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► **To cite this version:**

Guillaume Fertin, Samuele Giraud, Sylvie Hamel, Stéphane Vialette. Unshuffling Permutations: Trivial Bijections and Compositions. TAMC, Apr 2019, Kitakyushu, Japan. pp.242-261, 10.1007/978-3-030-14812-6\_15 . hal-02304028

**HAL Id: hal-02304028**

**<https://hal.science/hal-02304028>**

Submitted on 14 Feb 2023

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# Unshuffling permutations: Trivial bijections and compositions<sup>\*</sup>

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**Abstract.** Given permutations  $\pi$ ,  $\sigma_1$  and  $\sigma_2$ , the permutation  $\pi$  (viewed as a string) is said to be a *shuffle* of  $\sigma_1$  and  $\sigma_2$ , in symbols  $\pi \in \sigma_1 \sqcup \sigma_2$ , if  $\pi$  can be formed by interleaving the letters of two strings  $p_1$  and  $p_2$  that are order-isomorphic to  $\sigma_1$  and  $\sigma_2$ , respectively. Given a permutation  $\pi \in S_{2n}$  and a bijective mapping  $f : S_n \rightarrow S_n$ , the  $f$ -UNSHUFFLE-PERMUTATION problem is to decide whether there exists a permutation  $\sigma \in S_n$  such that  $\pi \in \sigma \sqcup f(\sigma)$ . We consider here this problem for the following bijective mappings: inversion, reverse, complementation, and all their possible compositions. In particular, we present combinatorial results about the permutations accepted by this problem. As main results, we obtain that this problem is NP-complete when  $f$  is the reverse, the complementation, or the composition of the reverse with the complementation.

**Keywords:** Permutation · Shuffle product · Computational complexity.

## 1 Introduction

Given permutations  $\pi$ ,  $\sigma_1$  and  $\sigma_2$ ,  $\pi$  is said to be a *shuffle* of  $\sigma_1$  and  $\sigma_2$ , in symbols  $\pi \in \sigma_1 \sqcup \sigma_2$ , if  $\pi$  is the disjoint union of two patterns  $p_1$  and  $p_2$  (*i.e.*, if, viewed as a string, it can be formed by interleaving the letters of  $p_1$  and  $p_2$ ) that are order-isomorphic to  $\sigma_1$  and  $\sigma_2$ , respectively. This shuffling operation  $\sqcup$  was introduced by Vargas [12] in an algebraic context and was called *supershuffle*. In case  $\sigma = \sigma_1 = \sigma_2$ , the permutation  $\pi$  is said to be a *square w.r.t. the shuffle product* (or simply a *square* when clear from the context), and  $\sigma$  is said to be a *square root of  $\pi$  w.r.t. the shuffle product* (or simply a *square root of  $\pi$*  when clear from the context). Note that a permutation may have several square roots. For example,  $\pi = 18346752 \in 1234 \sqcup 4321$  since  $\pi$  is a shuffle of the patterns  $p_1 = 1347$  and  $p_2 = 8652$  that are order-isomorphic to  $\sigma_1 = 1234$  and

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<sup>\*</sup> Partially supported by Laboratoire International Franco-Québécois de Recherche en Combinatoire (LIRCO) (G. Fertin and S. Vialette), supported by the Individual Discovery Grant RGPIN-2016-04576 from Natural Sciences and Engineering Research Council of Canada (NSERC) (S. Hamel)

$\sigma_2 = 4321$ , respectively (as shown in  $18^346^752$ ). However,  $\pi$  is not a square. Besides,  $\pi' = 24317856$  is a square as it is a shuffle of the patterns 2175 and 4386 that are both order-isomorphic to 2143 (as shown in  $243^178^56$ ). Note that 2143 is not the unique square root of  $\pi'$  since  $\pi'$  is also a shuffle of patterns 2156 and 4378 that are both order-isomorphic to 2134 (as shown in  $243^178^56$ ).

The above definitions are of course intended to be natural counterparts to the ordinary shuffle of words and languages. Given words  $u$ ,  $v_1$  and  $v_2$ ,  $u$  is said to be a *shuffle* of  $v_1$  and  $v_2$  (denoted  $u \in v_1 \sqcup v_2$ ), if  $u$  can be formed by interleaving the letters of  $v_1$  and  $v_2$  in a way that maintains the left-to-right ordering of the letter from each word. For example,  $u = abccbabacbb \in abcab \sqcup cbabcb$  (as shown in  $ab_ccbab_a_cbb$ ). Similarly, in case  $v = v_1 = v_2$ , the word  $u$  is said to be a *square w.r.t. the shuffle product*, and  $v$  is said to be a *square root of  $u$  w.r.t. the shuffle product*. For example,  $abaaabaabb$  is a square w.r.t. the shuffle product since it belongs to the shuffle of  $abaab$  with itself (as shown in  $ab_aaa_baa^bb$ ). Given words  $u$ ,  $v_1$  and  $v_2$ , deciding whether  $u \in v_1 \sqcup v_2$  is  $O(|u|^2 / \log(|u|))$  time solvable [6]. To the best of our knowledge, the first  $O(|u|^2)$  time algorithm for this problem appeared in [7]. However, deciding whether a given word  $u$  is a square w.r.t. the shuffle product is NP-complete [9, 2]. Finally, for a given word  $u$ , deciding whether there exists a word  $v$  such that  $u$  is in the shuffle of  $v$  with its reverse  $v^R$  (i.e.,  $u \in v \sqcup v^R$ ) is NP-complete as well [9]. However, the problem is polynomial-time solvable for words built from a binary alphabet.

Coming back to permutations, by using a pattern avoidance criterion on directed perfect matchings, it is proved in [4] that recognizing permutations that are squares is NP-complete, and a bijection between (213, 231)-avoiding square permutations and square binary words is presented. Given (3142, 2413)-avoiding (a.k.a. *separable*) permutations  $\pi$ ,  $\sigma_1$  and  $\sigma_2$ , it can be decided in polynomial time whether  $\pi \in \sigma_1 \sqcup \sigma_2$  [8].

We finally observe that deciding whether a permutation is a shuffle of two monotone permutations is in P:

(i) *merge permutations* are the union of two increasing subsequences and are characterized by the fact that they contain no decreasing subsequence of length 3 [5], whereas (ii) *skew-merged permutations* are the union of an increasing subsequence with a decreasing subsequence and are characterized by the fact that they contain no subsequence ordered in the same way as 2143 or 3412 [11].

In this paper, we are interested in a generalization of the problem consisting in recognizing square permutations. We call this new problem the  $f$ -UNSHUFFLE-PERMUTATION problem, which is defined as follows. Given a permutation  $\pi \in S_{2n}$  and a bijective mapping  $f : S_n \rightarrow S_n$ , the  $f$ -UNSHUFFLE-PERMUTATION problem asks whether there exists a permutation  $\sigma \in S_n$  such that  $\pi \in \sigma \sqcup f(\sigma)$ . We say in this case that  $\pi$  is a *generalized square permutation*. In case  $f = \text{id}$ , the id-UNSHUFFLE-PERMUTATION problem reduces to recognizing square permutations, and hence is NP-complete. This paper is devoted to studying the  $f$ -UNSHUFFLE-PERMUTATION problem in case  $f$  is either a trivial bijection (identity, complement, reverse or inverse) or obtained by composing trivial bijections. These bijections act on permutations by performing a transformation on their permutation matrices

and can hence be seen as elements of the Dihedral group  $D_4$ , as discussed in the next section. The paper is organized as follows. In Section 2, we provide the needed definitions. Section 3 is devoted to presenting generalized square permutations and some associated enumerative properties. Finally, in Section 4, we show hardness of recognizing some generalized square permutations.

## 2 Definitions

For any nonnegative integer  $n$ ,  $[n]$  is the set  $\{1, \dots, n\}$ . We follow the usual terminology on words [3]. Let us recall here the most important ones. Let  $u$  be a word. The length of  $u$  is denoted by  $|u|$ . The *empty word*, the only word of null length, is denoted by  $\epsilon$ . For any  $i \in [|u|]$ , the  $i$ -th letter of  $u$  is denoted by  $u(i)$ . If  $I$  is a subset of  $[|u|]$ ,  $u|_I$  is the subword of  $u$  consisting in the letters of  $u$  at the positions specified by the elements of  $I$ . A *permutation* of size  $n$  is a word  $\pi$  of length  $n$  on the alphabet  $[n]$  such that each letter admits exactly one occurrence. The set of all permutations of size  $n$  is denoted by  $S_n$ .

**Definition 1 (Reduced form).** *If  $\lambda$  is a list of distinct integers, the reduced form of  $\lambda$ , denoted  $\text{reduce}(\lambda)$ , is the permutation obtained from  $\lambda$  by replacing its  $i$ -th smallest entry with  $i$ . For instance,  $\text{reduce}(31845) = 21534$ .*

**Definition 2 (Order-isomorphism).** *Let  $\lambda_1$  and  $\lambda_2$  be two words of distinct integers such that  $\text{reduce}(\lambda_1) = \text{reduce}(\lambda_2)$ . We say that  $\lambda_1$  and  $\lambda_2$  are order-isomorphic and we denote it by  $\lambda_1 \simeq \lambda_2$ .*

**Definition 3 (Pattern containment).** *A permutation  $\sigma$  is said to be contained in, or to be a subpermutation of, another permutation  $\pi$ , written  $\sigma \preceq \pi$ , if  $\pi$  has a (not necessarily contiguous) subsequence whose terms are order-isomorphic to  $\sigma$ . We also say that  $\pi$  admits an occurrence of the pattern  $\sigma$ .*

Thus,  $\sigma \preceq \pi$  if there is a set  $I$  of positions in  $\pi$  such that  $\text{reduce}(\pi|_I) = \sigma$ . For example,  $1423 \preceq 149362785$  since  $\text{reduce}(149362785|_{\{2,3,5,8\}}) = \text{reduce}(4968) = 1423$ .

**Definition 4 (Trivial bijections).** *Let  $\pi = \pi(1)\pi(2) \dots \pi(n)$  be a permutation of size  $n$ . The reverse of  $\pi$  is the permutation  $r(\pi) = \pi(n)\pi(n-1) \dots \pi(1)$ . The complement of  $\pi$  is the permutation  $c(\pi) = \pi'(1)\pi'(2) \dots \pi'(n)$ , where  $\pi'(i) = n - \pi(i) + 1$ . The inverse is the regular group theoretical inverse on permutations, that is the  $\pi(i)$ -th position of the inverse  $i(\pi)$  is occupied by  $i$ . The reverse, complement and inverse are called the trivial bijections from  $S_n$  to itself.*

Abusing notations, for any  $S \subseteq S_n$ , we let  $c(S)$ ,  $i(S)$  and  $r(S)$  stand respectively for  $\{c(\pi) : \pi \in S\}$ ,  $\{i(\pi) : \pi \in S\}$  and  $\{r(\pi) : \pi \in S\}$ .

**Definition 5 (Shuffle).** *Let  $\sigma_1 \in S_{k_1}$  and  $\sigma_2 \in S_{k_2}$  be two permutations. A permutation  $\pi = \pi(1)\pi(2) \dots \pi(k_1 + k_2)$  is a shuffle of  $\sigma_1$  and  $\sigma_2$  if there is a subset  $I$  of  $[n]$  such that  $\pi|_I \simeq \sigma_1$  and  $\pi|_{\bar{I}} \simeq \sigma_2$ , where  $\bar{I} = [n] \setminus I$  and  $n = k_1 + k_2$ .*

In other terms,  $\pi$  is obtained by interleaving the letters of two words respectively order-isomorphic with  $\sigma_1$  and  $\sigma_2$ . We denote by  $\sigma_1 \sqcup \sigma_2$  the set of all shuffles of  $\sigma_1$  and  $\sigma_2$ .

For example,

$$12 \sqcup 21 = \{1243, 1324, 1342, 1423, 1432, 2134, 2314, 2341, 2413, 2431, \\ 3124, 3142, 3214, 3241, 3421, 4123, 4132, 4213, 4231, 4312\}.$$

We are now ready to define the  $f$ -UNSHUFFLE-PERMUTATION problem we focus on in this paper.

**Definition 6 ( $f$ -Unshuffle-Permutation).** *Given a permutation  $\pi \in S_{2n}$  and a bijective mapping  $f : S_n \rightarrow S_n$ , the  $f$ -UNSHUFFLE-PERMUTATION problem is to decide whether there exists a permutation  $\sigma \in S_n$  such that  $\pi \in \sigma \sqcup f(\sigma)$ .*

### 3 Generalized square permutations and enumerative properties

The bijections  $c$ ,  $i$ , and  $r$  are involutions and can be seen as particular elements of the dihedral group  $D_4$  encoding all the symmetries of the square. Before explaining why, let us recall a definition for  $D_4$ .

The group  $D_4$  is generated by two elements  $\mathbf{a}$  and  $\mathbf{b}$  subjected exactly to the nontrivial relations

$$\mathbf{a}\mathbf{a} = \epsilon, \quad \mathbf{b}\mathbf{b}\mathbf{b}\mathbf{b} = \epsilon, \quad \mathbf{a}\mathbf{b}\mathbf{a}\mathbf{b} = \epsilon, \tag{1}$$

where  $\epsilon$  is the unit of  $D_4$ . One can think of element  $\mathbf{a}$  (resp.  $\mathbf{b}$ ) acting on a square by performing a symmetry through the vertical axis (resp. a 90° clockwise rotation). Now, we can regard, for any  $n \in \mathbb{N}$ , each element of  $D_4$  as a map  $\phi : S_n \rightarrow S_n$  such that  $\phi(\pi)$ ,  $\pi \in S_n$ , is the permutation obtained by performing on its permutation matrix the transformations specified by  $\phi$ . The maps  $c$ ,  $i$ , and  $r$  can thus be expressed as

$$\mathbf{c} = \mathbf{b}\mathbf{b}\mathbf{a}, \quad \mathbf{i} = \mathbf{b}\mathbf{a}, \quad \mathbf{r} = \mathbf{a}. \tag{2}$$

Figure 1 shows the Cayley graph of this group. The main interest of seeing the three trivial bijections on  $S_n$  as elements of  $D_4$  lies in the fact that all the other bijections obtained by composing the trivial ones can be expressed by compositions of the two elements  $\mathbf{a}$  and  $\mathbf{b}$ . This will allow us to gain concision in some proofs presented in the sequel.

We now turn to defining *identity squares*, *complement squares*, *reverse squares* and *inverse squares* that are natural generalizations of square permutations to trivial bijections from  $S_n$  to itself.

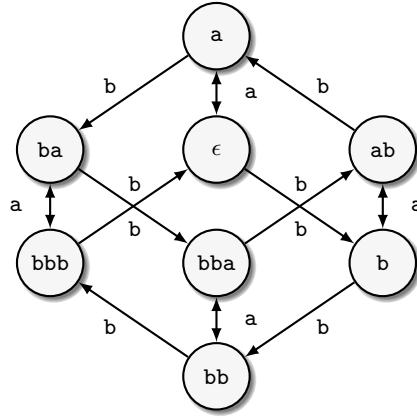


Fig. 1. The Cayley graph of the group  $D_4$ .

**Definition 7 (Square permutations for trivial bijections).** For  $n \in \mathbb{N}$ , define

$$\begin{aligned} \mathcal{S}_{2n}^\epsilon &= \{\pi \in S_{2n} : \exists \sigma \in S_n, \text{ such that } \pi \in \sigma \sqcup \sigma\}, \\ \mathcal{S}_{2n}^c &= \{\pi \in S_{2n} : \exists \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup c(\sigma)\}, \\ \mathcal{S}_{2n}^r &= \{\pi \in S_{2n} : \exists \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup r(\sigma)\}, \\ \mathcal{S}_{2n}^i &= \{\pi \in S_{2n} : \exists \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup i(\sigma)\}. \end{aligned}$$

We begin by proving that applying any trivial bijection is in some sense compatible with the shuffle operator.

**Lemma 1 (Trivial bijections and shuffle).** Let  $\pi, \sigma_1$  and  $\sigma_2$  be permutations such that  $\pi \in \sigma_1 \sqcup \sigma_2$ . Then

- $c(\pi) \in c(\sigma_1) \sqcup c(\sigma_2)$ ;
- $r(\pi) \in r(\sigma_1) \sqcup r(\sigma_2)$ ;
- $i(\pi) \in i(\sigma_1) \sqcup i(\sigma_2)$ .

*Proof.* Let  $n$  be the size of  $\pi$ . Since  $\pi$  is a shuffle of  $\sigma_1$  and  $\sigma_2$ , there is a subset  $I = \{i_1 < \dots < i_k\}$  of  $[n]$  such that  $\pi|_I \simeq \sigma_1$  and  $\pi|_{\bar{I}} \simeq \sigma_2$ .

By setting  $J = \{n - i_k + 1 < \dots < n - i_1 + 1\}$ , we have

$$r(\pi)|_J = (\pi(n) \dots \pi(1))|_J = \pi(i_k) \dots \pi(i_1) \simeq r(\sigma_1). \quad (3)$$

For the same reason, we have also  $r(\pi)|_{\bar{J}} \simeq r(\sigma_2)$ . Hence,  $r(\pi)$  is a shuffle of  $r(\sigma_1)$  and  $r(\sigma_2)$ .

Now, set  $J = \{\pi(i) : i \in I\} = \{\pi(\ell_1) < \dots < \pi(\ell_k)\}$  where  $\ell_1, \dots, \ell_k$  are elements of  $I$ . We have

$$i(\pi)|_J = (\pi^{-1}(1) \dots \pi^{-1}(n))|_J = \pi^{-1}(\pi(\ell_1)) \dots \pi^{-1}(\pi(\ell_k)) = \ell_1 \dots \ell_k \simeq i(\sigma_1). \quad (4)$$

For the same reason, we have also  $i(\pi)_{|\bar{j}} \simeq i(\sigma_2)$ . Hence,  $i(\pi)$  is also shuffle of  $i(\sigma_1)$  and  $i(\sigma_2)$ .

Finally, since  $\mathbf{b} = \mathbf{i} \circ \mathbf{r}$  in  $D_4$ , the map  $c$  can be expressed as compositions involving  $\mathbf{r}$  and  $\mathbf{i}$ . Hence,  $c(\pi)$  is a shuffle of  $c(\pi_1)$  and  $c(\pi_2)$ .  $\blacksquare$

The following lemma is well-known and is a consequence of interpreting  $c$ ,  $\mathbf{r}$ , and  $\mathbf{i}$  as compositions involving  $\mathbf{a}$  and  $\mathbf{b}$  according to (2) and the relations (1) between  $\mathbf{a}$  and  $\mathbf{b}$ .

**Lemma 2 (Compositions of trivial bijections).** *For any permutation  $\pi$ ,*

- $(c \circ \mathbf{r})(\pi) = (\mathbf{r} \circ c)(\pi)$ ;
- $(\mathbf{i} \circ c)(\pi) = (\mathbf{r} \circ \mathbf{i})(\pi)$ ;
- $(\mathbf{i} \circ \mathbf{r})(\pi) = (c \circ \mathbf{i})(\pi)$ .

According to Lemma 2 (and the fact that the group  $D_4$  has exactly eight elements), we thus only need to consider the following compositions in the rest of the paper.

**Definition 8 (Square permutations for compositions of trivial bijections).** *For  $n \in \mathbb{N}$ , define*

$$\begin{aligned} \mathcal{S}_{2n}^{\mathbf{rc}} &= \{\pi \in S_{2n} : \exists \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup (\mathbf{r} \circ c)(\sigma)\}, \\ \mathcal{S}_{2n}^{\mathbf{ci}} &= \{\pi \in S_{2n} : \exists \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup (c \circ \mathbf{i})(\sigma)\}, \\ \mathcal{S}_{2n}^{\mathbf{ic}} &= \{\pi \in S_{2n} : \exists \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup (\mathbf{i} \circ c)(\sigma)\}, \\ \mathcal{S}_{2n}^{\mathbf{irc}} &= \{\pi \in S_{2n} : \exists \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup (\mathbf{i} \circ \mathbf{r} \circ c)(\sigma)\}. \end{aligned}$$

We hence obtain the eight sets  $\mathcal{S}_{2n}^{\epsilon}$ ,  $\mathcal{S}_{2n}^{\mathbf{c}}$ ,  $\mathcal{S}_{2n}^{\mathbf{r}}$ ,  $\mathcal{S}_{2n}^{\mathbf{i}}$ ,  $\mathcal{S}_{2n}^{\mathbf{rc}}$ ,  $\mathcal{S}_{2n}^{\mathbf{ci}}$ ,  $\mathcal{S}_{2n}^{\mathbf{ic}}$ , and  $\mathcal{S}_{2n}^{\mathbf{irc}}$  of square permutations provided by Definitions 7 and 8. Let us now investigate some easy bijective and enumerative properties satisfied by these sets.

**Proposition 1 (Bijection between  $\mathcal{S}_{2n}^{\mathbf{c}}$  and  $\mathcal{S}_{2n}^{\mathbf{r}}$ ).** *For any  $n \in \mathbb{N}$ ,  $i(\mathcal{S}_{2n}^{\mathbf{c}}) = \mathcal{S}_{2n}^{\mathbf{r}}$ .*

*Proof.* Let  $\pi \in \mathcal{S}_{2n}$ . We have

$$\begin{aligned} \pi \in \mathcal{S}_{2n}^{\mathbf{c}} &\Leftrightarrow \exists \sigma \in S_n, \pi \in \sigma \sqcup c(\sigma) \\ &\Leftrightarrow \exists \sigma \in S_n, i(\pi) \in i(\sigma) \sqcup (\mathbf{i} \circ c)(\sigma) && \text{(Lemma 1)} \\ &\Leftrightarrow \exists \sigma \in S_n, i(\pi) \in i(\sigma) \sqcup (\mathbf{r} \circ \mathbf{i})(\sigma) && \text{(Lemma 2)} \\ &\Leftrightarrow \exists \sigma \in S_n, i(\pi) \in \sigma \sqcup \mathbf{r}(\sigma) \\ &\Leftrightarrow i(\pi) \in \mathcal{S}_{2n}^{\mathbf{r}}. && \blacksquare \end{aligned}$$

**Proposition 2 (Bijection between  $\mathcal{S}_{2n}^{\mathbf{ic}}$  and  $\mathcal{S}_{2n}^{\mathbf{ci}}$ ).** *For any  $n \in \mathbb{N}$ ,  $c(\mathcal{S}_{2n}^{\mathbf{ic}}) = \mathcal{S}_{2n}^{\mathbf{ci}}$ ,  $i(\mathcal{S}_{2n}^{\mathbf{ic}}) = \mathcal{S}_{2n}^{\mathbf{ci}}$ , and  $\mathbf{r}(\mathcal{S}_{2n}^{\mathbf{ic}}) = \mathcal{S}_{2n}^{\mathbf{ci}}$ .*

*Proof.* Let  $\pi \in S_{2n}$ . We have

$$\begin{aligned}
\pi \in \mathcal{S}_{2n}^{\text{ic}} &\Leftrightarrow \exists \sigma \in S_n, \pi \in \sigma \sqcup (\text{i} \circ \text{c})(\sigma) \\
&\Leftrightarrow \exists \sigma \in S_n, c(\pi) \in c(\sigma) \sqcup (c \circ \text{i} \circ \text{c})(\sigma) && \text{(Lemma 1)} \\
&\Leftrightarrow \exists \sigma \in S_n, c(\pi) \in \sigma \sqcup (c \circ \text{i})(\sigma) \\
&\Leftrightarrow c(\pi) \in \mathcal{S}_{2n}^{\text{ci}}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\pi \in \mathcal{S}_{2n}^{\text{ic}} &\Leftrightarrow \exists \sigma \in S_n, \pi \in \sigma \sqcup (\text{i} \circ \text{c})(\sigma) \\
&\Leftrightarrow \exists \sigma \in S_n, \text{i}(\pi) \in \text{i}(\sigma) \sqcup (\text{i} \circ \text{i} \circ \text{c})(\sigma) && \text{(Lemma 1)} \\
&\Leftrightarrow \exists \sigma \in S_n, \text{i}(\pi) \in \text{i}(\sigma) \sqcup (\text{i} \circ \text{r} \circ \text{i})(\sigma) && \text{(Lemma 2)} \\
&\Leftrightarrow \exists \sigma \in S_n, \text{i}(\pi) \in \sigma \sqcup (\text{i} \circ \text{r})(\sigma) \\
&\Leftrightarrow \exists \sigma \in S_n, \text{i}(\pi) \in \sigma \sqcup (c \circ \text{i})(\sigma) && \text{(Lemma 2)} \\
&\Leftrightarrow \text{i}(\pi) \in \mathcal{S}_{2n}^{\text{ci}}.
\end{aligned}$$

Finally, the fact that  $\text{i} \circ \text{c} \circ \text{i} = \text{r}$  implies that  $\text{r}$  is also a bijection between  $\mathcal{S}_{2n}^{\text{ic}}$  and  $\mathcal{S}_{2n}^{\text{ci}}$ .  $\blacksquare$

**Proposition 3 (Bijection between  $\mathcal{S}_{2n}^{\text{i}}$  and  $\mathcal{S}_{2n}^{\text{irc}}$ ).** For any  $n \in \mathbb{N}$ ,  $(\text{i} \circ \text{c})(\mathcal{S}_{2n}^{\text{irc}}) = \mathcal{S}_{2n}^{\text{i}}$ .

*Proof.* Let  $\pi \in S_{2n}$ . We have

$$\begin{aligned}
\pi \in \mathcal{S}_{2n}^{\text{irc}} &\Leftrightarrow \exists \sigma \in S_n, \pi \in \sigma \sqcup (\text{i} \circ \text{r} \circ \text{c})(\sigma) \\
&\Leftrightarrow \exists \sigma \in S_n, \text{i}(\pi) \in \text{i}(\sigma) \sqcup (\text{r} \circ \text{c})(\sigma) && \text{(Lemma 1)} \\
&\Leftrightarrow \exists \sigma \in S_n, (\text{r} \circ \text{i})(\pi) \in (\text{r} \circ \text{i})(\sigma) \sqcup c(\sigma) && \text{(Lemma 1)} \\
&\Leftrightarrow \exists \sigma \in S_n, (\text{i} \circ \text{c})(\pi) \in (\text{i} \circ \text{c})(\sigma) \sqcup c(\sigma) && \text{(Lemma 2)} \\
&\Leftrightarrow \exists \sigma \in S_n, (\text{i} \circ \text{c})(\pi) \in \text{i}(\sigma) \sqcup \sigma \\
&\Leftrightarrow (\text{i} \circ \text{c})(\pi) \in \mathcal{S}_{2n}^{\text{i}}.
\end{aligned}$$

$\blacksquare$

To simplify notations, we write  $s_{2n}^{\epsilon}$ ,  $s_{2n}^{\text{c}}$ ,  $s_{2n}^{\text{r}}$ ,  $s_{2n}^{\text{i}}$ ,  $s_{2n}^{\text{rc}}$ ,  $s_{2n}^{\text{ci}}$ ,  $s_{2n}^{\text{ic}}$ , and  $s_{2n}^{\text{irc}}$  for the cardinalities of the sets  $\mathcal{S}_{2n}^{\epsilon}$ ,  $\mathcal{S}_{2n}^{\text{c}}$ ,  $\mathcal{S}_{2n}^{\text{r}}$ ,  $\mathcal{S}_{2n}^{\text{i}}$ ,  $\mathcal{S}_{2n}^{\text{rc}}$ ,  $\mathcal{S}_{2n}^{\text{ci}}$ ,  $\mathcal{S}_{2n}^{\text{ic}}$ , and  $\mathcal{S}_{2n}^{\text{irc}}$ , respectively. Table 1 shows the first cardinalities. We note that, at this time, none of the corresponding integer sequence does appear in OEIS [10].

**Lemma 3 (Upper bound for the number of generalized squares).** For any  $n \in \mathbb{N}$ ,

$$|Q_{2n}| \leq \binom{2n-1}{n-1} \binom{2n}{n} n!, \quad (5)$$

where  $Q_{2n}$  is any of the sets  $\mathcal{S}_{2n}^{\epsilon}$ ,  $\mathcal{S}_{2n}^{\text{c}}$ ,  $\mathcal{S}_{2n}^{\text{r}}$ ,  $\mathcal{S}_{2n}^{\text{i}}$ ,  $\mathcal{S}_{2n}^{\text{rc}}$ ,  $\mathcal{S}_{2n}^{\text{ci}}$ ,  $\mathcal{S}_{2n}^{\text{ic}}$ , or  $\mathcal{S}_{2n}^{\text{irc}}$ .



$n$	0	1	2	3	4	5
$s_{2n}^{\text{rc}}$	1	2	20	472	18 988	1 112 688
$s_{2n}^{\text{c}} = s_{2n}^{\text{r}}$	1	2	20	480	19 744	1 185 264
$s_{2n}^{\text{i}} = s_{2n}^{\text{irc}}$	1	2	20	488	20 250	1 229 858
$s_{2n}^{\text{e}}$	1	2	20	504	21 032	1 293 418
$s_{2n}^{\text{ci}} = s_{2n}^{\text{ic}}$	1	2	20	586	27 990	2 044 596
$(2n)!$	1	2	24	720	40 320	3 628 800

**Table 1.** Cardinalities of the sets of square permutations of sizes  $0 \leq 2n \leq 10$ .

*Proof.* This is a consequence of the fact that to construct a permutation  $\pi$  of  $Q_{2n}$ , we can proceed by first choosing a set of  $n$  letters among the alphabet  $[2n]$  ( $\binom{2n}{n}$  choices), then by specifying an order from left to right for these letters ( $n!$  choices), and finally by deploying the chosen letters onto a set  $I$  of  $n$  positions among the set of all possible positions  $[2n]$  ( $\binom{2n}{n}$  choices). The empty positions are filled by the unique authorized completion such that  $\pi|_I \simeq \phi(\pi|_{\bar{I}})$  where  $\phi$  is the considered bijection on  $S_n$ . For this reason, there is no more than  $\binom{2n}{n}^2 n!$  elements in  $Q_{2n}$ . The tighter bound (5) is the consequence of the fact that we can assume that the first position can always be fixed. ■

Let us denote by  $\mathcal{S}^{\bullet}_{2n}$  the union of all the sets  $\mathcal{S}_{2n}^{\text{e}}$ ,  $\mathcal{S}_{2n}^{\text{c}}$ ,  $\mathcal{S}_{2n}^{\text{r}}$ ,  $\mathcal{S}_{2n}^{\text{i}}$ ,  $\mathcal{S}_{2n}^{\text{rc}}$ ,  $\mathcal{S}_{2n}^{\text{ci}}$ ,  $\mathcal{S}_{2n}^{\text{ic}}$ , and  $\mathcal{S}_{2n}^{\text{irc}}$ . We show that  $S_{2n}$  and  $\mathcal{S}^{\bullet}_{2n}$  do not coincide.

**Proposition 4 (Existence of non-square permutations).** *For any  $n \in \mathbb{N}, n \geq 9, S_{2n} \neq \mathcal{S}^{\bullet}_{2n}$ .*

*Proof.* By Lemma 3 (and using its notations), we have

$$|\mathcal{S}^{\bullet}_{2n}| \leq 8|Q_{2n}| \leq 8 \binom{2n-1}{n-1} \binom{2n}{n} n!. \quad (6)$$

The proportion of the elements of  $\mathcal{S}^{\bullet}_{2n}$  among all the permutations of  $S_{2n}$  admits as upper bound, when  $n \geq 1$ ,

$$\frac{|\mathcal{S}^{\bullet}_{2n}|}{|S_{2n}|} \leq \frac{8 \binom{2n-1}{n-1} \binom{2n}{n} n!}{(2n)!} = 4 \binom{2n}{n} \frac{1}{n!}. \quad (7)$$

But  $4 \binom{2n}{n} \frac{1}{n!} < 1$  for  $n \geq 9$ , thereby proving the result. ■

**Proposition 5 (Square permutations in some intersections).** *Let  $n \in \mathbb{N}$  such that  $2n \equiv 0 \pmod{4}$ . Then,*

*the permutation  $\pi = 12 \dots (n-1)(n) (2n)(2n-1) \dots (n+2)(n+1)$  belongs to  $\mathcal{S}_{2n}^{\text{e}} \cap \mathcal{S}_{2n}^{\text{r}} \cap \mathcal{S}_{2n}^{\text{c}} \cap \mathcal{S}_{2n}^{\text{i}} \cap \mathcal{S}_{2n}^{\text{ci}} \cap \mathcal{S}_{2n}^{\text{ic}}$ .*

*Proof.* Let  $I = \{1, 3, 5, \dots, 2n-1\}$  and  $\bar{I} = [2n] \setminus I$ . Then, since

$$\pi|_I \simeq 12 \dots \frac{n}{2} n(n-1) \dots \left(\frac{n}{2} + 1\right) \simeq \pi|_{\bar{I}}, \quad (8)$$

we have  $\pi|_J \simeq \pi|\bar{J} \simeq i(\pi|\bar{J})$ . This shows that  $\pi$  belongs to  $\mathcal{S}_{2n}^\epsilon \cap \mathcal{S}_{2n}^i$ .

Now, let  $J = \{1, 2, \dots, n\}$  and  $\bar{J} = [2n] \setminus J$ . Then, since

$$\pi|_J \simeq 12 \dots n \quad \text{and} \quad \pi|\bar{J} \simeq n \dots 21, \quad (9)$$

we have  $\pi|_J \simeq r(\pi|\bar{J}) \simeq c(\pi|\bar{J}) \simeq (i \circ c)(\pi|\bar{J}) \simeq (c \circ i)(\pi|\bar{J})$ . This shows that  $\pi$  belongs to  $\mathcal{S}_{2n}^i \cap \mathcal{S}_{2n}^c \cap \mathcal{S}_{2n}^{ic} \cap \mathcal{S}_{2n}^{ci}$ . ■

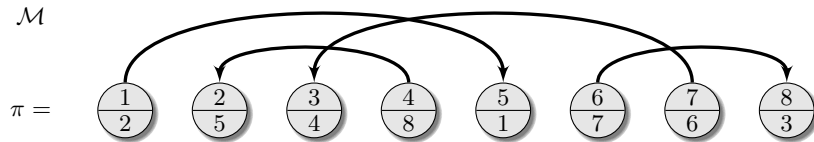
## 4 Recognizing generalized square permutations

This section is devoted to proving that deciding membership to  $\mathcal{S}_{2n}^i$ ,  $\mathcal{S}_{2n}^c$  and  $\mathcal{S}_{2n}^{ic}$  is NP-complete. The approach relies on constraint directed matchings on permutations.

**Definition 9 (Directed matching on permutations).** Let  $\pi \in S_n$ . A directed matching on  $\pi$  is a set  $\mathcal{M}$  of pairwise disjoint arcs  $(i, j)$ ,  $1 \leq i, j \leq n$  and  $i \neq j$ , that connect pairs of elements of  $\pi$ . The directed matching  $\mathcal{M}$  is perfect if every element of  $\pi$  is the source or the sink of an arc of  $\mathcal{M}$ .

**Definition 10 (Two-arcs pattern occurrences).** A two-arc pattern is a perfect directed matching on the set  $[4]$ . A perfect directed matching  $\mathcal{M}$  on a permutation  $\pi$  of size  $2n$  admits an occurrence of a two-arc pattern  $\mathcal{A}$  if there is an increasing map  $\phi : [4] \rightarrow [2n]$  (i.e.,  $i < i'$  implies  $\phi(i) < \phi(i')$ ) such that, if  $(i, i')$  is an arc of  $\mathcal{A}$ , then  $(\phi(i), \phi(i'))$  is an arc of  $\mathcal{M}$ . We say moreover that  $\mathcal{M}$  avoids  $\mathcal{A}$  if  $\mathcal{A}$  does not admits any occurrence of  $\mathcal{A}$ .

We shall draw two-arcs patterns by unlabeled graphs wherein vertices are implicitly indexed from 1 to 4 from left to right. Figure 4 shows an example of a directed perfect matching on a permutation. The directed matching does contain the patterns and does avoid the patterns



**Fig. 2.** A directed perfect matching  $\mathcal{M} = \{(1, 5), (4, 2), (7, 3), (6, 8)\}$  on the permutation  $\pi = 25481763$ . Recall that arcs refer to positions in  $\pi$ .

In case  $\pi$  is written as a concatenation of contiguous patterns  $\pi = \pi_1 \pi_2 \dots \pi_k$ , we write  $\mathcal{M}_{\pi_i \rightarrow \pi_j}$  for the subset of arcs of  $\mathcal{M}$  with source index in  $\pi_i$  and sink index in  $\pi_j$ . Hence,

$$\mathcal{M} = \bigsqcup_{\substack{i \in [k] \\ j \in [k]}} \mathcal{M}_{\pi_i \rightarrow \pi_j}. \quad (10)$$

The following properties will prove useful for defining equivalences between squares and restricted directed perfect matchings.

**Definition 11 (Property  $\mathbf{P}_1$  — Order isomorphism).** *Let  $\pi$  be a permutation. A directed perfect matching  $\mathcal{M}$  on  $\pi$  is said to have property  $\mathbf{P}_1$  if, for any two distinct arcs  $(i, i')$  and  $(j, j')$  of  $\mathcal{M}$ , we have  $\pi(i) < \pi(j)$  if and only if  $\pi(i') < \pi(j')$ .*

**Definition 12 (Property  $\mathbf{P}_2$  — Order anti-isomorphism).** *Let  $\pi$  be a permutation. A directed perfect matching  $\mathcal{M}$  on  $\pi$  is said to have property  $\mathbf{P}_2$  if, for any two distinct arcs  $(i, i')$  and  $(j, j')$  of  $\mathcal{M}$ , we have  $\pi(i) < \pi(j)$  if and only if  $\pi(i') > \pi(j')$ .*

**Definition 13 (Property  $\mathbf{Q}_1$  — First start first terminal).** *Let  $\pi$  be a permutation. A directed perfect matching  $\mathcal{M}$  on  $\pi$  is said to have property  $\mathbf{Q}_1$  if it avoids the following set of two-arcs patterns:*

$$\mathcal{Q}_1 = \left\{ \begin{array}{c} \curvearrowright \\ \bullet \bullet \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowright \\ \bullet \bullet \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowright \\ \bullet \bullet \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowright \\ \bullet \bullet \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowright \\ \bullet \bullet \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowright \\ \bullet \bullet \bullet \bullet \bullet \bullet \end{array} \right\}.$$

**Definition 14 (Property  $\mathbf{Q}_2$  — First start last terminal).** *Let  $\pi$  be a permutation. A directed perfect matching  $\mathcal{M}$  on  $\pi$  is said to have property  $\mathbf{Q}_2$  if it avoids the following set of two-arcs patterns:*

$$\mathcal{Q}_2 = \left\{ \begin{array}{c} \curvearrowleft \\ \bullet \bullet \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowleft \\ \bullet \bullet \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowleft \\ \bullet \bullet \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowleft \\ \bullet \bullet \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowleft \\ \bullet \bullet \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowleft \\ \bullet \bullet \bullet \bullet \bullet \bullet \end{array} \right\}.$$

Observe that  $\mathcal{Q}_1 \sqcup \mathcal{Q}_2$  is the set of all two-arcs patterns.

**Theorem 1 ([4]).** *Let  $\pi \in S_{2n}$ . The two following statements are equivalent.*






1.  $\pi \in S_{2n}^e$ .
2. *There exists a directed perfect matching  $\mathcal{M}$  on  $\pi$  that satisfies properties  $\mathbf{P}_1$  and  $\mathbf{Q}_1$ .*


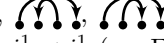
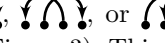
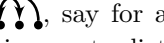
The following two definitions will intervene in the next constructions.

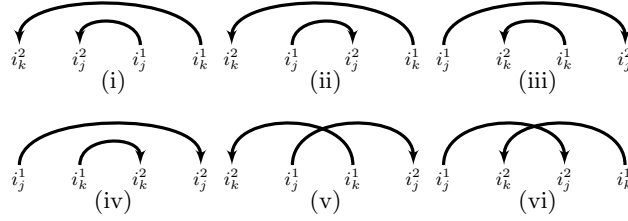
**Definition 15 (Lifting).** *Let  $\pi = \pi(1) \pi(2) \dots \pi(n)$  be a permutation of size  $n$  and  $k$  be a positive integer. We call  $k$ -lifting of  $\pi$ , denoted  $\pi[k]$ , the permutation  $(k + \pi(1)) (k + \pi(2)) \dots (k + \pi(n))$  on the alphabet  $\{k + 1, \dots, k + n\}$ .*

**Definition 16 (Monotone).** *For any positive integer  $k$ , we let  $\nearrow_k$  stand for the increasing permutation  $1 2 \dots k$  and  $\searrow_k$  stand for the decreasing permutation  $k (k - 1) \dots 1$ .*


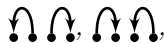



**Lemma 4 (One shot lemma).** *Let  $\pi \in S_{2n}$  and  $\mathcal{M}$  be a perfect directed matching on  $\pi$  that satisfies properties  $\mathbf{P}_i$  and  $\mathbf{Q}_j$ ,  $i \in \{1, 2\}$  and  $j \in \{1, 2\}$ . Then, the perfect directed matching  $\mathcal{M}'$  obtained by reversing each arc of  $\mathcal{M}$  satisfies properties  $\mathbf{P}_i$  and  $\mathbf{Q}_j$ .*


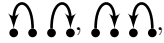
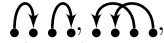

**Lemma 5.** Let  $I^1 = \{i_j^1 : j \in [n]\}$ ,  $i_1^1 < i_2^1 < \dots < i_n^1$ , and  $I^2 = \{i_j^2 : j \in [n]\}$ ,  $i_1^2 < i_2^2 < \dots < i_n^2$ , be such that  $I^1 \cap I^2 = \emptyset$ . The directed perfect matching  $\mathcal{M} = ([2n], E)$  defined by  $E = \{(i_j^1, i_j^2) : j \in [n]\}$  avoids the patterns , , , , and .

*Proof.* Suppose, aiming at a contradiction, that  $\mathcal{M}$  does contain the pattern , , , or , say for arcs  $(i_j^1, i_j^2)$  and  $(i_k^1, i_k^2)$ . Wlog, assume  $i_j^1 < i_k^1$  (see Figure 3). This is a contradiction since  $i_j^1 < i_k^1$  implies  $j < k$ , and hence,  $i_j^2 < i_k^2$ . ■



**Fig. 3.** Arcs  $(i_j^1, i_j^2)$  and  $(i_k^1, i_k^2)$  with  $i_j^1 < i_k^1$ .

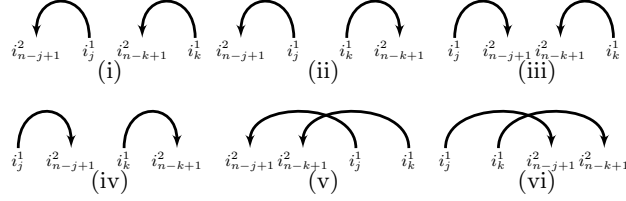
**Lemma 6.** Let  $I^1 = \{i_j^1 : j \in [n]\}$ ,  $i_1^1 < i_2^1 < \dots < i_n^1$ , and  $I^2 = \{i_j^2 : j \in [n]\}$ ,  $i_1^2 < i_2^2 < \dots < i_n^2$ , be such that  $I^1 \cap I^2 = \emptyset$ . The directed perfect matching  $\mathcal{M} = ([2n], E)$  defined by  $E = \{(i_j^1, i_{n-j+1}^2) : j \in [n]\}$  avoids the patterns , , , , and .

*Proof.* Suppose, aiming at a contradiction, that  $\mathcal{M}$  does contain the pattern , , , or , say for arcs  $(i_j^1, i_{n-j+1}^2)$  and  $(i_k^1, i_{n-k+1}^2)$ . Wlog, assume  $i_j^1 < i_k^1$  (see Figure 4). This is a contradiction since  $i_j^1 < i_k^1$  implies  $j < k$ , and hence,  $i_{n-j+1}^2 > i_{n-k+1}^2$ . ■

**Proposition 6.** Let  $\pi \in S_{2n}$ . The two following statements are equivalent.

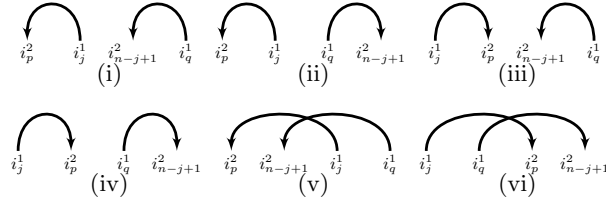
1.  $\pi \in \mathcal{S}_{2n}^r$ .
2. There exists a directed perfect matching  $\mathcal{M}$  on  $\pi$  that satisfies properties **P**<sub>1</sub> and **Q**<sub>2</sub>.

*Proof.* (**1**  $\Rightarrow$  **2**) Let  $\pi \in \mathcal{S}_{2n}^r$ . Then, there exists  $\sigma \in S_n$  such that  $\pi \in \sigma \sqcup r(\sigma)$ , and hence there exist two disjoint set of indexes  $I^1 = \{i_j^1 : j \in [n]\}$ ,  $i_1^1 < i_2^1 < \dots < i_n^1$ , and  $I^2 = \{i_j^2 : j \in [n]\}$ ,  $i_1^2 < i_2^2 < \dots < i_n^2$ , such that



**Fig. 4.** Arcs  $(i_j^1, i_{n-j+1}^2)$  and  $(i_k^1, i_{n-k+1}^2)$  with  $i_j^1 < i_k^2$ .

$\pi(i_1^1)\pi(i_2^1)\cdots\pi(i_n^1)$  is order-isomorphic to  $\sigma$  and  $\pi(i_1^2)\pi(i_2^2)\cdots\pi(i_n^2)$  is order-isomorphic to  $r(\sigma)$ . Let  $\mathcal{M} = (V, E)$  be the directed graph defined by  $V = [2n]$  and  $E = \{(i_j^1, i_{n-j+1}^2) : j \in [n]\}$ . Clearly,  $\mathcal{M}$  is a directed perfect matching since  $I^1 \cap I^2 = \emptyset$  and  $I^1 \cup I^2 = [2n]$ . According to Lemma 6,  $\mathcal{M}$  does avoid the patterns Finally, for any two distinct arcs  $(i_j^1, i_{n-j+1}^2)$  and  $(i_k^1, i_{n-k+1}^2)$  of  $\mathcal{M}$ , we have  $\pi(i_j^1) < \pi(i_k^1)$  if and only if  $\pi(i_{n-j+1}^2) < \pi(i_{n-k+1}^2)$  since  $\pi(i_1^1)\pi(i_2^1)\cdots\pi(i_n^1)$  is order-isomorphic to  $\sigma$  and  $\pi(i_1^2)\pi(i_2^2)\cdots\pi(i_n^2)$  is order-isomorphic to  $r(\sigma)$ . Therefore,  $\mathcal{M}$  satisfies properties **P<sub>1</sub>** and **Q<sub>2</sub>**.







**Fig. 5.** All possible configurations considered in Proof of Proposition 6 (**2**  $\Rightarrow$  **1**).

**(2  $\Rightarrow$  1)** Let  $\mathcal{M}$  be a directed perfect matching that satisfies properties **P<sub>1</sub>** and **Q<sub>2</sub>**. Let  $I^1 = \{i_j^1 : j \in [n]\}$ ,  $i_1^1 < i_2^1 < \cdots < i_n^1$ , be the set of the sources of the arcs of  $\mathcal{M}$  and let  $I^2 = \{i_j^2 : j \in [n]\}$ ,  $i_1^2 < i_2^2 < \cdots < i_n^2$ , be the set of the sinks of the arcs of  $\mathcal{M}$ . We first show that, for every  $j \in [n]$ ,  $(i_j^1, i_{n-j+1}^2)$  is an arc of  $\mathcal{M}$ . The proof is by induction on  $j = 1, 2, \dots, n$ .

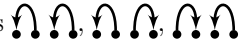
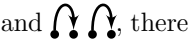


- Base. Suppose, aiming at a contradiction that  $(i_1^1, i_n^2)$  is not an arc of  $\mathcal{M}$ . Then, there exist  $1 < p < n$  and  $1 < q < n$  such that  $(i_1^1, i_p^2)$  and  $(i_q^1, i_n^2)$  are two arcs of  $\mathcal{M}$ . But  $i_1^1 < i_q^1$  and  $i_p^2 < i_n^2$ , and hence one of the configurations of Figure 5 (with  $j = 1$  and  $k = n$ ) does occur in  $\mathcal{M}$ . This is a contradiction since  $\mathcal{M}$  satisfies Property **Q<sub>2</sub>**, and hence  $(i_1^1, i_n^2)$  is an arc of  $\mathcal{M}$ .
- Induction step. Assume that  $(i_k^1, i_{n-k+1}^2)$  is an arc of  $\mathcal{M}$  for  $1 \leq k < j$ , and suppose, aiming at a contradiction, that  $(i_j^1, i_{n-j+1}^2)$  is not an arc of  $\mathcal{M}$ .

Then, there exist  $1 \leq p < n - j + 1$  and  $j < q \leq n$  such that  $(i_j^1, i_p^2)$  and  $(i_q^1, i_{n-j+1}^2)$  are two arcs of  $\mathcal{M}$ . But  $i_j^1 < i_q^1$  and  $i_p^2 < i_{n-j+1}^2$ , and hence one of the configurations of Figure 5 does occur in  $\mathcal{M}$ . This is a contradiction since  $\mathcal{M}$  satisfies Property **Q<sub>2</sub>**, and hence  $(i_j^1, i_{n-j+1}^2)$  is an arc of  $\mathcal{M}$ .

Now, let  $\sigma_1$  be the pattern of  $\pi$  induced by the sources of  $\mathcal{M}$  and  $\sigma_2$  be the pattern of  $\pi$  induced by the sinks of  $\mathcal{M}$  (i.e.,  $\sigma_1 = \pi(i_1^1) \pi(i_2^1) \dots \pi(i_n^1)$  and  $\sigma_2 = \pi(j_1^2) \pi(j_2^2) \dots \pi(i_n^2)$ ). Clearly  $\sigma_1$  and  $\sigma_2$  are disjoint in  $\pi$  (since  $\mathcal{M}$  is a matching) and cover  $\pi$  (since  $\mathcal{M}$  is perfect). Moreover, the fact that  $\mathcal{M}$  satisfies **P<sub>1</sub>** implies immediately that  $\text{reduce}(\sigma_1)$  is order-isomorphic to  $(\text{reduce} \circ r)(\sigma_2)$ , and hence this shows that  $\pi \in \mathcal{S}_{2n}^r$ . ■

**Lemma 7.** *Let  $\pi \in S_{2n}$  and  $\mathcal{M}$  be a perfect directed matching on  $\pi$  that avoids the patterns , , , and . Then, for any arc  $(i, j) \in \mathcal{M}$ , either  $1 \leq i \leq n < j \leq 2n$  or  $1 \leq j \leq n < i \leq 2n$ .*

*Proof.* We only prove the case  $1 \leq i \leq n < j \leq 2n$  since the proof for  $1 \leq j \leq n < i \leq 2n$  is similar.

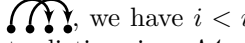
Suppose, aiming at a contradiction, that there exists an arc  $(i, j) \in \mathcal{M}$  with  $1 \leq i < j \leq n$ . Since  $\mathcal{M}$  avoids the patterns , , , and , there is no arc  $(k, \ell) \in \mathcal{M}$  with  $j < k \leq 2n$  and  $j < \ell \leq 2n$ ,  $k \neq \ell$ . Then it follows that there exist  $2n - j \geq n$  distinct arcs  $(k, \ell) \in \mathcal{M}$  with  $1 \leq \min\{k, \ell\} \leq j - 1$  and  $j + 1 \leq \max\{k, \ell\} \leq 2n$ ,  $k \neq \ell$ . This is a contradiction since  $(i, j) \in \mathcal{M}$  with  $1 \leq i < j \leq n$ . ■

The following technical lemma is needed to simplify the proof of upcoming Proposition 7.

**Lemma 8.** *Let  $k, \ell_1, \ell_2, \ell_3$  and  $\ell_4$  be positive integers and  $\pi \in \mathcal{S}_{2k+\ell_1+\ell_2+\ell_3+\ell_4}^r$  be the permutation defined by  $\pi = X_1 L X_2 X_3 R X_4$ , where  $X_1 = \mu_1[k + \ell_2]$ ,  $L = \nearrow_k[\ell_2]$ ,  $X_2 = \mu_2$ ,  $X_3 = \mu_3[k + \ell_1 + \ell_2]$ ,  $R = \searrow_k[k + \ell_1 + \ell_2 + \ell_3]$  and  $X_4 = \mu_4[2k + \ell_1 + \ell_2 + \ell_3]$  for some permutations  $\mu_i \in S_{\ell_i}$ ,  $1 \leq i \leq 4$  (see Figure 6). Let  $\mathcal{M}$  be a directed perfect matching on  $\pi$  that satisfies properties **P<sub>1</sub>** and **Q<sub>2</sub>**. If  $\mathcal{M}_{L \rightarrow R} \neq \emptyset$  or  $\mathcal{M}_{R \rightarrow L} \neq \emptyset$ , then either  $|\mathcal{M}_{L \rightarrow R}| = k$  or  $|\mathcal{M}_{R \rightarrow L}| = k$ .*

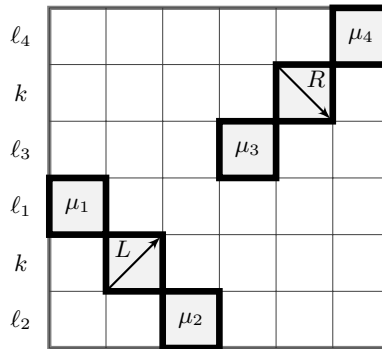
*Proof.* Suppose first, aiming at a contradiction, that  $\mathcal{M}_{L \rightarrow R} \neq \emptyset$  and  $\mathcal{M}_{R \rightarrow L} \neq \emptyset$ . Let  $(i, j) \in \mathcal{M}_{L \rightarrow R}$  and  $(i', j') \in \mathcal{M}_{R \rightarrow L}$ . By construction, we have  $\pi(i) < \pi(i')$  and  $\pi(j) > \pi(j')$  thereby contradicting Property **P<sub>1</sub>**. Then it follows that either  $\mathcal{M}_{L \rightarrow R} \neq \emptyset$  or  $\mathcal{M}_{R \rightarrow L} \neq \emptyset$ , but not both.

Suppose now that  $\mathcal{M}_{L \rightarrow R} \neq \emptyset$  and  $\mathcal{M}_{R \rightarrow L} = \emptyset$  (the case  $\mathcal{M}_{L \rightarrow R} = \emptyset$  and  $\mathcal{M}_{R \rightarrow L} \neq \emptyset$  can be proved with the same arguments). Suppose, aiming at a contradiction, that  $|\mathcal{M}_{L \rightarrow R}| \neq k$  and let  $(i, j) \in \mathcal{M}_{L \rightarrow R}$ . According to Lemma 7, we are left with considering the four following cases.

- There exists  $(i', j') \in \mathcal{M}_{L \rightarrow \mu_3}$ . Since  $\mathcal{M}$  avoids , we have  $i < i'$ , and hence  $\pi(i) < \pi(i')$  and  $\pi(j) > \pi(j')$ . This is a contradiction since  $\mathcal{M}$  satisfies Property **P<sub>1</sub>**.

- There exists  $(i', j') \in \mathcal{M}_{\mu_3 \rightarrow L}$  and hence  $\pi(i) < \pi(i')$  and  $\pi(j) > \pi(j')$ . This is a contradiction since  $\mathcal{M}$  satisfies Property  $\mathbf{P}_1$ .
- There exists  $(i', j') \in \mathcal{M}_{L \rightarrow \mu_4}$ . Since  $\mathcal{M}$  avoids  $\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array}$ , we have  $i' < i$ , and hence  $\pi(i) > \pi(i')$  and  $\pi(j) < \pi(j')$ . This is a contradiction since  $\mathcal{M}$  satisfies Property  $\mathbf{P}_1$ .
- There exists  $(i', j') \in \mathcal{M}_{\mu_4 \rightarrow L}$  and hence  $\pi(i) < \pi(i')$  and  $\pi(j) > \pi(j')$ . This is a contradiction since  $\mathcal{M}$  satisfies Property  $\mathbf{P}_1$ .

Therefore,  $|\mathcal{M}_{L \rightarrow R}| = k$ . ■



**Fig. 6.** Illustration of Lemma 8, where  $\pi = X_1 \ L \ X_2 \ X_3 \ R \ X_4$ .

**Proposition 7.**  $r$ -UNSHUFFLE-PERMUTATION is NP-complete.

*Proof.* We reduce from PERMUTATION PATTERN which, given two permutations  $\pi \in S_n$  and  $\sigma \in S_k$  with  $k \leq n$ , asks whether  $\sigma$  is a pattern of  $\pi$ . PERMUTATION PATTERN is known to be NP-complete [1]. Let  $\pi \in S_n$  and  $k \in S_k$  be an instance of PERMUTATION PATTERN. Set  $N_2 = n + k + 1$  and  $N_1 = 2N_2$ . Define a target permutation  $\mu \in S_{2n+2k+2N_1+2N_2}$  by  $\mu = \mu_\pi \ \mu_2 \ \mu_1 \ \mu_\sigma \ \mu'_2 \ \mu'_\pi \ \mu'_1 \ \mu'_\sigma$ , where

$$\begin{array}{ll}
 \mu_\sigma = \sigma, & \mu_\pi = \pi[k + N_1 + N_2], \\
 \mu'_\pi = r(\pi)[2n + k + 2N_1 + 2N_2] & \mu'_\sigma = r(\sigma)[n + k + N_1 + 2N_2], \\
 \mu_1 = \nearrow_{N_1} [k], & \mu'_1 = \searrow_{N_1} [2n + k + N_1 + 2N_2], \\
 \mu_2 = \nearrow_{N_2} [k + N_1], & \mu'_2 = \searrow_{N_2} [n + k + N_1 + N_2].
 \end{array}$$

Clearly, the construction can be carried out in polynomial time. We claim that  $\sigma$  is a pattern of  $\pi$  if and only if  $\mu \in \mathcal{S}_{2n+2k+2N_1+2N_2}^r$ .

( $\Rightarrow$ ) Suppose that  $\sigma$  is a pattern of  $\pi$ . Then there exist indices  $P = \{p_1, p_2, \dots, p_k\}$ ,  $1 \leq p_1 < p_2 < \dots < p_k \leq n$ , such that  $\sigma$  and  $\pi(p_1)\pi(p_2)\dots\pi(p_k)$  are order-isomorphic. For the sake of simplification, let  $Q = [n] \setminus P$  and write

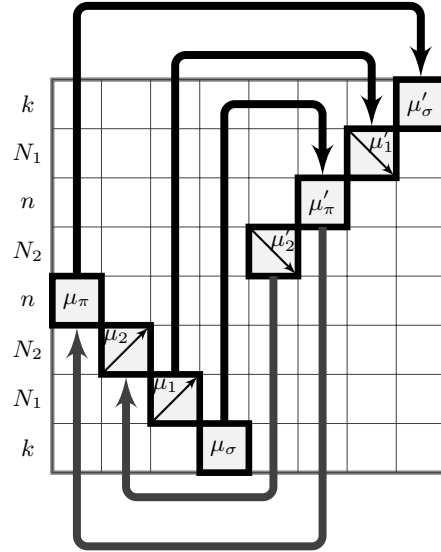


Fig. 7. Schematic representation of the construction used in Proposition 7.

$Q = \{q_1, q_2, \dots, q_{n-k}\}$ ,  $1 \leq q_1 < q_2 < \dots < q_{n-k} \leq n$ . Define a directed matching

$$\mathcal{M} = \mathcal{M}_{\mu_\sigma \rightarrow \mu'_\pi} \uplus \mathcal{M}_{\mu'_\pi \rightarrow \mu_\pi} \uplus \mathcal{M}_{\mu_\pi \rightarrow \mu'_\sigma} \uplus \mathcal{M}_{\mu_1 \rightarrow \mu'_1} \uplus \mathcal{M}_{\mu'_2 \rightarrow \mu_2}$$

on  $\mu$  as follows.

$$\begin{aligned} \mathcal{M}_{\mu_\sigma \rightarrow \mu'_\pi} &= \{(n + N_1 + N_2 + i, n + k + N_1 + 2N_2 + p_i) : 1 \leq i \leq k\} \\ \mathcal{M}_{\mu'_\pi \rightarrow \mu_\pi} &= \{(2n + k + N_1 + 2N_2 - q_i + 1, q_i) : 1 \leq i \leq n - k\} \\ \mathcal{M}_{\mu_\pi \rightarrow \mu'_\sigma} &= \{(p_i, 2n + 2k + 2N_1 + 2N_2 - i) : 1 \leq i \leq k\} \\ \mathcal{M}_{\mu_1 \rightarrow \mu'_1} &= \{(n + N_2 + i, 2n + k + 2N_1 + 2N_2 - i + 1) : 1 \leq i \leq N_1\} \\ \mathcal{M}_{\mu'_2 \rightarrow \mu_2} &= \{(n + k + N_1 + 2N_2 - i + 1, n + i) : 1 \leq i \leq N_2\}. \end{aligned}$$

Informally,

- $\mathcal{M}_{\mu_\sigma \rightarrow \mu'_\pi}$  is the directed (left-to-right) matching describing the (reversed) occurrence of  $\sigma$  in  $r(\pi)$ ,
- $\mathcal{M}_{\mu'_\pi \rightarrow \mu_\pi}$  is the directed (right-to-left) matching connecting the elements of  $r(\pi)$  and  $\pi$  that are not part of the occurrence of  $\sigma$  in  $\pi$ ,
- $\mathcal{M}_{\mu_\pi \rightarrow \mu'_\sigma}$  is the directed (left-to-right) matching describing the (reversed) occurrence of  $r(\sigma)$  in  $\pi$ ,
- $\mathcal{M}_{\mu_1 \rightarrow \mu'_1}$  is a directed (left-to-right) matching that fully connects  $\mu_1$  to  $\mu'_1$  and
- $\mathcal{M}_{\mu'_2 \rightarrow \mu_2}$  is a directed (right-to-left) matching that fully connects  $\mu'_2$  to  $\mu_2$ .




It is now a simple matter to check that  $\mathcal{M}$  is a directed matching on  $\mu$  that satisfies properties  $\mathbf{P}_1$  and  $\mathbf{Q}_2$ . Therefore, according to Proposition 6,  $\mu \in \mathcal{S}_{2n+2k+2N_1+2N_2}^r$ .


( $\Leftarrow$ ) Suppose that  $\mu \in \mathcal{S}_{2n}^r$ . Therefore, according to Proposition 6, there exists a directed perfect matching  $\mathcal{M}$  on  $\mu$  that satisfies properties  $\mathbf{P}_1$  and  $\mathbf{Q}_2$ . We begin with a sequence of claims that help defining the general structure of  $\mathcal{M}$ .

*Claim.* We may assume that  $|\mathcal{M}_{\mu_1 \rightarrow \mu'_1}| = N_1$ .

*Proof.* Combining Lemma 7 and  $N_1 = 2N_2 > n + k + N_2$ , we conclude that  $\mathcal{M}_{\mu_1 \rightarrow \mu'_1} \neq \emptyset$  or  $\mathcal{M}_{\mu'_1 \rightarrow \mu_1} \neq \emptyset$ . Thus, applying Lemma 8, we obtain  $|\mathcal{M}_{\mu_1 \rightarrow \mu'_1}| = N_1$  or  $|\mathcal{M}_{\mu'_1 \rightarrow \mu_1}| = N_1$ . Therefore, by Lemma 4 (One shot lemma), we may assume that  $\mathcal{M}_{\mu_1 \rightarrow \mu'_1} \neq \emptyset$ . ■

*Claim.* Assuming  $|\mathcal{M}_{\mu_1 \rightarrow \mu'_1}| = N_1$ , we have  $|\mathcal{M}_{\mu'_2 \rightarrow \mu_2}| = N_2$ .

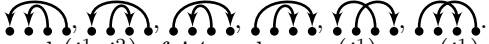
*Proof.* Combining Claim 4, Lemma 7 and  $N_2 > n + k$ , we conclude that  $\mathcal{M}_{\mu_2 \rightarrow \mu'_2} \neq \emptyset$  or  $\mathcal{M}_{\mu'_2 \rightarrow \mu_2} \neq \emptyset$ . Applying Lemma 8, we obtain  $|\mathcal{M}_{\mu_2 \rightarrow \mu'_2}| = N_2$  or  $|\mathcal{M}_{\mu'_2 \rightarrow \mu_2}| = N_2$ . The claim now follows from  $\mathcal{M}_{\mu_1 \rightarrow \mu'_1} \neq \emptyset$  (Claim 4) and the fact that  $\mathcal{M}$  avoids  (Property  $\mathbf{Q}_2$ ). ■

From the above two claims and the fact that  $\mathcal{M}$  avoids  (Property  $\mathbf{Q}_2$ ), we conclude that  $|\mathcal{M}_{\mu_\sigma \rightarrow \mu'_\pi}| = k$ . But  $\mathcal{M}$  satisfies Property  $\mathbf{P}_1$  and hence there exists an occurrence of  $r(\sigma)$  in  $r(\pi)$ . It follows that  $\sigma$  is a pattern of  $\pi$ . ■

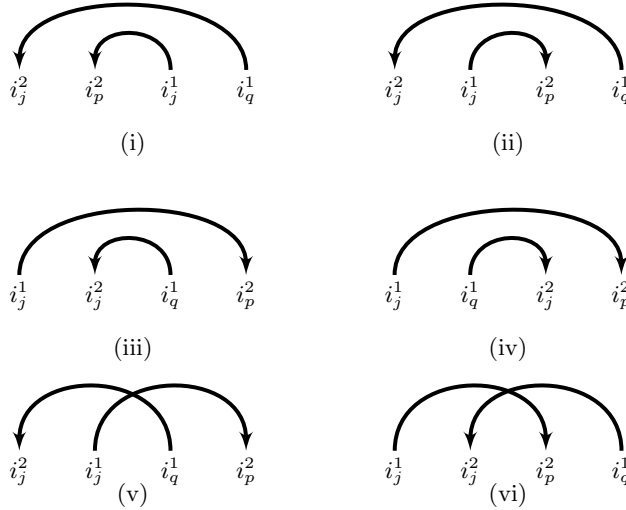
**Proposition 8.** *Let  $\pi \in S_{2n}$ . The two following statements are equivalent.*

1.  $\pi \in \mathcal{S}_{2n}^c$ .
2. *There exists a directed perfect matching  $\mathcal{M}$  that satisfies properties  $\mathbf{P}_2$  and  $\mathbf{Q}_1$ .*

*Proof.* (**1**  $\Rightarrow$  **2**) Let  $\pi \in \mathcal{S}_{2n}^c$ . Then, there exists  $\sigma \in S_n$  such that  $\pi \in \sigma \sqcup c(\sigma)$ , and hence there exist two disjoint set of indices  $I^1 = \{i_j^1 : j \in [n]\}$ ,  $i_1^1 < i_2^1 < \dots < i_n^1$ , and  $I^2 = \{i_j^2 : j \in [n]\}$ ,  $i_1^2 < i_2^2 < \dots < i_n^2$ , such that  $\pi(i_1^1)\pi(i_2^1)\dots\pi(i_n^1)$  is order-isomorphic to  $\sigma$  and  $\pi(i_1^2)\pi(i_2^2)\dots\pi(i_n^2)$  is order-isomorphic to  $c(\sigma)$ . Let  $\mathcal{M} = (V, E)$  be the directed graph defined by  $V = [2n]$  and  $E = \{(i_j^1, i_j^2) : j \in [n]\}$ . Clearly,  $\mathcal{M}$  is a directed perfect matching since  $I^1 \cap I^2 = \emptyset$  and  $I^1 \cup I^2 = [2n]$ . According to Lemma 5, the directed perfect

matching  $\mathcal{M}$  does avoid the patterns . Finally, for any two distinct arcs  $(i_j^1, i_j^2)$  and  $(i_k^1, i_k^2)$  of  $\mathcal{M}$ , we have  $\pi(i_j^1) < \pi(i_k^1)$  if and only if  $\pi(i_j^2) > \pi(i_k^2)$  since  $\pi(i_1^1)\pi(i_2^1)\dots\pi(i_n^1)$  is order-isomorphic to  $\sigma$  and  $\pi(i_1^2)\pi(i_2^2)\dots\pi(i_n^2)$  is order-isomorphic to  $c(\sigma)$  (and hence  $\pi(i_j^2) = n - \pi(i_j^1)$  and  $\pi(i_k^2) = n - \pi(i_k^1)$ ). Therefore,  $\mathcal{M}$  satisfies properties  $\mathbf{P}_2$  and  $\mathbf{Q}_1$ .

(**2**  $\Rightarrow$  **1**) Let  $\mathcal{M}$  be a directed perfect matching that satisfies properties  $\mathbf{P}_2$  and  $\mathbf{Q}_1$ . Let  $I^1 = \{i_j^1 : j \in [n]\}$ ,  $i_1^1 < i_2^1 < \dots < i_n^1$ , be the set the sources of the arcs of  $\mathcal{M}$  and let  $I^2 = \{i_j^2 : j \in [n]\}$ ,  $i_1^2 < i_2^2 < \dots < i_n^2$ , be the set of the sinks of the arcs of  $\mathcal{M}$ . We first show that, for every  $j \in [n]$ ,  $(i_j^1, i_j^2)$  is an arc of  $\mathcal{M}$ . The proof is by induction on  $j = 1, 2, \dots, n$ .



**Fig. 8.** All possible configurations considered in Proof of Proposition 8 ( $\mathbf{2} \Rightarrow \mathbf{1}$ ).

- Base. Suppose, aiming at a contradiction that  $(i_1^1, i_1^2)$  is not an arc of  $\mathcal{M}$ . Then, there exist  $1 < p \leq n$  and  $1 < q \leq n$  such that  $(i_1^1, i_p^2)$  and  $(i_q^1, i_1^2)$  are two arcs of  $\mathcal{M}$ . But  $i_1^1 < i_q^1$  and  $i_1^2 < i_p^2$ , and hence one of the configurations of Figure 8 (with  $j = 1$ ) does occur in  $\mathcal{M}$ . This is a contradiction since  $\mathcal{M}$  satisfies Property  $\mathbf{Q}_1$ , and hence  $(i_1^1, i_1^2)$  is an arc of  $\mathcal{M}$ .
- Induction step. Assume that  $(i_k^1, i_{n-k+1}^2)$  is an arc of  $\mathcal{M}$  for  $1 \leq k < j$ , and suppose, aiming at a contradiction, that  $(i_j^1, i_j^2)$  is not an arc of  $\mathcal{M}$ . Then, there exist  $j < p \leq n$  and  $j < q \leq n$  such that  $(i_j^1, i_p^2)$  and  $(i_q^1, i_{n-j+1}^2)$  are two arcs of  $\mathcal{M}$ . But  $i_j^1 < i_q^1$  and  $i_j^2 < i_p^2$ , and hence one of the configurations of Figure 8 does occur in  $\mathcal{M}$ . This is a contradiction since  $\mathcal{M}$  satisfies Property  $\mathbf{Q}_1$ , and hence  $(i_j^1, i_j^2)$  is an arc of  $\mathcal{M}$ .

Now, let  $\sigma_1$  be the pattern of  $\pi$  induced by the sources of  $\mathcal{M}$  and  $\sigma_2$  be the pattern of  $\pi$  induced by the sinks of  $\mathcal{M}$  (i.e.,  $\sigma_1 = \pi(i_1^1) \pi(i_2^1) \dots \pi(i_n^1)$  and  $\sigma_2 = \pi(j_1^2) \pi(j_2^2) \dots \pi(i_n^2)$ ). Clearly  $\sigma_1$  and  $\sigma_2$  are disjoint in  $\pi$  (since  $\mathcal{M}$  is a matching) and cover  $\pi$  (since  $\mathcal{M}$  is perfect). Moreover, the fact that  $\mathcal{M}$  satisfies  $\mathbf{P}_2$  implies immediately that  $\text{reduce}(\sigma_1)$  is order-isomorphic to  $(\text{reduce} \circ c)(\sigma_2)$ , and hence this shows that  $\pi \in \mathcal{S}_{2n}^c$ .  $\blacksquare$

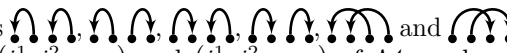
**Corollary 1.**  $c$ -UNSHUFFLE-PERMUTATION is NP-complete.

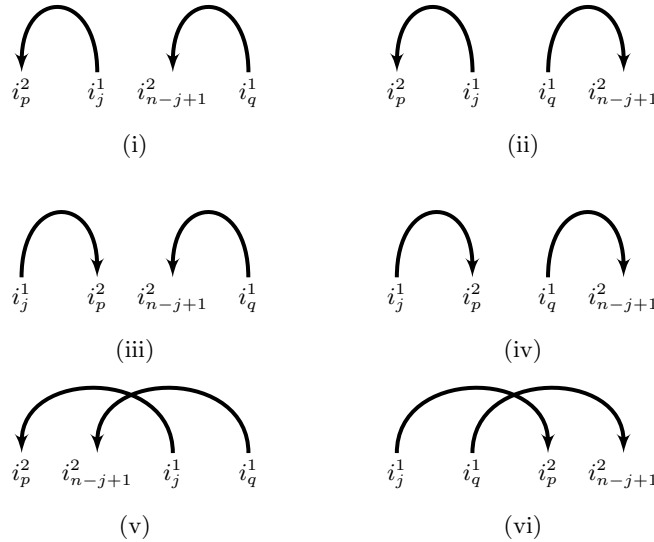
*Proof.* Combine Proposition 7 with Proposition 1.  $\blacksquare$

**Proposition 9.** Let  $\pi \in S_{2n}$ . The two following statements are equivalent.

1.  $\pi \in \mathcal{S}_{2n}^{\text{rc}}$ .

2. There exists a directed perfect matching  $\mathcal{M}$  that satisfies properties **P<sub>2</sub>** and **Q<sub>2</sub>**.

*Proof.* (**1**  $\Rightarrow$  **2**) Let  $\pi \in S_{2n}^{\text{rc}}$ . Then, there exists  $\sigma \in S_n$  such that  $\pi \in \sigma \sqcup (\text{r} \circ \text{c})(\sigma)$ , and hence there exist two disjoint set of indices  $I^1 = \{i_j^1 : j \in [n]\}$ ,  $i_1^1 < i_2^1 < \dots < i_n^1$ , and  $I^2 = \{i_j^2 : j \in [n]\}$ ,  $i_1^2 < i_2^2 < \dots < i_n^2$ , such that  $\pi(i_1^1)\pi(i_2^1)\dots\pi(i_n^1)$  is order-isomorphic to  $\sigma$  and  $\pi(i_1^2)\pi(i_2^2)\dots\pi(i_n^2)$  is order-isomorphic to  $(\text{r} \circ \text{c})(\sigma)$ . Let  $\mathcal{M} = (V, E)$  be the directed graph defined by  $V = [2n]$  and  $E = \{(i_j^1, i_{n-j+1}^2) : j \in [n]\}$ . Clearly,  $\mathcal{M}$  is a directed perfect matching since  $I^1 \cap I^2 = \emptyset$  and  $I^1 \cup I^2 = [2n]$ . According to Lemma 6, the directed perfect matching  $\mathcal{M}$  does avoid the patterns . Finally, for any two distinct arcs  $(i_j^1, i_{n-j+1}^2)$  and  $(i_k^1, i_{n-k+1}^2)$  of  $\mathcal{M}$ , we have  $\pi(i_j^1) < \pi(i_k^1)$  if and only if  $\pi(i_{n-j+1}^2) > \pi(i_{n-k+1}^2)$  since  $\pi(i_1^1)\pi(i_2^1)\dots\pi(i_n^1)$  is order-isomorphic to  $\sigma$  and  $\pi(i_1^2)\pi(i_2^2)\dots\pi(i_n^2)$  is order-isomorphic to  $(\text{r} \circ \text{c})(\sigma)$ . Therefore,  $\mathcal{M}$  satisfies properties **P<sub>1</sub>** and **Q<sub>2</sub>**.



**Fig. 9.** All possible configurations considered in Proof of Proposition 9 (**2**  $\Rightarrow$  **1**).

(**2**  $\Rightarrow$  **1**) Let  $\mathcal{M}$  be a directed perfect matching that satisfies properties **P<sub>1</sub>** and **Q<sub>2</sub>**. Let  $I^1 = \{i_j^1 : j \in [n]\}$ ,  $i_1^1 < i_2^1 < \dots < i_n^1$ , be the set the sources of the arcs of  $\mathcal{M}$  and let  $I^2 = \{i_j^2 : j \in [n]\}$ ,  $i_1^2 < i_2^2 < \dots < i_n^2$ , be the set of the sinks of the arcs of  $\mathcal{M}$ . We first show that, for every  $j \in [n]$ ,  $(i_j^1, i_{n-j+1}^2)$  is an arc of  $\mathcal{M}$ . The proof is by induction on  $j = 1, 2, \dots, n$ .

- Base. Suppose, aiming at a contradiction that  $(i_1^1, i_n^2)$  is not an arc of  $\mathcal{M}$ . Then, there exist  $1 \leq p < n$  and  $1 < q \leq n$  such that  $(i_1^1, i_p^2)$  and  $(i_q^1, i_n^2)$  are two arcs of  $\mathcal{M}$ . But  $i_1^1 < i_q^1$  and  $i_p^2 < i_n^2$ , and hence one of the configurations of Figure 9 (with  $j = 1$  and  $k = n$ ) does occur in  $\mathcal{M}$ . This is a contradiction since  $\mathcal{M}$  satisfies Property **Q<sub>2</sub>**, and hence  $(i_1^1, i_n^2)$  is an arc of  $\mathcal{M}$ .
- Induction step. Assume that  $(i_k^1, i_{n-k+1}^2)$  is an arc of  $\mathcal{M}$  for  $1 \leq k < j$ , and suppose, aiming at a contradiction, that  $(i_j^1, i_{n-j+1}^2)$  is not an arc of  $\mathcal{M}$ . Then, there exist  $1 \leq p < n - j + 1$  and  $j < q \leq n$  such that  $(i_j^1, i_p^2)$  and  $(i_q^1, i_{n-j+1}^2)$  are two arcs of  $\mathcal{M}$ . But  $i_j^1 < i_q^1$  and  $i_p^2 < i_{n-j+1}^2$ , and hence one of the configurations of Figure 9 does occur in  $\mathcal{M}$ . This is a contradiction since  $\mathcal{M}$  satisfies Property **Q<sub>2</sub>**, and hence  $(i_j^1, i_{n-j+1}^2)$  is an arc of  $\mathcal{M}$ .

Now, let  $\sigma_1$  be the pattern of  $\pi$  induced by the sources of  $\mathcal{M}$  and  $\sigma_2$  be the pattern of  $\pi$  induced by the sinks of  $\mathcal{M}$  (i.e.,  $\sigma_1 = \pi(i_1^1) \pi(i_2^1) \dots \pi(i_n^1)$  and  $\sigma_2 = \pi(j_1^2) \pi(j_2^2) \dots \pi(i_n^2)$ ). Clearly  $\sigma_1$  and  $\sigma_2$  are disjoint in  $\pi$  (since  $\mathcal{M}$  is a matching) and cover  $\pi$  (since  $\mathcal{M}$  is perfect). Moreover, the fact that  $\mathcal{M}$  satisfies **P<sub>2</sub>** implies immediately that  $\text{reduce}(\sigma_1)$  is order-isomorphic to  $(\text{reduce} \circ \text{r} \circ \text{c})(\sigma_2)$ , and hence this shows that  $\pi \in \mathcal{S}_{2n}^{\text{rc}}$ . ■

**Proposition 10.** (r o c)-UNSHUFFLE-PERMUTATION is NP-complete.

*Proof (sketch).* The proof is similar to Proposition 7, replacing  $\mu'_1$  by  $\nearrow_{N_1}$   $[2n+k+N_1+2N_2]$ ,  $\mu'_2$  by  $\nearrow_{N_2}$   $[n+k+N_1+N_2]$ ,  $\mu'_\pi$  by  $(\text{r o c})(\pi)[2n+k+2N_1+2N_2]$  and  $\mu'_\sigma$  by  $(\text{r o c})(\sigma)[n+k+N_1+2N_2]$ . ■

## 5 Conclusion and perspectives

In this paper we have proposed to investigate the problem of recognizing those permutations  $\pi$  for which there exists a permutation  $\sigma$  such that  $\pi \in \sigma \sqcup f(\sigma)$  for some bijection  $f : S_n \rightarrow S_n$ . Quite a number of problems are left open by the results presented here. For instance, can we efficiently decide membership to  $\mathcal{S}_{2n}^{\text{i}}$  (i.e., the permutations  $\pi$  for which there exists a permutation  $\sigma$  such that  $\pi \in \sigma \sqcup \text{i}(\sigma)$ )? To conclude, we wish to mention two conjectures.

*Conjecture 1 (Enumeration).* For any  $n \in \mathbb{N}$ ,  $s_{2n}^{\text{rc}} \leq s_{2n}^{\text{c}} = s_{2n}^{\text{r}} \leq s_{2n}^{\text{irc}} = s_{2n}^{\text{i}} \leq s_{2n}^{\text{e}} \leq s_{2n}^{\text{ci}} = s_{2n}^{\text{ic}}$ .

*Conjecture 2 (Ubiquitous generalized squares).* For any  $n \in \mathbb{N}$ ,  $\mathcal{S}_{2n}^{\text{e}} \cap \mathcal{S}_{2n}^{\text{r}} \cap \mathcal{S}_{2n}^{\text{c}} \cap \mathcal{S}_{2n}^{\text{i}} \cap \mathcal{S}_{2n}^{\text{ci}} \cap \mathcal{S}_{2n}^{\text{ic}} \cap \mathcal{S}_{2n}^{\text{rc}} \cap \mathcal{S}_{2n}^{\text{irc}} \neq \emptyset$ .

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