ , such that at every point  $x \in N$ , defines an irreducible representation of the corresponding Clifford algebra. Hence, the complex rank of  $\Sigma N$  is  $2^{\lfloor \frac{n+1}{2} \rfloor}$ . Given a  $\text{Spin}^c$  structure on  $(N^{n+1}, \bar{g})$ , one can prove that the determinant line bundle  $\det(\Sigma N)$  has a root of index  $2^{\lfloor \frac{n+1}{2} \rfloor - 1}$ . We denote by  $L^N$  this root line bundle over  $N$  and call it the auxiliary line bundle associated with the  $\text{Spin}^c$  structure. Locally, a  $\text{Spin}$  structure always exists. We denote by  $\Sigma' N$  the (possibly globally non-existent) spinor bundle. Moreover, the square root of the auxiliary line bundle  $L^N$  always exists locally. But,  $\Sigma N = \Sigma' N \otimes (L^N)^{\frac{1}{2}}$  exists globally. This

essentially means that, while the spinor bundle and  $(L^N)^{\frac{1}{2}}$  may not exist globally, their tensor product (the  $\text{Spin}^c$  bundle) is defined globally. Thus, the connection  $\nabla^N$  on  $\Sigma N$  is the twisted connection of the one on the spinor bundle (coming from the Levi-Civita connection, also denoted by  $\nabla^N$ ) and a fixed connection on  $L^N$ .

We may now define the Dirac operator  $D^N$  acting on the space of smooth sections of  $\Sigma N$  by the composition of the metric connection and the Clifford multiplication. In local coordinates this reads as

$$D^N = \sum_{j=1}^{n+1} e_j \cdot \nabla_{e_j}^N,$$

where  $\{e_1, \dots, e_{n+1}\}$  is a local oriented orthonormal tangent frame. It is a first order elliptic operator, formally self-adjoint with respect to the  $L^2$ -scalar product and satisfies, for any spinor field  $\psi$ , the Schrödinger-Lichnerowicz formula

$$(D^N)^2 \psi = (\nabla^N)^* \nabla^N \psi + \frac{1}{4} S \psi + \frac{i}{2} \Omega^N \cdot \psi, \quad (2)$$

where  $S$  is the scalar curvature of  $N$ ,  $(\nabla^N)^*$  is the adjoint of  $\nabla^N$  with respect to the  $L^2$  scalar product,  $i\Omega^N$  is the curvature of the fixed connection on the auxiliary line bundle  $L^N$  ( $\Omega^N$  is a real 2-form on  $N$ ) and  $\Omega^N \cdot$  is the extension of the Clifford multiplication to differential forms. For any  $X \in \Gamma(TN)$ , the Ricci identity is given by

$$\sum_{k=1}^{n+1} e_k \cdot \mathcal{R}^N(e_k, X) \psi = \frac{1}{2} \text{Ric}^N(X) \cdot \psi - \frac{i}{2} (X \lrcorner \Omega^N) \cdot \psi, \quad (3)$$

where  $\text{Ric}^N$  is the Ricci curvature of  $(N^{n+1}, \bar{g})$  and  $\mathcal{R}^N$  is the curvature tensor of the spinorial connection  $\nabla^N$ . In odd dimension, the volume form  $\omega_{\mathbb{C}} := i^{\lfloor \frac{n+2}{2} \rfloor} e_1 \cdot \dots \cdot e_{n+1}$  acts on  $\Sigma N$  as the identity, i.e.,  $\omega_{\mathbb{C}} \cdot \psi = \psi$  for any spinor  $\psi \in \Gamma(\Sigma N)$ . Besides, in even dimension, we have  $\omega_{\mathbb{C}}^2 = 1$ . We denote by  $\Sigma^{\pm} N$  the eigenbundles corresponding to the eigenvalues  $\pm 1$ , hence  $\Sigma N = \Sigma^+ N \oplus \Sigma^- N$  and a spinor field  $\psi$  can be written  $\psi = \psi^+ + \psi^-$ . The conjugate  $\bar{\psi}$  of  $\psi$  is defined by  $\bar{\psi} = \psi^+ - \psi^-$ .

Every Spin manifold has a trivial  $\text{Spin}^c$  structure [6]. In fact, we choose the trivial line bundle with the trivial connection whose curvature is zero. Also every Kähler manifold  $(N, J, \bar{g})$  of complex dimension  $m$  ( $n+1 = 2m$ ) has a canonical  $\text{Spin}^c$  structure coming from the complex structure  $J$ . Let  $\times$  be the Kähler form defined by the complex structure  $J$ , i.e.  $\times(X, Y) = \bar{g}(JX, Y)$  for all vector fields  $X, Y \in \Gamma(TN)$ . The complexified tangent bundle  $T^{\mathbb{C}}N = TN \otimes_{\mathbb{R}} \mathbb{C}$  decomposes into

$$T^{\mathbb{C}}N = T_{1,0}N \oplus T_{0,1}N,$$

where  $T_{1,0}N$  (resp.  $T_{0,1}N$ ) is the  $i$ -eigenbundle (resp.  $-i$ -eigenbundle) of the complex linear extension of the complex structure. Indeed,

$$T_{1,0}N = \overline{T_{0,1}N} = \{X - iJX \mid X \in \Gamma(TN)\}.$$

Thus, the spinor bundle of the canonical  $\text{Spin}^c$  structure is given by

$$\Sigma N = \Lambda^{0,*}N = \bigoplus_{r=0}^m \Lambda^r(T_{0,1}^*N),$$

where  $T_{0,1}^*N$  is the dual space of  $T_{0,1}N$ . The auxiliary bundle of this canonical  $\text{Spin}^c$  structure is given by  $L^N = (K_N)^{-1} = \Lambda^m(T_{0,1}^*N)$ , where  $K_N = \Lambda^m(T_{1,0}^*N)$  is the

canonical bundle of  $N$  [6]. This line bundle  $L^N$  has a canonical holomorphic connection induced from the Levi-Civita connection whose curvature form is given by  $i\Omega^N = -i\rho$ , where  $\rho$  is the Ricci form given by  $\rho(X, Y) = \text{Ric}^N(JX, Y)$  for all  $X, Y \in \Gamma(TN)$ . Hence, this  $\text{Spin}^c$  structure carries parallel spinors (the constant complex functions) lying in the set of complex functions  $\Lambda^{0,0}N \subset \Lambda^{0,*}N$  [23]. Of course, we can define another  $\text{Spin}^c$  structure for which the spinor bundle is given by  $\Lambda^{*,0}N = \bigoplus_{r=0}^m \Lambda^r(T_{1,0}^*N)$  and the auxiliary line bundle by  $K_N$ . This  $\text{Spin}^c$  structure will be called the anti-canonical  $\text{Spin}^c$  structure [6] and it carries also parallel spinors (the constant complex functions) lying in the set of complex functions  $\Lambda^{0,0}N \subset \Lambda^{0,*}N$  [23].

For any other  $\text{Spin}^c$  structure on the Kähler manifold  $N$ , the spinorial bundle can be written as [6, 12]:

$$\Sigma N = \Lambda^{0,*}N \otimes \mathfrak{L},$$

where  $\mathfrak{L}^2 = K_N \otimes L^N$  and  $L^N$  is the auxiliary bundle associated with this  $\text{Spin}^c$  structure. In this case, the 2-form  $\times$  can be considered as an endomorphism of  $\Sigma N$  via Clifford multiplication and it acts on a spinor field  $\psi$  locally by [14, 6]:

$$\times \cdot \psi = \frac{1}{2} \sum_{j=1}^m e_j \cdot J e_j \cdot \psi.$$

Hence, we have the well-known orthogonal splitting

$$\Sigma N = \bigoplus_{r=0}^m \Sigma_r N, \quad (4)$$

where  $\Sigma_r N$  denotes the eigensubbundle corresponding to the eigenvalue  $i(m - 2r)$  of  $\times$ , with complex rank  $\binom{m}{k}$ . The bundle  $\Sigma_r N$  corresponds to  $\Lambda^{0,r}N \otimes \mathfrak{L}$ . Moreover,

$$\Sigma^+ N = \bigoplus_{r \text{ even}} \Sigma_r N \quad \text{and} \quad \Sigma^- N = \bigoplus_{r \text{ odd}} \Sigma_r N.$$

For the canonical (resp. the anti-canonical)  $\text{Spin}^c$  structure, the subbundle  $\Sigma_0 N$  (resp.  $\Sigma_m N$ ) is trivial, i.e.,  $\Sigma_0 N = \Lambda^{0,0}N \subset \Sigma^+ N$  (resp.  $\Sigma_m N = \Lambda^{0,0}N$  which is in  $\Sigma^+ N$  if  $m$  is even and in  $\Sigma^- N$  if  $m$  is odd).

The product  $N_1 \times N_2$  of two Kähler  $\text{Spin}^c$  manifolds is again a  $\text{Spin}^c$  manifold. We denote by  $m_1$  (resp.  $m_2$ ) the complex dimension of  $N_1$  (resp.  $N_2$ ). The spinor bundle is identified by

$$\Sigma(N_1 \times N_2) \simeq \Sigma N_1 \otimes \Sigma N_2,$$

via the Clifford multiplication denoted also by “ $\cdot$ ”:

$$(X_1 + X_2) \cdot (\psi_1 \otimes \psi_2) = X_1 \cdot \psi_1 \otimes \psi_2 + \bar{\psi}_1 \otimes X_2 \cdot \psi_2,$$

where  $X_1 \in \Gamma(TM_1)$ ,  $X_2 \in \Gamma(TM_2)$ ,  $\psi_1 \in \Gamma(\Sigma M_1)$  and  $\psi_2 \in \Gamma(\Sigma M_2)$ . We consider the decomposition (4) of  $\Sigma N_1$  and  $\Sigma N_2$  with respect to their Kähler forms  $\times^{N_1}$  and  $\times^{N_2}$ . Then, the corresponding decomposition of  $\Sigma(N_1 \times N_2)$  into eigenbundles of  $\times^{N_1 \times N_2} = \times^{N_1} + \times^{N_2}$  is:

$$\Sigma(N_1 \times N_2) \simeq \bigoplus_{k=0}^{m_1+m_2} \Sigma_k(N_1 \times N_2),$$

with

$$\Sigma_r(N_1 \times N_2) \simeq \bigoplus_{k=0}^r \Sigma_k N_1 \otimes \Sigma_{r-k} N_2,$$

since the Kähler form  $\kappa^{N_1 \times N_2}$  acts on a section of  $\Sigma_k N_1 \otimes \Sigma_{r-k} N_2$  as

$$\kappa^{N_1 \times N_2}(\psi_1 \otimes \psi_2) = \kappa^{N_1} \cdot \psi_1 \otimes \psi_2 + \psi_1 \otimes \kappa^{N_2} \cdot \psi_2 = i(m_1 + m_2 - 2r)\psi_1 \otimes \psi_2.$$

**Spin<sup>c</sup> hypersurfaces and the Gauss formula:** Let  $N$  be an oriented  $(n+1)$ -dimensional Riemannian Spin<sup>c</sup> manifold and  $M \subset N$  be an oriented hypersurface. The manifold  $M$  inherits a Spin<sup>c</sup> structure induced from the one on  $N$ , and we have [27]

$$\Sigma M \simeq \begin{cases} \Sigma N|_M & \text{if } n \text{ is even,} \\ \Sigma^+ N|_M \text{ or } \Sigma^- N|_M & \text{if } n \text{ is odd.} \end{cases}$$

Moreover the Clifford multiplication by a vector field  $X$ , tangent to  $M$ , is denoted by  $\gamma$  and given by

$$\gamma(X)\phi = (X \cdot \nu \cdot \psi)|_M, \quad (5)$$

where  $\psi \in \Gamma(\Sigma N)$  (or  $\psi \in \Gamma(\Sigma^+ N)$  if  $n$  is odd),  $\phi$  is the restriction of  $\psi$  to  $M$ , “ $\cdot$ ” is the Clifford multiplication on  $N$ ,  $\gamma$  that on  $M$  and  $\nu$  is the unit inner normal vector. If  $\psi \in \Gamma(\Sigma^- N)$  when  $n$  is odd, then we have

$$\gamma(X)\phi = -(X \cdot \nu \cdot \psi)|_M. \quad (6)$$

The curvature 2-form  $i\Omega$  on the auxiliary line bundle  $L = L|_M^N$  defining the Spin<sup>c</sup> structure on  $M$  is given by  $i\Omega = i\Omega|_M^N$ . For every  $\psi \in \Gamma(\Sigma N)$  ( $\psi \in \Gamma(\Sigma^+ N)$  if  $n$  is odd), the real 2-forms  $\Omega$  and  $\Omega^N$  are related by [27]:

$$(\Omega^N \cdot \psi)|_M = \gamma(\Omega)\phi - \gamma(\nu \lrcorner \Omega^N)\phi. \quad (7)$$

When  $\psi \in \Gamma(\Sigma^- N)$  if  $n$  is odd, we have

$$(\Omega^N \cdot \psi)|_M = \gamma(\Omega)\phi + \gamma(\nu \lrcorner \Omega^N)\phi. \quad (8)$$

We denote by  $\nabla$  the spinorial Levi-Civita connection on  $\Sigma M$ . For all  $X \in \Gamma(TM)$  and  $\psi \in \Gamma(\Sigma^+ N)$ , we have the Spin<sup>c</sup> Gauss formula [27]:

$$(\nabla_X^{\Sigma^+ N} \psi)|_M = \nabla_X \phi + \frac{1}{2}\gamma(II X)\phi, \quad (9)$$

where  $II$  denotes the Weingarten map of the hypersurface. If  $\psi \in \Gamma(\Sigma^- N)$ , we have

$$(\nabla_X^{\Sigma^- N} \psi)|_M = \nabla_X \phi - \frac{1}{2}\gamma(II X)\phi, \quad (10)$$

for all  $X \in \Gamma(TM)$ .

## 2.2 Basic facts about $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$ and their real hypersurfaces

Let  $(\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2), \bar{g})$  be the Riemannian product of  $\mathbb{M}_1(c_1)$  and  $\mathbb{M}_2(c_2)$ , where  $M_i(c_i)$  denotes the space form of constant sectional curvature  $c_i$  and  $\bar{g}$  denotes the product metric. Consider  $(M^3, g = (\cdot, \cdot))$  an oriented real hypersurface of  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  endowed with the metric  $g := (\cdot, \cdot)$  induced by  $\bar{g}$ . The product structure of  $\mathbb{P} := \mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  is given by the map  $F : T\mathbb{P} \rightarrow T\mathbb{P}$  defined, for  $X_1 \in \Gamma(T\mathbb{M}_1(c_1))$  and  $X_2 \in \Gamma(T\mathbb{M}_2(c_2))$ , by

$$F(X_1 + X_2) = X_1 - X_2. \quad (11)$$

The map  $F$  satisfies  $F^2 = \text{Id}_{T\mathbb{P}}$ ,  $F \neq \text{Id}_{T\mathbb{P}}$ , where  $\text{Id}_{T\mathbb{P}}$  denotes the identity map on  $T\mathbb{P}$ . Denoting the Levi-Civita connection on  $\mathbb{P}$  by  $\nabla^{\mathbb{P}}$ , we have  $\nabla^{\mathbb{P}} F = 0$  and for any

$X, Y \in \Gamma(T\mathbb{P})$ , we have  $\bar{g}(FX, Y) = \bar{g}(X, FY)$ . The product structure  $F$  induces the existence on  $M$  of a vector  $V \in \Gamma(TM)$ , a function  $h : M \rightarrow \mathbb{R}$  and an endomorphism  $f : TM \rightarrow TM$  such that, for all  $X \in \Gamma(TM)$ ,

$$FX = fX + (V, X)\nu \quad \text{and} \quad F\nu = V + h\nu, \quad (12)$$

where  $\nu$  is the unit normal vector of the immersion.

**Lemma 2.1.** *The function  $f$  is symmetric. Moreover, for any  $X \in \Gamma(TM)$ , we have*

$$f^2X + (V, X)V = X, \quad (13)$$

$$fV = -hV, \quad (14)$$

$$h^2 + \|V\|^2 = 1. \quad (15)$$

*Proof.* First of all, for any  $X, Y \in \Gamma(TM)$ , we have

$$\begin{aligned} (fX, Y) &= \bar{g}(fX, Y) = \bar{g}(FX - (V, X)\nu, Y) = \bar{g}(FX, Y) \\ &= \bar{g}(X, FY) = \bar{g}(X, fY + (V, Y)\nu) = (X, fY). \end{aligned}$$

Hence  $f$  is symmetric. For any  $X \in \Gamma(TM)$ ,  $F^2X = X$ . This means that

$$(f + (V, X)\nu)^2(X) = X,$$

and hence

$$\begin{cases} f^2X + (V, X)V = X, \\ (V, fX) + h(V, X) = 0, \end{cases}$$

which are Equation (13) and Equation (14). We also have  $F^2\nu = \nu$ . Thus,

$$V + (V, V)\nu + hV + h^2\nu = \nu.$$

This gives  $\|V\|^2 + h^2 = 1$  which is Equation (15).  $\square$

Moreover, the complex structure  $J = J_1 + J_2$  on  $\mathbb{P}$  (where  $J_i$  denotes the complex structure on  $\mathbb{M}_i(c_i)$ ) induces on  $M$  an almost contact metric structure  $(\mathfrak{X}, \xi, \eta, g = (\cdot, \cdot))$ , where  $\mathfrak{X}$  is the  $(1, 1)$ -tensor defined, for all  $X, Y \in \Gamma(TM)$  by

$$(\mathfrak{X}X, Y) = \bar{g}(JX, Y).$$

The tangent vector field  $\xi$  and the 1-form  $\eta$  associated with  $\xi$  satisfy

$$\xi = -J\nu \quad \text{and} \quad \eta(X) = (\xi, X),$$

for all  $X \in \Gamma(TM)$ . Then, we can easily see that, for all  $X \in \Gamma(TM)$ , the following holds:

$$JX = \mathfrak{X}X + \eta(X)\nu, \quad (16)$$

$$\mathfrak{X}^2X = -X + \eta(X)\xi, \quad g(\xi, \xi) = 1, \quad \text{and} \quad \mathfrak{X}\xi = 0. \quad (17)$$

Here, we recall that given an almost contact metric structure  $(\mathfrak{X}, \xi, \eta, g)$  one can define a 2-form  $\Theta$  by  $\Theta(X, Y) = g(X, \mathfrak{X}Y)$  for all  $X, Y \in \Gamma(TM)$ . Now,  $(\mathfrak{X}, \xi, \eta, g)$  is said to satisfy the contact condition if  $-2\Theta = d\eta$  and if it is the case,  $(\mathfrak{X}, \xi, \eta, g)$  is called a contact metric structure on  $M$ . A contact metric structure  $(\mathfrak{X}, \xi, \eta, g)$  is called a Sasakian structure (and  $M$  a Sasaki manifold) if  $\xi$  is a Killing vector field (or equivalently,  $\mathfrak{X} = \nabla\xi$ ) and

$$(\nabla_X \mathfrak{X})Y = \eta(Y)X - g(X, Y)\xi, \quad \text{for all } X, Y \in \Gamma(TM).$$

For  $\mathbb{P}$ , one can choose a local orthonormal frame  $\{e_1, e_2 = \mathfrak{X}e_1, \xi, \nu\}$  where  $\{e_1, e_2 = \mathfrak{X}e_1, \xi\}$  denotes a local orthonormal frame of  $M$ .

**Lemma 2.2.** *We have*

- (i)  $\mathfrak{X}$  is antisymmetric on  $TM$ , i.e.  $(\mathfrak{X}e_1, e_2) = -(e_1, \mathfrak{X}e_2)$  and  $\mathfrak{X}\xi = 0$
- (ii)  $J \circ F = F \circ J$
- (iii)  $(V, \mathfrak{X}X) + \eta(X)h = \eta(fX)$
- (iv)  $f\mathfrak{X}X + \eta(X)V = \mathfrak{X}fX - (V, X)\xi$
- (v)  $\eta(V) = 0$
- (vi)  $f\xi = h\xi - \mathfrak{X}V$
- (vii)  $\eta(fV) = 0$
- (viii)  $(fe_1, e_2) = 0$  and  $(fe_1, e_1) = (fe_2, e_2) = -h$
- (ix)  $JV = \mathfrak{X}V$
- (x)  $F\xi = f\xi$

*Proof.* For any  $X, Y \in \Gamma(TM)$ , we have  $(\mathfrak{X}X, \mathfrak{X}Y) = (X, Y) - \eta(X)\eta(Y)$ . Thus,  $(\mathfrak{X}e_1, e_2) = (\mathfrak{X}^2e_1, \mathfrak{X}e_2) = -(e_1, \mathfrak{X}e_2)$ . It is evident that  $\mathfrak{X}\xi = 0$ . This proves (i). Now, for any  $X_1 + X_2 \in \Gamma(T\mathbb{P})$ , we have

$$\begin{aligned} J \circ F(X_1 + X_2) &= J(X_1 - X_2) = J_1X_1 - J_2X_2 \\ &= F(J_1X_1 + J_2X_2) = F \circ J(X_1 + X_2). \end{aligned}$$

This proves (ii). From  $J \circ F = F \circ J$ , and using that  $f$  is symmetric and (17), we have for any  $X \in \Gamma(TM)$ ,

$$\begin{cases} f\mathfrak{X}X + \eta(X)V = \mathfrak{X}fX - (V, X)\xi, \\ (V, \mathfrak{X}X) + h\eta(X) = \eta(fX). \end{cases}$$

This proves (iii) and (iv). We also have  $J(F\nu) = F(J\nu)$ . Thus,

$$\mathfrak{X}V + \eta(V)\nu - h\xi = -f\xi - (V, \xi)\nu.$$

This implies

$$\begin{cases} f\xi = h\xi - \mathfrak{X}V, \\ \eta(V) = -(V, \xi) = 0. \end{cases}$$

This proves (v) and (vi). From  $(V, \mathfrak{X}X) + h\eta(X) = \eta(fX)$  and for  $X = V$ , we get

$$\eta(fV) = h\eta(V) + \underbrace{(V, \mathfrak{X}V)}_{=0} = 0,$$

which is (vii). We calculate

$$\begin{aligned} (fe_1, e_2) &= -(f\mathfrak{X}e_2, e_2) = (-\mathfrak{X}fe_2 + (V, e_2)\xi, e_2) \\ &= -(\mathfrak{X}fe_2, e_2) = (fe_2, \mathfrak{X}e_2) = -(fe_2, e_1). \end{aligned}$$

Since  $f$  symmetric, it implies that  $(fe_1, e_2) = 0$ . Moreover, we have

$$\begin{aligned} (fe_1, e_1) &= -(f\mathfrak{X}e_2, e_1) = (-\mathfrak{X}fe_2 + (V, e_2)\xi, e_1) \\ &= -(\mathfrak{X}fe_2, e_1) = (fe_2, \mathfrak{X}e_1) = (fe_2, e_2). \end{aligned}$$

We know that  $\text{tr}(F) = 0$ . Thus,

$$\begin{aligned} 0 &= (Fe_1, e_1) + (Fe_2, e_2) + (F\xi, \xi) + (F\nu, \nu) \\ &= (fe_1, e_1) + (fe_2, e_2) + \underbrace{(f\xi, \xi)}_{=h} + \underbrace{(V + h\nu, \nu)}_{h+0=h} \\ &= (fe_1, e_1) + (fe_2, e_2). \end{aligned}$$

Thus,  $(fe_1, e_1) = (fe_2, e_2) = -h$ . This proves (viii). Since  $(V, \xi) = 0$ , it is clear that  $F\xi = f\xi$  and from  $J = \mathfrak{X} + \eta(\cdot)\nu$ , we have  $JV = \mathfrak{X}V$ . This proves (ix) and (x).  $\square$

For all  $X, Y, Z \in \Gamma(TM)$ , the Gauss equation for the hypersurface  $M$  of  $\mathbb{P}$  can be written as

$$\begin{aligned} \mathcal{R}(X, Y)Z &= \frac{c_1}{4} \left( (X + fX) \wedge (Y + fY) \right) + \frac{c_2}{4} \left( (X - fX) \wedge (Y - fY) \right) \\ &\quad + g(IY, Z)IX - g(IX, Z)IY, \end{aligned} \quad (18)$$

where  $\mathcal{R}$  denotes the Riemann curvature tensor. The Codazzi equation is

$$\begin{aligned} d^\nabla II(X, Y) &= \frac{c_1}{4} \left( g(fY, Z)g(V, X) - g(fX, Z)g(V, Y) \right. \\ &\quad \left. + g(Y, Z)g(V, X) - g(X, Z)g(V, Y) \right) \\ &\quad - \frac{c_2}{4} \left( g(Y, Z)g(V, X) - g(Y, fZ)g(V, X) \right. \\ &\quad \left. - g(X, Z)g(Y, V) + g(X, fZ)g(V, Y) \right) \end{aligned} \quad (19)$$

Now, we ask if the Gauss equation (18) and the Codazzi equation (19) are sufficient to get an isometric immersion of  $(M, g)$  into  $\mathbb{P} = \mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$ .

**Definition 2.3 (Compatibility equations).** *Let  $(M^3, g)$  be a simply connected oriented Riemannian manifold endowed with an almost contact metric structure  $(\mathfrak{X}, \xi, \eta)$  and  $E$  be a field of symmetric endomorphisms on  $M$ . We say that  $(M, g, E)$  satisfies the compatibility equations for  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  if and only if for any  $X, Y, Z \in \Gamma(TM)$ , we have*

$$\begin{aligned} \mathcal{R}(X, Y)Z &= \frac{c_1}{4} \left( (X + fX) \wedge (Y + fY) \right) + \frac{c_2}{4} \left( (X - fX) \wedge (Y - fY) \right) \\ &\quad + g(EY, Z)EX - g(EX, Z)EY, \end{aligned} \quad (20)$$

$$\begin{aligned} d^\nabla E(X, Y) &= \frac{c_1}{4} \left( g(fY, Z)g(V, X) - g(fX, Z)g(V, Y) \right. \\ &\quad \left. + g(Y, Z)g(V, X) - g(X, Z)g(V, Y) \right) \\ &\quad - \frac{c_2}{4} \left( g(Y, Z)g(V, X) - g(Y, fZ)g(V, X) \right. \\ &\quad \left. - g(X, Z)g(Y, V) + g(X, fZ)g(V, Y) \right), \end{aligned} \quad (21)$$

$$(\nabla_X f)Y = g(Y, V)EX + g(EX, Y)V, \quad (22)$$

$$\nabla_X V = -f(EX) + hEX, \quad (23)$$

$$\nabla f = -2EV. \quad (24)$$

In [16, 5], Kowalczyk and De Lira-Tojeiro-Vitório proved independently that the Gauss equation (20) and the Codazzi equation (21) together with (13), (14), (15), (22), (23), (24) and if  $\frac{F \pm Id}{2}$  are of rank 2, where  $F$  is given by  $F = \begin{pmatrix} f & V \\ V & h \end{pmatrix}$ , are necessary and sufficient for the existence of an isometric immersion from  $M$  into  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  such that the complex structure of  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  over  $M$  is given by  $J = \mathfrak{X} + \eta(\cdot)\nu$ ,  $E$  as second fundamental form and such that the product structure coincides with  $F$  over  $M$ . This immersion is global if  $M$  is simply connected. Note that this was previously proven in a more abstract way by Piccione and Tausk [29].

### 3 Isometric immersions into $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$ via spinors

In this section, we consider two different  $\text{Spin}^c$  structures on  $\mathbb{P} = \mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  carrying parallel spinor fields. For the first structure, the parallel spinor  $\psi$  is lying in  $\Sigma^+\mathbb{P}$  and for the second  $\text{Spin}^c$  structure the parallel spinor field  $\Psi$  is lying in  $\Sigma^-\mathbb{P}$ . The restriction of these two  $\text{Spin}^c$  structures to any hypersurface  $M^3$  defines two  $\text{Spin}^c$  structures on  $M$ , each one with a generalized Killing spinor field. These spinor fields will characterize the isometric immersion of  $M$  into  $\mathbb{P} = \mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$ .

We denote by  $\pi_i(X)$  the projection of a vector  $X$  on  $T\mathbb{M}_i(c_i)$ . We have

$$\left\{ \begin{array}{l} \pi_1(V) = \frac{(1-h)V + \|V\|^2\nu}{2} \\ \pi_2(V) = \frac{(h+1)V - \|V\|^2\nu}{2} \\ \pi_1(\xi) = -\pi_1(J\nu) = -J(\pi_1(\nu)) \\ \pi_2(\xi) = -\pi_2(J\nu) = -J(\pi_2(\nu)) \\ \pi_1(\nu) = \frac{(h+1)\nu + V}{2} \\ \pi_2(\nu) = \frac{(1-h)\nu - V}{2} \end{array} \right. \quad (25)$$

#### 3.1 A first $\text{Spin}^c$ structure on $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$ and its restriction to hypersurfaces

Assume that there exists an isometric immersion of  $(M^3, g)$  into  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  with shape operator  $II$ . By Section 2.2, we know that  $M$  has an almost contact metric structure  $(\mathfrak{X}, \xi, \eta)$  such that  $\mathfrak{X}X = JX - \eta(X)\nu$  for every  $X \in \Gamma(TM)$  and the product structure  $F$  on  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  will be restricted via  $f, V$  and  $h$ . Consider the product of the canonical  $\text{Spin}^c$  structure on  $\mathbb{M}_1(c_1)$  with the canonical one on  $\mathbb{M}_2(c_2)$ . It has a parallel spinor  $\psi = \psi_1^+ \otimes \psi_2^+$  lying in  $\Sigma_0(\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)) = \Sigma_0(\mathbb{M}_1(c_1)) \otimes \Sigma_0(\mathbb{M}_2(c_2)) \subset \Sigma^+(\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2))$ . First of all, using (4), we have for any  $X \in \Gamma(TM)$ ,

$$J(\pi_2(X)) \cdot \pi_2(X) \cdot \psi_2^+ = i|\pi_2(X)|^2\psi_2^+ \quad \text{and} \quad J(\pi_1(X)) \cdot \pi_1(X) \cdot \psi_1^+ = i|\pi_1(X)|^2\psi_1^+.$$

**Lemma 3.1.** *We have*

$$-\pi_1(\nu) \cdot \psi_1^+ \otimes \pi_2(\xi) \cdot \psi_2^+ + \pi_1(\xi) \cdot \psi_1^+ \otimes \pi_2(\nu) \cdot \psi_2^+ = 0$$

*Proof.* Using that  $i\pi_2(\nu) \cdot \psi_2^+ = J(\pi_2(\nu)) \cdot \psi_2^+$  and  $i\pi_1(\nu) \cdot \psi_1^+ = J(\pi_1(\nu)) \cdot \psi_1^+$ , we have

$$\begin{aligned} & -\pi_1(\nu) \cdot \psi_1^+ \otimes \pi_2(\xi) \cdot \psi_2^+ + \pi_1(\xi) \cdot \psi_1^+ \otimes \pi_2(\nu) \cdot \psi_2^+ \\ &= i\pi_1(\nu) \cdot \psi_1^+ \otimes \pi_2(\nu) \cdot \psi_2^+ - i\pi_1(\nu) \cdot \psi_1^+ \otimes \pi_2(\nu) \cdot \psi_2^+ \\ &= 0. \end{aligned}$$

□

**Lemma 3.2.** *The restriction  $\varphi_1$  of the parallel spinor  $\psi$  on  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  is a solution of the generalized Killing equation*

$$\nabla_X^1 \varphi_1 + \frac{1}{2} \gamma_1(II X) \varphi_1 = 0, \quad (26)$$

where  $\nabla^1$  (resp.  $\gamma_1$ ) denotes the  $\text{Spin}^c$  Levi-Civita connection (resp. the Clifford multiplication) on the induced  $\text{Spin}^c$  bundle. Moreover,  $\varphi_1$  satisfies  $\gamma_1(\xi)\varphi_1 = -i\varphi_1$ . The

curvature 2-form  $i\Omega^1$  of the auxiliary line bundle associated with the induced  $\text{Spin}^c$  structure is given in the basis  $\{e_1, e_2 = \mathfrak{X}e_1, \xi\}$  by

$$\begin{aligned}\Omega^1(e_1, e_2) &= \frac{c_1}{2}(h-1) - \frac{c_2}{2}(h+1), \\ \Omega^1(e_1, \xi) &= \frac{c_1 - c_2}{2}(e_1, V), \\ \Omega^1(e_2, \xi) &= \frac{c_1 - c_2}{2}(e_2, V).\end{aligned}$$

*Proof.* By the Gauss formula (9), the restriction  $\varphi_1$  of the parallel spinor  $\psi$  on  $\mathbb{P}$  satisfies

$$\nabla_X^1 \varphi_1 = -\frac{1}{2} \gamma_1(II)\varphi_1.$$

Now, for any  $X, Y \in \Gamma(TM)$ , we have

$$\begin{aligned}\Omega^1(X, Y) &= \Omega^{\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)}(X, Y) \\ &= -\text{Ric}^{\mathbb{M}_1(c_1)}(J\pi_1 X, \pi_1 Y) - \text{Ric}^{\mathbb{M}_2(c_2)}(J\pi_2 X, \pi_2 Y) \\ &= -\frac{c_1}{4}g(\mathfrak{X}X + \eta(X)\nu + \mathfrak{X}fX + \eta(fX)\nu - (V, X)\xi, Y + fY + (V, Y)\nu) \\ &\quad - \frac{c_2}{4}g(\mathfrak{X}X + \eta(X)\nu - \mathfrak{X}fX - \eta(fX)\nu + (V, X)\xi, Y - fY - (V, Y)\nu).\end{aligned}$$

Using Lemma 2.2, we have

$$\begin{aligned}\Omega^1(e_1, e_2) &= \frac{c_1}{2}(h-1) - \frac{c_2}{2}(h+1), \\ \Omega^1(e_1, \xi) &= \frac{c_1 - c_2}{2}(V, e_1), \\ \Omega^1(e_2, \xi) &= \frac{c_1 - c_2}{2}(V, e_2).\end{aligned}$$

Now, we have

$$\begin{aligned}\gamma_1(\xi)(\varphi_1) &= \xi \cdot \nu \cdot (\psi_1^+ \otimes \psi_2^+)|_M \\ &= [\pi_1(\xi) \cdot \pi_1(\nu) \cdot \psi_1^+ \otimes \psi_2^+ - \pi_1(\nu) \cdot \psi_1^+ \otimes \pi_2(\xi) \cdot \psi_2^+]|_M \\ &\quad + [\pi_1(\xi) \cdot \psi_1^+ \otimes \pi_2(\nu) \cdot \psi_2^+ + \psi_1^+ \otimes \pi_2(\xi) \cdot \pi_2(\nu) \cdot \psi_2^+]|_M\end{aligned}$$

Thus,

$$\begin{aligned}\gamma_1(\xi)(\varphi_1) &= \xi \cdot \nu \cdot (\psi_1^+ \otimes \psi_2^+)|_M \\ &= [-i|\pi_1(\nu)|^2 - i|\pi_2(\nu)|^2]\varphi_1 - [\pi_1(\nu) \cdot \psi_1^+ \otimes \pi_2(\xi) \cdot \psi_2^+ + \pi_1(\xi) \cdot \psi_1^+ \otimes \pi_2(\nu) \cdot \psi_2^+]|_M \\ &= -i\varphi_1 + \underbrace{[-\pi_1(\nu) \cdot \psi_1^+ \otimes \pi_2(\xi) \cdot \psi_2^+ + \pi_1(\xi) \cdot \psi_1^+ \otimes \pi_2(\nu) \cdot \psi_2^+]}_{=0 \text{ by Lemma 3.1}}|_M \\ &= -i\varphi_1.\end{aligned}$$

□

### 3.2 A second $\text{Spin}^c$ structure on $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$ and its restriction to hypersurfaces

One can also endow  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  with another  $\text{Spin}^c$  structure. Mainly, the one coming from the product of the anticanonical  $\text{Spin}^c$  on  $\mathbb{M}_1(c_1)$  with the canonical  $\text{Spin}^c$  structure on  $\mathbb{M}_2(c_2)$  which carries also a parallel spinor  $\Psi = \psi_1^- \otimes \psi_2^+$ . The parallel spinor  $\Psi$  lies in  $\Sigma_1(\mathbb{M}_1(c_1)) \otimes \Sigma_0(\mathbb{M}_2(c_2)) \subset \Sigma^-(\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2))$ . Using (4), we have for any  $X \in \Gamma(TM)$

$$J(\pi_2(X)) \cdot \pi_2(X) \cdot \psi_2^+ = i|\pi_2(X)|^2 \psi_2^+ \quad \text{and} \quad J(\pi_1(X)) \cdot \pi_1(X) \cdot \psi_1^- = -|\pi_1(X)|^2 i\psi_1^-.$$

**Lemma 3.3.** *We have*

$$\pi_1(\nu) \cdot \psi_1^- \otimes (\pi_2(V) + i\pi_2(\xi)) \cdot \psi_2^+ - (\pi_1(V) + i\pi_1(\xi)) \cdot \psi_1^- \otimes \pi_2(\nu) \cdot \psi_2^+ = 0$$

*Proof.* Using that  $i\pi_2(\nu) \cdot \psi_2^+ = J(\pi_2(\nu)) \cdot \psi_2^+$  and  $i\pi_1(\nu) \cdot \psi_1^- = -J(\pi_1(\nu)) \cdot \psi_1^-$ , we have

$$\begin{aligned} & \pi_1(\nu) \cdot \psi_1^- \otimes (\pi_2(V) + i\pi_2(\xi)) \cdot \psi_2^+ - (\pi_1(V) + i\pi_1(\xi)) \cdot \psi_1^- \otimes \pi_2(\nu) \cdot \psi_2^+ \\ = & \underbrace{2\pi_1(\nu) \cdot \psi_1^- \otimes \pi_2(\nu) \cdot \psi_2^+}_A + \underbrace{\pi_1(\nu) \cdot \psi_1^- \otimes \pi_2(V) \cdot \psi_2^+}_B - \underbrace{\pi_1(V) \cdot \psi_1^- \otimes \pi_2(\nu) \cdot \psi_2^+}_C \end{aligned}$$

Let's calculate each term of the last identity. First we have

$$\begin{aligned} A &= 2\pi_1(\nu) \cdot \psi_1^- \otimes \pi_2(\nu) \cdot \psi_2^+ \\ &= \frac{1-h^2}{2}(\nu \cdot \psi_1^- \otimes \nu \cdot \psi_2^+) - \frac{h+1}{2}(\nu \cdot \psi_1^- \otimes V \cdot \psi_2^+) \\ &\quad + \frac{1-h}{2}(V \cdot \psi_1^- \otimes \nu \cdot \psi_2^+) - \frac{1}{2}(V \cdot \psi_1^- \otimes V \cdot \psi_2^+). \end{aligned}$$

Next, we have

$$\begin{aligned} B &= \pi_1(\nu) \cdot \psi_1^- \otimes \pi_2(V) \cdot \psi_2^+ \\ &= \frac{(h+1)^2}{4}(\nu \cdot \psi_1^- \otimes V \cdot \psi_2^+) - \frac{h+1}{4}\|V\|^2(\nu \cdot \psi_1^- \otimes \nu \cdot \psi_2^+) \\ &\quad + \frac{h+1}{4}(V \cdot \psi_1^- \otimes V \cdot \psi_2^+) - \frac{1}{4}\|V\|^2(V \cdot \psi_1^- \otimes \nu \cdot \psi_2^+), \end{aligned}$$

and

$$\begin{aligned} C &= \pi_1(V) \cdot \psi_1^- \otimes \pi_2(\nu) \cdot \psi_2^+ \\ &= \frac{(1-h)^2}{4}(V \cdot \psi_1^- \otimes \nu \cdot \psi_2^+) - \frac{1-h}{4}(V \cdot \psi_1^- \otimes V \cdot \psi_2^+) \\ &\quad + \frac{1-h}{4}\|V\|^2(\nu \cdot \psi_1^- \otimes \nu \cdot \psi_2^+) - \frac{1}{4}\|V\|^2(\nu \cdot \psi_1^- \otimes V \cdot \psi_2^+). \end{aligned}$$

It's clear that  $A + B + C = 0$ .  $\square$

**Lemma 3.4.** *The restriction  $\varphi_2$  of the parallel spinor  $\Psi$  (for the  $\text{Spin}^c$  structure described above) on  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  is a solution of the generalized Killing equation*

$$\nabla_X^2 \varphi_2 = \frac{1}{2} \gamma_2(IX) \varphi_2, \quad (27)$$

where  $\nabla^2$  (resp.  $\gamma_2$ ) denotes the  $\text{Spin}^c$  connection (resp. the Clifford multiplication) on the induced  $\text{Spin}^c$  bundle. Moreover,  $\varphi_2$  satisfies  $\gamma_2(V)\varphi_2 = -i\gamma_2(\xi)\varphi_2 + h\varphi_2$ . The curvature 2-form of the auxiliary line bundle associated with the induced  $\text{Spin}^c$  structure is given in the basis  $\{e_1, e_2 = \mathfrak{X}e_1, \xi\}$  by

$$\Omega^2(e_1, e_2) = -\frac{c_1}{2}(h-1) - \frac{c_2}{2}(h+1),$$

$$\Omega^2(e_1, \xi) = -\frac{c_1 + c_2}{2}(e_1, V),$$

$$\Omega^2(e_2, \xi) = -\frac{c_1 + c_2}{2}(e_2, V).$$

Moreover, we have

$$0 = (\gamma_2(V)\varphi_2, \varphi_2), \quad (28)$$

$$g(V, e_1) = -i(\gamma_2(e_2)\varphi_2, \varphi_2), \quad (29)$$

$$g(V, e_2) = i(\gamma_2(e_1)\varphi_2, \varphi_2), \quad (30)$$

$$h = i(\gamma_2(\xi)\varphi_2, \varphi_2). \quad (31)$$

*Proof.* By the Gauss formula (9), the restriction  $\varphi_2$  of the parallel spinor  $\Psi$  on  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  satisfies

$$\nabla_X^2 \varphi_2 = \frac{1}{2} \gamma_2(II X) \varphi_2.$$

Now, for any  $X, Y \in \Gamma(TM)$ , we have

$$\begin{aligned} \Omega(X, Y) &= \Omega^{\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)}(X, Y) \\ &= \text{Ric}^{\mathbb{M}_1(c_1)}(J\pi_1 X, \pi_1 Y) - \text{Ric}^{\mathbb{M}_2(c_2)}(J\pi_2 X, \pi_2 Y) \\ &= \frac{c_1}{4} g(\mathfrak{X}X + \eta(X)\nu + \mathfrak{X}fX + \eta(fX)\nu - (V, X)\xi, Y + fY + (V, Y)\nu) \\ &\quad - \frac{c_2}{4} g(\mathfrak{X}X + \eta(X)\nu - \mathfrak{X}fX - \eta(fX)\nu + (V, X)\xi, Y - fY - (V, Y)\nu). \end{aligned}$$

In the basis  $\{e_1, e_2 = \mathfrak{X}e_1, \xi\}$ , we have

$$\Omega(e_1, e_2) = -\frac{c_1}{2}(h-1) - \frac{c_2}{2}(h+1),$$

$$\Omega(e_1, \xi) = -\frac{c_1 + c_2}{2}(V, e_1),$$

$$\Omega(e_2, \xi) = -\frac{c_1 + c_2}{2}(V, e_2).$$

Now, let's calculate

$$\begin{aligned} -\gamma_2(\xi)(\varphi_2) &= [\xi \cdot \nu \cdot (\psi_1^- \otimes \psi_2^+)]_{|_M} \\ &= [\pi_1(\xi) \cdot \pi_1(\nu) \cdot \psi_1^- \otimes \psi_2^+ + \pi_1(\nu) \cdot \psi_1^- \otimes \pi_2(\xi) \cdot \psi_2^+]_{|_M} \\ &\quad - [\pi_1(\xi) \cdot \psi_1^- \otimes \pi_2(\nu) \cdot \psi_2^+ + \psi_1^- \otimes \pi_2(\xi) \cdot \pi_2(\nu) \cdot \psi_2^+]_{|_M}. \end{aligned}$$

Thus, we get

$$\begin{aligned} -\gamma_2(\xi)(\varphi_2) &= [\xi \cdot \nu \cdot (\psi_1^- \otimes \psi_2^+)]_{|_M} \\ &= i(|\pi_1(\nu)|^2 - |\pi_2(\nu)|^2)\varphi_2 + [\pi_1(\nu) \cdot \psi_1^- \otimes \pi_2(\xi) \cdot \psi_2^+ - \pi_1(\xi) \cdot \psi_1^- \otimes \pi_2(\nu) \cdot \psi_2^+]_{|_M} \\ &= ih\varphi_2 + [\pi_1(\nu) \cdot \psi_1^- \otimes \pi_2(\xi) \cdot \psi_2^+ - \pi_1(\xi) \cdot \psi_1^- \otimes \pi_2(\nu) \cdot \psi_2^+]_{|_M}. \end{aligned}$$

In a similar way, we have

$$\begin{aligned} -\gamma_2(V)(\varphi_2) &= [V \cdot \nu \cdot (\psi_1^- \otimes \psi_2^+)]_{|_M} \\ &= [\pi_1(V) \cdot \pi_1(\nu) \cdot \psi_1^- \otimes \psi_2^+ + \pi_1(\nu) \cdot \psi_1^- \otimes \pi_2(V) \cdot \psi_2^+]_{|_M} \\ &\quad - [\pi_1(V) \cdot \psi_1^- \otimes \pi_2(\nu) \cdot \psi_2^+ + \psi_1^- \otimes \pi_2(V) \cdot \pi_2(\nu) \cdot \psi_2^+]_{|_M} \\ &= [\pi_1(\nu) \cdot \psi_1^- \otimes \pi_2(V) \cdot \psi_2^+ - \pi_1(V) \cdot \psi_1^- \otimes \pi_2(\nu) \cdot \psi_2^+]_{|_M}. \end{aligned}$$

Now, we have

$$\begin{aligned} &[-\gamma_2(V) - i\gamma_2(\xi)](\varphi_2) \\ &= [\pi_1(\nu) \cdot \psi_1^- \otimes \pi_2(V) \cdot \psi_2^+ - \pi_1(V) \cdot \psi_1^- \otimes \pi_2(\nu) \cdot \psi_2^+]_{|_M} \\ &\quad - h\varphi_2 + [i\pi_1(\nu) \cdot \psi_1^- \otimes \pi_2(\xi) \cdot \psi_2^+ - i\pi_1(\xi) \cdot \psi_1^- \otimes \pi_2(\nu) \cdot \psi_2^+]_{|_M} \\ &= -h\varphi_2 \\ &\quad + \underbrace{[\pi_1(\nu) \cdot \psi_1^- \otimes (\pi_2(V) + i\pi_2(\xi)) \cdot \psi_2^+ - (\pi_1(V) + i\pi_1(\xi)) \cdot \psi_1^- \otimes \pi_2(\nu) \cdot \psi_2^+]_{|_M}}_{=0 \text{ by Lemma 3.3}} \\ &= -h\varphi_2. \end{aligned}$$

Taking the scalar product of the last identity with  $\varphi_2$ , then the real part of the scalar product with  $\gamma_2(e_1)\varphi_2$ , then with  $\gamma_2(e_2)\varphi_2$ , we get (28), (29), (30) and (31).  $\square$

## 4 Generalized Killing $\text{Spin}^c$ spinors and isometric immersions

**Lemma 4.1.** [26] *Let  $E$  be a field of symmetric endomorphisms on a  $\text{Spin}^c$  manifold  $M^3$  of dimension 3, then*

$$\begin{aligned} \gamma(E(e_i))\gamma(E(e_j)) - \gamma(E(e_j))\gamma(E(e_i)) &= 2(a_{j3}a_{i2} - a_{j2}a_{i3})e_1 \\ &+ 2(a_{i3}a_{j1} - a_{i1}a_{j3})e_2 \\ &+ 2(a_{i1}a_{j2} - a_{i2}a_{j1})e_3, \end{aligned} \quad (32)$$

where  $(a_{ij})_{i,j}$  is the matrix of  $E$  written in any local orthonormal frame of  $TM$ .

**Proposition 4.2.** *Let  $(M^3, g)$  be a Riemannian  $\text{Spin}^c$  manifold endowed with an almost contact metric structure  $(\mathfrak{X}, \xi, \eta)$ . Assume that there exists a vector  $V$  and a function  $h$  and a  $\text{Spin}^c$  structure with non-trivial spinor  $\varphi_1$  satisfying*

$$\nabla_X^1 \varphi_1 = -\frac{1}{2}\gamma_1(EX)\varphi_1 \quad \text{and} \quad \gamma_1(\xi)\varphi_1 = -i\varphi_1,$$

where  $E$  is a field of symmetric endomorphisms on  $M$ . Moreover, we suppose that the curvature 2-form of the connection on the auxiliary line bundle associated with the  $\text{Spin}^c$  structure is given by

$$\begin{aligned} \Omega^1(e_1, e_2) &= \frac{c_1}{2}(h-1) - \frac{c_2}{2}(h+1), \\ \Omega^1(e_1, \xi) &= \frac{c_1 - c_2}{2}(V, e_1), \\ \Omega^1(e_2, \xi) &= \frac{c_1 - c_2}{2}(V, e_2), \end{aligned}$$

in the basis  $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$ . Hence, the Gauss equation is satisfied for  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  if and only if the Codazzi equation for  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  is satisfied.

*Proof.* We compute the spinorial curvature  $\mathcal{R}^1$  on  $\varphi_1$ , we get

$$\mathcal{R}_{X,Y}^1 \varphi_1 = -\frac{1}{2}\gamma_1(d^\nabla E(X, Y))\varphi_1 + \frac{1}{4}(\gamma_1(EY)\gamma_1(EX) - \gamma_1(EX)\gamma_1(EY))\varphi_1.$$

In the basis  $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$ , the Ricci identity (3) gives that

$$\begin{aligned} \frac{1}{2}\gamma_1(\text{Ric}(X))\varphi_1 - \frac{i}{2}\gamma_1(X \lrcorner \Omega^1)\varphi_1 &= \frac{1}{4} \sum_{k=1}^3 \gamma_1(e_k)(\gamma_1(EX)\gamma_1(Ee_k) - \gamma_1(Ee_k)\gamma_1(EX))\varphi_1 \\ &\quad - \frac{1}{2} \sum_{k=1}^3 \gamma_1(e_k)\gamma_1(d^\nabla E(e_k, X))\varphi_1. \end{aligned}$$

By Lemma 4.1 and for  $X = e_1$ , the last identity becomes

$$\begin{aligned} &(\mathbf{R}_{1221} + \mathbf{R}_{1331} - a_{11}a_{33} - a_{11}a_{22} + a_{13}^2 + a_{12}^2 + \frac{c_1}{2}(h-1) - \frac{c_2}{2}(h+1))\gamma_1(e_1)\varphi_1 \\ &+ (\mathbf{R}_{1332} - a_{12}a_{33} + a_{32}a_{13})\gamma_1(e_2)\varphi_1 \\ &+ (\mathbf{R}_{1223} - a_{22}a_{13} + a_{32}a_{12})\gamma_1(e_3)\varphi_1 \\ &- \frac{c_1 - c_2}{2}(V, e_1)\varphi_1 \end{aligned} \quad (33)$$

$$= -\gamma_1(e_2)\gamma_1(d^\nabla E(e_2, e_1))\varphi_1 - \gamma_1(e_3)\gamma_1(d^\nabla E(e_3, e_1))\varphi_1.$$

Since  $|\varphi|$  is constant ( $|\varphi| = 1$ ), the set  $\{\varphi_1, \gamma_1(e_1)\varphi_1, \gamma_1(e_2)\varphi_1, \gamma_1(e_3)\varphi_1\}$  is an orthonormal frame of  $\Sigma M$  with respect to the real scalar product  $\Re(\cdot, \cdot)$ . Hence, from Equation (35) we deduce

$$\begin{aligned} R_{1221} + R_{1331} - (a_{11}a_{33} + a_{11}a_{22} - a_{13}^2 - a_{12}^2) &+ \frac{c_2}{2}(h-1) - \frac{c_2}{2}(h+1) \\ &= g(d^\nabla E(e_1, e_2), e_3) - g(d^\nabla E(e_1, e_3), e_2) \end{aligned}$$

$$\begin{aligned} R_{1332} - (a_{12}a_{33} - a_{32}a_{13}) &= g(d^\nabla E(e_1, e_3), e_1) \\ R_{1223} - (a_{22}a_{13} - a_{32}a_{12}) &= -g(d^\nabla E(e_1, e_2), e_1) \\ -\frac{(c_1 - c_2)}{2}(V, e_1) &= g(d^\nabla E(e_2, e_1), e_2) + g(d^\nabla E(e_3, e_1), e_3) \end{aligned}$$

The same computation holds for the unit vector fields  $e_2$  and  $e_3$  and we get

$$\begin{aligned} R_{2331} - (a_{12}a_{33} - a_{13}a_{23}) &= -g(d^\nabla E(e_2, e_3), e_2) \\ R_{2332} + R_{2112} - (a_{22}a_{33} + a_{22}a_{11} - a_{13}^2 - a_{12}^2) &+ \frac{c_1}{2}(h-1) - \frac{c_2}{2}(h+1) \\ &= g(d^\nabla E(e_2, e_3), e_1) + g(d^\nabla E(e_1, e_2), e_3) \\ R_{2113} - (a_{23}a_{11} - a_{12}a_{13}) &= -g(d^\nabla E(e_1, e_2), e_2) \\ -\frac{(c_1 - c_2)}{2}(V, e_2) &= g(d^\nabla E(e_1, e_2), e_1) + g(d^\nabla E(e_3, e_2), e_3) \\ R_{3221} - (a_{13}a_{22} - a_{23}a_{21}) - \frac{(c_1 - c_2)}{2}(V, e_2) &= -g(d^\nabla E(e_2, e_3), e_3) \\ R_{3112} - (a_{32}a_{11} - a_{31}a_{12}) + \frac{(c_1 - c_2)}{2}(V, e_1) &= g(d^\nabla E(e_1, e_3), e_3) \\ R_{3113} + R_{3223} - (a_{22}a_{33} - a_{11}a_{33} + a_{13}^2 + a_{23}^2) &= g(d^\nabla E(e_2, e_3), e_1) - g(d^\nabla E(e_1, e_3), e_2) \\ g(d^\nabla E(e_2, e_3), e_2) &= -g(d^\nabla E(e_1, e_3), e_1) \end{aligned}$$

The last twelve equations will be called System 1 and it is clear that the Gauss equation for  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  is satisfied if and only if the Codazzi equation for  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  is satisfied.  $\square$

**Lemma 4.3.** *Under the same condition as Proposition 4.2, we have  $\nabla_X \xi = \mathfrak{X}EX$ .*

*Proof.* In fact, we simply compute the derivative of  $\gamma_1(\xi)\varphi_1 = -i\varphi_1$  in the direction of  $X \in \Gamma(TM)$  to get

$$\gamma_1(\nabla_X \xi)\varphi = \frac{i}{2}\gamma_1(EX)\varphi_1 + \frac{1}{2}\gamma_1(\xi)\gamma_1(EX)\varphi_1$$

Using that  $-i\gamma_1(e_2)\varphi_1 = \gamma_1(e_1)\varphi_1$ , the last equation reduces to

$$\gamma_1(\nabla_X \xi)\varphi_1 - g(EX, e_1)\gamma_1(e_2)\varphi_1 + g(EX, e_2)\gamma_1(e_1)\varphi_1 = 0.$$

Finally  $\nabla_X \xi = \mathfrak{X}EX$ .  $\square$

**Proposition 4.4.** *Let  $(M^3, g)$  be a Riemannian  $\text{Spin}^c$  manifold endowed with an almost contact metric structure  $(\mathfrak{X}, \xi, \eta)$ . Assume that there exist a nonzero vector field  $V$  and a function  $h$  such that there exists a  $\text{Spin}^c$  structure with non-trivial spinor  $\varphi$  satisfying*

$$\nabla_X^2 \varphi = \frac{1}{2}\gamma_2(EX)\varphi \quad \text{and} \quad \gamma_2(V)\varphi_2 = -i\gamma_2(\xi)\varphi_2 + h\varphi_2,$$

where  $E$  is a field of symmetric endomorphisms on  $M$ . Moreover, we suppose that the curvature 2-form of the connection on the auxiliary line bundle associated with the  $\text{Spin}^c$  structure is given by

$$\Omega^2(e_1, e_2) = -\frac{c_1}{2}(h-1) - \frac{c_2}{2}(h+1)$$

$$\Omega^2(e_1, \xi) = -\frac{(c_1 + c_2)}{2}(e_1, V)$$

$$\Omega^2(e_2, \xi) = -\frac{(c_1 + c_2)}{2}(e_2, V)$$

in the basis  $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$ . The Gauss equation for  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  is satisfied if and only if the Codazzi equation for  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  is satisfied.

*Proof.* First, from  $\gamma_2(V)\varphi_2 = -i\gamma_2(\xi)\varphi_2 + h\varphi_2$ , we have that (28), (29), (30) and (31) are satisfied. We compute the spinorial curvature  $\mathcal{R}^2$  on  $\varphi_2$ , we get

$$\mathcal{R}_{X,Y}^2\varphi_2 = \frac{1}{2}\gamma_2(d^\nabla E(X, Y))\varphi_2 + \frac{1}{4}(\gamma_2(EY)\gamma_2(EX) - \gamma_2(EX)\gamma_2(EY))\varphi_2.$$

In the basis  $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$ , the Ricci identity (3) gives that

$$\begin{aligned} \frac{1}{2}\gamma_2(\text{Ric}(X))\varphi_2 - \frac{i}{2}\gamma_2(X \lrcorner \Omega^2)\varphi_2 &= \frac{1}{4} \sum_{k=1}^3 \gamma_2(e_k)(\gamma_2(EX)\gamma_2(Ee_k) - \gamma_2(Ee_k)\gamma_2(EX))\varphi_2 \\ &+ \frac{1}{2} \sum_{k=1}^3 \gamma_2(e_k)\gamma_2(d^\nabla E(e_k, X))\varphi_2. \end{aligned}$$

By Lemma 4.1 and for  $X = e_1$ , the last identity becomes

$$\begin{aligned} &(\mathbf{R}_{1221} + \mathbf{R}_{1331} - a_{11}a_{33} - a_{11}a_{22} + a_{13}^2 + a_{12}^2)\gamma_2(e_1)\varphi_2 \quad (34) \\ &+ \frac{i}{2}(c_1 + c_2)(V, e_1)\gamma_2(\xi)\varphi_2 + \frac{i}{2}[c_1(h-1) + c_2(h+1)]\gamma_2(e_2)\varphi_2 \\ &+ (\mathbf{R}_{1332} - a_{12}a_{33} + a_{32}a_{13})\gamma_2(e_2)\varphi_2 \\ &+ (\mathbf{R}_{1223} - a_{22}a_{13} + a_{32}a_{12})\gamma_2(e_3)\varphi_2 \\ &= \gamma_2(e_2)\gamma_2(d^\nabla E(e_2, e_1))\varphi_2 + \gamma_2(e_3)\gamma_2(d^\nabla E(e_3, e_1))\varphi_2. \end{aligned}$$

Since  $|\varphi|$  is constant ( $|\varphi| = 1$ ), the set  $\{\varphi_2, \gamma_2(e_1)\varphi_2, \gamma_2(e_2)\varphi_2, \gamma_2(e_3)\varphi_2\}$  is an orthonormal frame of  $\Sigma M$  with respect to the real scalar product  $\Re(\cdot, \cdot)$ . Hence, from Equation (35) we deduce

$$\begin{aligned} &\mathbf{R}_{1221} + \mathbf{R}_{1331} - (a_{11}a_{33} + a_{11}a_{22} - a_{13}^2 - a_{12}^2) \\ &- \frac{1}{2}(c_1 + c_2)(V, e_1)^2 - \frac{1}{2}h(c_1(h-1) + c_2(h+1)) \\ &= g(d^\nabla E(e_2, e_1), e_3) - g(d^\nabla E(e_3, e_1), e_2) \\ \\ &\mathbf{R}_{1332} - (a_{12}a_{33} - a_{32}a_{13}) - \frac{1}{2}(c_1 + c_2)(V, e_1)(V, e_2) = -g(d^\nabla E(e_1, e_3), e_1) \\ &\mathbf{R}_{1223} - (a_{22}a_{13} - a_{32}a_{12}) + \frac{1}{2}(c_1(h-1) + c_2(h+1))(V, e_2) = -g(d^\nabla E(e_2, e_1), e_1) \\ &\frac{1}{2}(c_1 + c_2)h(V, e_1) - \frac{1}{2}(c_1(h-1) + c_2(h+1))(V, e_1) = -g(d^\nabla E(e_2, e_1), e_2) \\ &\quad -g(d^\nabla E(e_3, e_1), e_3) \end{aligned}$$

The same computation holds for the unit vector fields  $e_2$  and  $e_3$  and we get

$$\begin{aligned}
R_{2331} - (a_{12}a_{33} - a_{13}a_{23}) - \frac{1}{2}(c_1 + c_2)(V, e_2)(V, e_1) &= g(d^\nabla E(e_2, e_3), e_2) \\
R_{2332} + R_{2112} - (a_{22}a_{33} + a_{22}a_{11} - a_{13}^2 - a_{12}^2) \\
- \frac{1}{2}(c_1 + c_2)(e_2, V)^2 - \frac{1}{2}h(c_1(h-1) + c_2(h+1)) \\
&= -g(d^\nabla E(e_2, e_3), e_1) - g(d^\nabla E(e_1, e_2), e_3) \\
R_{2113} - (a_{23}a_{11} - a_{12}a_{13}) - \frac{1}{2}(c_1(h-1) + c_2(h+1))(V, e_1) &= g(d^\nabla E(e_1, e_2), e_2) \\
-\frac{1}{2}(c_1(h-1) + c_2(h+1))(V, e_2) + \frac{1}{2}(c_1 + c_2)h(V, e_2) &= -g(d^\nabla E(e_1, e_2), e_1) \\
&\quad -g(d^\nabla E(e_3, e_2), e_3) \\
R_{3221} - (a_{13}a_{22} - a_{23}a_{21}) + \frac{1}{2}(c_1 + c_2)h(V, e_2) &= g(d^\nabla E(e_2, e_3), e_3) \\
R_{3112} - (a_{32}a_{11} - a_{31}a_{12}) - \frac{1}{2}(c_1 + c_2)h(V, e_1) &= -g(d^\nabla E(e_1, e_3), e_3) \\
R_{3113} + R_{3223} - (a_{22}a_{33} - a_{11}a_{33} + a_{13}^2 + a_{23}^2) \\
- \frac{1}{2}(c_1 + c_2)(e_1, V)^2 - \frac{1}{2}(c_1 + c_2)(V, e_2)^2 \\
&= -g(d^\nabla E(e_2, e_3), e_1) + g(d^\nabla E(e_1, e_3), e_2) \\
\underbrace{-\frac{i}{2}(e_1, V)(\gamma_2(e_1)\varphi_2, \varphi_2) - \frac{i}{2}(V, e_2)(\gamma_2(e_2)\varphi_2, \varphi_2)}_{(\gamma_2(V)\varphi_2, \varphi_2)=0} &= -g(d^\nabla E(e_2, e_3), e_2) - g(d^\nabla E(e_1, e_3), e_1)
\end{aligned}$$

The last twelve equations will be called System 2 and it is clear that the Gauss equation for  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  is satisfied if and only if the Codazzi equation for  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  is satisfied.  $\square$

#### 4.1 Spinorial characterization of hypersurfaces of $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$

The main goal of this section is to prove Theorem 1.

*Proof of Theorem 1.* It is clear from the previous section that Assertrion 1 implies Assertrion 2. Now, Assume that Assertrion 2 holds. We have to establish the compatibility equation (20), (21), (13), (14), (15), (22), (23) and (24). First we define  $f : TM \rightarrow TM$  by  $(fe_1, e_1) = (fe_2, e_2) = -h$ ,  $(fe_1, e_2) = 0$  and  $(f\xi, e_1) = (V, e_2)$ ,  $(f\xi, e_2) = -(V, e_1)$ . Since  $\gamma_2(V)\varphi_2 = -i\gamma_2(\xi)\varphi_2 + h\varphi_2$ . It is clear that

$$\begin{cases}
h^2 + \|V\|^2 = 1 \\
(V, e_1) = -i(\gamma_2(e_2)\varphi_2, \varphi_2) \\
(V, e_2) = i(\gamma_2(e_1)\varphi_2, \varphi_2) \\
(V, \xi) = 0 \\
fV = -hV \\
f^2 = \text{Id} - (V, \cdot)V
\end{cases}$$

So Equations (13), (14), (15) are satisfied. Moreover, by Lemma 4.3, we have

$$\begin{aligned}
(\nabla_{e_1} V, \xi) &= (V, e_1)(\nabla_{e_1} e_1, \xi) + (V, e_2)(\nabla_{e_1} e_2, \xi) \\
&= -(V, e_1)(e_1 \mathfrak{X} E e_1) - (V, e_2)(e_2, \mathfrak{X} E e_1) \\
&= (V, e_1)E_{12} - (V, e_2)E_{11} + E_{11}(V, e_1) - E_{12}(V, e_2).
\end{aligned}$$

By a similar computation, we get that Equation (23) is satisfied. Now, Equation (24) is also satisfied because

$$\begin{aligned}
X(h) &= (i\gamma_2(\underbrace{\nabla_X \xi}_{=\mathfrak{X}EX})\varphi_2, \varphi_2) + \frac{i}{2}(\gamma_2(\xi)\gamma_2(EX)\varphi_2, \varphi_2) - \frac{i}{2}(\gamma_2(EX)\gamma_2(\xi)\varphi_2, \varphi_2) \\
&= iE(X, e_1)(\gamma_2(e_2)\varphi_2, \varphi_2) - iE(X, e_2)(\gamma_2(e_1)\varphi_2, \varphi_2) \\
&\quad + \frac{i}{2}E(X, e_2)(\gamma_2(\xi)\gamma_2(e_1)\varphi_2, \varphi_2) - \frac{i}{2}E(X, e_1)(\gamma_2(e_1)\gamma_2(\xi)\varphi_2, \varphi_2) \\
&\quad + \frac{i}{2}E(X, e_2)(\gamma_2(\xi)\gamma_2(e_2)\varphi_2, \varphi_2) - \frac{i}{2}E(X, e_2)(\gamma_2(e_2)\gamma_2(\xi)\varphi_2, \varphi_2) \\
&\quad - \frac{i}{2}E(X, \xi)|\varphi_2|^2 + \frac{i}{2}E(X, \xi)|\varphi_2|^2 \\
&= -(V, e_1)E(X, e_1) - E(X, e_2)(V, e_2) \\
&\quad + \frac{i}{2}E(X, e_1)(\gamma_2(e_2)\varphi_2, \varphi_2) + \frac{i}{2}E(X, e_1)(\gamma_2(e_2)\varphi_2, \varphi_2) \\
&\quad - \frac{i}{2}E(X, e_2)(\gamma_2(e_1)\varphi_2, \varphi_2) - \frac{i}{2}E(X, e_2)(\gamma_2(e_1)\varphi_2, \varphi_2) \\
&= -2E(X, e_2)(V, e_2) - 2E(X, e_1)(V, e_1) \\
&= -2(EV, X).
\end{aligned}$$

Now, we have

$$\begin{aligned}
(\nabla_{e_1} f e_1, e_1) &= e_1(f e_1, e_1) + (f e_1, \xi)(\nabla_{e_1} \xi, e_1) \\
&= e_1(-h) + (V, e_2)(\mathfrak{X}E e_1, e_1) \\
&= 2(EV, e_1) - (V, e_2)E_{12} \\
&= 2E_{11}(e_1, V) - (e_2, V)E_{12},
\end{aligned}$$

and

$$\begin{aligned}
(f(\nabla_{e_1} e_1), e_1) &= (\nabla_{e_1} e_1, f e_1) \\
&= (f e_1, e_2)(\nabla_{e_1} e_1, e_2) + (f e_1, \xi)(\nabla_{e_1} e_1, \xi) \\
&= -(\nabla_{e_1} \xi, e_1)(V, e_2) = -(\mathfrak{X}E e_1, e_1)(V, e_2) = E_{12}(V, e_2)
\end{aligned}$$

Thus,  $((\nabla_{e_1} f) e_1, e_1) = 2E_{11}(e_1, V)$ . By a similar computation, one can get (22). Solving System 1 and System 2 simultaneously gives the Gauss and the Codazzi equations. Finally, we have to check that  $\frac{F+\text{Id}}{2}$  and  $\frac{F-\text{Id}}{2}$  are of rank 2. In fact, in the basis  $\{e_1, e_2 = \mathfrak{X}e_1, \xi, \nu\}$ , the matrix  $\frac{F+\text{Id}}{2}$  can be written as

$$\frac{1}{2} \begin{pmatrix} -h+1 & 0 & (V, e_2) & (V, e_1) \\ 0 & -h+1 & (V, e_1) & (V, e_2) \\ (V, e_2) & -(V, e_1) & h+1 & 0 \\ (V, e_1) & (V, e_2) & 0 & h+1 \end{pmatrix}$$

Using that  $h^2 + \|V\|^2 = 1$ , one can check that it is of rank 2. Same holds for  $\frac{F-\text{Id}}{2}$ .  $\square$

**Remark 1.** Before giving some applications, we want to mention that both equivalent assertions of Theorem 1 are also equivalent to a third one described in terms of the Dirac operators  $D^1$  and  $D^2$ , and the energy-momentum tensors associated to  $\varphi_1$  and  $\varphi_2$ . We recall that the energy-momentum tensors  $Q_{\varphi_j}$ ,  $j = 1, 2$ , associated to the spinors field  $\varphi_j$  are the  $(2, 0)$ -tensors respectively defined by

$$Q_{\varphi_j}(X, Y) = \frac{1}{2} \Re e(\gamma_j(X) \cdot \nabla_Y^j \varphi + \gamma_j(Y) \cdot \nabla_X^j \varphi, \frac{\varphi}{|\varphi|^2}).$$

This third assertion can be written as:

3. There exists 2  $\text{Spin}^c$  structures on  $M$  carrying each one a non-trivial spinor  $\varphi_1$  and  $\varphi_2$  of constant norms and satisfying

$$D^1\varphi_1 = \frac{3}{2}H\varphi_1 \quad \text{and} \quad \gamma_1(\xi)\varphi_1 = -i\varphi_1,$$

$$D^2\varphi_2 = -\frac{3}{2}H\varphi_2 \quad \text{and} \quad \gamma_2(V)\varphi_2 = -i\gamma_2(\xi)\varphi_2 + h\varphi_2,$$

so that their energy-momentum tensors  $Q_{\varphi_1}$  and  $Q_{\varphi_2}$  are the same. Moreover, the curvature 2-form of the connection on the auxiliary bundle associated with these two  $\text{Spin}^c$  structure are given by ( $j \in \{1, 2\}$ )

$$\begin{cases} \Omega^j(e_1, e_2) = \frac{1}{2}(-1)^{j-1}c_1(h-1) - \frac{1}{2}c_2(h+1), \\ \Omega^j(e_1, \xi) = \frac{1}{2}\left((-1)^{j-1}c_1 - c_2\right)(e_1, V), \\ \Omega^j(e_2, \xi) = \frac{1}{2}\left((-1)^{j-1}c_1 - c_2\right)(e_2, V), \end{cases}$$

in the basis  $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$ .

Indeed, clearly, the second assertion of Theorem 1 implies assertion 3.. Reciprocally, as proven in [19],  $D^1\varphi_1 = \frac{3}{2}H\varphi_1$  with  $\varphi_1$  of constant norm implies that  $\nabla_X^1\varphi_1 = -\frac{1}{2}\gamma_1(E_1X)\varphi_1$  with  $E_1 = Q_{\varphi_1}$ . Similarly, we also get  $\nabla_X^2\varphi_2 = \frac{1}{2}\gamma_1(E_2X)\varphi_2$  with  $E_2 = Q_{\varphi_2}$ . Now, since  $Q_{\varphi_1} = Q_{\varphi_2}$ , this gives assertion 2 of Theorem 1.

## 5 Totally geodesic and totally umbilical hypersurfaces of $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$

In this section, we use our main result, Theorem 1, to give some geometric results on totally geodesic and umbilical hypersurfaces of  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$ .

**Lemma 5.1.** *Let  $(M^3, g = (\cdot, \cdot))$  be a totally umbilical hypersurface of  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$ . Then,*

$$\|V\| |c_1 - c_2| = 4\|dH\| \quad (35)$$

**Proof.** From Theorem 1, we know that there exists 2  $\text{Spin}^c$  structures on  $M$  carrying each one a non-trivial spinor  $\varphi_1$  and  $\varphi_2$  satisfying

$$\nabla_X^1\varphi_1 = -\frac{1}{2}\gamma_1(EX)\varphi_1 = -\frac{H}{2}\gamma_1(X)\varphi_1 \quad \text{and} \quad \gamma_1(\xi)\varphi_1 = -i\varphi_1.$$

$$\nabla_X^2\varphi_2 = \frac{1}{2}\gamma_2(EX)\varphi_2 = \frac{H}{2}\gamma_2(X)\varphi_2 \quad \text{and} \quad \gamma_2(V)\varphi_2 = -i\gamma_2(\xi)\varphi_2 + h\varphi_2.$$

The curvature 2-form of the connection on the auxiliary bundle associated with these two  $\text{Spin}^c$  structure are given by ( $j \in \{1, 2\}$ )

$$\Omega^j(e_1, e_2) = \frac{1}{2}(-1)^{j-1}c_1(h-1) - \frac{1}{2}c_2(h+1), \quad (36)$$

$$\Omega^j(e_1, \xi) = \frac{1}{2}\left((-1)^{j-1}c_1 - c_2\right)(e_1, V), \quad (37)$$

$$\Omega^j(e_2, \xi) = \frac{1}{2}\left((-1)^{j-1}c_1 - c_2\right)(e_2, V), \quad (38)$$

in the basis  $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$ .

**For the second  $\text{Spin}^c$  structure:** The Ricci identity for  $M$  can be written as

$$\frac{1}{2}\gamma_2(\text{Ric}^M(X))\varphi_2 - \frac{i}{2}\gamma_2(X \lrcorner \Omega^2)\varphi_2 = \frac{1}{2}\gamma_2(dH)\gamma_2(X)\varphi_2 + \frac{3}{2}dH(X)\varphi_2 + H^2\gamma_2(X)\varphi_2.$$

For  $X = \xi$ , the of the scalar product of the previous identity with  $\varphi_2$  gives

$$\begin{aligned} -\frac{i}{2}\Omega^2(\xi, e_1)(\gamma_2(e_1)\varphi_2, \varphi_2) &= -\frac{i}{2}\Omega^2(\xi, e_2)(\gamma_2(e_2)\varphi_2, \varphi_2) \\ &= \frac{1}{2}\Re(\gamma_2(dH)\gamma_2(\xi)\varphi_2, \varphi_2) + \frac{3}{2}dH(\xi) \end{aligned}$$

Using that  $-\gamma_2(e_1)\gamma_2(e_2)\gamma_2(\xi)\varphi_2 = \varphi_2$ , we get

$$-\frac{i}{2}\Omega^2(\xi, e_1)(\gamma_2(e_1)\varphi_2, \varphi_2) - \frac{i}{2}\Omega^2(\xi, e_2)(\gamma_2(e_2)\varphi_2, \varphi_2) = dH(\xi)$$

Finally, using (29), (30), (37) and (38), we obtain  $dH(\xi) = 0$ . In a similar way, for  $X = e_1$  the real part of the scalar product with  $\varphi_2$  of the Ricci identity gives

$$\begin{aligned} -\frac{i}{2}\Omega^2(e_1, e_2)(\gamma_2(e_2)\varphi_2, \varphi_2) &= -\frac{i}{2}\Omega^2(e_1, \xi)(\gamma_2(\xi)\varphi_2, \varphi_2) \\ &= \frac{1}{2}\Re(\gamma_2(dH)\gamma_2(e_1)\varphi_2, \varphi_2) + \frac{3}{2}dH(e_1). \end{aligned}$$

Using that  $-\gamma_2(e_1)\gamma_2(e_2)\gamma_2(\xi)\varphi_2 = \varphi_2$  and  $dH(\xi) = 0$ , we get

$$-\frac{i}{2}\Omega^2(e_1, e_2)(\gamma_2(e_2)\varphi_2, \varphi_2) - \frac{i}{2}\Omega^2(e_1, \xi)(\gamma_2(\xi)\varphi_2, \varphi_2) = dH(e_1).$$

Finally, using (29), (30), (37) and (36), we obtain  $dH(e_1) = \frac{c_1 - c_2}{4}g(V, e_1)$ . In a similar way we can get  $dH(e_2) = \frac{c_1 - c_2}{4}g(V, e_2)$ . Hence, we have  $\|dH\|^2 = \frac{(c_1 - c_2)^2}{16}\|V\|^2$ . For consistency, one can also take the first  $\text{Spin}^c$  structure and check that a similar identity can be obtained.

**Proposition 5.2.** *Let  $M$  be a totally umbilical hypersurface of in  $\mathbb{M}_1(c_1) \times \mathbb{M}_1(c_1)$ . Then  $M$  is totally geodesic or an extrinsic hypersphere. Moreover, if  $c_1 \neq 0$ , the universal cover of  $M$  is a Non-Einstein Sasaki manifold or a product of a Kähler manifold (of complex dimension 1) with  $\mathbb{R}$ . If  $c_1 = 0$ , then  $M$  is a Spin manifold with a parallel or Killing spin spinor.*

*Proof.* Let  $M$  be a totally umbilical hypersurface of  $\mathbb{M}_1(c_1) \times \mathbb{M}_1(c_1)$ . We have from (35) that  $dH = 0$ , so  $H$  is constant. Assume that  $c_1 \neq 0$ . If this constant  $H$  is 0,  $M$  has a parallel  $\text{Spin}^c$  spinor and if  $H \neq 0$ , then  $M$  has a Killing  $\text{Spin}^c$  spinor. From the classification of parallel and Killing  $\text{Spin}^c$  spinors [23], we get the desired result. If  $c_1 = 0$ , then the curvature of the auxiliary line bundle defining the  $\text{Spin}^c$  structure is zero and hence  $M$  is a Spin manifold with parallel or Killing spin spinor.  $\square$

**Proposition 5.3.** *Let  $M$  be a totally umbilical hypersurface of in  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$  ( $c_1 \neq c_2$ ) having a local product structure. Then  $M$  is totally geodesic or an extrinsic hypersphere. If  $c_1 \neq c_2 \neq 0$ , the universal cover of  $M$  is a non-Einstein Sasaki manifold or a product of a Kähler manifold (of complex dimension 1) with  $\mathbb{R}$*

*Proof.* Let  $M$  be a totally umbilical hypersurface of  $\mathbb{M}_1(c_1) \times \mathbb{M}_2(c_2)$ . Since  $V = 0$ , we have from (35) that  $dH = 0$ , so  $H$  is constant. If this constant is 0,  $M$  has a parallel  $\text{Spin}^c$  spinor and if  $H \neq 0$ , then  $M$  has a Killing  $\text{Spin}^c$  spinor. From the classification of parallel and Killing  $\text{Spin}^c$  spinors [23], we get the desired result.  $\square$

Using also Theorem 1, one can also prove the following:

**Proposition 5.4.** *Simply connected 3-dimensional homogeneous manifolds  $\mathbb{E}(\kappa, \tau)$  ( $\tau \neq 0$ ), with 4-dimensional isometry group cannot be immersed in  $\mathbb{M}_1(c_1) \times \mathbb{M}_1(c_1)$  as totally umbilical hypersurfaces.*

*Proof.* For  $c_1 = 0$ , this has been proved by Lawn and Roth [19], even without assuming the umbilicity. Assume that  $c_1 \neq 0$  and  $\mathbb{E}(\kappa, \tau)$  can be immersed in a totally umbilical way in  $\mathbb{M}_1(c_1) \times \mathbb{M}_1(c_1)$ . By Proposition 5.2, we have that  $H$  is constant and by Theorem 1,  $\mathbb{E}(\kappa, \tau)$  has two  $\text{Spin}^c$  structures carrying each one a non-trivial spinor  $\varphi_1$  and  $\varphi_2$  satisfying

$$\nabla_X^1 \varphi_1 = -\frac{H}{2} \gamma_1(X) \varphi_1 \quad \text{and} \quad \gamma_1(\xi) \varphi_1 = -i \varphi_1.$$

$$\nabla_X^2 \varphi_2 = \frac{H}{2} \gamma_2(X) \varphi_2 \quad \text{and} \quad \gamma_2(V) \varphi_2 = -i \gamma_2(\xi) \varphi_2 + h \varphi_2.$$

The curvature 2-form of the connection on the auxiliary bundle associated with these two  $\text{Spin}^c$  structure are given by ( $j \in \{1, 2\}$ )

$$\Omega^j(e_1, e_2) = \frac{1}{2} (-1)^{j-1} c_1 (h-1) - \frac{1}{2} c_2 (h+1),$$

$$\Omega^j(e_1, \xi) = \frac{1}{2} \left( (-1)^{j-1} c_1 - c_2 \right) (e_1, V),$$

$$\Omega^j(e_2, \xi) = \frac{1}{2} \left( (-1)^{j-1} c_1 - c_2 \right) (e_2, V),$$

in the basis  $\{e_1, e_2 = \mathfrak{X}e_1, e_3 = \xi\}$ . We will call the first one  $\text{Spin}^c$  structure  $t_1$  and the second one  $t_2$ . Since  $H$  is constant, these two spinors are in fact real Killing spinors. But, it is known [26] that the manifold  $\mathbb{E}(\kappa, \tau)$  has only two  $\text{Spin}^c$  structures carrying Killing spinors. The first one (call it  $t_3$ ) carries a Killing spinor  $\varphi$  with Killing constant  $\frac{\tau}{2}$  and for which  $\gamma_3(\xi) \varphi = -i \varphi$ ,  $\Omega^3(e_1, e_2) = -(\kappa - 4\tau^2)$  and  $\xi \lrcorner \Omega^3 = 0$ , where we denote by  $\gamma_3$  and  $i\Omega^3$  the Clifford multiplication and the curvature 2-form of the auxiliary line bundle associated to the structure  $t_3$ . The second one (let's call it  $t_4$ ) also carries a Killing spinor  $\varphi$  with Killing constant  $\frac{\tau}{2}$  for which  $\gamma_4(\xi) \varphi = i \varphi$ ,  $\Omega^4(e_1, e_2) = (\kappa - 4\tau^2)$  and  $\xi \lrcorner \Omega^4 = 0$ , where we denote by  $\gamma_4$  and  $i\Omega^4$  the Clifford multiplication and the curvature 2-form of the auxiliary line bundle associated to the structure  $t_4$ . By comparison, we must have that  $t_1 = t_3$  and  $t_2 = t_4$ . Thus we get

$$\gamma_2(V) \varphi_2 = -i \gamma_2(\xi) \varphi_2 + h \varphi_2 = (h+1) \varphi_2.$$

Hence  $V = 0$ . This means that  $\mathbb{E}(\kappa, \tau)$  has a local product, which is a contradiction.  $\square$

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