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# Consistency Measures, Inconsistency Measures, and Mix Measures (Preliminary Report)

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## Résumé

Nous proposons une tentative d'exploration du concept de mesures de cohérence. Par celles-ci, il s'agit d'attribuer un degré de cohérence à des ensembles finis de formules logiques, comme un pendant au concept bien connu de mesures d'incohérence qui attribuent un degré d'incohérence à des ensembles finis de formules logiques. Nous introduisons un ensemble primitif de postulats pour des mesures de cohérence. Nous nous penchons sur quelques correspondances avec les mesures d'incohérence. Nous posons également les bases d'une dualité entre les deux univers. Finalement, nous examinons de façon préliminaire ce que pourrait être une mesure mixte, à savoir, une mesure qui détermine un degré, sur un même référentiel, pour la cohérence (valeur positive) ainsi que pour l'incohérence (valeur négative). Nous abordons aussi, en comparaison, la question des super-modèles de Ginsberg et col., ainsi que ce qui peut en être considéré comme une généralisation, les morpho-logiques.

## Abstract

We give some insight into a preliminary attempt at investigating a notion of consistency measures. These would provide a consistency degree for any finite collection of logical formulas, on a par with the well-known notion of inconsistency measures, that aim at assigning degrees of inconsistency to finite sets of logical formulas. We first propose a basic set of postulates for consistency measures. We look at a couple of relationships with inconsistency measures. We even lay grounds for a formal duality between these two domains. Lastly, we have a look at what would be a mix measure, that is, a measure that gives a degree, on the same scale, for consistency (positive values) and inconsistency (negative values). We also mention supermodels, as defined by Ginsberg et al., as well as a theory that can be regarded as a generalization of super-models, namely morpho-logics.

## 1 Introduction

In some sense, a formula can be viewed as more consistent than another. An illustration on such an idea is as follows. For a formula  $\varphi$  and a propositional variable  $a$ , define

$$va.\varphi \stackrel{\text{def}}{=} \varphi[\top \leftarrow a] \wedge \varphi[\perp \leftarrow a].$$

Then, assuming  $\text{Var}(\varphi) = \{a_1, \dots, a_n\}$ , and  $\rho$  a permutation (for the  $a_i$ 's), an example of a “consistency measure” is

$$\mathbf{C}(\varphi) = \inf\{k \mid \exists \rho \text{ s.t. } va_{\rho(1)} \cdots a_{\rho(k)}.\varphi \vdash \perp\}.$$

In a more explicit way,  $\min\{k \mid \exists \rho \text{ s.t. } va_{\rho(1)} \cdots a_{\rho(k)}.\varphi \vdash \perp\}$  is the value of  $\mathbf{C}(\varphi)$  in the case that there exist  $k$  and  $\rho$  such that  $va_{\rho(1)} \cdots a_{\rho(k)}.\varphi$  is inconsistent. Otherwise,  $\mathbf{C}(\varphi) = \infty$ .

**Example 1** In view of  $va.\neg a = (\neg\top \wedge \neg\perp)$ , it holds that  $\mathbf{C}(\neg a) = 1$ . Similarly,  $\mathbf{C}(a) = 1$ . However,  $va.(a \vee \neg a) = (\top \vee \neg\top) \wedge (\perp \vee \neg\perp)$  so that  $\mathbf{C}(a \vee \neg a) = \infty$ .

Generalizing the above idea, a consistency measure is a total function that maps every finite set of formulas  $K$  to a value in  $\mathbb{R}^+ \cup \{\infty\}$ .

## 2 Preliminaries

All the formal matters will refer to propositional logic  $\vdash$  with a language  $\mathcal{L}$  based on a set of propositional variables denoted  $\text{Var}(\mathcal{L})$  as well as the propositional constants  $\perp$  and  $\top$ . The symbols for the connectives are  $\neg$  (negation),  $\wedge$  (conjunction),  $\vee$  (disjunction),  $\rightarrow$  (material implication). Logical equivalence (not the equivalence connective) is denoted by means of  $\equiv$ . For the sake of clarity, we write  $\&$  (and),  $\Rightarrow$  (if ... then),  $\Leftrightarrow$  (if and only if) in meta-level statements where  $\varphi, \psi, \dots$  denote formulas of  $\mathcal{L}$  while  $K, K'$ ,

... are called belief bases and denote finite sets of formulas of  $\mathcal{L}$ . As a final word about notation,  $\mathcal{K}_{\mathcal{L}}$  is comprised of all belief bases over  $\mathcal{L}$ .

For our purposes, a consistency measure is a map

$$\mathbf{C} : \mathcal{K}_{\mathcal{L}} \rightarrow \mathbb{R}^+ \cup \{\infty\}.$$

The paper is organized as follows: in section 3 below, basic postulates for consistency measures are proposed, and the duality with inconsistency measures is investigated. In section 4, the notion of mixed measure is introduced. Section 5.1 exhibits a link with supermodels such as defined by Ginsberg et al.

### 3 Postulates for consistency measures

Not all such functions  $\mathbf{C}$  can do! A list of requirements over  $\mathbf{C}$  is needed. To this end, postulates can ensure  $\mathbf{C}$  to *make sense* for the purpose of consistency measuring.

This section is an investigation into such requirements as postulates formulated on the following grounds:

- The context is classical logic  $\vdash$  over a language  $\mathcal{L}$ .
- Belief bases are finite sets of formulas of  $\mathcal{L}$ .
- $\mathbf{C}$  maps all finite sets of formulas of  $\mathcal{L}$  to values in  $\mathbb{R}^+ \cup \{\infty\}$ .

The first postulates that come to mind are those specifying what cases must be ascribed the lowest, respectively highest, consistency degree. Thus arise the following postulates.

#### Inconsistency Null

$$\mathbf{C}(K) = 0 \Leftrightarrow K \vdash \perp$$

Inconsistent bases, and only them, have the lowest consistency degree: 0.

#### Tautology Top

$$\vdash K \Rightarrow \mathbf{C}(K) \geq \mathbf{C}(K')$$

Tautologies have the highest consistency degree.

#### Equivalence

$$K \equiv K' \Rightarrow \mathbf{C}(K) = \mathbf{C}(K')$$

Logically equivalent bases have the same consistency degree.<sup>1</sup>

#### Variant Equality

$$\sigma K = K' \ \& \ \sigma' K = K \Rightarrow \mathbf{C}(K) = \mathbf{C}(K')$$

Renaming does not change the consistency degree.

1. Due to Inconsistency Null, introducing the extra condition  $K \not\vdash \perp$  (or, equivalently,  $K' \not\vdash \perp$ ) would be otiose.

**Proposition 1** *Inconsistency Null is equivalent to the conjunction of the conditions below.*

$$(\forall K' \mathbf{C}(K) \leq \mathbf{C}(K')) \Rightarrow K \vdash \perp \quad (1)$$

$$K \vdash \perp \Rightarrow \mathbf{C}(K) \leq \mathbf{C}(K') \quad (2)$$

$$\exists K \mathbf{C}(K) = 0 \quad (3)$$

Two (equivalent in the context of (1)–(3)) consequences are:

$$K \vdash \perp \ \& \ K' \not\vdash \perp \Rightarrow \mathbf{C}(K) < \mathbf{C}(K') \quad (4)$$

$$K \not\vdash \perp \Rightarrow 0 < \mathbf{C}(K) \quad (5)$$

**Proof** Let us start by showing that Inconsistency Null entails (1)–(3). To start with, (3) is a trivial consequence of Inconsistency Null. As regards (2), it is a direct consequence of Inconsistency Null because the codomain of  $\mathbf{C}$  is  $\mathbb{R}^+ \cup \{\infty\}$ . Now, in order to prove (1), assume  $\forall K' \mathbf{C}(K) \leq \mathbf{C}(K')$ . By (3),  $\mathbf{C}(K_0) = 0$  for some  $K_0$ . Due to Inconsistency Null,  $K_0 \vdash \perp$ . An instance of (2) is  $K \vdash \perp \Rightarrow \mathbf{C}(K) \leq \mathbf{C}(K_0)$  which gives  $K \vdash \perp \Rightarrow \mathbf{C}(K) \leq 0$  hence  $K \vdash \perp \Rightarrow \mathbf{C}(K) = 0$  (the codomain of  $\mathbf{C}$  is  $\mathbb{R}^+ \cup \{\infty\}$ ). Applying Inconsistency Null,  $K \vdash \perp$  and (1) is proven. Let us now show that (1)–(3) entail Inconsistency Null. An obvious consequence of (1)–(2) is  $(\forall K' \mathbf{C}(K) \leq \mathbf{C}(K')) \Leftrightarrow K \vdash \perp$  i.e.  $\mathbf{C}(K) = \min\{\mathbf{C}(K') \mid K' \subseteq_f \mathcal{L}\} \Leftrightarrow K \vdash \perp$ .<sup>2</sup> Now, (3) means that  $\min\{\mathbf{C}(K') \mid K' \subseteq_f \mathcal{L}\} = 0$  since the codomain of  $\mathbf{C}$  is  $\mathbb{R}^+ \cup \{\infty\}$ . Therefore,  $(\forall K' \mathbf{C}(K) \leq \mathbf{C}(K')) \Leftrightarrow K \vdash \perp$  is equivalent with Inconsistency Null.

We now show that (1) and (2) imply (4). Due to (1),  $(\forall K'' \mathbf{C}(K') \leq \mathbf{C}(K'')) \Rightarrow K' \vdash \perp$ . In contrapositive form,  $K' \not\vdash \perp \Rightarrow \exists K'' \mathbf{C}(K') > \mathbf{C}(K'')$ . Since an immediate instance of (2) is  $K \vdash \perp \Rightarrow \mathbf{C}(K) \leq \mathbf{C}(K'')$ , it ensues that  $K \vdash \perp \ \& \ K' \not\vdash \perp$  gives  $\exists K'' \mathbf{C}(K'') < \mathbf{C}(K')$  as well as  $\forall K'' \mathbf{C}(K) \leq \mathbf{C}(K'')$  hence  $\exists K'' \mathbf{C}(K) \leq \mathbf{C}(K'') < \mathbf{C}(K')$ . Summing it up,  $K \vdash \perp \ \& \ K' \not\vdash \perp \Rightarrow \mathbf{C}(K) < \mathbf{C}(K')$ .

Lastly, we show that (4) and (5) are equivalent in the context of (1)–(3). Assume first that  $\mathbf{C}$  satisfies  $K \vdash \perp \ \& \ K' \not\vdash \perp \Rightarrow \mathbf{C}(K) < \mathbf{C}(K')$ . However,  $\{\perp\} \vdash \perp$ . Accordingly,  $K' \not\vdash \perp \Rightarrow \mathbf{C}(\{\perp\}) < \mathbf{C}(K')$ . Since the codomain of  $\mathbf{C}$  is  $\mathbb{R}^+ \cup \{\infty\}$ , it then follows that  $K' \not\vdash \perp \Rightarrow 0 < \mathbf{C}(K')$ . As to the converse, assume that  $\mathbf{C}$  satisfies  $K' \not\vdash \perp \Rightarrow 0 < \mathbf{C}(K')$ . A consequence of (2) and (3) (together with the fact that the codomain of  $\mathbf{C}$  is  $\mathbb{R}^+ \cup \{\infty\}$ ) is  $K \vdash \perp \Rightarrow \mathbf{C}(K) = 0$ . It follows that  $K \vdash \perp \ \& \ K' \not\vdash \perp \Rightarrow \mathbf{C}(K) < \mathbf{C}(K')$ . ■

Conditions (1)–(2) express that inconsistent formulas get the lowest consistency degree and induce that consistent formulas are ascribed a strictly greater consistency degree.

2. We write  $X \subseteq_f Y$  to denote that  $X$  is a finite subset of  $Y$ .

### 3.1 Postulate chopping

Do we wish to endorse the general principle that  $K' \vdash K$  entails  $C(K) \geq C(K')$ ? Under the reading of “being more consistent” as “having more models”, the answer must be positive.

Adopting a positive answer actually amounts to supplementing Inconsistency Null with the following postulate.

*Entailment Decrease*

$$K \vdash K' \Rightarrow C(K) \leq C(K')$$

**Proposition 2** *Entailment Decrease entails Tautology Top, Equivalence, Variant Equality, condition (2) from Proposition 1 and the following property.*

$$\sigma K = K' \Rightarrow C(K) \geq C(K') \quad (6)$$

**Proof** The case of Equivalence is trivial. As regards Tautology Top, if  $\vdash K$  then  $K' \vdash K$ . By Entailment Decrease,  $C(K') \leq C(K)$ . As to Variant Equality, it is a special case of (6) that is taken care of as the last item in this proof. As regards (2), assume  $K \vdash \perp$ . Thus,  $K \vdash K'$ . Applying Entailment Decrease,  $C(K) \leq C(K')$ . As regards (6), assume  $\sigma K = K'$ . Now,  $K' = \sigma K$  entails  $K' \equiv K \cup \{a \rightarrow b \mid [b \leftarrow a] \in \sigma\}$ . Obviously,  $K' \vdash K$ . In view of Entailment Decrease,  $C(K') \leq C(K)$  ensues. ■

In (6), the significant case is  $K'$  consistent (otherwise,  $C(K') = 0$  by Inconsistency Null and  $C(K) \geq C(K')$  trivially ensues). Thus, (6) expresses that  $K$  has a higher (or equal) consistency degree than any instance of  $K$ .

**Example 2** *Let  $K = \{p \vee q\}$  and  $K' = \{p \vee p\}$ . Thus,  $\sigma K = K'$  for  $\sigma = [p \leftarrow q]$ . In view of (6),  $C(K) \geq C(K')$  i.e.  $C(p \vee q) \geq C(p \vee p)$ .*

Obviously, Entailment Decrease realizes the idea that “having more models” does imply “being more consistent”: If  $\text{Mod}(K) \subseteq \text{Mod}(K')$  then  $C(K) \leq C(K')$ . The converse is untrue because  $\leq$  is a total order whereas  $\subseteq$  (as ranging over models of finite subsets of  $\mathcal{L}$ ) is only a partial order. In particular, for  $\text{Var}(K) \cap \text{Var}(K') = \emptyset$ , it must still be the case that either  $C(K) \leq C(K')$  or  $C(K') \leq C(K)$  but neither  $\text{Mod}(K) \subseteq \text{Mod}(K')$  nor  $\text{Mod}(K') \subseteq \text{Mod}(K)$  hold.

**Note.** A strict version of Entailment Decrease is of interest, in the form of condition (4) from Proposition 1, that is:

*Strict Entailment Decrease*

$$K \vdash K' \ \& \ K' \not\vdash K \Rightarrow C(K) < C(K')$$

The codomain of a consistency measure might be restricted to  $[0, 1]$ . Equivalently, such a consistency measure can be viewed as satisfying the following postulate.

*Normalization*

$$0 \leq C(K) \leq 1$$

Trivially, Normalization suggests alternative postulates, e.g., Tautology Top could be replaced by the following (stronger even in the case that the codomain of  $C$  is  $[0, 1]$ ) postulate.

*Tautology 1-Top*

$$\vdash K \Rightarrow C(K) = 1$$

### 3.2 Relationship with inconsistency measures

Inconsistency measures (they have received a great deal of attention [9]) are meant to indicate to what extent a finite set of formulas  $K$  is inconsistent. Formally, an inconsistency measure  $I$  maps every finite set of formulas  $K$  to a value in  $\mathbf{R}^+ \cup \{\infty\}$ . Interestingly, various postulates have been proposed for inconsistency measuring, here is the most important one (see the survey [18]).

*Consistency Null*

$$I(K) = 0 \Leftrightarrow K \not\vdash \perp$$

No more is needed to show that every pair  $(C, I)$ , *no matter how arbitrary*, conveys some duality between  $C$  and  $I$ :

**Proposition 3** *Let  $C$  enjoy Inconsistency Null and  $I$  enjoy Consistency Null. Then,*

$$C(K) = 0 \Leftrightarrow K \vdash \perp \Leftrightarrow I(K) > 0$$

$$C(K) > 0 \Leftrightarrow K \not\vdash \perp \Leftrightarrow I(K) = 0$$

**Proof** ( $C(K) = 0 \Rightarrow K \vdash \perp$ ) Assuming  $C(K) = 0$ , Inconsistency Null then gives  $K \vdash \perp$ . ( $K \vdash \perp \Rightarrow I(K) > 0$ ) Assuming  $K \vdash \perp$ , Consistency Null gives  $I(K) \neq 0$ . As the codomain of  $I$  is  $\mathbf{R}^+ \cup \{\infty\}$ , it follows that  $I(K) > 0$ . ( $I(K) > 0 \Rightarrow C(K) = 0$ ) Assuming  $I(K) > 0$ , Consistency Null gives  $K \vdash \perp$ . Applying Inconsistency Null,  $C(K) = 0$ .

$C(K) > 0 \Leftrightarrow K \not\vdash \perp \Leftrightarrow I(K) = 0$  is proven in a similar way. ■

The *Tautology Top* postulate is to be put in regard with the following property (induced from Consistency Null and the fact that  $I$  has codomain  $\mathbf{R}^+ \cup \{\infty\}$ ):

$$K \not\vdash \perp \Rightarrow I(K) \leq I(K')$$

That is, the lowest value for  $I$  holds for the case that  $K$  is consistent. Duality appears as *Tautology Top* actually means that the highest value for  $C$  holds for  $K$  being tautological, intuitively the most extreme<sup>3</sup> form of consistency:

$$\underbrace{K \in \max_{\geq_{\text{cons}}} \{K \mid K \not\vdash \perp\}}_{\vdash K} \Rightarrow C(K) \geq C(K')$$

3. We use  $\geq_{\text{cons}}$  to intuitively denote a pre-order for “is at least as consistent as”.

According to this scheme, it is expected that the following consequence of Inconsistency Null

$$K \vdash \perp \Rightarrow C(K) \leq C(K')$$

gets a dual so that the highest value for  $I$  holds for the most extreme<sup>4</sup> form of inconsistency:

$$\underbrace{K \in \max_{\succeq_{inc}} \{K \mid K \vdash \perp\}}_{???} \Rightarrow I(K) \geq I(K')$$

Duality breaks down as no such notion of “maximal inconsistent” arises in classical logic.

### 3.2.1 Formal duality

The aim here is to set a duality formally between postulates (and more generally properties) about consistency measures and inconsistency measures.

**Definition 1** *Recursively set  $[A \mathcal{R} B]^* = A^* \mathcal{R}^* B^*$  where the relation  $\mathcal{R}$  as well as the expressions  $A$  and  $B$  have the following primitive cases:*

$C(X)$	$\xleftrightarrow{*}$	$I(X)$
<i>is inconsistent</i>	$\xleftrightarrow{*}$	<i>is consistent</i>
$\Rightarrow$	$\xleftrightarrow{*}$	$\Rightarrow$
$\leq$	$\xleftrightarrow{*}$	$\leq$
$>$	$\xleftrightarrow{*}$	$>$
$=$	$\xleftrightarrow{*}$	$=$
$\perp$	$\xleftrightarrow{*}$	$\perp$
$0$	$\xleftrightarrow{*}$	$0$
$K$	$\xleftrightarrow{*}$	$K$
$K'$	$\xleftrightarrow{*}$	$K'$
	$\vdots$	

**Example 3** *Here are a few examples of dual properties (we write  $K \vdash \perp$  to stand for “ $K$  is inconsistent” but the latter expression is what is used for determining the dual expression, and similarly for  $K \not\vdash \perp$  standing for “ $K$  is consistent”).*

$$[C(K) = 0 \Rightarrow K \vdash \perp]^* = [C(K) = 0]^* \Rightarrow [K \vdash \perp]^* \\ = I(K) = 0 \Rightarrow K \not\vdash \perp$$

$$[K \vdash \perp \Rightarrow I(K) > 0]^* = [K \vdash \perp]^* \Rightarrow [I(K) > 0]^* \\ = K \not\vdash \perp \Rightarrow C(K) > 0$$

$$[K \vdash \perp \Rightarrow C(K) \leq C(K')]^* = [K \vdash \perp]^* \Rightarrow [C(K) \leq C(K')]^* \\ = K \not\vdash \perp \Rightarrow I(K) \leq I(K')$$

4. We use  $\succeq_{inc}$  to intuitively denote a pre-order for “is at least as inconsistent as”.

The duality fails with

$$\vdash K \Rightarrow C(K) \geq C(K')$$

because  $\vdash K$  is not mapped to  $\not\vdash K$  (there is no “is maximally unsatisfiable” —which is expected as dual to “is valid” understood as “is maximally satisfiable”).

The duality also fails as Definition 1 has no entry for  $\vdash^*$ . In fact, the idea that “having more models” implies “being more consistent” has no counterpart in the universe of inconsistency measures: by strong completeness of propositional logic, having more models is equivalent with propositional entailment but this is useless for inconsistent sets of formulas because they all entail each other. In symbols,  $[K \vdash K' \Rightarrow C(K) \leq C(K')]^*$  is undefined. The corresponding item would be  $K \vdash K' \Rightarrow I(K) \leq I(K')$  which would make  $I$  to collapse because  $K \vdash K'$  holds for every inconsistent  $K$  and  $K'$ .

## 4 Mixed measures

Consider extending  $C$  so that not all inconsistent bases get the same consistency degree. An idea is to extend the codomain to  $\mathbb{R} \cup \{-\infty, +\infty\}$ . The main postulate would become

*Inconsistency below zero*

$$C(K) \leq 0 \Leftrightarrow K \vdash \perp$$

The negative values can as well come from a given inconsistency measure  $I$ , by letting  $C(K) = -I(K)$  whenever  $K \vdash \perp$ .<sup>5</sup>

The original (i.e., non-extended)  $C$  ascribes 0 to every inconsistent  $K$  whereas  $I$  ascribes 0 to every consistent  $K$ . Since no  $K$  is both consistent and inconsistent, the extended  $C$  is to ascribe 0 to no  $K$  at all: the codomain of the extended  $C$  is not to include 0.

**Definition 2** *A mixed measure  $M$  assigns any finite  $K$  a value in  $\mathbb{R}^* \cup \{-\infty, +\infty\}$ .*

For mixed measures, the fundamental postulate therefore is

*Inconsistency*

$$M(K) < 0 \Leftrightarrow K \vdash \perp$$

**Lemma 1** *The Inconsistency postulate is equivalent to*

$$M(K) > 0 \Leftrightarrow K \not\vdash \perp$$

**Proof** Trivial. ■

5. Careful: Equivalently,  $I(K) = -C(K)$  but keep in mind that  $K \vdash \perp$  must hold hence  $I$  actually fails to be defined from  $C$ .

**Lemma 2** *The Inconsistency postulate implies that  $M(K) > M(K')$  whenever  $K \not\vdash \perp$  and  $K' \vdash \perp$ .*

**Proof** Trivial. ■

Another major postulate is

*Tautology Top*

$$\vdash K \Leftrightarrow \forall K' M(K) \geq M(K')$$

That is, tautologies are ascribed the highest degree (they are regarded as more consistent than any non-tautologies).

## 5 Consistency Measures based on Operations over Models

Entailment Decrease (or the weaker Equivalence postulate) allows us to take advantage of operations over the set of models of a belief base  $K$  to ascribe a consistency degree to  $K$  using the set of models of  $K$ . The next two sections present such examples: supermodels and morpho-logics.

### 5.1 Supermodels

Ginsberg et al. [6] introduced the notion of supermodels. In a nutshell, these act as would be models for a formula  $\varphi$  as follows: if the truth-value of  $n$  atoms (*not* occurrences of atoms) are switched, does switching the truth-value of  $m$  other atoms yield a model of  $\varphi$ ? This can be viewed as an assessment of consistency, indicative of the extent to which the model can be distorted and still remain a model. To be self-contained, here is a short presentation of supermodels.

**Definition 3** *An  $(n, m)$ -supermodel of  $\varphi$  is a model  $\mathfrak{M}$  of  $\varphi$  such that whenever the truth-value of at most  $n$  variables is switched in  $\mathfrak{M}$  (yielding an interpretation  $\mathfrak{M}'$ ), we can [still] obtain a model of  $\varphi$  by changing the truth-value of at most  $m$  other variables in  $\mathfrak{M}'$ .*

**Remark.** Should there be less than  $n$  or  $m$  variables occurring in  $\varphi$ , the definition means that the extra variables whose value may happen to be varied are taken in the rest of the infinite supply (remember that an interpretation is defined over the set of all variables).

**Notation.** The set of all  $(n, m)$ -supermodels of  $\varphi$  is denoted  $\text{Supmod}_\varphi(n, m)$ .

Obviously, the classical models of  $\varphi$  always are the  $(0, 0)$ -supermodels of  $\varphi$ , i.e.  $\text{Supmod}_\varphi(0, 0) = \text{Mod}(\varphi)$ .

**Example 4** *The interpretation  $pq$  is a  $(1, 0)$ -supermodel of  $p \vee q$  as well as a classical model of  $p \vee q$ .*

Making a restricted use of supermodels is enough to induce a consistency measure as follows.

$$C(K) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } \text{Supmod}_K(0, 0) = \emptyset \\ 1 + \sup\{n \mid \text{Supmod}_K(n, 0) \neq \emptyset\} & \text{otherwise} \end{cases}$$

Detailing,  $C(K) = 1 + \max\{n \mid \text{Supmod}_K(n, 0) \neq \emptyset\}$  when  $K$  is neither tautologous nor inconsistent, while  $C(K) = \infty$  for  $K$  tautologous.

Please observe that  $C$  satisfies Inconsistency Null and Entailment Decrease hence Tautology Top, Equivalence, Variant Equality. Actually,  $C$  satisfies Strict Entailment Decrease. Trivially,  $C$  fails Tautology 1-Top and Normalization.

### 5.2 Morpho-Logics

Mathematical morphology, which is based on set theory, deals with shapes and transformations. It has been introduced in logic by Bloch and Lang [4]. Two basic operations in mathematical morphology are dilation (meant to enlarge a shape) and erosion (meant to contract a shape). They are based on a auxiliary structure, captured by a function that maps each point (of the shape) to a set of points (meant to serve as a neighborhood but other options are possible).

This gives us, for  $\Omega$  a set of interpretations on  $\text{Var}(\mathcal{L})$ , and  $\omega$  varying over the set of all interpretations on  $\text{Var}(\mathcal{L})$ ,

$$D_f(\Omega) = \{\omega \mid f(\omega) \cap \Omega \neq \emptyset\}$$

and

$$E_f(\Omega) = \{\omega \mid f(\omega) \subseteq \Omega\}$$

Clearly, properties of  $f$  are crucial for  $D_f$  and  $E_f$  to be faithful to the idea of *actually* enlarging and contracting: e.g.,  $f$  must be extensive in the sense that  $\omega \in f(\omega)$ , ...

Both  $D_f$  and  $E_f$  are monotone wrt set inclusion of sets of interpretations, e.g.,

$$\text{if } \Gamma \subseteq \Omega \text{ then } E_f(\Gamma) \subseteq E_f(\Omega)$$

With respect to the auxiliary structure, however, dilation is monotone but erosion is anti-monotone:<sup>6</sup>

$$\text{if } f \leq g \text{ then } E_g(\Omega) \subseteq E_f(\Omega)$$

A host of consistency measures can be cast by resorting to distance-based functions. For example, a family  $(f_i)_{i \in \mathbb{N}}$  can be defined by

$$f_i(\omega) = \{\omega' \mid \delta(\omega, \omega') \leq i\}$$

where  $\delta$  is an integer-valued distance over the set of all interpretations on  $\text{Var}(\mathcal{L})$ .

<sup>6</sup>  $\leq$  is the usual order over functions with codomain a set of sets:  $f \leq f'$  iff  $\forall \omega, f(\omega) \subseteq f'(\omega)$ .

A collection of consistency measures then arises from

$$C(K) \stackrel{\text{def}}{=} \sup \{n \mid E_{f_n}(\text{Mod}(K)) \neq \emptyset\}$$

Please observe that the consistency measure previously defined with supermodels is the same as the consistency measure obtained here for the case that  $\delta$  is the Hamming<sup>7</sup> distance  $d_H$  defined as

$$d_H(\omega, \omega') = \sum_{a \in \text{Var}(K)} |\omega(a) - \omega'(a)|$$

## 6 Interim Conclusion

Indeed, there is an extensive body of literature on inconsistency measures: From the seminal article [7], with milestones such as [13, 11, 17] up to more recent contributions in a special issue [14] and an anniversary book [9], as well as, in the meantime, specific measures [19, 15, 1, 5, 12] or analysis of measures through properties and postulates [10, 2, 3, 18] and even measures defined with positive/negative values [16] (also, [8] by a measure comparing inconsistency in two sets of formulas). From a dual perspective, a similar development would seem natural. It certainly makes sense to rank logical bases from more consistent to less consistent in the context of getting conclusions on firmer grounds.

We have shown that a study of consistent measures could reflect work on inconsistency measures. A number of correspondences show up. Though, even with such a preliminary investigation, a caveat has been exhibited: While a notion of “maximally satisfiable” makes sense, there is no corresponding notion of “maximally unsatisfiable”. On the more positive side, a finding of interest is that very little (actually, the pair of Null postulates) is needed to establish the correspondence between consistency/inconsistency and positive/negative values for consistency measures and inconsistency measures (Proposition 3).

Also, we have initiated a process of merging a consistency and inconsistency measure into a single measure. Here, a perhaps unexpected byproduct is that the null value disappears. . . Hopefully, further such surprises are afoot in the area of consistency measures.

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7. Intuitively, the Hamming distance counts how many atoms have a different value (0 or 1) in  $\omega$  and  $\omega'$ .

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