



HAL
open science

Consistency Measures, Inconsistency Measures, and Mix Measures (Preliminary Report)

Philippe Besnard, Vincent Risch

► **To cite this version:**

Philippe Besnard, Vincent Risch. Consistency Measures, Inconsistency Measures, and Mix Measures (Preliminary Report). 13èmes Journées d'Intelligence Artificielle Fondamentale (JIAF 2019), Jul 2019, Toulouse, France. pp.37–43. hal-02302956

HAL Id: hal-02302956

<https://hal.science/hal-02302956v1>

Submitted on 1 Oct 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Consistency Measures, Inconsistency Measures, and Mix Measures (Preliminary Report)

Philippe Besnard¹ Vincent Risch²

¹ IRIT, CNRS, Univ. Toulouse, France

² LIS, Université Aix-Marseille, France

besnard@irit.fr vincent.risch@univ-amu.fr

Résumé

Nous proposons une tentative d'exploration du concept de mesures de cohérence. Par celles-ci, il s'agit d'attribuer un degré de cohérence à des ensembles finis de formules logiques, comme un pendant au concept bien connu de mesures d'incohérence qui attribuent un degré d'incohérence à des ensembles finis de formules logiques. Nous introduisons un ensemble primitif de postulats pour des mesures de cohérence. Nous nous penchons sur quelques correspondances avec les mesures d'incohérence. Nous posons également les bases d'une dualité entre les deux univers. Finalement, nous examinons de façon préliminaire ce que pourrait être une mesure mixte, à savoir, une mesure qui détermine un degré, sur un même référentiel, pour la cohérence (valeur positive) ainsi que pour l'incohérence (valeur négative). Nous abordons aussi, en comparaison, la question des super-modèles de Ginsberg et col., ainsi que ce qui peut en être considéré comme une généralisation, les morpho-logiques.

Abstract

We give some insight into a preliminary attempt at investigating a notion of consistency measures. These would provide a consistency degree for any finite collection of logical formulas, on a par with the well-known notion of inconsistency measures, that aim at assigning degrees of inconsistency to finite sets of logical formulas. We first propose a basic set of postulates for consistency measures. We look at a couple of relationships with inconsistency measures. We even lay grounds for a formal duality between these two domains. Lastly, we have a look at what would be a mix measure, that is, a measure that gives a degree, on the same scale, for consistency (positive values) and inconsistency (negative values). We also mention supermodels, as defined by Ginsberg et al., as well as a theory that can be regarded as a generalization of super-models, namely morpho-logics.

1 Introduction

In some sense, a formula can be viewed as more consistent than another. An illustration on such an idea is as follows. For a formula φ and a propositional variable a , define

$$va.\varphi \stackrel{\text{def}}{=} \varphi[\top \leftarrow a] \wedge \varphi[\perp \leftarrow a].$$

Then, assuming $\text{Var}(\varphi) = \{a_1, \dots, a_n\}$, and ρ a permutation (for the a_i 's), an example of a “consistency measure” is

$$\mathbf{C}(\varphi) = \inf\{k \mid \exists \rho \text{ s.t. } va_{\rho(1)} \dots a_{\rho(k)}.\varphi \vdash \perp\}.$$

In a more explicit way, $\min\{k \mid \exists \rho \text{ s.t. } va_{\rho(1)} \dots a_{\rho(k)}.\varphi \vdash \perp\}$ is the value of $\mathbf{C}(\varphi)$ in the case that there exist k and ρ such that $va_{\rho(1)} \dots a_{\rho(k)}.\varphi$ is inconsistent. Otherwise, $\mathbf{C}(\varphi) = \infty$.

Example 1 In view of $va.\neg a = (\neg\top \wedge \neg\perp)$, it holds that $\mathbf{C}(\neg a) = 1$. Similarly, $\mathbf{C}(a) = 1$. However, $va.(a \vee \neg a) = (\top \vee \neg\top) \wedge (\perp \vee \neg\perp)$ so that $\mathbf{C}(a \vee \neg a) = \infty$.

Generalizing the above idea, a consistency measure is a total function that maps every finite set of formulas K to a value in $\mathbb{R}^+ \cup \{\infty\}$.

2 Preliminaries

All the formal matters will refer to propositional logic \vdash with a language \mathcal{L} based on a set of propositional variables denoted $\text{Var}(\mathcal{L})$ as well as the propositional constants \perp and \top . The symbols for the connectives are \neg (negation), \wedge (conjunction), \vee (disjunction), \rightarrow (material implication). Logical equivalence (not the equivalence connective) is denoted by means of \equiv . For the sake of clarity, we write $\&$ (and), \Rightarrow (if ... then), \Leftrightarrow (if and only if) in meta-level statements where φ, ψ, \dots denote formulas of \mathcal{L} while K, K' ,

... are called belief bases and denote finite sets of formulas of \mathcal{L} . As a final word about notation, $\mathcal{K}_{\mathcal{L}}$ is comprised of all belief bases over \mathcal{L} .

For our purposes, a consistency measure is a map

$$\mathbf{C} : \mathcal{K}_{\mathcal{L}} \rightarrow \mathbb{R}^+ \cup \{\infty\}.$$

The paper is organized as follows: in section 3 below, basic postulates for consistency measures are proposed, and the duality with inconsistency measures is investigated. In section 4, the notion of mixed measure is introduced. Section 5.1 exhibits a link with supermodels such as defined by Ginsberg et al.

3 Postulates for consistency measures

Not all such functions \mathbf{C} can do! A list of requirements over \mathbf{C} is needed. To this end, postulates can ensure \mathbf{C} to *make sense* for the purpose of consistency measuring.

This section is an investigation into such requirements as postulates formulated on the following grounds:

- The context is classical logic \vdash over a language \mathcal{L} .
- Belief bases are finite sets of formulas of \mathcal{L} .
- \mathbf{C} maps all finite sets of formulas of \mathcal{L} to values in $\mathbb{R}^+ \cup \{\infty\}$.

The first postulates that come to mind are those specifying what cases must be ascribed the lowest, respectively highest, consistency degree. Thus arise the following postulates.

Inconsistency Null

$$\mathbf{C}(K) = 0 \Leftrightarrow K \vdash \perp$$

Inconsistent bases, and only them, have the lowest consistency degree: 0.

Tautology Top

$$\vdash K \Rightarrow \mathbf{C}(K) \geq \mathbf{C}(K')$$

Tautologies have the highest consistency degree.

Equivalence

$$K \equiv K' \Rightarrow \mathbf{C}(K) = \mathbf{C}(K')$$

Logically equivalent bases have the same consistency degree.¹

Variant Equality

$$\sigma K = K' \ \& \ \sigma' K = K \Rightarrow \mathbf{C}(K) = \mathbf{C}(K')$$

Renaming does not change the consistency degree.

1. Due to Inconsistency Null, introducing the extra condition $K \not\vdash \perp$ (or, equivalently, $K' \not\vdash \perp$) would be otiose.

Proposition 1 *Inconsistency Null is equivalent to the conjunction of the conditions below.*

$$(\forall K' \mathbf{C}(K) \leq \mathbf{C}(K')) \Rightarrow K \vdash \perp \quad (1)$$

$$K \vdash \perp \Rightarrow \mathbf{C}(K) \leq \mathbf{C}(K') \quad (2)$$

$$\exists K \mathbf{C}(K) = 0 \quad (3)$$

Two (equivalent in the context of (1)–(3)) consequences are:

$$K \vdash \perp \ \& \ K' \not\vdash \perp \Rightarrow \mathbf{C}(K) < \mathbf{C}(K') \quad (4)$$

$$K \not\vdash \perp \Rightarrow 0 < \mathbf{C}(K) \quad (5)$$

Proof Let us start by showing that Inconsistency Null entails (1)–(3). To start with, (3) is a trivial consequence of Inconsistency Null. As regards (2), it is a direct consequence of Inconsistency Null because the codomain of \mathbf{C} is $\mathbb{R}^+ \cup \{\infty\}$. Now, in order to prove (1), assume $\forall K' \mathbf{C}(K) \leq \mathbf{C}(K')$. By (3), $\mathbf{C}(K_0) = 0$ for some K_0 . Due to Inconsistency Null, $K_0 \vdash \perp$. An instance of (2) is $K \vdash \perp \Rightarrow \mathbf{C}(K) \leq \mathbf{C}(K_0)$ which gives $K \vdash \perp \Rightarrow \mathbf{C}(K) \leq 0$ hence $K \vdash \perp \Rightarrow \mathbf{C}(K) = 0$ (the codomain of \mathbf{C} is $\mathbb{R}^+ \cup \{\infty\}$). Applying Inconsistency Null, $K \vdash \perp$ and (1) is proven. Let us now show that (1)–(3) entail Inconsistency Null. An obvious consequence of (1)–(2) is $(\forall K' \mathbf{C}(K) \leq \mathbf{C}(K')) \Leftrightarrow K \vdash \perp$ i.e. $\mathbf{C}(K) = \min\{\mathbf{C}(K') \mid K' \subseteq_f \mathcal{L}\} \Leftrightarrow K \vdash \perp$.² Now, (3) means that $\min\{\mathbf{C}(K') \mid K' \subseteq_f \mathcal{L}\} = 0$ since the codomain of \mathbf{C} is $\mathbb{R}^+ \cup \{\infty\}$. Therefore, $(\forall K' \mathbf{C}(K) \leq \mathbf{C}(K')) \Leftrightarrow K \vdash \perp$ is equivalent with Inconsistency Null.

We now show that (1) and (2) imply (4). Due to (1), $(\forall K'' \mathbf{C}(K') \leq \mathbf{C}(K'')) \Rightarrow K' \vdash \perp$. In contrapositive form, $K' \not\vdash \perp \Rightarrow \exists K'' \mathbf{C}(K') > \mathbf{C}(K'')$. Since an immediate instance of (2) is $K \vdash \perp \Rightarrow \mathbf{C}(K) \leq \mathbf{C}(K'')$, it ensues that $K \vdash \perp \ \& \ K' \not\vdash \perp$ gives $\exists K'' \mathbf{C}(K'') < \mathbf{C}(K')$ as well as $\forall K'' \mathbf{C}(K) \leq \mathbf{C}(K'')$ hence $\exists K'' \mathbf{C}(K) \leq \mathbf{C}(K'') < \mathbf{C}(K')$. Summing it up, $K \vdash \perp \ \& \ K' \not\vdash \perp \Rightarrow \mathbf{C}(K) < \mathbf{C}(K')$.

Lastly, we show that (4) and (5) are equivalent in the context of (1)–(3). Assume first that \mathbf{C} satisfies $K \vdash \perp \ \& \ K' \not\vdash \perp \Rightarrow \mathbf{C}(K) < \mathbf{C}(K')$. However, $\{\perp\} \vdash \perp$. Accordingly, $K' \not\vdash \perp \Rightarrow \mathbf{C}(\{\perp\}) < \mathbf{C}(K')$. Since the codomain of \mathbf{C} is $\mathbb{R}^+ \cup \{\infty\}$, it then follows that $K' \not\vdash \perp \Rightarrow 0 < \mathbf{C}(K')$. As to the converse, assume that \mathbf{C} satisfies $K' \not\vdash \perp \Rightarrow 0 < \mathbf{C}(K')$. A consequence of (2) and (3) (together with the fact that the codomain of \mathbf{C} is $\mathbb{R}^+ \cup \{\infty\}$) is $K \vdash \perp \Rightarrow \mathbf{C}(K) = 0$. It follows that $K \vdash \perp \ \& \ K' \not\vdash \perp \Rightarrow \mathbf{C}(K) < \mathbf{C}(K')$. ■

Conditions (1)–(2) express that inconsistent formulas get the lowest consistency degree and induce that consistent formulas are ascribed a strictly greater consistency degree.

2. We write $X \subseteq_f Y$ to denote that X is a finite subset of Y .

3.1 Postulate chopping

Do we wish to endorse the general principle that $K' \vdash K$ entails $C(K) \geq C(K')$? Under the reading of “being more consistent” as “having more models”, the answer must be positive.

Adopting a positive answer actually amounts to supplementing Inconsistency Null with the following postulate.

Entailment Decrease

$$K \vdash K' \Rightarrow C(K) \leq C(K')$$

Proposition 2 *Entailment Decrease entails Tautology Top, Equivalence, Variant Equality, condition (2) from Proposition 1 and the following property.*

$$\sigma K = K' \Rightarrow C(K) \geq C(K') \quad (6)$$

Proof The case of Equivalence is trivial. As regards Tautology Top, if $\vdash K$ then $K' \vdash K$. By Entailment Decrease, $C(K') \leq C(K)$. As to Variant Equality, it is a special case of (6) that is taken care of as the last item in this proof. As regards (2), assume $K \vdash \perp$. Thus, $K \vdash K'$. Applying Entailment Decrease, $C(K) \leq C(K')$. As regards (6), assume $\sigma K = K'$. Now, $K' = \sigma K$ entails $K' \equiv K \cup \{a \rightarrow b \mid [b \leftarrow a] \in \sigma\}$. Obviously, $K' \vdash K$. In view of Entailment Decrease, $C(K') \leq C(K)$ ensues. ■

In (6), the significant case is K' consistent (otherwise, $C(K') = 0$ by Inconsistency Null and $C(K) \geq C(K')$ trivially ensues). Thus, (6) expresses that K has a higher (or equal) consistency degree than any instance of K .

Example 2 *Let $K = \{p \vee q\}$ and $K' = \{p \vee p\}$. Thus, $\sigma K = K'$ for $\sigma = [p \leftarrow q]$. In view of (6), $C(K) \geq C(K')$ i.e. $C(p \vee q) \geq C(p \vee p)$.*

Obviously, Entailment Decrease realizes the idea that “having more models” does imply “being more consistent”: If $\text{Mod}(K) \subseteq \text{Mod}(K')$ then $C(K) \leq C(K')$. The converse is untrue because \leq is a total order whereas \subseteq (as ranging over models of finite subsets of \mathcal{L}) is only a partial order. In particular, for $\text{Var}(K) \cap \text{Var}(K') = \emptyset$, it must still be the case that either $C(K) \leq C(K')$ or $C(K') \leq C(K)$ but neither $\text{Mod}(K) \subseteq \text{Mod}(K')$ nor $\text{Mod}(K') \subseteq \text{Mod}(K)$ hold.

Note. A strict version of Entailment Decrease is of interest, in the form of condition (4) from Proposition 1, that is:

Strict Entailment Decrease

$$K \vdash K' \ \& \ K' \not\vdash K \Rightarrow C(K) < C(K')$$

The codomain of a consistency measure might be restricted to $[0, 1]$. Equivalently, such a consistency measure can be viewed as satisfying the following postulate.

Normalization

$$0 \leq C(K) \leq 1$$

Trivially, Normalization suggests alternative postulates, e.g., Tautology Top could be replaced by the following (stronger even in the case that the codomain of C is $[0, 1]$) postulate.

Tautology 1-Top

$$\vdash K \Rightarrow C(K) = 1$$

3.2 Relationship with inconsistency measures

Inconsistency measures (they have received a great deal of attention [9]) are meant to indicate to what extent a finite set of formulas K is inconsistent. Formally, an inconsistency measure I maps every finite set of formulas K to a value in $\mathbf{R}^+ \cup \{\infty\}$. Interestingly, various postulates have been proposed for inconsistency measuring, here is the most important one (see the survey [18]).

Consistency Null

$$I(K) = 0 \Leftrightarrow K \not\vdash \perp$$

No more is needed to show that every pair (C, I) , *no matter how arbitrary*, conveys some duality between C and I :

Proposition 3 *Let C enjoy Inconsistency Null and I enjoy Consistency Null. Then,*

$$C(K) = 0 \Leftrightarrow K \vdash \perp \Leftrightarrow I(K) > 0$$

$$C(K) > 0 \Leftrightarrow K \not\vdash \perp \Leftrightarrow I(K) = 0$$

Proof ($C(K) = 0 \Rightarrow K \vdash \perp$) Assuming $C(K) = 0$, Inconsistency Null then gives $K \vdash \perp$. ($K \vdash \perp \Rightarrow I(K) > 0$) Assuming $K \vdash \perp$, Consistency Null gives $I(K) \neq 0$. As the codomain of I is $\mathbf{R}^+ \cup \{\infty\}$, it follows that $I(K) > 0$. ($I(K) > 0 \Rightarrow C(K) = 0$) Assuming $I(K) > 0$, Consistency Null gives $K \vdash \perp$. Applying Inconsistency Null, $C(K) = 0$.

$C(K) > 0 \Leftrightarrow K \not\vdash \perp \Leftrightarrow I(K) = 0$ is proven in a similar way. ■

The *Tautology Top* postulate is to be put in regard with the following property (induced from Consistency Null and the fact that I has codomain $\mathbf{R}^+ \cup \{\infty\}$):

$$K \not\vdash \perp \Rightarrow I(K) \leq I(K')$$

That is, the lowest value for I holds for the case that K is consistent. Duality appears as *Tautology Top* actually means that the highest value for C holds for K being tautological, intuitively the most extreme³ form of consistency:

$$\underbrace{K \in \max_{\geq_{\text{cons}}} \{K \mid K \not\vdash \perp\}}_{\vdash K} \Rightarrow C(K) \geq C(K')$$

3. We use \geq_{cons} to intuitively denote a pre-order for “is at least as consistent as”.

According to this scheme, it is expected that the following consequence of Inconsistency Null

$$K \vdash \perp \Rightarrow C(K) \leq C(K')$$

gets a dual so that the highest value for I holds for the most extreme⁴ form of inconsistency:

$$\underbrace{K \in \max_{\succeq_{inc}} \{K \mid K \vdash \perp\}}_{???} \Rightarrow I(K) \geq I(K')$$

Duality breaks down as no such notion of “maximal inconsistent” arises in classical logic.

3.2.1 Formal duality

The aim here is to set a duality formally between postulates (and more generally properties) about consistency measures and inconsistency measures.

Definition 1 *Recursively set $[A \mathcal{R} B]^* = A^* \mathcal{R}^* B^*$ where the relation \mathcal{R} as well as the expressions A and B have the following primitive cases:*

$C(X)$	$\overset{*}{\longleftrightarrow}$	$I(X)$
<i>is inconsistent</i>	$\overset{*}{\longleftrightarrow}$	<i>is consistent</i>
\Rightarrow	$\overset{*}{\longleftrightarrow}$	\Rightarrow
\leq	$\overset{*}{\longleftrightarrow}$	\leq
$>$	$\overset{*}{\longleftrightarrow}$	$>$
$=$	$\overset{*}{\longleftrightarrow}$	$=$
\perp	$\overset{*}{\longleftrightarrow}$	\perp
0	$\overset{*}{\longleftrightarrow}$	0
K	$\overset{*}{\longleftrightarrow}$	K
K'	$\overset{*}{\longleftrightarrow}$	K'
	\vdots	

Example 3 *Here are a few examples of dual properties (we write $K \vdash \perp$ to stand for “ K is inconsistent” but the latter expression is what is used for determining the dual expression, and similarly for $K \not\vdash \perp$ standing for “ K is consistent”).*

$$[C(K) = 0 \Rightarrow K \vdash \perp]^* = [C(K) = 0]^* \Rightarrow [K \vdash \perp]^* \\ = I(K) = 0 \Rightarrow K \not\vdash \perp$$

$$[K \vdash \perp \Rightarrow I(K) > 0]^* = [K \vdash \perp]^* \Rightarrow [I(K) > 0]^* \\ = K \not\vdash \perp \Rightarrow C(K) > 0$$

$$[K \vdash \perp \Rightarrow C(K) \leq C(K')]^* = [K \vdash \perp]^* \Rightarrow [C(K) \leq C(K')]^* \\ = K \not\vdash \perp \Rightarrow I(K) \leq I(K')$$

4. We use \succeq_{inc} to intuitively denote a pre-order for “is at least as inconsistent as”.

The duality fails with

$$\vdash K \Rightarrow C(K) \geq C(K')$$

because $\vdash K$ is not mapped to $\not\vdash K$ (there is no “is maximally unsatisfiable” —which is expected as dual to “is valid” understood as “is maximally satisfiable”).

The duality also fails as Definition 1 has no entry for \vdash^* . In fact, the idea that “having more models” implies “being more consistent” has no counterpart in the universe of inconsistency measures: by strong completeness of propositional logic, having more models is equivalent with propositional entailment but this is useless for inconsistent sets of formulas because they all entail each other. In symbols, $[K \vdash K' \Rightarrow C(K) \leq C(K')]^*$ is undefined. The corresponding item would be $K \vdash K' \Rightarrow I(K) \leq I(K')$ which would make I to collapse because $K \vdash K'$ holds for every inconsistent K and K' .

4 Mixed measures

Consider extending C so that not all inconsistent bases get the same consistency degree. An idea is to extend the codomain to $\mathbb{R} \cup \{-\infty, +\infty\}$. The main postulate would become

Inconsistency below zero

$$C(K) \leq 0 \Leftrightarrow K \vdash \perp$$

The negative values can as well come from a given inconsistency measure I , by letting $C(K) = -I(K)$ whenever $K \vdash \perp$.⁵

The original (i.e., non-extended) C ascribes 0 to every inconsistent K whereas I ascribes 0 to every consistent K . Since no K is both consistent and inconsistent, the extended C is to ascribe 0 to no K at all: the codomain of the extended C is not to include 0.

Definition 2 *A mixed measure M assigns any finite K a value in $\mathbb{R}^* \cup \{-\infty, +\infty\}$.*

For mixed measures, the fundamental postulate therefore is

Inconsistency

$$M(K) < 0 \Leftrightarrow K \vdash \perp$$

Lemma 1 *The Inconsistency postulate is equivalent to*

$$M(K) > 0 \Leftrightarrow K \not\vdash \perp$$

Proof Trivial. ■

5. Careful: Equivalently, $I(K) = -C(K)$ but keep in mind that $K \vdash \perp$ must hold hence I actually fails to be defined from C .

Lemma 2 *The Inconsistency postulate implies that $M(K) > M(K')$ whenever $K \not\vdash \perp$ and $K' \vdash \perp$.*

Proof Trivial. ■

Another major postulate is

Tautology Top

$$\vdash K \Leftrightarrow \forall K' M(K) \geq M(K')$$

That is, tautologies are ascribed the highest degree (they are regarded as more consistent than any non-tautologies).

5 Consistency Measures based on Operations over Models

Entailment Decrease (or the weaker Equivalence postulate) allows us to take advantage of operations over the set of models of a belief base K to ascribe a consistency degree to K using the set of models of K . The next two sections present such examples: supermodels and morpho-logics.

5.1 Supermodels

Ginsberg et al. [6] introduced the notion of supermodels. In a nutshell, these act as would be models for a formula φ as follows: if the truth-value of n atoms (*not* occurrences of atoms) are switched, does switching the truth-value of m other atoms yield a model of φ ? This can be viewed as an assessment of consistency, indicative of the extent to which the model can be distorted and still remain a model. To be self-contained, here is a short presentation of supermodels.

Definition 3 *An (n, m) -supermodel of φ is a model \mathfrak{M} of φ such that whenever the truth-value of at most n variables is switched in \mathfrak{M} (yielding an interpretation \mathfrak{M}'), we can [still] obtain a model of φ by changing the truth-value of at most m other variables in \mathfrak{M}' .*

Remark. Should there be less than n or m variables occurring in φ , the definition means that the extra variables whose value may happen to be varied are taken in the rest of the infinite supply (remember that an interpretation is defined over the set of all variables).

Notation. The set of all (n, m) -supermodels of φ is denoted $\text{Supmod}_\varphi(n, m)$.

Obviously, the classical models of φ always are the $(0, 0)$ -supermodels of φ , i.e. $\text{Supmod}_\varphi(0, 0) = \text{Mod}(\varphi)$.

Example 4 *The interpretation pq is a $(1, 0)$ -supermodel of $p \vee q$ as well as a classical model of $p \vee q$.*

Making a restricted use of supermodels is enough to induce a consistency measure as follows.

$$C(K) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } \text{Supmod}_K(0, 0) = \emptyset \\ 1 + \sup\{n \mid \text{Supmod}_K(n, 0) \neq \emptyset\} & \text{otherwise} \end{cases}$$

Detailing, $C(K) = 1 + \max\{n \mid \text{Supmod}_K(n, 0) \neq \emptyset\}$ when K is neither tautologous nor inconsistent, while $C(K) = \infty$ for K tautologous.

Please observe that C satisfies Inconsistency Null and Entailment Decrease hence Tautology Top, Equivalence, Variant Equality. Actually, C satisfies Strict Entailment Decrease. Trivially, C fails Tautology 1-Top and Normalization.

5.2 Morpho-Logics

Mathematical morphology, which is based on set theory, deals with shapes and transformations. It has been introduced in logic by Bloch and Lang [4]. Two basic operations in mathematical morphology are dilation (meant to enlarge a shape) and erosion (meant to contract a shape). They are based on a auxiliary structure, captured by a function that maps each point (of the shape) to a set of points (meant to serve as a neighborhood but other options are possible).

This gives us, for Ω a set of interpretations on $\text{Var}(\mathcal{L})$, and ω varying over the set of all interpretations on $\text{Var}(\mathcal{L})$,

$$D_f(\Omega) = \{\omega \mid f(\omega) \cap \Omega \neq \emptyset\}$$

and

$$E_f(\Omega) = \{\omega \mid f(\omega) \subseteq \Omega\}$$

Clearly, properties of f are crucial for D_f and E_f to be faithful to the idea of *actually* enlarging and contracting: e.g., f must be extensive in the sense that $\omega \in f(\omega)$, ...

Both D_f and E_f are monotone wrt set inclusion of sets of interpretations, e.g.,

$$\text{if } \Gamma \subseteq \Omega \text{ then } E_f(\Gamma) \subseteq E_f(\Omega)$$

With respect to the auxiliary structure, however, dilation is monotone but erosion is anti-monotone:⁶

$$\text{if } f \leq g \text{ then } E_g(\Omega) \subseteq E_f(\Omega)$$

A host of consistency measures can be cast by resorting to distance-based functions. For example, a family $(f_i)_{i \in \mathbb{N}}$ can be defined by

$$f_i(\omega) = \{\omega' \mid \delta(\omega, \omega') \leq i\}$$

where δ is an integer-valued distance over the set of all interpretations on $\text{Var}(\mathcal{L})$.

⁶ \leq is the usual order over functions with codomain a set of sets: $f \leq f'$ iff $\forall \omega, f(\omega) \subseteq f'(\omega)$.

A collection of consistency measures then arises from

$$C(K) \stackrel{\text{def}}{=} \sup \{n \mid E_{f_n}(\text{Mod}(K)) \neq \emptyset\}$$

Please observe that the consistency measure previously defined with supermodels is the same as the consistency measure obtained here for the case that δ is the Hamming⁷ distance d_H defined as

$$d_H(\omega, \omega') = \sum_{a \in \text{Var}(K)} |\omega(a) - \omega'(a)|$$

6 Interim Conclusion

Indeed, there is an extensive body of literature on inconsistency measures: From the seminal article [7], with milestones such as [13, 11, 17] up to more recent contributions in a special issue [14] and an anniversary book [9], as well as, in the meantime, specific measures [19, 15, 1, 5, 12] or analysis of measures through properties and postulates [10, 2, 3, 18] and even measures defined with positive/negative values [16] (also, [8] by a measure comparing inconsistency in two sets of formulas). From a dual perspective, a similar development would seem natural. It certainly makes sense to rank logical bases from more consistent to less consistent in the context of getting conclusions on firmer grounds.

We have shown that a study of consistent measures could reflect work on inconsistency measures. A number of correspondences show up. Though, even with such a preliminary investigation, a caveat has been exhibited: While a notion of “maximally satisfiable” makes sense, there is no corresponding notion of “maximally unsatisfiable”. On the more positive side, a finding of interest is that very little (actually, the pair of Null postulates) is needed to establish the correspondence between consistency/inconsistency and positive/negative values for consistency measures and inconsistency measures (Proposition 3).

Also, we have initiated a process of merging a consistency and inconsistency measure into a single measure. Here, a perhaps unexpected byproduct is that the null value disappears. . . Hopefully, further such surprises are afoot in the area of consistency measures.

References

- [1] Ammoura, Meriem, Badran Raddaoui, Yakoub Salhi et Brahim Oukacha: *On an MCS-based inconsistency measure*. *Journal of Approximate Reasoning*, 80:443–459, 2017.
- [2] Besnard, Philippe: *Revisiting postulates for inconsistency measures*. Dans Fermé, Eduardo et João Leite (rédacteurs): *14th European Conference on Logics in Artificial Intelligence (JELIA'14)*, tome 8761 de *Lecture Notes in Computer Science*, pages 383–396, Funchal, Madeira, Portugal, September 24–26, 2014. Springer.
- [3] Besnard, Philippe: *Basic postulates for inconsistency measures*. Dans Hameurlain, Abdelkader, Josef Küng, Roland Wagner et Hendrik Decker (rédacteurs): *Transactions on Large-Scale Data- and Knowledge-Centered Systems XXXIV*, tome 10620 de *Lecture Notes in Computer Science*, pages 1–12. Springer, 2017.
- [4] Bloch, Isabelle et Jérôme Lang: *Towards mathematical morpho-logics*. Dans Bouchon-Meunier, Bernadette, Julio Gutiérrez-Ríos, Luis Magdalena et Ronald R. Yager (rédacteurs): *Technologies for Constructing Intelligent Systems. 2 – Tools*, tome 90 de *Studies in Fuzziness and Soft Computing*, pages 367–380. Springer, 2002.
- [5] De Bona, Glauber et Anthony Hunter: *Localising iceberg inconsistencies*. *Artificial Intelligence*, 246:118–151, 2017.
- [6] Ginsberg, Matthew L., Andrew J. Parkes et Amitabha Roy: *Supermodels and robustness*. Dans *Proceedings of the 15th National Conference on Artificial Intelligence (AAAI-1998)*, pages 334–339. AAAI Press / The MIT Press, 1998.
- [7] Grant, John: *Classifications for inconsistent theories*. *Notre Dame Journal of Formal Logic*, 19(3):435–444, 1978.
- [8] Grant, John et Anthony Hunter: *Measuring the good and the bad in inconsistent information*. Dans Walsh, Toby (rédacteur): *22nd International Joint Conference on Artificial Intelligence (IJCAI'11)*, pages 2632–2637, Barcelona, Catalonia, Spain, July 16–22, 2011. AAAI Press.
- [9] Grant, John et Maria Vanina Martinez (rédacteurs): *Measuring Inconsistency in Information*. College Publications, 2018.
- [10] Hunter, Anthony et Sébastien Konieczny: *Measuring inconsistency through minimal inconsistent sets*. Dans Brewka, Gerhard et Jérôme Lang (rédacteurs): *11th Conference on Principles of Knowledge Representation and Reasoning (KR'08)*, pages 358–366, Sydney, Australia, September 16–19, 2008. AAAI Press.
- [11] Hunter, Anthony et Sébastien Konieczny: *On the measure of conflicts: Shapley inconsistency values*. *Artificial Intelligence*, 174(14):1007–1026, 2010.
- [12] Jabbour, Saïd, Yue Ma, Badran Raddaoui et Lakhdar Saïd: *Quantifying conflicts in propositional logic*

7. Intuitively, the Hamming distance counts how many atoms have a different value (0 or 1) in ω and ω' .

- through prime implicates*. Journal of Approximate Reasoning, 89:27–40, 2017.
- [13] Knight, Kevin: *Measuring inconsistency*. Journal of Philosophical Logic, 31(1):77–98, 2002.
- [14] Liu, Weiru et Kedian Mu (rédacteurs): *Special Issue on Theories of Inconsistency Measures and their Applications*, tome 89 de *Journal of Approximate Reasoning*. Elsevier, 2017.
- [15] McAreavey, Kevin, Weiru Liu et Paul C. Miller: *Computational approaches to finding and measuring inconsistency in arbitrary knowledge bases*. Journal of Approximate Reasoning, 55(8):1659–1693, 2014.
- [16] Mu, Kedian, Jun Hong, Zhi Jin et Weiru Liu: *From inconsistency handling to non-canonical requirements management*. Journal of Approximate Reasoning, 54(1):109–131, 2013.
- [17] Thimm, Matthias: *Inconsistency measures for probabilistic logics*. Artificial Intelligence, 197:1–24, 2013.
- [18] Thimm, Matthias: *On the evaluation of inconsistency measures*. Dans Grant, John et Maria Vanina Martinez (rédacteurs): *Measuring Inconsistency in Information*, pages 19–60. College Publications, 2018.
- [19] Xiao, Guohui et Yue Ma: *Inconsistency measurement based on variables in minimal unsatisfiable subsets*. Dans De Raedt, Luc, Christian Bessière, Didier Dubois, Patrick Doherty, Paolo Frasconi, Fredrik Heintz et Peter J. F. Lucas (rédacteurs): *20th European Conference on Artificial Intelligence (ECAI'12)*, tome 242 de *Frontiers in Artificial Intelligence and Applications*, pages 864–869, Montpellier, France, August 27-31, 2012. IOS Press.