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► **To cite this version:**

Guillaume Fertin, Samuele Giraud, Sylvie Hamel, Stéphane Vialette. Unshuffling Permutations. TAMC, Apr 2019, Kitakyushu, Japan. pp.242-261, 10.1007/978-3-030-14812-6_15 . hal-02304028

HAL Id: hal-02304028

<https://hal.science/hal-02304028v1>

Submitted on 14 Feb 2023

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Unshuffling permutations: Trivial bijections and compositions^{*}

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Abstract. Given permutations π , σ_1 and σ_2 , the permutation π (viewed as a string) is said to be a *shuffle* of σ_1 and σ_2 , in symbols $\pi \in \sigma_1 \sqcup \sigma_2$, if π can be formed by interleaving the letters of two strings p_1 and p_2 that are order-isomorphic to σ_1 and σ_2 , respectively. Given a permutation $\pi \in S_{2n}$ and a bijective mapping $f : S_n \rightarrow S_n$, the f -UNSHUFFLE-PERMUTATION problem is to decide whether there exists a permutation $\sigma \in S_n$ such that $\pi \in \sigma \sqcup f(\sigma)$. We consider here this problem for the following bijective mappings: inversion, reverse, complementation, and all their possible compositions. In particular, we present combinatorial results about the permutations accepted by this problem. As main results, we obtain that this problem is NP-complete when f is the reverse, the complementation, or the composition of the reverse with the complementation.

Keywords: Permutation · Shuffle product · Computational complexity.

1 Introduction

Given permutations π , σ_1 and σ_2 , π is said to be a *shuffle* of σ_1 and σ_2 , in symbols $\pi \in \sigma_1 \sqcup \sigma_2$, if π is the disjoint union of two patterns p_1 and p_2 (*i.e.*, if, viewed as a string, it can be formed by interleaving the letters of p_1 and p_2) that are order-isomorphic to σ_1 and σ_2 , respectively. This shuffling operation \sqcup was introduced by Vargas [12] in an algebraic context and was called *supershuffle*. In case $\sigma = \sigma_1 = \sigma_2$, the permutation π is said to be a *square w.r.t. the shuffle product* (or simply a *square* when clear from the context), and σ is said to be a *square root of π w.r.t. the shuffle product* (or simply a *square root of π* when clear from the context). Note that a permutation may have several square roots. For example, $\pi = 18346752 \in 1234 \sqcup 4321$ since π is a shuffle of the patterns $p_1 = 1347$ and $p_2 = 8652$ that are order-isomorphic to $\sigma_1 = 1234$ and

^{*} Partially supported by Laboratoire International Franco-Québécois de Recherche en Combinatoire (LIRCO) (G. Fertin and S. Vialette), supported by the Individual Discovery Grant RGPIN-2016-04576 from Natural Sciences and Engineering Research Council of Canada (NSERC) (S. Hamel)

$\sigma_2 = 4321$, respectively (as shown in 18^346^752). However, π is not a square. Besides, $\pi' = 24317856$ is a square as it is a shuffle of the patterns 2175 and 4386 that are both order-isomorphic to 2143 (as shown in 243^178^56). Note that 2143 is not the unique square root of π' since π' is also a shuffle of patterns 2156 and 4378 that are both order-isomorphic to 2134 (as shown in 243^178^56).

The above definitions are of course intended to be natural counterparts to the ordinary shuffle of words and languages. Given words u , v_1 and v_2 , u is said to be a *shuffle* of v_1 and v_2 (denoted $u \in v_1 \sqcup v_2$), if u can be formed by interleaving the letters of v_1 and v_2 in a way that maintains the left-to-right ordering of the letter from each word. For example, $u = abccbabacbb \in abcab \sqcup cbabcb$ (as shown in $ab_ccbab_a_cbb$). Similarly, in case $v = v_1 = v_2$, the word u is said to be a *square w.r.t. the shuffle product*, and v is said to be a *square root of u w.r.t. the shuffle product*. For example, $abaaabaabb$ is a square w.r.t. the shuffle product since it belongs to the shuffle of $abaab$ with itself (as shown in $ab_aaa_baa^bb$). Given words u , v_1 and v_2 , deciding whether $u \in v_1 \sqcup v_2$ is $O(|u|^2 / \log(|u|))$ time solvable [6]. To the best of our knowledge, the first $O(|u|^2)$ time algorithm for this problem appeared in [7]. However, deciding whether a given word u is a square w.r.t. the shuffle product is NP-complete [9, 2]. Finally, for a given word u , deciding whether there exists a word v such that u is in the shuffle of v with its reverse v^R (i.e., $u \in v \sqcup v^R$) is NP-complete as well [9]. However, the problem is polynomial-time solvable for words built from a binary alphabet.

Coming back to permutations, by using a pattern avoidance criterion on directed perfect matchings, it is proved in [4] that recognizing permutations that are squares is NP-complete, and a bijection between (213, 231)-avoiding square permutations and square binary words is presented. Given (3142, 2413)-avoiding (a.k.a. *separable*) permutations π , σ_1 and σ_2 , it can be decided in polynomial time whether $\pi \in \sigma_1 \sqcup \sigma_2$ [8].

We finally observe that deciding whether a permutation is a shuffle of two monotone permutations is in P:

(i) *merge permutations* are the union of two increasing subsequences and are characterized by the fact that they contain no decreasing subsequence of length 3 [5], whereas (ii) *skew-merged permutations* are the union of an increasing subsequence with a decreasing subsequence and are characterized by the fact that they contain no subsequence ordered in the same way as 2143 or 3412 [11].

In this paper, we are interested in a generalization of the problem consisting in recognizing square permutations. We call this new problem the f -UNSHUFFLE-PERMUTATION problem, which is defined as follows. Given a permutation $\pi \in S_{2n}$ and a bijective mapping $f : S_n \rightarrow S_n$, the f -UNSHUFFLE-PERMUTATION problem asks whether there exists a permutation $\sigma \in S_n$ such that $\pi \in \sigma \sqcup f(\sigma)$. We say in this case that π is a *generalized square permutation*. In case $f = \text{id}$, the id-UNSHUFFLE-PERMUTATION problem reduces to recognizing square permutations, and hence is NP-complete. This paper is devoted to studying the f -UNSHUFFLE-PERMUTATION problem in case f is either a trivial bijection (identity, complement, reverse or inverse) or obtained by composing trivial bijections. These bijections act on permutations by performing a transformation on their permutation matrices

and can hence be seen as elements of the Dihedral group D_4 , as discussed in the next section. The paper is organized as follows. In Section 2, we provide the needed definitions. Section 3 is devoted to presenting generalized square permutations and some associated enumerative properties. Finally, in Section 4, we show hardness of recognizing some generalized square permutations.

2 Definitions

For any nonnegative integer n , $[n]$ is the set $\{1, \dots, n\}$. We follow the usual terminology on words [3]. Let us recall here the most important ones. Let u be a word. The length of u is denoted by $|u|$. The *empty word*, the only word of null length, is denoted by ϵ . For any $i \in [|u|]$, the i -th letter of u is denoted by $u(i)$. If I is a subset of $[|u|]$, $u|_I$ is the subword of u consisting in the letters of u at the positions specified by the elements of I . A *permutation* of size n is a word π of length n on the alphabet $[n]$ such that each letter admits exactly one occurrence. The set of all permutations of size n is denoted by S_n .

Definition 1 (Reduced form). *If λ is a list of distinct integers, the reduced form of λ , denoted $\text{reduce}(\lambda)$, is the permutation obtained from λ by replacing its i -th smallest entry with i . For instance, $\text{reduce}(31845) = 21534$.*

Definition 2 (Order-isomorphism). *Let λ_1 and λ_2 be two words of distinct integers such that $\text{reduce}(\lambda_1) = \text{reduce}(\lambda_2)$. We say that λ_1 and λ_2 are order-isomorphic and we denote it by $\lambda_1 \simeq \lambda_2$.*

Definition 3 (Pattern containment). *A permutation σ is said to be contained in, or to be a subpermutation of, another permutation π , written $\sigma \preceq \pi$, if π has a (not necessarily contiguous) subsequence whose terms are order-isomorphic to σ . We also say that π admits an occurrence of the pattern σ .*

Thus, $\sigma \preceq \pi$ if there is a set I of positions in π such that $\text{reduce}(\pi|_I) = \sigma$. For example, $1423 \preceq 149362785$ since $\text{reduce}(149362785|_{\{2,3,5,8\}}) = \text{reduce}(4968) = 1423$.

Definition 4 (Trivial bijections). *Let $\pi = \pi(1)\pi(2) \dots \pi(n)$ be a permutation of size n . The reverse of π is the permutation $r(\pi) = \pi(n)\pi(n-1) \dots \pi(1)$. The complement of π is the permutation $c(\pi) = \pi'(1)\pi'(2) \dots \pi'(n)$, where $\pi'(i) = n - \pi(i) + 1$. The inverse is the regular group theoretical inverse on permutations, that is the $\pi(i)$ -th position of the inverse $i(\pi)$ is occupied by i . The reverse, complement and inverse are called the trivial bijections from S_n to itself.*

Abusing notations, for any $S \subseteq S_n$, we let $c(S)$, $i(S)$ and $r(S)$ stand respectively for $\{c(\pi) : \pi \in S\}$, $\{i(\pi) : \pi \in S\}$ and $\{r(\pi) : \pi \in S\}$.

Definition 5 (Shuffle). *Let $\sigma_1 \in S_{k_1}$ and $\sigma_2 \in S_{k_2}$ be two permutations. A permutation $\pi = \pi(1)\pi(2) \dots \pi(k_1 + k_2)$ is a shuffle of σ_1 and σ_2 if there is a subset I of $[n]$ such that $\pi|_I \simeq \sigma_1$ and $\pi|_{\bar{I}} \simeq \sigma_2$, where $\bar{I} = [n] \setminus I$ and $n = k_1 + k_2$.*

In other terms, π is obtained by interleaving the letters of two words respectively order-isomorphic with σ_1 and σ_2 . We denote by $\sigma_1 \sqcup \sigma_2$ the set of all shuffles of σ_1 and σ_2 .

For example,

$$12 \sqcup 21 = \{1243, 1324, 1342, 1423, 1432, 2134, 2314, 2341, 2413, 2431, \\ 3124, 3142, 3214, 3241, 3421, 4123, 4132, 4213, 4231, 4312\}.$$

We are now ready to define the f -UNSHUFFLE-PERMUTATION problem we focus on in this paper.

Definition 6 (f -Unshuffle-Permutation). *Given a permutation $\pi \in S_{2n}$ and a bijective mapping $f : S_n \rightarrow S_n$, the f -UNSHUFFLE-PERMUTATION problem is to decide whether there exists a permutation $\sigma \in S_n$ such that $\pi \in \sigma \sqcup f(\sigma)$.*

3 Generalized square permutations and enumerative properties

The bijections c , i , and r are involutions and can be seen as particular elements of the dihedral group D_4 encoding all the symmetries of the square. Before explaining why, let us recall a definition for D_4 .

The group D_4 is generated by two elements \mathbf{a} and \mathbf{b} subjected exactly to the nontrivial relations

$$\mathbf{a}\mathbf{a} = \epsilon, \quad \mathbf{b}\mathbf{b}\mathbf{b}\mathbf{b} = \epsilon, \quad \mathbf{a}\mathbf{b}\mathbf{a}\mathbf{b} = \epsilon, \tag{1}$$

where ϵ is the unit of D_4 . One can think of element \mathbf{a} (resp. \mathbf{b}) acting on a square by performing a symmetry through the vertical axis (resp. a 90° clockwise rotation). Now, we can regard, for any $n \in \mathbb{N}$, each element of D_4 as a map $\phi : S_n \rightarrow S_n$ such that $\phi(\pi)$, $\pi \in S_n$, is the permutation obtained by performing on its permutation matrix the transformations specified by ϕ . The maps c , i , and r can thus be expressed as

$$\mathbf{c} = \mathbf{b}\mathbf{b}\mathbf{a}, \quad \mathbf{i} = \mathbf{b}\mathbf{a}, \quad \mathbf{r} = \mathbf{a}. \tag{2}$$

Figure 1 shows the Cayley graph of this group. The main interest of seeing the three trivial bijections on S_n as elements of D_4 lies in the fact that all the other bijections obtained by composing the trivial ones can be expressed by compositions of the two elements \mathbf{a} and \mathbf{b} . This will allow us to gain concision in some proofs presented in the sequel.

We now turn to defining *identity squares*, *complement squares*, *reverse squares* and *inverse squares* that are natural generalizations of square permutations to trivial bijections from S_n to itself.

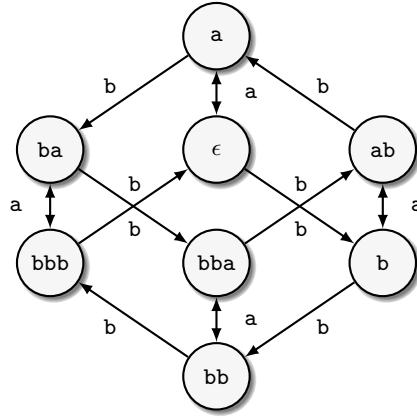


Fig. 1. The Cayley graph of the group D_4 .

Definition 7 (Square permutations for trivial bijections). For $n \in \mathbb{N}$, define

$$\begin{aligned} \mathcal{S}_{2n}^\epsilon &= \{\pi \in S_{2n} : \exists \sigma \in S_n, \text{ such that } \pi \in \sigma \sqcup \sigma\}, \\ \mathcal{S}_{2n}^c &= \{\pi \in S_{2n} : \exists \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup c(\sigma)\}, \\ \mathcal{S}_{2n}^r &= \{\pi \in S_{2n} : \exists \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup r(\sigma)\}, \\ \mathcal{S}_{2n}^i &= \{\pi \in S_{2n} : \exists \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup i(\sigma)\}. \end{aligned}$$

We begin by proving that applying any trivial bijection is in some sense compatible with the shuffle operator.

Lemma 1 (Trivial bijections and shuffle). Let π, σ_1 and σ_2 be permutations such that $\pi \in \sigma_1 \sqcup \sigma_2$. Then

- $c(\pi) \in c(\sigma_1) \sqcup c(\sigma_2)$;
- $r(\pi) \in r(\sigma_1) \sqcup r(\sigma_2)$;
- $i(\pi) \in i(\sigma_1) \sqcup i(\sigma_2)$.

Proof. Let n be the size of π . Since π is a shuffle of σ_1 and σ_2 , there is a subset $I = \{i_1 < \dots < i_k\}$ of $[n]$ such that $\pi|_I \simeq \sigma_1$ and $\pi|_{\bar{I}} \simeq \sigma_2$.

By setting $J = \{n - i_k + 1 < \dots < n - i_1 + 1\}$, we have

$$r(\pi)|_J = (\pi(n) \dots \pi(1))|_J = \pi(i_k) \dots \pi(i_1) \simeq r(\sigma_1). \quad (3)$$

For the same reason, we have also $r(\pi)|_{\bar{J}} \simeq r(\sigma_2)$. Hence, $r(\pi)$ is a shuffle of $r(\sigma_1)$ and $r(\sigma_2)$.

Now, set $J = \{\pi(i) : i \in I\} = \{\pi(\ell_1) < \dots < \pi(\ell_k)\}$ where ℓ_1, \dots, ℓ_k are elements of I . We have

$$i(\pi)|_J = (\pi^{-1}(1) \dots \pi^{-1}(n))|_J = \pi^{-1}(\pi(\ell_1)) \dots \pi^{-1}(\pi(\ell_k)) = \ell_1 \dots \ell_k \simeq i(\sigma_1). \quad (4)$$

For the same reason, we have also $i(\pi)_{|\bar{j}} \simeq i(\sigma_2)$. Hence, $i(\pi)$ is also shuffle of $i(\sigma_1)$ and $i(\sigma_2)$.

Finally, since $\mathbf{b} = \mathbf{i} \circ \mathbf{r}$ in D_4 , the map c can be expressed as compositions involving \mathbf{r} and \mathbf{i} . Hence, $c(\pi)$ is a shuffle of $c(\pi_1)$ and $c(\pi_2)$. \blacksquare

The following lemma is well-known and is a consequence of interpreting c , \mathbf{r} , and \mathbf{i} as compositions involving \mathbf{a} and \mathbf{b} according to (2) and the relations (1) between \mathbf{a} and \mathbf{b} .

Lemma 2 (Compositions of trivial bijections). *For any permutation π ,*

- $(c \circ \mathbf{r})(\pi) = (\mathbf{r} \circ c)(\pi)$;
- $(\mathbf{i} \circ c)(\pi) = (\mathbf{r} \circ \mathbf{i})(\pi)$;
- $(\mathbf{i} \circ \mathbf{r})(\pi) = (c \circ \mathbf{i})(\pi)$.

According to Lemma 2 (and the fact that the group D_4 has exactly eight elements), we thus only need to consider the following compositions in the rest of the paper.

Definition 8 (Square permutations for compositions of trivial bijections). *For $n \in \mathbb{N}$, define*

$$\begin{aligned} \mathcal{S}_{2n}^{\mathbf{rc}} &= \{\pi \in S_{2n} : \exists \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup (\mathbf{r} \circ c)(\sigma)\}, \\ \mathcal{S}_{2n}^{\mathbf{ci}} &= \{\pi \in S_{2n} : \exists \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup (c \circ \mathbf{i})(\sigma)\}, \\ \mathcal{S}_{2n}^{\mathbf{ic}} &= \{\pi \in S_{2n} : \exists \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup (\mathbf{i} \circ c)(\sigma)\}, \\ \mathcal{S}_{2n}^{\mathbf{irc}} &= \{\pi \in S_{2n} : \exists \sigma \in S_n \text{ such that } \pi \in \sigma \sqcup (\mathbf{i} \circ \mathbf{r} \circ c)(\sigma)\}. \end{aligned}$$

We hence obtain the eight sets $\mathcal{S}_{2n}^{\epsilon}$, $\mathcal{S}_{2n}^{\mathbf{c}}$, $\mathcal{S}_{2n}^{\mathbf{r}}$, $\mathcal{S}_{2n}^{\mathbf{i}}$, $\mathcal{S}_{2n}^{\mathbf{rc}}$, $\mathcal{S}_{2n}^{\mathbf{ci}}$, $\mathcal{S}_{2n}^{\mathbf{ic}}$, and $\mathcal{S}_{2n}^{\mathbf{irc}}$ of square permutations provided by Definitions 7 and 8. Let us now investigate some easy bijective and enumerative properties satisfied by these sets.

Proposition 1 (Bijection between $\mathcal{S}_{2n}^{\mathbf{c}}$ and $\mathcal{S}_{2n}^{\mathbf{r}}$). *For any $n \in \mathbb{N}$, $i(\mathcal{S}_{2n}^{\mathbf{c}}) = \mathcal{S}_{2n}^{\mathbf{r}}$.*

Proof. Let $\pi \in \mathcal{S}_{2n}$. We have

$$\begin{aligned} \pi \in \mathcal{S}_{2n}^{\mathbf{c}} &\Leftrightarrow \exists \sigma \in S_n, \pi \in \sigma \sqcup c(\sigma) \\ &\Leftrightarrow \exists \sigma \in S_n, i(\pi) \in i(\sigma) \sqcup (\mathbf{i} \circ c)(\sigma) && \text{(Lemma 1)} \\ &\Leftrightarrow \exists \sigma \in S_n, i(\pi) \in i(\sigma) \sqcup (\mathbf{r} \circ \mathbf{i})(\sigma) && \text{(Lemma 2)} \\ &\Leftrightarrow \exists \sigma \in S_n, i(\pi) \in \sigma \sqcup \mathbf{r}(\sigma) \\ &\Leftrightarrow i(\pi) \in \mathcal{S}_{2n}^{\mathbf{r}}. \end{aligned} \quad \blacksquare$$

Proposition 2 (Bijection between $\mathcal{S}_{2n}^{\mathbf{ic}}$ and $\mathcal{S}_{2n}^{\mathbf{ci}}$). *For any $n \in \mathbb{N}$, $c(\mathcal{S}_{2n}^{\mathbf{ic}}) = \mathcal{S}_{2n}^{\mathbf{ci}}$, $i(\mathcal{S}_{2n}^{\mathbf{ic}}) = \mathcal{S}_{2n}^{\mathbf{ci}}$, and $\mathbf{r}(\mathcal{S}_{2n}^{\mathbf{ic}}) = \mathcal{S}_{2n}^{\mathbf{ci}}$.*

Proof. Let $\pi \in S_{2n}$. We have

$$\begin{aligned}
\pi \in \mathcal{S}_{2n}^{\text{ic}} &\Leftrightarrow \exists \sigma \in S_n, \pi \in \sigma \sqcup (\text{i} \circ \text{c})(\sigma) \\
&\Leftrightarrow \exists \sigma \in S_n, c(\pi) \in c(\sigma) \sqcup (c \circ \text{i} \circ \text{c})(\sigma) && \text{(Lemma 1)} \\
&\Leftrightarrow \exists \sigma \in S_n, c(\pi) \in \sigma \sqcup (c \circ \text{i})(\sigma) \\
&\Leftrightarrow c(\pi) \in \mathcal{S}_{2n}^{\text{ci}}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\pi \in \mathcal{S}_{2n}^{\text{ic}} &\Leftrightarrow \exists \sigma \in S_n, \pi \in \sigma \sqcup (\text{i} \circ \text{c})(\sigma) \\
&\Leftrightarrow \exists \sigma \in S_n, \text{i}(\pi) \in \text{i}(\sigma) \sqcup (\text{i} \circ \text{i} \circ \text{c})(\sigma) && \text{(Lemma 1)} \\
&\Leftrightarrow \exists \sigma \in S_n, \text{i}(\pi) \in \text{i}(\sigma) \sqcup (\text{i} \circ \text{r} \circ \text{i})(\sigma) && \text{(Lemma 2)} \\
&\Leftrightarrow \exists \sigma \in S_n, \text{i}(\pi) \in \sigma \sqcup (\text{i} \circ \text{r})(\sigma) \\
&\Leftrightarrow \exists \sigma \in S_n, \text{i}(\pi) \in \sigma \sqcup (c \circ \text{i})(\sigma) && \text{(Lemma 2)} \\
&\Leftrightarrow \text{i}(\pi) \in \mathcal{S}_{2n}^{\text{ci}}.
\end{aligned}$$

Finally, the fact that $\text{i} \circ \text{c} \circ \text{i} = \text{r}$ implies that r is also a bijection between $\mathcal{S}_{2n}^{\text{ic}}$ and $\mathcal{S}_{2n}^{\text{ci}}$. \blacksquare

Proposition 3 (Bijection between $\mathcal{S}_{2n}^{\text{i}}$ and $\mathcal{S}_{2n}^{\text{irc}}$). For any $n \in \mathbb{N}$, $(\text{i} \circ \text{c})(\mathcal{S}_{2n}^{\text{irc}}) = \mathcal{S}_{2n}^{\text{i}}$.

Proof. Let $\pi \in S_{2n}$. We have

$$\begin{aligned}
\pi \in \mathcal{S}_{2n}^{\text{irc}} &\Leftrightarrow \exists \sigma \in S_n, \pi \in \sigma \sqcup (\text{i} \circ \text{r} \circ \text{c})(\sigma) \\
&\Leftrightarrow \exists \sigma \in S_n, \text{i}(\pi) \in \text{i}(\sigma) \sqcup (\text{r} \circ \text{c})(\sigma) && \text{(Lemma 1)} \\
&\Leftrightarrow \exists \sigma \in S_n, (\text{r} \circ \text{i})(\pi) \in (\text{r} \circ \text{i})(\sigma) \sqcup c(\sigma) && \text{(Lemma 1)} \\
&\Leftrightarrow \exists \sigma \in S_n, (\text{i} \circ \text{c})(\pi) \in (\text{i} \circ \text{c})(\sigma) \sqcup c(\sigma) && \text{(Lemma 2)} \\
&\Leftrightarrow \exists \sigma \in S_n, (\text{i} \circ \text{c})(\pi) \in \text{i}(\sigma) \sqcup \sigma \\
&\Leftrightarrow (\text{i} \circ \text{c})(\pi) \in \mathcal{S}_{2n}^{\text{i}}.
\end{aligned}$$

\blacksquare

To simplify notations, we write s_{2n}^{ϵ} , s_{2n}^{c} , s_{2n}^{r} , s_{2n}^{i} , s_{2n}^{rc} , s_{2n}^{ci} , s_{2n}^{ic} , and s_{2n}^{irc} for the cardinalities of the sets $\mathcal{S}_{2n}^{\epsilon}$, $\mathcal{S}_{2n}^{\text{c}}$, $\mathcal{S}_{2n}^{\text{r}}$, $\mathcal{S}_{2n}^{\text{i}}$, $\mathcal{S}_{2n}^{\text{rc}}$, $\mathcal{S}_{2n}^{\text{ci}}$, $\mathcal{S}_{2n}^{\text{ic}}$, and $\mathcal{S}_{2n}^{\text{irc}}$, respectively. Table 1 shows the first cardinalities. We note that, at this time, none of the corresponding integer sequence does appear in OEIS [10].

Lemma 3 (Upper bound for the number of generalized squares). For any $n \in \mathbb{N}$,

$$|Q_{2n}| \leq \binom{2n-1}{n-1} \binom{2n}{n} n!, \quad (5)$$

where Q_{2n} is any of the sets $\mathcal{S}_{2n}^{\epsilon}$, $\mathcal{S}_{2n}^{\text{c}}$, $\mathcal{S}_{2n}^{\text{r}}$, $\mathcal{S}_{2n}^{\text{i}}$, $\mathcal{S}_{2n}^{\text{rc}}$, $\mathcal{S}_{2n}^{\text{ci}}$, $\mathcal{S}_{2n}^{\text{ic}}$, or $\mathcal{S}_{2n}^{\text{irc}}$.

n	0	1	2	3	4	5
s_{2n}^{rc}	1	2	20	472	18 988	1 112 688
$s_{2n}^{\text{c}} = s_{2n}^{\text{r}}$	1	2	20	480	19 744	1 185 264
$s_{2n}^{\text{i}} = s_{2n}^{\text{irc}}$	1	2	20	488	20 250	1 229 858
s_{2n}^{e}	1	2	20	504	21 032	1 293 418
$s_{2n}^{\text{ci}} = s_{2n}^{\text{ic}}$	1	2	20	586	27 990	2 044 596
$(2n)!$	1	2	24	720	40 320	3 628 800

Table 1. Cardinalities of the sets of square permutations of sizes $0 \leq 2n \leq 10$.

Proof. This is a consequence of the fact that to construct a permutation π of Q_{2n} , we can proceed by first choosing a set of n letters among the alphabet $[2n]$ ($\binom{2n}{n}$ choices), then by specifying an order from left to right for these letters ($n!$ choices), and finally by deploying the chosen letters onto a set I of n positions among the set of all possible positions $[2n]$ ($\binom{2n}{n}$ choices). The empty positions are filled by the unique authorized completion such that $\pi|_I \simeq \phi(\pi|_{\bar{I}})$ where ϕ is the considered bijection on S_n . For this reason, there is no more than $\binom{2n}{n}^2 n!$ elements in Q_{2n} . The tighter bound (5) is the consequence of the fact that we can assume that the first position can always be fixed. ■

Let us denote by $\mathcal{S}^{\bullet}_{2n}$ the union of all the sets $\mathcal{S}_{2n}^{\text{e}}$, $\mathcal{S}_{2n}^{\text{c}}$, $\mathcal{S}_{2n}^{\text{r}}$, $\mathcal{S}_{2n}^{\text{i}}$, $\mathcal{S}_{2n}^{\text{rc}}$, $\mathcal{S}_{2n}^{\text{ci}}$, $\mathcal{S}_{2n}^{\text{ic}}$, and $\mathcal{S}_{2n}^{\text{irc}}$. We show that S_{2n} and $\mathcal{S}^{\bullet}_{2n}$ do not coincide.

Proposition 4 (Existence of non-square permutations). *For any $n \in \mathbb{N}, n \geq 9, S_{2n} \neq \mathcal{S}^{\bullet}_{2n}$.*

Proof. By Lemma 3 (and using its notations), we have

$$|\mathcal{S}^{\bullet}_{2n}| \leq 8|Q_{2n}| \leq 8 \binom{2n-1}{n-1} \binom{2n}{n} n!. \quad (6)$$

The proportion of the elements of $\mathcal{S}^{\bullet}_{2n}$ among all the permutations of S_{2n} admits as upper bound, when $n \geq 1$,

$$\frac{|\mathcal{S}^{\bullet}_{2n}|}{|S_{2n}|} \leq \frac{8 \binom{2n-1}{n-1} \binom{2n}{n} n!}{(2n)!} = 4 \binom{2n}{n} \frac{1}{n!}. \quad (7)$$

But $4 \binom{2n}{n} \frac{1}{n!} < 1$ for $n \geq 9$, thereby proving the result. ■

Proposition 5 (Square permutations in some intersections). *Let $n \in \mathbb{N}$ such that $2n \equiv 0 \pmod{4}$. Then,*

the permutation $\pi = 12 \dots (n-1)(n) (2n)(2n-1) \dots (n+2)(n+1)$ belongs to $\mathcal{S}_{2n}^{\text{e}} \cap \mathcal{S}_{2n}^{\text{r}} \cap \mathcal{S}_{2n}^{\text{c}} \cap \mathcal{S}_{2n}^{\text{i}} \cap \mathcal{S}_{2n}^{\text{ci}} \cap \mathcal{S}_{2n}^{\text{ic}}$.

Proof. Let $I = \{1, 3, 5, \dots, 2n-1\}$ and $\bar{I} = [2n] \setminus I$. Then, since

$$\pi|_I \simeq 12 \dots \frac{n}{2} n(n-1) \dots \left(\frac{n}{2} + 1\right) \simeq \pi|_{\bar{I}}, \quad (8)$$

we have $\pi|_J \simeq \pi|\bar{J} \simeq i(\pi|\bar{J})$. This shows that π belongs to $\mathcal{S}_{2n}^\epsilon \cap \mathcal{S}_{2n}^i$.

Now, let $J = \{1, 2, \dots, n\}$ and $\bar{J} = [2n] \setminus J$. Then, since

$$\pi|_J \simeq 12 \dots n \quad \text{and} \quad \pi|\bar{J} \simeq n \dots 21, \quad (9)$$

we have $\pi|_J \simeq r(\pi|\bar{J}) \simeq c(\pi|\bar{J}) \simeq (i \circ c)(\pi|\bar{J}) \simeq (c \circ i)(\pi|\bar{J})$. This shows that π belongs to $\mathcal{S}_{2n}^i \cap \mathcal{S}_{2n}^c \cap \mathcal{S}_{2n}^{ic} \cap \mathcal{S}_{2n}^{ci}$. ■

4 Recognizing generalized square permutations

This section is devoted to proving that deciding membership to \mathcal{S}_{2n}^i , \mathcal{S}_{2n}^c and \mathcal{S}_{2n}^{ic} is NP-complete. The approach relies on constraint directed matchings on permutations.

Definition 9 (Directed matching on permutations). Let $\pi \in S_n$. A directed matching on π is a set \mathcal{M} of pairwise disjoint arcs (i, j) , $1 \leq i, j \leq n$ and $i \neq j$, that connect pairs of elements of π . The directed matching \mathcal{M} is perfect if every element of π is the source or the sink of an arc of \mathcal{M} .

Definition 10 (Two-arcs pattern occurrences). A two-arc pattern is a perfect directed matching on the set $[4]$. A perfect directed matching \mathcal{M} on a permutation π of size $2n$ admits an occurrence of a two-arc pattern \mathcal{A} if there is an increasing map $\phi : [4] \rightarrow [2n]$ (i.e., $i < i'$ implies $\phi(i) < \phi(i')$) such that, if (i, i') is an arc of \mathcal{A} , then $(\phi(i), \phi(i'))$ is an arc of \mathcal{M} . We say moreover that \mathcal{M} avoids \mathcal{A} if \mathcal{A} does not admits any occurrence of \mathcal{A} .

We shall draw two-arcs patterns by unlabeled graphs wherein vertices are implicitly indexed from 1 to 4 from left to right. Figure 4 shows an example of a directed perfect matching on a permutation. The directed matching does contain the patterns and does avoid the patterns

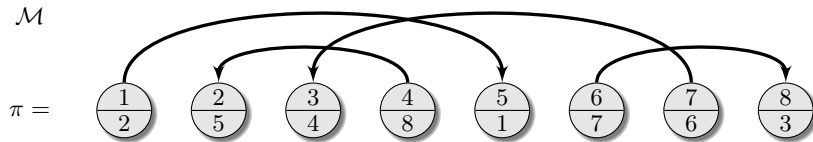


Fig. 2. A directed perfect matching $\mathcal{M} = \{(1, 5), (4, 2), (7, 3), (6, 8)\}$ on the permutation $\pi = 25481763$. Recall that arcs refer to positions in π .

In case π is written as a concatenation of contiguous patterns $\pi = \pi_1 \pi_2 \dots \pi_k$, we write $\mathcal{M}_{\pi_i \rightarrow \pi_j}$ for the subset of arcs of \mathcal{M} with source index in π_i and sink index in π_j . Hence,

$$\mathcal{M} = \bigsqcup_{\substack{i \in [k] \\ j \in [k]}} \mathcal{M}_{\pi_i \rightarrow \pi_j}. \quad (10)$$

The following properties will prove useful for defining equivalences between squares and restricted directed perfect matchings.

Definition 11 (Property \mathbf{P}_1 — Order isomorphism). *Let π be a permutation. A directed perfect matching \mathcal{M} on π is said to have property \mathbf{P}_1 if, for any two distinct arcs (i, i') and (j, j') of \mathcal{M} , we have $\pi(i) < \pi(j)$ if and only if $\pi(i') < \pi(j')$.*

Definition 12 (Property \mathbf{P}_2 — Order anti-isomorphism). *Let π be a permutation. A directed perfect matching \mathcal{M} on π is said to have property \mathbf{P}_2 if, for any two distinct arcs (i, i') and (j, j') of \mathcal{M} , we have $\pi(i) < \pi(j)$ if and only if $\pi(i') > \pi(j')$.*

Definition 13 (Property \mathbf{Q}_1 — First start first terminal). *Let π be a permutation. A directed perfect matching \mathcal{M} on π is said to have property \mathbf{Q}_1 if it avoids the following set of two-arcs patterns:*

$$\mathcal{Q}_1 = \left\{ \begin{array}{c} \curvearrowright \\ \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowright \\ \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowright \\ \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowright \\ \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowright \\ \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowright \\ \bullet \bullet \bullet \bullet \end{array} \right\}.$$

Definition 14 (Property \mathbf{Q}_2 — First start last terminal). *Let π be a permutation. A directed perfect matching \mathcal{M} on π is said to have property \mathbf{Q}_2 if it avoids the following set of two-arcs patterns:*

$$\mathcal{Q}_2 = \left\{ \begin{array}{c} \curvearrowleft \\ \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowleft \\ \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowleft \\ \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowleft \\ \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowleft \\ \bullet \bullet \bullet \bullet \end{array}, \begin{array}{c} \curvearrowleft \\ \bullet \bullet \bullet \bullet \end{array} \right\}.$$

Observe that $\mathcal{Q}_1 \sqcup \mathcal{Q}_2$ is the set of all two-arcs patterns.

Theorem 1 ([4]). *Let $\pi \in S_{2n}$. The two following statements are equivalent.*





1. $\pi \in S_{2n}^e$.
2. *There exists a directed perfect matching \mathcal{M} on π that satisfies properties \mathbf{P}_1 and \mathbf{Q}_1 .*


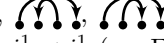
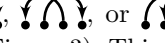
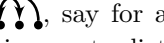
The following two definitions will intervene in the next constructions.

Definition 15 (Lifting). *Let $\pi = \pi(1) \pi(2) \dots \pi(n)$ be a permutation of size n and k be a positive integer. We call k -lifting of π , denoted $\pi[k]$, the permutation $(k + \pi(1)) (k + \pi(2)) \dots (k + \pi(n))$ on the alphabet $\{k + 1, \dots, k + n\}$.*

Definition 16 (Monotone). *For any positive integer k , we let \nearrow_k stand for the increasing permutation $1 2 \dots k$ and \searrow_k stand for the decreasing permutation $k (k - 1) \dots 1$.*

Lemma 4 (One shot lemma). *Let $\pi \in S_{2n}$ and \mathcal{M} be a perfect directed matching on π that satisfies properties \mathbf{P}_i and \mathbf{Q}_j , $i \in \{1, 2\}$ and $j \in \{1, 2\}$. Then, the perfect directed matching \mathcal{M}' obtained by reversing each arc of \mathcal{M} satisfies properties \mathbf{P}_i and \mathbf{Q}_j .*

Lemma 5. Let $I^1 = \{i_j^1 : j \in [n]\}$, $i_1^1 < i_2^1 < \dots < i_n^1$, and $I^2 = \{i_j^2 : j \in [n]\}$, $i_1^2 < i_2^2 < \dots < i_n^2$, be such that $I^1 \cap I^2 = \emptyset$. The directed perfect matching $\mathcal{M} = ([2n], E)$ defined by $E = \{(i_j^1, i_j^2) : j \in [n]\}$ avoids the patterns , , , and .

Proof. Suppose, aiming at a contradiction, that \mathcal{M} does contain the pattern , , , or , say for arcs (i_j^1, i_j^2) and (i_k^1, i_k^2) . Wlog, assume $i_j^1 < i_k^1$ (see Figure 3). This is a contradiction since $i_j^1 < i_k^1$ implies $j < k$, and hence, $i_j^2 < i_k^2$. ■

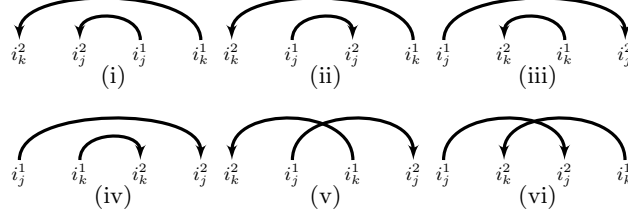

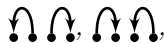



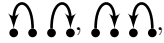
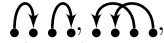



Fig. 3. Arcs (i_j^1, i_j^2) and (i_k^1, i_k^2) with $i_j^1 < i_k^1$.

Lemma 6. Let $I^1 = \{i_j^1 : j \in [n]\}$, $i_1^1 < i_2^1 < \dots < i_n^1$, and $I^2 = \{i_j^2 : j \in [n]\}$, $i_1^2 < i_2^2 < \dots < i_n^2$, be such that $I^1 \cap I^2 = \emptyset$. The directed perfect matching $\mathcal{M} = ([2n], E)$ defined by $E = \{(i_j^1, i_{n-j+1}^2) : j \in [n]\}$ avoids the patterns , , , and .

Proof. Suppose, aiming at a contradiction, that \mathcal{M} does contain the pattern , , , or , say for arcs (i_j^1, i_{n-j+1}^2) and (i_k^1, i_{n-k+1}^2) . Wlog, assume $i_j^1 < i_k^1$ (see Figure 4). This is a contradiction since $i_j^1 < i_k^1$ implies $j < k$, and hence, $i_{n-j+1}^2 > i_{n-k+1}^2$. ■

Proposition 6. Let $\pi \in S_{2n}$. The two following statements are equivalent.

1. $\pi \in \mathcal{S}_{2n}^r$.
2. There exists a directed perfect matching \mathcal{M} on π that satisfies properties **P**₁ and **Q**₂.

Proof. (**1** \Rightarrow **2**) Let $\pi \in \mathcal{S}_{2n}^r$. Then, there exists $\sigma \in S_n$ such that $\pi \in \sigma \sqcup r(\sigma)$, and hence there exist two disjoint set of indexes $I^1 = \{i_j^1 : j \in [n]\}$, $i_1^1 < i_2^1 < \dots < i_n^1$, and $I^2 = \{i_j^2 : j \in [n]\}$, $i_1^2 < i_2^2 < \dots < i_n^2$, such that

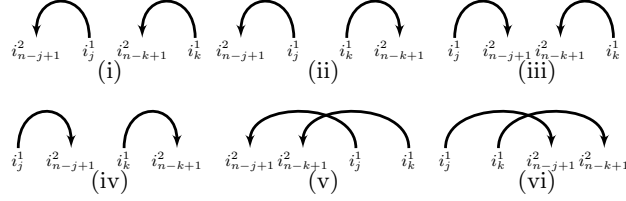


Fig. 4. Arcs (i_j^1, i_{n-j+1}^2) and (i_k^1, i_{n-k+1}^2) with $i_j^1 < i_k^2$.

$\pi(i_1^1)\pi(i_2^1)\cdots\pi(i_n^1)$ is order-isomorphic to σ and $\pi(i_1^2)\pi(i_2^2)\cdots\pi(i_n^2)$ is order-isomorphic to $r(\sigma)$. Let $\mathcal{M} = (V, E)$ be the directed graph defined by $V = [2n]$ and $E = \{(i_j^1, i_{n-j+1}^2) : j \in [n]\}$. Clearly, \mathcal{M} is a directed perfect matching since $I^1 \cap I^2 = \emptyset$ and $I^1 \cup I^2 = [2n]$. According to Lemma 6, \mathcal{M} does avoid the patterns Finally, for any two distinct arcs (i_j^1, i_{n-j+1}^2) and (i_k^1, i_{n-k+1}^2) of \mathcal{M} , we have $\pi(i_j^1) < \pi(i_k^1)$ if and only if $\pi(i_{n-j+1}^2) < \pi(i_{n-k+1}^2)$ since $\pi(i_1^1)\pi(i_2^1)\cdots\pi(i_n^1)$ is order-isomorphic to σ and $\pi(i_1^2)\pi(i_2^2)\cdots\pi(i_n^2)$ is order-isomorphic to $r(\sigma)$. Therefore, \mathcal{M} satisfies properties **P₁** and **Q₂**.

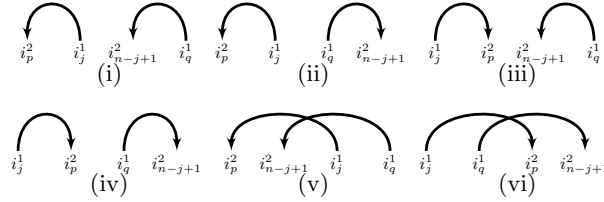





Fig. 5. All possible configurations considered in Proof of Proposition 6 (**2** \Rightarrow **1**).

(2 \Rightarrow 1) Let \mathcal{M} be a directed perfect matching that satisfies properties **P₁** and **Q₂**. Let $I^1 = \{i_j^1 : j \in [n]\}$, $i_1^1 < i_2^1 < \cdots < i_n^1$, be the set of the sources of the arcs of \mathcal{M} and let $I^2 = \{i_j^2 : j \in [n]\}$, $i_1^2 < i_2^2 < \cdots < i_n^2$, be the set of the sinks of the arcs of \mathcal{M} . We first show that, for every $j \in [n]$, (i_j^1, i_{n-j+1}^2) is an arc of \mathcal{M} . The proof is by induction on $j = 1, 2, \dots, n$.

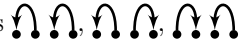
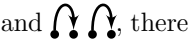

- Base. Suppose, aiming at a contradiction that (i_1^1, i_n^2) is not an arc of \mathcal{M} . Then, there exist $1 < p < n$ and $1 < q < n$ such that (i_1^1, i_p^2) and (i_q^1, i_n^2) are two arcs of \mathcal{M} . But $i_1^1 < i_q^1$ and $i_p^2 < i_n^2$, and hence one of the configurations of Figure 5 (with $j = 1$ and $k = n$) does occur in \mathcal{M} . This is a contradiction since \mathcal{M} satisfies Property **Q₂**, and hence (i_1^1, i_n^2) is an arc of \mathcal{M} .
- Induction step. Assume that (i_k^1, i_{n-k+1}^2) is an arc of \mathcal{M} for $1 \leq k < j$, and suppose, aiming at a contradiction, that (i_j^1, i_{n-j+1}^2) is not an arc of \mathcal{M} .

Then, there exist $1 \leq p < n - j + 1$ and $j < q \leq n$ such that (i_j^1, i_p^2) and (i_q^1, i_{n-j+1}^2) are two arcs of \mathcal{M} . But $i_j^1 < i_q^1$ and $i_p^2 < i_{n-j+1}^2$, and hence one of the configurations of Figure 5 does occur in \mathcal{M} . This is a contradiction since \mathcal{M} satisfies Property **Q₂**, and hence (i_j^1, i_{n-j+1}^2) is an arc of \mathcal{M} .

Now, let σ_1 be the pattern of π induced by the sources of \mathcal{M} and σ_2 be the pattern of π induced by the sinks of \mathcal{M} (i.e., $\sigma_1 = \pi(i_1^1) \pi(i_2^1) \dots \pi(i_n^1)$ and $\sigma_2 = \pi(j_1^2) \pi(j_2^2) \dots \pi(i_n^2)$). Clearly σ_1 and σ_2 are disjoint in π (since \mathcal{M} is a matching) and cover π (since \mathcal{M} is perfect). Moreover, the fact that \mathcal{M} satisfies **P₁** implies immediately that $\text{reduce}(\sigma_1)$ is order-isomorphic to $(\text{reduce} \circ r)(\sigma_2)$, and hence this shows that $\pi \in \mathcal{S}_{2n}^r$. ■

Lemma 7. *Let $\pi \in S_{2n}$ and \mathcal{M} be a perfect directed matching on π that avoids the patterns , , and . Then, for any arc $(i, j) \in \mathcal{M}$, either $1 \leq i \leq n < j \leq 2n$ or $1 \leq j \leq n < i \leq 2n$.*

Proof. We only prove the case $1 \leq i \leq n < j \leq 2n$ since the proof for $1 \leq j \leq n < i \leq 2n$ is similar.

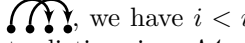
Suppose, aiming at a contradiction, that there exists an arc $(i, j) \in \mathcal{M}$ with $1 \leq i < j \leq n$. Since \mathcal{M} avoids the patterns , , and , there is no arc $(k, \ell) \in \mathcal{M}$ with $j < k \leq 2n$ and $j < \ell \leq 2n$, $k \neq \ell$. Then it follows that there exist $2n - j \geq n$ distinct arcs $(k, \ell) \in \mathcal{M}$ with $1 \leq \min\{k, \ell\} \leq j - 1$ and $j + 1 \leq \max\{k, \ell\} \leq 2n$, $k \neq \ell$. This is a contradiction since $(i, j) \in \mathcal{M}$ with $1 \leq i < j \leq n$. ■

The following technical lemma is needed to simplify the proof of upcoming Proposition 7.

Lemma 8. *Let $k, \ell_1, \ell_2, \ell_3$ and ℓ_4 be positive integers and $\pi \in \mathcal{S}_{2k+\ell_1+\ell_2+\ell_3+\ell_4}^r$ be the permutation defined by $\pi = X_1 L X_2 X_3 R X_4$, where $X_1 = \mu_1[k + \ell_2]$, $L = \nearrow_k[\ell_2]$, $X_2 = \mu_2$, $X_3 = \mu_3[k + \ell_1 + \ell_2]$, $R = \searrow_k[k + \ell_1 + \ell_2 + \ell_3]$ and $X_4 = \mu_4[2k + \ell_1 + \ell_2 + \ell_3]$ for some permutations $\mu_i \in S_{\ell_i}$, $1 \leq i \leq 4$ (see Figure 6). Let \mathcal{M} be a directed perfect matching on π that satisfies properties **P₁** and **Q₂**. If $\mathcal{M}_{L \rightarrow R} \neq \emptyset$ or $\mathcal{M}_{R \rightarrow L} \neq \emptyset$, then either $|\mathcal{M}_{L \rightarrow R}| = k$ or $|\mathcal{M}_{R \rightarrow L}| = k$.*

Proof. Suppose first, aiming at a contradiction, that $\mathcal{M}_{L \rightarrow R} \neq \emptyset$ and $\mathcal{M}_{R \rightarrow L} \neq \emptyset$. Let $(i, j) \in \mathcal{M}_{L \rightarrow R}$ and $(i', j') \in \mathcal{M}_{R \rightarrow L}$. By construction, we have $\pi(i) < \pi(i')$ and $\pi(j) > \pi(j')$ thereby contradicting Property **P₁**. Then it follows that either $\mathcal{M}_{L \rightarrow R} \neq \emptyset$ or $\mathcal{M}_{R \rightarrow L} \neq \emptyset$, but not both.

Suppose now that $\mathcal{M}_{L \rightarrow R} \neq \emptyset$ and $\mathcal{M}_{R \rightarrow L} = \emptyset$ (the case $\mathcal{M}_{L \rightarrow R} = \emptyset$ and $\mathcal{M}_{R \rightarrow L} \neq \emptyset$ can be proved with the same arguments). Suppose, aiming at a contradiction, that $|\mathcal{M}_{L \rightarrow R}| \neq k$ and let $(i, j) \in \mathcal{M}_{L \rightarrow R}$. According to Lemma 7, we are left with considering the four following cases.

- There exists $(i', j') \in \mathcal{M}_{L \rightarrow \mu_3}$. Since \mathcal{M} avoids , we have $i < i'$, and hence $\pi(i) < \pi(i')$ and $\pi(j) > \pi(j')$. This is a contradiction since \mathcal{M} satisfies Property **P₁**.

- There exists $(i', j') \in \mathcal{M}_{\mu_3 \rightarrow L}$ and hence $\pi(i) < \pi(i')$ and $\pi(j) > \pi(j')$. This is a contradiction since \mathcal{M} satisfies Property \mathbf{P}_1 .
- There exists $(i', j') \in \mathcal{M}_{L \rightarrow \mu_4}$. Since \mathcal{M} avoids $\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array}$, we have $i' < i$, and hence $\pi(i) > \pi(i')$ and $\pi(j) < \pi(j')$. This is a contradiction since \mathcal{M} satisfies Property \mathbf{P}_1 .
- There exists $(i', j') \in \mathcal{M}_{\mu_4 \rightarrow L}$ and hence $\pi(i) < \pi(i')$ and $\pi(j) > \pi(j')$. This is a contradiction since \mathcal{M} satisfies Property \mathbf{P}_1 .

Therefore, $|\mathcal{M}_{L \rightarrow R}| = k$. ■

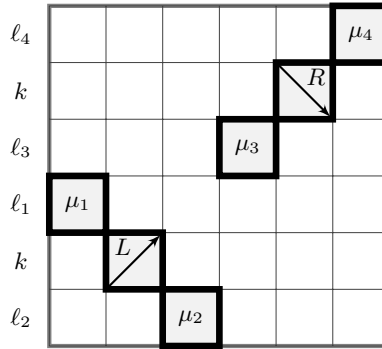


Fig. 6. Illustration of Lemma 8, where $\pi = X_1 \ L \ X_2 \ X_3 \ R \ X_4$.

Proposition 7. r -UNSHUFFLE-PERMUTATION is NP-complete.

Proof. We reduce from PERMUTATION PATTERN which, given two permutations $\pi \in S_n$ and $\sigma \in S_k$ with $k \leq n$, asks whether σ is a pattern of π . PERMUTATION PATTERN is known to be NP-complete [1]. Let $\pi \in S_n$ and $k \in S_k$ be an instance of PERMUTATION PATTERN. Set $N_2 = n + k + 1$ and $N_1 = 2N_2$. Define a target permutation $\mu \in S_{2n+2k+2N_1+2N_2}$ by $\mu = \mu_\pi \ \mu_2 \ \mu_1 \ \mu_\sigma \ \mu'_2 \ \mu'_\pi \ \mu'_1 \ \mu'_\sigma$, where

$$\begin{array}{ll}
 \mu_\sigma = \sigma, & \mu_\pi = \pi[k + N_1 + N_2], \\
 \mu'_\pi = r(\pi)[2n + k + 2N_1 + 2N_2] & \mu'_\sigma = r(\sigma)[n + k + N_1 + 2N_2], \\
 \mu_1 = \nearrow_{N_1} [k], & \mu'_1 = \searrow_{N_1} [2n + k + N_1 + 2N_2], \\
 \mu_2 = \nearrow_{N_2} [k + N_1], & \mu'_2 = \searrow_{N_2} [n + k + N_1 + N_2].
 \end{array}$$

Clearly, the construction can be carried out in polynomial time. We claim that σ is a pattern of π if and only if $\mu \in \mathcal{S}_{2n+2k+2N_1+2N_2}^r$.

(\Rightarrow) Suppose that σ is a pattern of π . Then there exist indices $P = \{p_1, p_2, \dots, p_k\}$, $1 \leq p_1 < p_2 < \dots < p_k \leq n$, such that σ and $\pi(p_1)\pi(p_2)\dots\pi(p_k)$ are order-isomorphic. For the sake of simplification, let $Q = [n] \setminus P$ and write

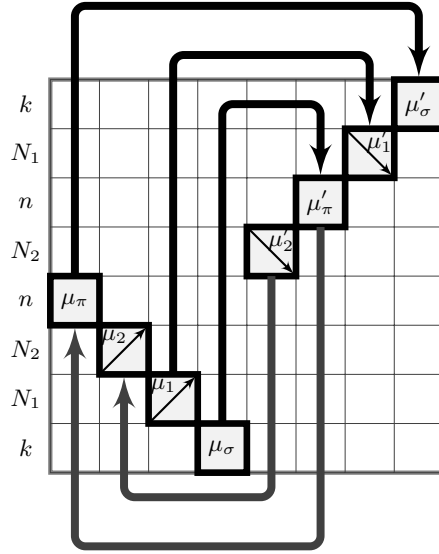


Fig. 7. Schematic representation of the construction used in Proposition 7.

$Q = \{q_1, q_2, \dots, q_{n-k}\}$, $1 \leq q_1 < q_2 < \dots < q_{n-k} \leq n$. Define a directed matching

$$\mathcal{M} = \mathcal{M}_{\mu_\sigma \rightarrow \mu'_\pi} \uplus \mathcal{M}_{\mu'_\pi \rightarrow \mu_\pi} \uplus \mathcal{M}_{\mu_\pi \rightarrow \mu'_\sigma} \uplus \mathcal{M}_{\mu_1 \rightarrow \mu'_1} \uplus \mathcal{M}_{\mu'_2 \rightarrow \mu_2}$$

on μ as follows.

$$\begin{aligned} \mathcal{M}_{\mu_\sigma \rightarrow \mu'_\pi} &= \{(n + N_1 + N_2 + i, n + k + N_1 + 2N_2 + p_i) : 1 \leq i \leq k\} \\ \mathcal{M}_{\mu'_\pi \rightarrow \mu_\pi} &= \{(2n + k + N_1 + 2N_2 - q_i + 1, q_i) : 1 \leq i \leq n - k\} \\ \mathcal{M}_{\mu_\pi \rightarrow \mu'_\sigma} &= \{(p_i, 2n + 2k + 2N_1 + 2N_2 - i) : 1 \leq i \leq k\} \\ \mathcal{M}_{\mu_1 \rightarrow \mu'_1} &= \{(n + N_2 + i, 2n + k + 2N_1 + 2N_2 - i + 1) : 1 \leq i \leq N_1\} \\ \mathcal{M}_{\mu'_2 \rightarrow \mu_2} &= \{(n + k + N_1 + 2N_2 - i + 1, n + i) : 1 \leq i \leq N_2\}. \end{aligned}$$

Informally,

- $\mathcal{M}_{\mu_\sigma \rightarrow \mu'_\pi}$ is the directed (left-to-right) matching describing the (reversed) occurrence of σ in $r(\pi)$,
- $\mathcal{M}_{\mu'_\pi \rightarrow \mu_\pi}$ is the directed (right-to-left) matching connecting the elements of $r(\pi)$ and π that are not part of the occurrence of σ in π ,
- $\mathcal{M}_{\mu_\pi \rightarrow \mu'_\sigma}$ is the directed (left-to-right) matching describing the (reversed) occurrence of $r(\sigma)$ in π ,
- $\mathcal{M}_{\mu_1 \rightarrow \mu'_1}$ is a directed (left-to-right) matching that fully connects μ_1 to μ'_1 and
- $\mathcal{M}_{\mu'_2 \rightarrow \mu_2}$ is a directed (right-to-left) matching that fully connects μ'_2 to μ_2 .


It is now a simple matter to check that \mathcal{M} is a directed matching on μ that satisfies properties \mathbf{P}_1 and \mathbf{Q}_2 . Therefore, according to Proposition 6, $\mu \in \mathcal{S}_{2n+2k+2N_1+2N_2}^r$.


(\Leftarrow) Suppose that $\mu \in \mathcal{S}_{2n}^r$. Therefore, according to Proposition 6, there exists a directed perfect matching \mathcal{M} on μ that satisfies properties \mathbf{P}_1 and \mathbf{Q}_2 . We begin with a sequence of claims that help defining the general structure of \mathcal{M} .

Claim. We may assume that $|\mathcal{M}_{\mu_1 \rightarrow \mu'_1}| = N_1$.

Proof. Combining Lemma 7 and $N_1 = 2N_2 > n + k + N_2$, we conclude that $\mathcal{M}_{\mu_1 \rightarrow \mu'_1} \neq \emptyset$ or $\mathcal{M}_{\mu'_1 \rightarrow \mu_1} \neq \emptyset$. Thus, applying Lemma 8, we obtain $|\mathcal{M}_{\mu_1 \rightarrow \mu'_1}| = N_1$ or $|\mathcal{M}_{\mu'_1 \rightarrow \mu_1}| = N_1$. Therefore, by Lemma 4 (One shot lemma), we may assume that $\mathcal{M}_{\mu_1 \rightarrow \mu'_1} \neq \emptyset$. ■

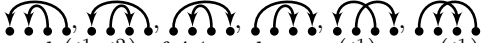
Claim. Assuming $|\mathcal{M}_{\mu_1 \rightarrow \mu'_1}| = N_1$, we have $|\mathcal{M}_{\mu'_2 \rightarrow \mu_2}| = N_2$.

Proof. Combining Claim 4, Lemma 7 and $N_2 > n + k$, we conclude that $\mathcal{M}_{\mu_2 \rightarrow \mu'_2} \neq \emptyset$ or $\mathcal{M}_{\mu'_2 \rightarrow \mu_2} \neq \emptyset$. Applying Lemma 8, we obtain $|\mathcal{M}_{\mu_2 \rightarrow \mu'_2}| = N_2$ or $|\mathcal{M}_{\mu'_2 \rightarrow \mu_2}| = N_2$. The claim now follows from $\mathcal{M}_{\mu_1 \rightarrow \mu'_1} \neq \emptyset$ (Claim 4) and the fact that \mathcal{M} avoids  (Property \mathbf{Q}_2). ■

From the above two claims and the fact that \mathcal{M} avoids  (Property \mathbf{Q}_2), we conclude that $|\mathcal{M}_{\mu_\sigma \rightarrow \mu'_\pi}| = k$. But \mathcal{M} satisfies Property \mathbf{P}_1 and hence there exists an occurrence of $r(\sigma)$ in $r(\pi)$. It follows that σ is a pattern of π . ■

Proposition 8. *Let $\pi \in S_{2n}$. The two following statements are equivalent.*

1. $\pi \in \mathcal{S}_{2n}^c$.
2. *There exists a directed perfect matching \mathcal{M} that satisfies properties \mathbf{P}_2 and \mathbf{Q}_1 .*

Proof. (**1** \Rightarrow **2**) Let $\pi \in \mathcal{S}_{2n}^c$. Then, there exists $\sigma \in S_n$ such that $\pi \in \sigma \sqcup c(\sigma)$, and hence there exist two disjoint set of indices $I^1 = \{i_j^1 : j \in [n]\}$, $i_1^1 < i_2^1 < \dots < i_n^1$, and $I^2 = \{i_j^2 : j \in [n]\}$, $i_1^2 < i_2^2 < \dots < i_n^2$, such that $\pi(i_1^1)\pi(i_2^1)\dots\pi(i_n^1)$ is order-isomorphic to σ and $\pi(i_1^2)\pi(i_2^2)\dots\pi(i_n^2)$ is order-isomorphic to $c(\sigma)$. Let $\mathcal{M} = (V, E)$ be the directed graph defined by $V = [2n]$ and $E = \{(i_j^1, i_j^2) : j \in [n]\}$. Clearly, \mathcal{M} is a directed perfect matching since $I^1 \cap I^2 = \emptyset$ and $I^1 \cup I^2 = [2n]$. According to Lemma 5, the directed perfect matching \mathcal{M} does avoid the patterns . Finally, for any two distinct arcs (i_j^1, i_j^2) and (i_k^1, i_k^2) of \mathcal{M} , we have $\pi(i_j^1) < \pi(i_k^1)$ if and only if $\pi(i_j^2) > \pi(i_k^2)$ since $\pi(i_1^1)\pi(i_2^1)\dots\pi(i_n^1)$ is order-isomorphic to σ and $\pi(i_1^2)\pi(i_2^2)\dots\pi(i_n^2)$ is order-isomorphic to $c(\sigma)$ (and hence $\pi(i_j^2) = n - \pi(i_j^1)$ and $\pi(i_k^2) = n - \pi(i_k^1)$). Therefore, \mathcal{M} satisfies properties \mathbf{P}_2 and \mathbf{Q}_1 .

(**2** \Rightarrow **1**) Let \mathcal{M} be a directed perfect matching that satisfies properties \mathbf{P}_2 and \mathbf{Q}_1 . Let $I^1 = \{i_j^1 : j \in [n]\}$, $i_1^1 < i_2^1 < \dots < i_n^1$, be the set the sources of the arcs of \mathcal{M} and let $I^2 = \{i_j^2 : j \in [n]\}$, $i_1^2 < i_2^2 < \dots < i_n^2$, be the set of the sinks of the arcs of \mathcal{M} . We first show that, for every $j \in [n]$, (i_j^1, i_j^2) is an arc of \mathcal{M} . The proof is by induction on $j = 1, 2, \dots, n$.

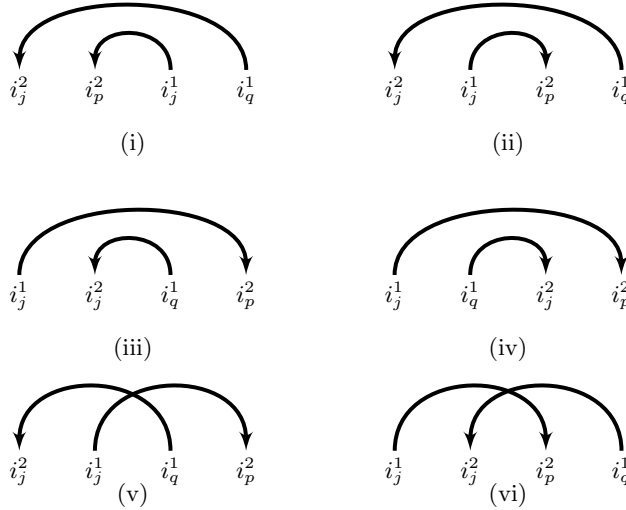


Fig. 8. All possible configurations considered in Proof of Proposition 8 ($\mathbf{2} \Rightarrow \mathbf{1}$).

- Base. Suppose, aiming at a contradiction that (i_1^1, i_1^2) is not an arc of \mathcal{M} . Then, there exist $1 < p \leq n$ and $1 < q \leq n$ such that (i_1^1, i_p^2) and (i_q^1, i_1^2) are two arcs of \mathcal{M} . But $i_1^1 < i_q^1$ and $i_1^2 < i_p^2$, and hence one of the configurations of Figure 8 (with $j = 1$) does occur in \mathcal{M} . This is a contradiction since \mathcal{M} satisfies Property \mathbf{Q}_1 , and hence (i_1^1, i_1^2) is an arc of \mathcal{M} .
- Induction step. Assume that (i_k^1, i_{n-k+1}^2) is an arc of \mathcal{M} for $1 \leq k < j$, and suppose, aiming at a contradiction, that (i_j^1, i_j^2) is not an arc of \mathcal{M} . Then, there exist $j < p \leq n$ and $j < q \leq n$ such that (i_j^1, i_p^2) and (i_q^1, i_{n-j+1}^2) are two arcs of \mathcal{M} . But $i_j^1 < i_q^1$ and $i_j^2 < i_p^2$, and hence one of the configurations of Figure 8 does occur in \mathcal{M} . This is a contradiction since \mathcal{M} satisfies Property \mathbf{Q}_1 , and hence (i_j^1, i_j^2) is an arc of \mathcal{M} .

Now, let σ_1 be the pattern of π induced by the sources of \mathcal{M} and σ_2 be the pattern of π induced by the sinks of \mathcal{M} (i.e., $\sigma_1 = \pi(i_1^1) \pi(i_2^1) \dots \pi(i_n^1)$ and $\sigma_2 = \pi(j_1^2) \pi(j_2^2) \dots \pi(i_n^2)$). Clearly σ_1 and σ_2 are disjoint in π (since \mathcal{M} is a matching) and cover π (since \mathcal{M} is perfect). Moreover, the fact that \mathcal{M} satisfies \mathbf{P}_2 implies immediately that $\text{reduce}(\sigma_1)$ is order-isomorphic to $(\text{reduce} \circ c)(\sigma_2)$, and hence this shows that $\pi \in \mathcal{S}_{2n}^c$. \blacksquare

Corollary 1. c -UNSHUFFLE-PERMUTATION is NP-complete.

Proof. Combine Proposition 7 with Proposition 1. \blacksquare

Proposition 9. Let $\pi \in S_{2n}$. The two following statements are equivalent.

1. $\pi \in \mathcal{S}_{2n}^{\text{rc}}$.

2. There exists a directed perfect matching \mathcal{M} that satisfies properties **P₂** and **Q₂**.

Proof. (**1** \Rightarrow **2**) Let $\pi \in S_{2n}^{\text{rc}}$. Then, there exists $\sigma \in S_n$ such that $\pi \in \sigma \sqcup (\text{r} \circ \text{c})(\sigma)$, and hence there exist two disjoint set of indices $I^1 = \{i_j^1 : j \in [n]\}$, $i_1^1 < i_2^1 < \dots < i_n^1$, and $I^2 = \{i_j^2 : j \in [n]\}$, $i_1^2 < i_2^2 < \dots < i_n^2$, such that $\pi(i_1^1)\pi(i_2^1)\dots\pi(i_n^1)$ is order-isomorphic to σ and $\pi(i_1^2)\pi(i_2^2)\dots\pi(i_n^2)$ is order-isomorphic to $(\text{r} \circ \text{c})(\sigma)$. Let $\mathcal{M} = (V, E)$ be the directed graph defined by $V = [2n]$ and $E = \{(i_j^1, i_{n-j+1}^2) : j \in [n]\}$. Clearly, \mathcal{M} is a directed perfect matching since $I^1 \cap I^2 = \emptyset$ and $I^1 \cup I^2 = [2n]$. According to Lemma 6, the directed perfect matching \mathcal{M} does avoid the patterns and . Finally, for any two distinct arcs (i_j^1, i_{n-j+1}^2) and (i_k^1, i_{n-k+1}^2) of \mathcal{M} , we have $\pi(i_j^1) < \pi(i_k^1)$ if and only if $\pi(i_{n-j+1}^2) > \pi(i_{n-k+1}^2)$ since $\pi(i_1^1)\pi(i_2^1)\dots\pi(i_n^1)$ is order-isomorphic to σ and $\pi(i_1^2)\pi(i_2^2)\dots\pi(i_n^2)$ is order-isomorphic to $(\text{r} \circ \text{c})(\sigma)$. Therefore, \mathcal{M} satisfies properties **P₁** and **Q₂**.

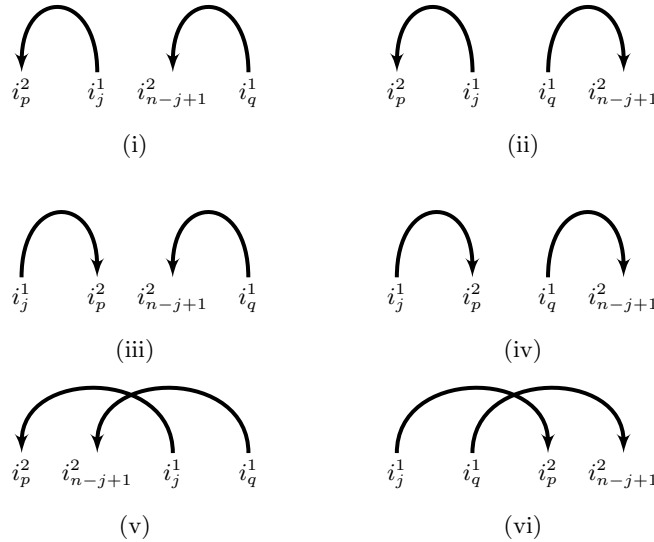


Fig. 9. All possible configurations considered in Proof of Proposition 9 (**2** \Rightarrow **1**).

(**2** \Rightarrow **1**) Let \mathcal{M} be a directed perfect matching that satisfies properties **P₁** and **Q₂**. Let $I^1 = \{i_j^1 : j \in [n]\}$, $i_1^1 < i_2^1 < \dots < i_n^1$, be the set the sources of the arcs of \mathcal{M} and let $I^2 = \{i_j^2 : j \in [n]\}$, $i_1^2 < i_2^2 < \dots < i_n^2$, be the set of the sinks of the arcs of \mathcal{M} . We first show that, for every $j \in [n]$, (i_j^1, i_{n-j+1}^2) is an arc of \mathcal{M} . The proof is by induction on $j = 1, 2, \dots, n$.

- Base. Suppose, aiming at a contradiction that (i_1^1, i_n^2) is not an arc of \mathcal{M} . Then, there exist $1 \leq p < n$ and $1 < q \leq n$ such that (i_1^1, i_p^2) and (i_q^1, i_n^2) are two arcs of \mathcal{M} . But $i_1^1 < i_q^1$ and $i_p^2 < i_n^2$, and hence one of the configurations of Figure 9 (with $j = 1$ and $k = n$) does occur in \mathcal{M} . This is a contradiction since \mathcal{M} satisfies Property **Q₂**, and hence (i_1^1, i_n^2) is an arc of \mathcal{M} .
- Induction step. Assume that (i_k^1, i_{n-k+1}^2) is an arc of \mathcal{M} for $1 \leq k < j$, and suppose, aiming at a contradiction, that (i_j^1, i_{n-j+1}^2) is not an arc of \mathcal{M} . Then, there exist $1 \leq p < n - j + 1$ and $j < q \leq n$ such that (i_j^1, i_p^2) and (i_q^1, i_{n-j+1}^2) are two arcs of \mathcal{M} . But $i_j^1 < i_q^1$ and $i_p^2 < i_{n-j+1}^2$, and hence one of the configurations of Figure 9 does occur in \mathcal{M} . This is a contradiction since \mathcal{M} satisfies Property **Q₂**, and hence (i_j^1, i_{n-j+1}^2) is an arc of \mathcal{M} .

Now, let σ_1 be the pattern of π induced by the sources of \mathcal{M} and σ_2 be the pattern of π induced by the sinks of \mathcal{M} (i.e., $\sigma_1 = \pi(i_1^1) \pi(i_2^1) \dots \pi(i_n^1)$ and $\sigma_2 = \pi(j_1^2) \pi(j_2^2) \dots \pi(i_n^2)$). Clearly σ_1 and σ_2 are disjoint in π (since \mathcal{M} is a matching) and cover π (since \mathcal{M} is perfect). Moreover, the fact that \mathcal{M} satisfies **P₂** implies immediately that $\text{reduce}(\sigma_1)$ is order-isomorphic to $(\text{reduce} \circ \text{r} \circ \text{c})(\sigma_2)$, and hence this shows that $\pi \in \mathcal{S}_{2n}^{\text{rc}}$. ■

Proposition 10. (r o c)-UNSHUFFLE-PERMUTATION is NP-complete.

Proof (sketch). The proof is similar to Proposition 7, replacing μ'_1 by $\nearrow_{N_1} [2n+k+N_1+2N_2]$, μ'_2 by $\nearrow_{N_2} [n+k+N_1+N_2]$, μ'_π by $(\text{r o c})(\pi)[2n+k+2N_1+2N_2]$ and μ'_σ by $(\text{r o c})(\sigma)[n+k+N_1+2N_2]$. ■

5 Conclusion and perspectives

In this paper we have proposed to investigate the problem of recognizing those permutations π for which there exists a permutation σ such that $\pi \in \sigma \sqcup f(\sigma)$ for some bijection $f : S_n \rightarrow S_n$. Quite a number of problems are left open by the results presented here. For instance, can we efficiently decide membership to \mathcal{S}_{2n}^i (i.e., the permutations π for which there exists a permutation σ such that $\pi \in \sigma \sqcup i(\sigma)$)? To conclude, we wish to mention two conjectures.

Conjecture 1 (Enumeration). For any $n \in \mathbb{N}$, $s_{2n}^{\text{rc}} \leq s_{2n}^{\text{c}} = s_{2n}^{\text{r}} \leq s_{2n}^{\text{irc}} = s_{2n}^{\text{i}} \leq s_{2n}^{\text{e}} \leq s_{2n}^{\text{ci}} = s_{2n}^{\text{ic}}$.

Conjecture 2 (Ubiquitous generalized squares). For any $n \in \mathbb{N}$, $\mathcal{S}_{2n}^{\text{e}} \cap \mathcal{S}_{2n}^{\text{r}} \cap \mathcal{S}_{2n}^{\text{c}} \cap \mathcal{S}_{2n}^{\text{i}} \cap \mathcal{S}_{2n}^{\text{ci}} \cap \mathcal{S}_{2n}^{\text{ic}} \cap \mathcal{S}_{2n}^{\text{rc}} \cap \mathcal{S}_{2n}^{\text{irc}} \neq \emptyset$.

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