



**HAL**  
open science

# Higher-order Hamiltonians for the trigonometric Gaudin model

Alexander Molev, Eric Ragoucy

► **To cite this version:**

Alexander Molev, Eric Ragoucy. Higher-order Hamiltonians for the trigonometric Gaudin model. *Lett.Math.Phys.*, 2019, 109 (9), pp.2035-2048. 10.1007/s11005-019-01170-2. hal-02302975

**HAL Id: hal-02302975**

**<https://hal.science/hal-02302975v1>**

Submitted on 6 Jan 2025

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Higher order Hamiltonians for the trigonometric Gaudin model

Alexander Molev and Eric Ragoucy

## Abstract

We consider the trigonometric classical  $r$ -matrix for  $\mathfrak{gl}_N$  and the associated quantum Gaudin model. We produce higher Hamiltonians in an explicit form by applying the limit  $q \rightarrow 1$  to elements of the Bethe subalgebra for the  $XXZ$  model.

Preprint LAPTH-004/18

School of Mathematics and Statistics  
University of Sydney, NSW 2006, Australia  
alexander.molev@sydney.edu.au

Laboratoire de Physique Théorique LAPTh, CNRS and Université de Savoie  
BP 110, 74941 Annecy-le-Vieux Cedex, France  
eric.ragoucy@lapth.cnrs.fr

## 1 Introduction

Explicit higher Hamiltonians for the rational Gaudin model associated with  $\mathfrak{gl}_N$  were produced by Talalaev [19] by making use of the Bethe subalgebra of the Yangian for  $\mathfrak{gl}_N$  and taking a classical limit; see also [13]. Some related families of higher Hamiltonians and their analogues for the orthogonal and symplectic Lie algebras were produced by using the *center at the critical level* following the general approach of Feigin, Frenkel and Reshetikhin [7]; see [3], [5] and also [14] for more details and references. In particular, such a family arises from the coefficients of the differential operators  $\text{tr}(\partial_u + E(u))^k$  with  $k = 1, 2, \dots$  which form a commutative subalgebra of  $U(t^{-1}\mathfrak{gl}_N[t^{-1}])$ . Here  $E(u) = [E_{ij}(u)]$  is the matrix with the entries

$$E_{ij}(u) = \sum_{n=0}^{\infty} E_{ij}[-n-1]u^n, \quad E_{ij}[-n-1] = E_{ij}t^{-n-1},$$

where we use the standard basis elements  $E_{ij}$  of  $\mathfrak{gl}_N$ . The commuting family of the coefficients of the power series  $\text{tr } E(u)^2$  can be regarded as a generating series of the quadratic Gaudin Hamiltonians as considered by Sklyanin [17].

Both rational and trigonometric Gaudin models were studied by Jurčo [11]. They are associated with the corresponding *classical  $r$ -matrix*  $r(x)$ . In the trigonometric case  $t^{-1}\mathfrak{gl}_N[t^{-1}]$  is replaced by the extended Lie algebra  $\widehat{\mathfrak{g}}^+ = \mathfrak{b}^+ \oplus t^{-1}\mathfrak{gl}_N[t^{-1}]$ , where  $\mathfrak{b}^+$  is the subalgebra of  $\mathfrak{gl}_N$  spanned by the elements  $E_{ij}$  with  $i \leq j$ . Accordingly,  $E(u)$  is replaced by the matrix  $\mathcal{L}^+(u) = [\mathcal{L}_{ij}^+(u)]$  with

$$\mathcal{L}_{ij}^+(u) = \sum_{n=0}^{\infty} \mathcal{L}_{ij}^+[-n] u^n, \quad (1.1)$$

where  $\mathcal{L}_{ij}^+[-n] = -2E_{ij}t^{-n}$  for  $n \geq 1$  and  $\mathcal{L}_{ij}^+[0] = -(1 + \text{sgn}(j-i))E_{ij}$  assuming  $\text{sgn}(0) = 0$ . It is shown in [11] that the coefficients of the series  $\text{tr } \mathcal{L}^+(u)^2$  are pairwise commuting elements of  $U(\widehat{\mathfrak{g}}^+)$ . Moreover, as with the rational Gaudin model, this series plays the role of the generating function for quadratic Hamiltonians. Namely, taking the image in the tensor product of the vector representations, we get

$$\mathcal{L}^+(u) \mapsto r_{01}(u/a_1) + \cdots + r_{0l}(u/a_l) \quad (1.2)$$

for some parameters  $a_i$ , where

$$r(x) = \sum_{i,j=1}^N \left( \frac{1+x}{1-x} + \text{sgn}(j-i) \right) e_{ij} \otimes e_{ji} \quad (1.3)$$

is a *trigonometric classical  $r$ -matrix*. It satisfies the *classical Yang–Baxter equation*

$$[r_{12}(x_1/x_2), r_{23}(x_2/x_3)] + [r_{23}(x_2/x_3), r_{31}(x_3/x_1)] + [r_{31}(x_3/x_1), r_{12}(x_1/x_2)] = 0$$

together with the *skew-symmetry condition*

$$r_{12}(x) + r_{21}(1/x) = 0.$$

Taking the residue at  $a_i$ , we recover the  $i$ -th Gaudin Hamiltonian

$$\text{res}_{u=a_i} \text{tr } \mathcal{L}^+(u)^2 = 2a_i \sum_{j \neq i} r_{ij}(a_i/a_j), \quad (1.4)$$

assuming the parameters  $a_i$  are all distinct and nonzero.

Our main result is a construction of higher order Hamiltonians for the trigonometric Gaudin model. They are obtained from a commuting family of elements of  $U(\widehat{\mathfrak{g}}^+)$  which occur as the coefficients of formal series written explicitly in terms of  $\mathcal{L}^+(u)$ . This commuting family is analogous to the one produced from the differential operators  $\text{tr}(\partial_u + E(u))^k$  because the highest degree term of the corresponding operator coincides with  $\text{tr } \mathcal{L}^+(u)^k$ . By using the representation (1.2) one gets a commuting family of higher order Hamiltonians which are pairwise commuting operators in the tensor product of the vector representations.

We are grateful to Nicolas Crampé for many useful discussions. This work was completed during the first author's visit to the *Laboratoire d'Annecy-le-Vieux de Physique Théorique*. He thanks the lab for the support and hospitality.

## 2 Trigonometric Gaudin model

The commutation relations of the Lie algebra  $\widehat{\mathfrak{g}}^+$  admit the matrix form

$$[\mathcal{L}_1^+(u), \mathcal{L}_2^+(v)] = [\mathcal{L}_1^+(u) + \mathcal{L}_2^+(v), r_{12}(u/v)],$$

where both sides take values in  $\text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N \otimes \text{U}(\widehat{\mathfrak{g}}^+)$  and the subscripts indicate the copies of the endomorphism algebra. For all  $s \geq 1$  consider the multiple tensor products

$$\underbrace{\text{End } \mathbb{C}^N \otimes \dots \otimes \text{End } \mathbb{C}^N}_s \otimes \text{U}(\widehat{\mathfrak{g}}^+). \quad (2.1)$$

Introduce the function  $T(y)$  in a variable  $y$  with values in  $\text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N$  by

$$T(y) = \sum_{i=1}^N e_{ii} \otimes e_{ii} + \frac{1}{1-y} \sum_{i < j} e_{ij} \otimes e_{ji} + \frac{1}{1+y} \sum_{i > j} e_{ij} \otimes e_{ji}. \quad (2.2)$$

For any  $1 \leq a < b \leq s$  we let  $T_{ab}(y)$  denote the function  $T(y)$  regarded as an element of (2.1) associated with the  $a$ -th and  $b$ -th copies of  $\text{End } \mathbb{C}^N$  and as the identity element in all the remaining tensor factors. Now define differential operators  $\theta_m \in \text{U}(\widehat{\mathfrak{g}}^+)[[u, \partial_u]]$  by means of the generating function

$$\sum_{m=1}^{\infty} \theta_m y^m = \sum_{s=1}^{\infty} y^s \text{tr}_{1, \dots, s} T_{s-1s}(y) \dots T_{12}(y) \mathcal{L}_1 \dots \mathcal{L}_s, \quad (2.3)$$

where  $\mathcal{L} = 2u\partial_u - \mathcal{L}^+(u)$  and the trace is taken over all  $s$  copies of  $\text{End } \mathbb{C}^N$ . We can write  $T(y)$  as the series

$$T(y) = P + \sum_{r=1}^{\infty} (\overline{\mathcal{T}} y^{2r} + \mathcal{T} y^{2r+1}), \quad (2.4)$$

where

$$P = \sum_{i,j=1}^N e_{ij} \otimes e_{ji}, \quad \mathcal{T} = \sum_{i,j=1}^N \text{sgn}(j-i) e_{ij} \otimes e_{ji} \quad \text{and} \quad \overline{\mathcal{T}} = \sum_{i \neq j} e_{ij} \otimes e_{ji}. \quad (2.5)$$

Note that  $\mathcal{T} = r(-1)$  is the value of the classical  $r$ -matrix (1.3) at  $x = -1$ . Taking the coefficient of  $y$  in (2.3) we get

$$\theta_1 = \text{tr } \mathcal{L} = 2Nu\partial_u - \text{tr } \mathcal{L}^+(u).$$

The coefficients of the series  $\text{tr } \mathcal{L}^+(u)$  are central in  $\text{U}(\widehat{\mathfrak{g}}^+)$ . Furthermore,

$$\begin{aligned} \theta_2 &= \text{tr}_{1,2} P_{12} \mathcal{L}_1 \mathcal{L}_2 = \text{tr } \mathcal{L}^2 = \text{tr} (2u\partial_u - \mathcal{L}^+(u))^2 \\ &= 4Nu^2 \partial_u^2 - 4u(\text{tr } \mathcal{L}^+(u) - N)\partial_u - 2u \text{tr } \mathcal{L}^+(u)' + \text{tr } \mathcal{L}^+(u)^2 \end{aligned}$$

and

$$\begin{aligned}\theta_3 &= \text{tr}_{1,2,3} P_{23} P_{12} \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3 + \text{tr}_{1,2} \mathcal{T}_{12} \mathcal{L}_1 \mathcal{L}_2 = \text{tr} \mathcal{L}^3 + \text{tr}_{1,2} \mathcal{T}_{12} \mathcal{L}_1 \mathcal{L}_2 \\ &= \text{tr} (2u \partial_u - \mathcal{L}^+(u))^3 + \sum_{i,j=1}^N \text{sgn}(i-j) \mathcal{L}_{ij}^+(u) \mathcal{L}_{ji}^+(u).\end{aligned}$$

For any  $m \geq 1$  the differential operator  $\theta_m$  takes the form

$$\theta_m = \theta_m^{(0)} \partial_u^m + \dots + \theta_m^{(m-1)} \partial_u + \theta_m^{(m)},$$

where each  $\theta_m^{(k)}$  is a power series in  $u$  with coefficients in the algebra  $\mathbb{U}(\widehat{\mathfrak{g}}^+)$ . In particular,

$$\theta_m^{(m)} = (-1)^m \text{tr} \mathcal{L}^+(u)^m + \text{lower degree terms.} \quad (2.6)$$

This follows from the expansion (2.4) and the relation

$$\text{tr}_{1,\dots,s} P_{s-1s} \dots P_{12} \mathcal{L}_1 \dots \mathcal{L}_s = \text{tr} \mathcal{L}^s = \text{tr} (2u \partial_u - \mathcal{L}^+(u))^s.$$

Our main result is the following theorem which we will prove in the next section.

**Theorem 2.1.** *The coefficients of all power series  $\theta_m^{(k)}$  generate a commutative subalgebra of  $\mathbb{U}(\widehat{\mathfrak{g}}^+)$ .*

By Theorem 2.1 the commuting family (2.6) quantizes the well-known Hamiltonians  $\text{tr} L(u)^m$  of the classical trigonometric Gaudin model; see [1], [18]. Note also that the above expressions for  $\theta_m$  with  $m = 1, 2, 3$  show that the commutative subalgebra provided by Theorem 2.1 contains the coefficients of the power series  $\text{tr} \mathcal{L}^+(u)^2$  (see (1.4)), as well as the coefficients of the power series

$$\text{tr} \mathcal{L}^+(u)^3 - 2u \text{tr} \mathcal{L}^+(u) \mathcal{L}^+(u)' + \sum_{i,j=1}^N \text{sgn}(j-i) \mathcal{L}_{ij}^+(u) \mathcal{L}_{ji}^+(u).$$

### 3 Proof of Theorem 2.1

We start by recalling the *Bethe subalgebra* of the quantum affine algebra  $\mathbb{U}_q(\widehat{\mathfrak{gl}}_N)$  over  $\mathbb{C}(q)$  associated with the *XXZ* model; see e.g. [16] for a review. This is a commutative subalgebra which lies within the *q-Yangian*  $Y_q(\mathfrak{gl}_N) \subset \mathbb{U}_q(\widehat{\mathfrak{gl}}_N)$ . The algebra  $Y_q(\mathfrak{gl}_N)$  is generated by elements

$$l_{ij}^+[-r], \quad 1 \leq i, j \leq N, \quad r = 0, 1, \dots,$$

with the conditions that  $l_{ij}^+[0] = 0$  for  $i > j$  and the elements  $l_{ii}^+[0]$  are invertible, subject to the defining relations

$$R(u/v) L_1^+(u) L_2^+(v) = L_2^+(v) L_1^+(u) R(u/v). \quad (3.1)$$

Here we use the matrix  $L^+(u) = [l_{ij}^+(u)]$ , whose entries are formal power series in  $u$ ,

$$l_{ij}^+(u) = \sum_{r=0}^{\infty} l_{ij}^+[-r] u^r$$

and regard it as the element

$$L^+(u) = \sum_{i,j=1}^n e_{ij} \otimes l_{ij}^+(u) \in \text{End } \mathbb{C}^N \otimes Y_q(\mathfrak{gl}_N)[[u]].$$

By a standard notation, subscripts are used to indicate copies of the matrix in the tensor product algebra

$$\text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N \otimes Y_q(\mathfrak{gl}_N)[[u]]$$

so that  $L_2^+(v) = I \otimes L^+(v)$  etc., where  $I$  is the identity matrix. The  $R$ -matrix is given by

$$\begin{aligned} R(x) = & \sum_i e_{ii} \otimes e_{ii} + \frac{1-x}{q-q^{-1}x} \sum_{i \neq j} e_{ii} \otimes e_{jj} \\ & + \frac{(q-q^{-1})x}{q-q^{-1}x} \sum_{i>j} e_{ij} \otimes e_{ji} + \frac{q-q^{-1}}{q-q^{-1}x} \sum_{i<j} e_{ij} \otimes e_{ji}. \end{aligned} \quad (3.2)$$

Consider the  $q$ -permutation  $P^q \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N) \cong \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N$  defined by

$$P^q = \sum_i e_{ii} \otimes e_{ii} + q \sum_{i>j} e_{ij} \otimes e_{ji} + q^{-1} \sum_{i<j} e_{ij} \otimes e_{ji}. \quad (3.3)$$

The symmetric group  $\mathfrak{S}_k$  acts on the tensor product space  $(\mathbb{C}^N)^{\otimes k}$  by  $s_a \mapsto P_{s_a}^q := P_{aa+1}^q$  for  $a = 1, \dots, k-1$ , where  $s_a$  denotes the transposition  $(a, a+1)$ . The operator  $P_{aa+1}^q$  acts as  $P^q$  in the tensor product of the  $a$ -th and  $(a+1)$ -th copies of  $\mathbb{C}^N$  and acts as the identity operator in the remaining copies. If  $\sigma = s_{a_1} \dots s_{a_l}$  is a reduced decomposition of an element  $\sigma \in \mathfrak{S}_k$  then we set  $P_\sigma^q = P_{s_{a_1}}^q \dots P_{s_{a_l}}^q$ . We denote by  $A^{(k)}$  the image of the normalized antisymmetrizer associated with the  $q$ -permutations:

$$A^{(k)} = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \text{sgn } \sigma \cdot P_\sigma^q. \quad (3.4)$$

For each  $k = 1, \dots, N$  consider the power series in  $u$  defined by

$$\text{tr}_{1, \dots, k} A^{(k)} L_1^+(u) \dots L_k^+(u q^{-2k+2}) \quad (3.5)$$

with the trace taken over all  $k$  copies of  $\text{End } \mathbb{C}^N$  in the tensor product algebra

$$\underbrace{\text{End } \mathbb{C}^N \otimes \dots \otimes \text{End } \mathbb{C}^N}_k \otimes Y_q(\mathfrak{gl}_N)[[u]]. \quad (3.6)$$

It is well-known that the coefficients of all power series (3.5) generate a commutative subalgebra  $\mathcal{B}_N$  of  $Y_q(\mathfrak{gl}_N)$ . Another family of generators of this subalgebra can be obtained from the *Newton identities*; see [4, Theorem 6.6]. Adapting to our settings, we find that the coefficients of all power series

$$\mathrm{tr}_{1,\dots,k} P_{(k,k-1,\dots,1)}^q L_1^+(u) \dots L_k^+(uq^{-2k+2}), \quad k = 1, 2, \dots \quad (3.7)$$

belong to  $\mathcal{B}_N$ .

Since the  $q$ -Yangian  $Y_q(\mathfrak{gl}_N)$  is a deformation of the universal enveloping algebra  $U(\widehat{\mathfrak{g}}^+)$ , the classical limit  $q \rightarrow 1$  takes  $\mathcal{B}_N$  to a commutative subalgebra of  $U(\widehat{\mathfrak{g}}^+)$ . To get its generators in an explicit form we will use the power series (3.7) and apply an argument similar to the one used in [12, Theorem 3.8]. We will use both the permutation  $P$  given in (2.5) and the  $q$ -permutation  $P^q$  defined in (3.3). Introduce the operator  $\delta$  which interacts with power series in  $u$  by the rule  $\delta g(u) = g(uq^{-2})\delta$ . Adjoining this element to the algebra  $Y_q(\mathfrak{gl}_N)[[u]]$ , set  $M = L^+(u)\delta$ . For each  $m \geq 1$  consider the expression

$$\mathcal{M}_m = \frac{1}{(q-1)^m} (1 - (M_m)^\rightarrow) \left( P_{m-1\ m} - P_{m-1\ m}^q (M_{m-1})^\rightarrow \right) \dots \left( P_{12} - P_{12}^q (M_1)^\rightarrow \right) 1, \quad (3.8)$$

where the arrow in the superscript indicates that the corresponding term is understood as the operator of right multiplication:

$$\left( P_{a\ a+1} - P_{a\ a+1}^q (M_a)^\rightarrow \right) X := P_{a\ a+1} X - P_{a\ a+1}^q X M_a. \quad (3.9)$$

The operators in (3.8) are meant to be applied consecutively from right to left. By taking the trace over all  $m$  copies of  $\mathrm{End}\ \mathbb{C}^N$  in (3.8), we get a polynomial in  $\delta$ ,

$$\mathrm{tr}_{1,\dots,m} \mathcal{M}_m \in Y_q(\mathfrak{gl}_N)[[u]][\delta],$$

whose coefficients are power series in  $u$ .

**Lemma 3.1.** *All coefficients of the polynomial  $\mathrm{tr}_{1,\dots,m} \mathcal{M}_m$  belong to  $\mathcal{B}_N[[u]]$ .*

*Proof.* Expand the product in (3.8) to get the expression

$$\mathcal{M}_m = \frac{1}{(q-1)^m} \sum_{k=0}^m \sum_{1 \leq a_1 < \dots < a_k \leq m} (-1)^k \Pi_{a_1, \dots, a_k} M_{a_1} \dots M_{a_k},$$

where

$$\Pi_{a_1, \dots, a_k} = P_{(m, m-1, \dots, a_k+1)} P_{a_k\ a_k+1}^q P_{(a_k, \dots, a_{k-1}+1)} P_{a_{k-1}\ a_{k-1}+1}^q \dots P_{(a_2, \dots, a_1+1)} P_{a_1\ a_1+1}^q P_{(a_1, \dots, 1)}.$$

As shown in [12, Theorem 3.8], for the partial trace we have

$$\mathrm{tr}_{\{1, \dots, m\} \setminus \{a_1, \dots, a_k\}} \Pi_{a_1, \dots, a_k} = P_{a_{k-1}\ a_k}^q P_{a_{k-2}\ a_{k-1}}^q \dots P_{a_1\ a_2}^q. \quad (3.10)$$

Hence we obtain

$$\begin{aligned}\mathrm{tr}_{1,\dots,m}\mathcal{M}_m &= \frac{1}{(q-1)^m} \sum_{k=0}^m \sum_{1 \leq a_1 < \dots < a_k \leq m} (-1)^k \mathrm{tr}_{a_1,\dots,a_k} P_{a_{k-1}a_k}^q \dots P_{a_1 a_2}^q M_{a_1} \dots M_{a_k} \\ &= \frac{1}{(q-1)^m} \sum_{k=0}^m (-1)^k \binom{m}{k} \mathrm{tr}_{1,\dots,k} P_{k-1k}^q \dots P_{12}^q M_1 \dots M_k.\end{aligned}$$

Since  $P_{k-1k}^q P_{k-2k-1}^q \dots P_{12}^q = P_{(k,k-1,\dots,1)}^q$  and

$$M_1 \dots M_k = L_1^+(u) \dots L_k^+(uq^{-2k+2}) \delta^k,$$

the claim follows.  $\square$

Lemma 3.1 provides a family of elements of the commutative subalgebra  $\mathcal{B}_N$  of  $Y_q(\mathfrak{gl}_N)$ . As a next step, we will calculate the classical limits  $q \rightarrow 1$  of these elements. They will form a commuting family of elements of the algebra  $U(\widehat{\mathfrak{g}}^+)$ . To this end, we will use expansions into power series in  $q-1$ . Write

$$\delta = 1 - 2(q-1)u\partial_u + \dots$$

We have

$$L^+(u) = 1 + (q-1)\mathcal{L}^+(u) + \dots \quad \text{and} \quad 1 - M = (q-1)\mathcal{L} + \dots$$

with  $\mathcal{L} = 2u\partial_u - \mathcal{L}^+(u)$ . Furthermore,

$$P - P^q = (q-1)\mathcal{T} + \dots$$

with  $\mathcal{T}$  defined in (2.5). For the expression (3.9) we then get

$$\left(P_{a_{a+1}} - P_{a_{a+1}}^q (M_a)^{\rightarrow}\right) X = (q-1) \left(\mathcal{T}_{a_{a+1}} + P_{a_{a+1}}(\mathcal{L}_a)^{\rightarrow}\right) X + \dots$$

Thus we arrive at the next lemma.

**Lemma 3.2.** *The classical limit of the polynomial  $\mathcal{M}_m$  is the differential operator*

$$\overline{\mathcal{M}}_m = (\mathcal{L}_m)^{\rightarrow} \left(\mathcal{T}_{m-1m} + P_{m-1m}(\mathcal{L}_{m-1})^{\rightarrow}\right) \dots \left(\mathcal{T}_{12} + P_{12}(\mathcal{L}_1)^{\rightarrow}\right) 1, \quad (3.11)$$

where we use the arrow notation as in (3.9).  $\square$

The expanded form of (3.11) is given by

$$\overline{\mathcal{M}}_m = \sum_{k=1}^m \sum_{1 \leq a_1 < \dots < a_k = m} \Gamma_{a_1,\dots,a_k} \mathcal{L}_{a_1} \dots \mathcal{L}_{a_k},$$



where

$$\Gamma_{a_1, \dots, a_k} = \mathcal{T}_{(m, m-1, \dots, a_k+1)} P_{a_k a_k+1} \mathcal{T}_{(a_k, \dots, a_{k-1}+1)} P_{a_{k-1} a_{k-1}+1} \cdots \mathcal{T}_{(a_2, \dots, a_1+1)} P_{a_1 a_1+1} \mathcal{T}_{(a_1, \dots, 1)}$$

and we set

$$\mathcal{T}_{(c_k, \dots, c_1)} = \mathcal{T}_{c_{k-1} c_k} \cdots \mathcal{T}_{c_1 c_2}$$

and  $\mathcal{T}_{(a_i, \dots, a_{i-1}+1)} = 1$  if  $a_i = a_{i-1} + 1$ .

**Lemma 3.3.** *For any  $k \geq 3$  we have*

$$\mathrm{tr}_{2, \dots, k-1} \mathcal{T}_{(k, k-1, \dots, 1)} = \begin{cases} \mathcal{T}_{1k} & \text{if } k \text{ is even,} \\ \overline{\mathcal{T}}_{1k} & \text{if } k \text{ is odd.} \end{cases}$$

*Proof.* This follows by a straightforward induction argument. □

As a next step, we use Lemma 3.3 and the identities

$$\mathrm{tr}_1 \mathcal{T}_{12} = \mathrm{tr}_1 \overline{\mathcal{T}}_{12} = 0,$$

to calculate the trace  $\mathrm{tr}_{1, \dots, m} \overline{\mathcal{M}}_m$ . We also have

$$\mathrm{tr}_2 \mathcal{T}_{23} P_{12} = \mathrm{tr}_2 P_{12} \mathcal{T}_{13} = \mathcal{T}_{13}$$

and the same relation holds for  $\mathcal{T}$  replaced by  $\overline{\mathcal{T}}$ . We thus obtain

$$\mathrm{tr}_{1, \dots, m} \overline{\mathcal{M}}_m = \sum_{k=1}^m \sum_{1=a_1 < \dots < a_k=m} \mathrm{tr}_{a_1, \dots, a_k} \mathcal{T}_{[a_{k-1} a_k]} \cdots \mathcal{T}_{[a_1 a_2]} \mathcal{L}_{a_1} \cdots \mathcal{L}_{a_k},$$

where

$$\mathcal{T}_{[ab]} = \begin{cases} \mathcal{T}_{ab} & \text{if } b-a \geq 3 \text{ is odd,} \\ \overline{\mathcal{T}}_{ab} & \text{if } b-a \geq 2 \text{ is even,} \\ P_{ab} & \text{if } b-a = 1. \end{cases}$$

We may thus conclude that the coefficients  $\theta_m$  defined in (2.3) are found by

$$\theta_m = \mathrm{tr}_{1, \dots, m} \overline{\mathcal{M}}_m.$$

The coefficients of these differential operators pairwise commute which completes the proof of Theorem 2.1.

**Shifted commutative subalgebra.** To make a connection with invariants of the vacuum module over the Lie algebra  $\widehat{\mathfrak{g}}^+$ , we also consider a modified version of the commutative subalgebra of  $U(\widehat{\mathfrak{g}}^+)$  provided by Theorem 2.1. It is well-known that the relations (3.1) remain valid after the replacement  $L^+(u) \mapsto L^+(u)D$ , where  $D$  is the diagonal matrix

$$D = \text{diag}[q^{N-1}, q^{N-3}, \dots, q^{-N+1}]. \quad (3.12)$$

Define differential operators  $\vartheta_m \in U(\widehat{\mathfrak{g}}^+)[[u, \partial_u]]$  by means of the generating function

$$\sum_{m=1}^{\infty} \vartheta_m y^m = \sum_{s=1}^{\infty} y^s \text{tr}_{1, \dots, s} T_{s-1s}(y) \dots T_{12}(y) \overline{\mathcal{L}}_1 \dots \overline{\mathcal{L}}_s, \quad (3.13)$$

where  $\overline{\mathcal{L}} = 2u\partial_u - \rho - \mathcal{L}^+(u)$  and  $\rho$  is the diagonal matrix

$$\rho = \text{diag}[N-1, N-3, \dots, -N+1]. \quad (3.14)$$

The differential operator  $\vartheta_m$  takes the form

$$\vartheta_m = \vartheta_m^{(0)} \partial_u^m + \dots + \vartheta_m^{(m-1)} \partial_u + \vartheta_m^{(m)}, \quad (3.15)$$

where each  $\vartheta_m^{(k)}$  is a power series in  $u$  with coefficients in the algebra  $U(\widehat{\mathfrak{g}}^+)$ . Repeating the arguments of this section for the matrix  $M = L^+(u)D\delta$ , we come to the following.

**Corollary 3.4.** *The coefficients of the power series  $\vartheta_m^{(k)}$  generate a commutative subalgebra of  $U(\widehat{\mathfrak{g}}^+)$ .*

*Proof.* The only additional observation is the power series expansion for the new matrix  $M$  given by

$$M = 1 - (q-1)(2u\partial_u - \rho - \mathcal{L}^+(u)) + \dots$$

implied by the expansion  $D = 1 + (q-1)\rho + \dots$  □

## 4 Invariants of the vacuum module

Now we consider the full quantum affine algebra  $U_q(\widehat{\mathfrak{gl}}_N)$  in its *RLL* presentation; see [10], [15]. We will need the normalized *R*-matrix

$$\overline{R}(x) = f(x)R(x), \quad (4.1)$$

where  $R(x)$  is defined in (3.2) and

$$f(x) = 1 + \sum_{k=1}^{\infty} f_k(q)x^k$$

is a formal power series in  $x$  whose coefficients  $f_k(q)$  are rational functions in  $q$  uniquely determined by the relation

$$f(xq^{2N}) = f(x) \frac{(1-xq^2)(1-xq^{2N-2})}{(1-x)(1-xq^{2N})}. \quad (4.2)$$

The *quantum affine algebra*  $U_q(\widehat{\mathfrak{gl}}_N)$  is generated by elements

$$l_{ij}^+[-r], \quad l_{ij}^-[r] \quad \text{with} \quad 1 \leq i, j \leq N, \quad r = 0, 1, \dots,$$

and the invertible central element  $q^c$ , subject to the defining relations

$$l_{ji}^+[0] = l_{ij}^-[0] = 0 \quad \text{for} \quad 1 \leq i < j \leq N, \quad (4.3)$$

$$l_{ii}^+[0] l_{ii}^-[0] = l_{ii}^-[0] l_{ii}^+[0] = 1 \quad \text{for} \quad i = 1, \dots, N, \quad (4.4)$$

and

$$R(u/v)L_1^\pm(u)L_2^\pm(v) = L_2^\pm(v)L_1^\pm(u)R(u/v), \quad (4.5)$$

$$\overline{R}(uq^{-c}/v)L_1^+(u)L_2^-(v) = L_2^-(v)L_1^+(u)\overline{R}(uq^c/v). \quad (4.6)$$

In the last two relations we consider the matrices  $L^\pm(u) = [l_{ij}^\pm(u)]$ , whose entries are formal power series in  $u$  and  $u^{-1}$ ,

$$l_{ij}^+(u) = \sum_{r=0}^{\infty} l_{ij}^+[-r]u^r, \quad l_{ij}^-(u) = \sum_{r=0}^{\infty} l_{ij}^-[r]u^{-r}. \quad (4.7)$$

The  $q$ -Yangian  $Y_q(\mathfrak{gl}_N)$  can be identified with the subalgebra of  $U_q(\widehat{\mathfrak{gl}}_N)$  generated by the coefficients of the series  $l_{ij}^+(u)$  with  $1 \leq i, j \leq N$ .

The *vacuum module at the critical level*  $c = -N$  over  $U_q(\widehat{\mathfrak{gl}}_N)$  is the universal module  $V_q(\mathfrak{gl}_N)$  generated by a nonzero vector  $\mathbf{1}$  subject to the conditions

$$L^-(u)\mathbf{1} = I\mathbf{1}, \quad q^c\mathbf{1} = q^{-N}\mathbf{1},$$

where  $I$  denotes the identity matrix. As a vector space,  $V_q(\mathfrak{gl}_N)$  can be identified with the subalgebra  $Y_q(\mathfrak{gl}_N)$  of  $U_q(\widehat{\mathfrak{gl}}_N)$  generated by the coefficients of all series  $l_{ij}^+(u)$  subject to the additional relations  $l_{ii}^+[0] = 1$ . The subspace of invariants of  $V_q(\mathfrak{gl}_N)$  is defined by

$$\mathfrak{z}_q(\widehat{\mathfrak{gl}}_N) = \{v \in V_q(\mathfrak{gl}_N) \mid L^-(u)v = Iv\}.$$

One can regard  $\mathfrak{z}_q(\widehat{\mathfrak{gl}}_N)$  as a subspace of  $Y_q(\mathfrak{gl}_N)$ . This subspace is closed under the multiplication in the quantum affine algebra and it can be identified with a subalgebra of  $Y_q(\mathfrak{gl}_N)$ . By [8, Corollary 3.3], for  $k = 1, \dots, N$  all coefficients of the series

$$\text{tr}_{1, \dots, k} A^{(k)} L_1^+(z) \dots L_k^+(zq^{-2k+2}) D_1 \dots D_k \mathbf{1} \quad (4.8)$$

belong to the algebra of invariants  $\mathfrak{z}_q(\widehat{\mathfrak{gl}}_N)$ . Moreover, the coefficients of all these series pairwise commute. As with the series (3.5), applying the Newton identities of [4, Theorem 6.6], we find that the coefficients of all power series

$$\mathrm{tr}_{1,\dots,k} P_{(k,k-1,\dots,1)}^q L_1^+(u) \dots L_k^+(uq^{-2k+2}) D_1 \dots D_k \mathbf{1}, \quad (4.9)$$

belong to  $\mathfrak{z}_q(\widehat{\mathfrak{gl}}_N)$  for all  $k \geq 1$ .

Under the limit  $q \rightarrow 1$  the algebra  $U_q(\widehat{\mathfrak{gl}}_N)$  turns into the universal enveloping algebra  $U(\widehat{\mathfrak{gl}}_N)$ . To make this statement more precise, consider the presentation of the affine Lie algebra  $\widehat{\mathfrak{gl}}_N$  associated with the classical  $r$ -matrix (1.3). Recall that the affine Kac–Moody algebra  $\widehat{\mathfrak{gl}}_N = \mathfrak{gl}_N[t, t^{-1}] \oplus \mathbb{C}K$  has the commutation relations

$$[E_{ij}[r], E_{kl}[s]] = \delta_{kj} E_{il}[r+s] - \delta_{il} E_{kj}[r+s] + r\delta_{r,-s} K \left( \delta_{kj} \delta_{il} - \frac{\delta_{ij} \delta_{kl}}{N} \right), \quad (4.10)$$

and the element  $K$  is central, where we set  $E_{ij}[r] = E_{ij}t^r$  for all  $r \in \mathbb{Z}$ . In addition to the matrix  $\mathcal{L}^+(u)$  with the entries (1.1) introduce the matrix  $\mathcal{L}^-(u) = [\mathcal{L}_{ij}^-(u)]$  with

$$\mathcal{L}_{ij}^-(u) = \sum_{n=0}^{\infty} \mathcal{L}_{ij}^-[n] u^{-n}, \quad (4.11)$$

where  $\mathcal{L}_{ij}^-[n] = 2E_{ij}[n]$  for  $n \geq 1$  and  $\mathcal{L}_{ij}^-[0] = (1 + \mathrm{sgn}(i-j))E_{ij}[0]$ . The defining relations of the algebra  $U(\widehat{\mathfrak{gl}}_N)$  can be written in the form

$$[\mathcal{L}_1^\pm(u), \mathcal{L}_2^\pm(v)] = [\mathcal{L}_1^\pm(u) + \mathcal{L}_2^\pm(v), r_{12}(u/v)], \quad (4.12)$$

$$[\mathcal{L}_1^+(u), \mathcal{L}_2^-(v)] = [\mathcal{L}_1^+(u) + \mathcal{L}_2^-(v), r_{12}(u/v)] + \frac{4uv}{(u-v)^2} \left( P_{12} - \frac{1}{N} \right) K, \quad (4.13)$$

where  $r(x)$  is defined in (1.3) and we write 1 for the tensor product of the identity matrices  $I \otimes I$ ; cf. [2]. We have the following well-known property.

**Proposition 4.1.** *The defining relations of  $U(\widehat{\mathfrak{gl}}_N)$  are recovered from those of  $U_q(\widehat{\mathfrak{gl}}_N)$  by the expansions into power series in  $q-1$ ,*

$$L^\pm(u) = I + (q-1)\mathcal{L}^\pm(u) + \dots$$

and setting  $c \mapsto K$ .

*Proof.* We will only demonstrate how the relation (4.13) is obtained from (4.6), which should explain the role of the normalized  $R$ -matrix (4.1). Relations (4.12) are verified in the same way with simpler calculations. Expanding into power series in  $q-1$  and identifying  $I \otimes I$  with 1 we get

$$R(x) = 1 + (q-1) \left( r(x) - \frac{1+x}{1-x} 1 \right) + \dots \quad (4.14)$$

and

$$f(x) = 1 + 2(q-1) \frac{(N-1)x}{N(1-x)} + \dots, \quad (4.15)$$

where the second expansion is implied e.g. by the calculations in [12, Sec. 2]. Now apply (4.6) to get

$$\begin{aligned} & \overline{R}(uq^{-c}/v)(L_1^+(u) - 1)(L_2^-(v) - 1) - (L_2^-(v) - 1)(L_1^+(u) - 1)\overline{R}(uq^c/v) \\ &= (L_1^+(u) - 1 + L_2^-(v) - 1)\overline{R}(uq^c/v) - \overline{R}(uq^{-c}/v)(L_1^+(u) - 1 + L_2^-(v) - 1) \\ & \quad + \overline{R}(uq^c/v) - \overline{R}(uq^{-c}/v). \end{aligned}$$

Dividing both sides by  $(q-1)^2$  and taking the limit  $q \rightarrow 1$  we get

$$[\mathcal{L}_1^+(u), \mathcal{L}_2^-(v)] = [\mathcal{L}_1^+(u) + \mathcal{L}_2^-(v), r_{12}(u/v)] + \frac{\overline{R}(uq^c/v) - \overline{R}(uq^{-c}/v)}{(q-1)^2} \Big|_{q=1}.$$

Using (4.14) and (4.15) we find that

$$\frac{\overline{R}(xq^c) - \overline{R}(xq^{-c})}{(q-1)^2} \Big|_{q=1} = \frac{4cx}{(1-x)^2} \left( P - \frac{1}{N} \right)$$

thus completing the proof.  $\square$

The (*trigonometric*) *vacuum module at the critical level* over the affine Lie algebra  $\widehat{\mathfrak{gl}}_N$  is the universal module  $V_{\text{tr}}(\mathfrak{gl}_N)$  generated by a nonzero vector  $\mathbf{1}$  subject to the conditions  $K\mathbf{1} = -N\mathbf{1}$  and

$$E_{ij}[n]\mathbf{1} = 0 \quad \text{for all } i, j \quad \text{and} \quad n \geq 1, \quad \text{and} \quad E_{ij}[0]\mathbf{1} = 0 \quad \text{for } i \geq j. \quad (4.16)$$

By (4.11) these conditions can be written in a matrix form as  $\mathcal{L}^-(u)\mathbf{1} = 0$ . The *subspace of invariants* of  $V_{\text{tr}}(\mathfrak{gl}_N)$  is defined by

$$\mathfrak{z}_{\text{tr}}(\widehat{\mathfrak{gl}}_N) = \{v \in V_{\text{tr}}(\mathfrak{gl}_N) \mid \mathcal{L}^-(u)v = 0\}. \quad (4.17)$$

By the Poincaré–Birkhoff–Witt theorem, the vacuum module is isomorphic to the universal enveloping algebra  $U(\widehat{\mathfrak{g}}^+)$ , as a vector space, so that we can regard  $\mathfrak{z}_{\text{tr}}(\widehat{\mathfrak{gl}}_N)$  as a subalgebra of  $U(\widehat{\mathfrak{g}}^+)$ .

Note that the above definitions are quite analogous to those of the standard vacuum module over  $\widehat{\mathfrak{gl}}_N$ , where the conditions (4.16) are replaced by

$$E_{ij}[n]\mathbf{1} = 0 \quad \text{for all } i, j \quad \text{and} \quad n \geq 0.$$

The corresponding vacuum module has a vertex algebra structure and its subspace of invariants defined by analogy with (4.17) coincides with the center of this vertex algebra. The center is a commutative associative algebra whose structure was described by a theorem of Feigin and Frenkel [6]; see also [9] for a detailed general proof and [14] for an explicit approach in the case of classical Lie algebras. Our next result can be regarded as a trigonometric analogue of the Sugawara operators in type *A*. Recall the power series  $\vartheta_m^{(k)} \in U(\widehat{\mathfrak{g}}^+)[[u]]$  defined in (3.13) and (3.15).

**Theorem 4.2.** *The coefficients of the power series  $\vartheta_m^{(k)}$  belong to  $\mathfrak{z}_{\text{tr}}(\widehat{\mathfrak{gl}}_N)$ .*

*Proof.* As we pointed out above, the coefficients of all power series (4.9) are invariants of the vacuum module  $V_q(\mathfrak{gl}_N)$  over the quantum affine algebra  $U_q(\widehat{\mathfrak{gl}}_N)$ . Therefore, the claim is derived from Proposition 4.1 by taking the limit  $q \rightarrow 1$  as in Section 3.  $\square$

## References

- [1] O. Babelon and C.-M. Viallet, *Hamiltonian structures and Lax equations*, Phys. Lett. B **237** (1990), 411–416.
- [2] P. Baseilhac, S. Belliard and N. Crampé, *FRT presentation of the Onsager algebras*, [arXiv:1709.08555](https://arxiv.org/abs/1709.08555).
- [3] A. Chervov and G. Falqui, *Manin matrices and Talalaev’s formula*, J. Phys. A: Math. Theor. **41** (2008), 194006 (28pp).
- [4] A. Chervov, G. Falqui, V. Rubtsov and A. Silantyev, *Algebraic properties of Manin matrices II:  $q$ -analogues and integrable systems*, Adv. in Appl. Math. **60** (2014), 25–89.
- [5] A. V. Chervov and A. I. Molev, *On higher order Sugawara operators*, Int. Math. Res. Not. (2009), 1612–1635.
- [6] B. Feigin and E. Frenkel, *Affine Kac–Moody algebras at the critical level and Gelfand–Dikii algebras*, Int. J. Mod. Phys. A7, Suppl. 1A (1992), 197–215.
- [7] B. Feigin, E. Frenkel and N. Reshetikhin, *Gaudin model, Bethe ansatz and critical level*, Comm. Math. Phys. **166** (1994), 27–62.
- [8] L. Frappat, N. Jing, A. Molev and E. Ragoucy, *Higher Sugawara operators for the quantum affine algebras of type A*, Commun. Math. Phys. **345** (2016), 631–657.
- [9] E. Frenkel, *Langlands correspondence for loop groups*, Cambridge Studies in Advanced Mathematics, 103. Cambridge University Press, Cambridge, 2007.
- [10] I. B. Frenkel and N. Yu. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Comm. Math. Phys. **146** (1992), 1–60.
- [11] B. Jurčo, *Classical Yang–Baxter equations and quantum integrable systems*, J. Math. Phys. **30** (1989), 1289–1293.
- [12] S. Kožić and A. Molev, *Center of the quantum affine vertex algebra associated with trigonometric  $R$ -matrix*, J. Phys. A: Math. Theor. **50** (2017), 325201 (21pp).
- [13] E. Mukhin, V. Tarasov and A. Varchenko, *Bethe eigenvectors of higher transfer matrices*, J. Stat. Mech. Theory Exp. 2006, no. 8, P08002, 44 pp.

- [14] A. Molev, *Sugawara operators for classical Lie algebras*. Mathematical Surveys and Monographs, 229. AMS, Providence, RI, 2018.
- [15] N. Yu. Reshetikhin and M. A. Semenov-Tian-Shansky, *Central extensions of quantum current groups*, Lett. Math. Phys. **19** (1990), 133–142.
- [16] R. Rimányi, V. Tarasov and A. Varchenko, *Trigonometric weight functions as  $K$ -theoretic stable envelope maps for the cotangent bundle of a flag variety*, J. Geom. Phys. **94** (2015), 81–119.
- [17] E. K. Sklyanin, *Separation of variables in the Gaudin model* (Russian), Zap. Nauchn. Sem. LOMI **164** (1987), *Differentsialnaya Geom. Gruppy Li i Mekh.* IX, 151–169; English translation in *J. Soviet Math.* **47** (1989), 2473–2488.
- [18] T. Skrypnyk, *Quantum integrable systems, non-skew-symmetric  $r$ -matrices and algebraic Bethe ansatz*, J. Math. Phys. **48** (2007), no. 2, 023506, 14 pp.
- [19] D. V. Talalaev, *The quantum Gaudin system*, Funct. Anal. Appl. **40** (2006), 73–77.