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An optimal control problem governed by heat equation with nonconvex constraints applied to selective laser melting process

Tonia-Maria Alam, Serge Nicaise and Luc Paquet*

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Abstract

This paper deals with a PDE-constraints optimal control problem applied to an Additive Manufacturing process, namely a selective laser melting. Here, we want to control the temperature gradient inside the domain during a fixed time of heating, by acting on the trajectory of the dynamic Gaussian heating source. The non convex set of admissible controls reflects the fact that the control must fill the part of the boundary irradiated by the laser.

AMS subject classifications. 49B22 · 74F05

Key Words. Laser trajectory optimization · Optimal control · Non-convex constraints · First-order necessary optimality condition

1 Introduction

Selective laser melting (SLM) is an Additive Layer Manufacturing process used to produce three-dimensional objects from metal powders by melting the material in a layer-by-layer manner. First, a thin layer of powder is spread onto a build platform and simultaneously levelled or compacted to the required thickness. The laser beam scans the powder surface at an appropriate speed, heating the surface according to the desired scanning pattern and part profile. The mechanisms of SLM have been discussed in [9, 17, 16].

Thermal distortion of the fabricated part is one serious problem in SLM process [6], because of its fast laser scan rates and material transformations (solidification and liquifation) in a very short time frame. The temperature field was found to be inhomogeneous by many previous researchers [21, 23]. Meanwhile, the temperature evolution history in SLM process has significant effects on the quality of the final parts, such as density, dimensions, mechanical properties, microstructure, etc. For metals, rapid repeated heating and cooling cycles of the powder during SLM build process is responsible for large temperature gradients result in high residual stresses and deformations, and may even lead to crack formation in the fabricated part.

The simulation results from [19, 4] show that thermal gradients are very different from one type of trajectory to another one, a similar observation can be made for thermal stresses [15, 10]. Also in [14, 3], fractal scanning strategies based upon mathematical fill curves, namely the Hilbert and

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Peano-Gosper curves were explored to steer the influence of laser trajectory on thermal stresses and temperature distribution. Finally in [1] the issue of thermal stresses in Additive Manufacturing is treated from a shape and topology optimization point of view.

However, to the best of our knowledge, a model incorporating trajectory optimization to minimize thermal gradients in SLM has not yet been studied in the literature. Our aim is to propose a mathematical model to find an optimal trajectory minimizing thermal gradients within the produced part using optimal control theory of PDE's. Thus, we introduce the appropriate cost functional and the set of admissible controls taking into account the constraints on laser trajectory.

We consider the optimal control of a linear heat equation that models the distribution of temperature within one layer Ω heated by a Gaussian laser source [20]:

$$(1.1) \quad \begin{cases} \rho c \partial_t y - \kappa \Delta y = 0 & \text{in } Q = \Omega \times]0, T[, \\ -\kappa \frac{\partial y}{\partial \nu} = h y - g_\gamma & \text{on } \Sigma_1 = \Gamma_1 \times]0, T[, \\ -\kappa \frac{\partial y}{\partial \nu} = h y & \text{on } \Sigma_2 = \Gamma_2 \times]0, T[, \\ y = 0 & \text{on } \Sigma_3 = \Gamma_3 \times]0, T[, \\ y(x, 0) = y_0(x) & \text{for } x \in \Omega. \end{cases}$$

In these equations $y(x, t)$ denotes the temperature at point $x \in \Omega$ and time $t \in]0, T[$. We may always suppose that the ambient temperature $y_a = 0$ by taking as new dependent variable $y - y_a$ so we are led to the system (1.1). Here, $\Omega \subset \mathbb{R}^3$ is a bounded and simply connected domain with a connected Lipschitz boundary $\partial\Omega$, that is supposed to be split up

$$\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3},$$

where Γ_i , $i = 1, 2, 3$, are disjoint open disjoint subsets of $\partial\Omega$. In practice Γ_1 corresponds to the additive layer and is supposed to be included in a plane, that without loss of generality can be assumed to be \mathbb{R}^2 . By $\nu(x)$, we denote the outward normal direction at the point $x \in \partial\Omega$. The positive constants ρ , c , κ and h are respectively mass density, heat capacity, thermal conductivity and the convective heat transfer coefficient. The heat source g_γ is in the form

$$(1.2) \quad g_\gamma(x, t) = \alpha \frac{2P}{\pi R^2} \exp\left(-2 \frac{|x - \gamma(t)|^2}{R^2}\right), \quad \forall (x, t) \in \Sigma_1,$$

where the absorbance of the material α , the laser power P , the radius of the laser spot R are positive constants. The control $\gamma : t \in [0, T] \rightarrow \Gamma_1$ represents the displacement of the laser beam center on Γ_1 with respect to time. Note that g_γ depends nonlinearly on γ .

As suggested before, our main goal is to find a trajectory γ (the control) in such a way as to minimize temperature gradients inside the layer Ω with the constraints that the laser spot runs over the whole surface Γ_1 and does not leave it. From a mathematical point of view, these constraints lead to a non convex admissible set of controls, which is the main difficulty to overcome.

The outline of this paper is as follows. In section 2, we introduce the optimal control problem. We explain the link between our non convex set of admissible controls and laser trajectory in SLM process. Then we prove existence of a solution to the optimal control problem. In section 3, we prove the differentiability of the control-to-state mapping, result from which we infer the differentiability of the reduced cost functional. In section 4, the adjoint state is introduced which allow us to compute the Fréchet derivative of this reduced cost functional. Therefore, we derive a first order necessary condition for a control to be optimal in the form of a variational inequality. The main difficulty is in the non convex constraints required on the control γ .

2 The optimal control problem

For further purposes, we introduce the following (Hilbert) spaces:

$$\begin{aligned} H_{\Gamma_3}^1(\Omega) &:= \{u \in H^1(\Omega) \text{ such that } u|_{\Gamma_3} = 0\}, \\ W(0, T) &:= \{u \in L^2(0, T; H_{\Gamma_3}^1(\Omega)) \text{ such that } \frac{du}{dt} \in L^2(0, T; (H_{\Gamma_3}^1(\Omega))^*)\}. \end{aligned}$$

Recall that by Theorem 3.12 in [18], if y_0 belongs to $L^2(Q)$ and $\gamma \in H^1(0, T; \mathbb{R}^2)$ then the initial-boundary value problem (1.1) has a unique solution y in $W(0, T) \cap L^\infty(0, T; L^2(\Omega))$ in the sense that

$$\begin{aligned} (2.1) \quad & \rho c \int_0^T \left\langle \frac{dy}{dt}(\cdot, t), \varphi(\cdot, t) \right\rangle_{(H_{\Gamma_3}^1(\Omega))^*, H_{\Gamma_3}^1(\Omega)} dt + \kappa \int_0^T \int_{\Omega} \nabla y(x, t) \cdot \nabla \varphi(x, t) dx dt \\ & + h \int_0^T \int_{\Gamma_1} y(x, t) \varphi(x, t) dS(x) dt + h \int_0^T \int_{\Gamma_2} y(x, t) \varphi(x, t) dS(x) dt \\ & - \alpha \frac{2P}{\pi R^2} \int_0^T \int_{\Gamma_1} g_\gamma(x, t) \varphi(x, t) dS(x) dt = 0, \end{aligned}$$

for all $\varphi \in L^2(0, T; H_{\Gamma_3}^1(\Omega))$.

Our goal is to find the trajectory γ (the control) in such a way as to minimize the temperature gradients inside the layer Ω . Also we want to choose the control in such a way that the corresponding temperature distribution y in Q (the state) is the best possible approximation to a given temperature distribution $y_Q \in L^2(Q)$. To meet all requirements, we define the following cost functional

$$\begin{aligned} (2.2) \quad J(y, \gamma) &:= \frac{1}{2} \int_0^T \int_{\Omega} |\nabla y(x, t)|^2 dx dt + \frac{\lambda_Q}{2} \int_0^T \int_{\Omega} |y(x, t) - y_Q(x, t)|^2 dx dt \\ &+ \frac{\lambda_\gamma}{2} \|\gamma\|_{H^1(0, T; \mathbb{R}^2)}^2, \end{aligned}$$

where $\lambda_Q \geq 0$ and $\lambda_\gamma > 0$ are constants, while $y_Q \in L^2(Q)$ is a given function. Note that λ_γ is a regularization parameter. The optimal control problem is

$$(OCP) \quad \min_{\gamma \in U_{ad}} J(y(\gamma), \gamma),$$

where $y(\gamma)$ denotes the weak solution of problem (1.1) associated with the control γ , and the set of admissible controls $U_{ad} \subset H^1(0, T; \mathbb{R}^2)$ is defined as follows. For $\epsilon \geq R$, and a fixed positive constant c , U_{ad} is defined by

$$\begin{aligned} U_{ad} &:= \{\gamma \in H^1(0, T; \mathbb{R}^2); R(\gamma) \subset \Gamma_{1, -\epsilon}, R_\epsilon(\gamma) = \Gamma_1 \\ &\text{and } \exists c > 0 \text{ s.t. } |\gamma'(t)| \leq c \text{ a.e. } t \in [0, T]\}, \end{aligned} \quad (2.3)$$

where $R(\gamma) := \gamma([0, T])$,

$$\Gamma_{1, -\epsilon} = \{x \in \Gamma_1; \text{dist}(x, \partial\Gamma_1) \geq \epsilon\}, \quad (2.4)$$

and

$$R_\epsilon(\gamma) = \{x \in \Gamma_1; \text{dist}(x, R(\gamma)) \leq \epsilon\}. \quad (2.5)$$

Note that the condition $\text{dist}(x, \partial\Gamma_1) \geq \epsilon$ has a physical meaning because the control $t \mapsto \gamma(t)$ is nothing else than the path traced by the laser beam center which has R for radius. Practically, ϵ should be chosen in function of R . The constraint $R(\gamma) \subset \Gamma_{1,-\epsilon}$ is chosen to describe that γ must stay from an ϵ distance from the boundary of Γ_1 . Moreover, $R_\epsilon(\gamma) = \Gamma_1$ is to constrain γ to cover Γ_1 . Note that U_{ad} is not convex due to the two constraints $R(\gamma) \subset \Gamma_{1,-\epsilon}$ and $R_\epsilon(\gamma) = \Gamma_1$. By the theory of "tubes" [2], if $\partial\Gamma_1 \in C^2(\mathbb{R}^2)$, U_{ad} will be non void if $\epsilon > 0$ is chosen sufficiently small and the constant c in definition (2.3) is chosen large enough.

Let us prove some preliminary results that will allow us to show that (OCP) has at least one optimal control.

Proposition 2.1. U_{ad} is a weakly closed subset of $H^1(0, T; \mathbb{R}^2)$.

Proof. Let $(\gamma_n)_{n \in \mathbb{N}} \subset U_{ad}$ be a weakly convergent sequence in $H^1(0, T; \mathbb{R}^2)$ and let us call γ its weak limit. As the embedding from $H^1(0, T; \mathbb{R}^2)$ into $C([0, T]; \mathbb{R}^2)$ is compact, the weak convergence in $H^1(0, T; \mathbb{R}^2)$ of $(\gamma_n)_{n \in \mathbb{N}}$ to γ implies the strong convergence of $(\gamma_n)_{n \in \mathbb{N}}$ in $C([0, T]; \mathbb{R}^2)$.

Given $x \in \Gamma_1$, there exists $(t_n)_{n \in \mathbb{N}} \subset [0, T]$ such that

$$\gamma_n(t_n) \in \Gamma_{1,-\epsilon} \quad \text{for every } n \in \mathbb{N},$$

$$|x - \gamma_n(t_n)| \leq \epsilon \quad \text{for every } n \in \mathbb{N}.$$

As $[0, T]$ is compact there exists a subsequence $(t_{n_j})_{j \in \mathbb{N}} \subset [0, T]$ convergent to some $t \in [0, T]$. Thus we have

$$\begin{aligned} |\gamma_{n_j}(t_{n_j}) - \gamma(t)| &\leq |\gamma_{n_j}(t_{n_j}) - \gamma(t_{n_j})| + |\gamma(t_{n_j}) - \gamma(t)| \\ &\leq \|\gamma_{n_j} - \gamma\|_\infty + |\gamma(t_{n_j}) - \gamma(t)| \rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned} \quad (2.6)$$

This implies that $|x - \gamma(t)| \leq \epsilon$. Thus, $R_\epsilon(\gamma) = \Gamma_1$ and $R(\gamma) \subset \Gamma_{1,-\epsilon}$ (recalling that $\Gamma_{1,-\epsilon}$ is closed).

Using Mazur's theorem [22] for $\eta = \frac{1}{j}$ there exists a convex combination

$$u_{n_j} = \sum_{k=1}^{n_j} \alpha_k \gamma_k, \left(\alpha_k \geq 0, \sum_{k=1}^{n_j} \alpha_k = 1 \right), \text{ such that } \|\gamma - u_{n_j}\|_{H^1(0, T; \mathbb{R}^2)} \leq \eta.$$

This implies that

$$\|\gamma' - u'_{n_j}\|_{L^2(0, T; \mathbb{R}^2)} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

In particular, there exists a subsequence $(u'_{n_{j_k}})_{k \in \mathbb{N}}$ such that

$$u'_{n_{j_k}}(t) \rightarrow \gamma'(t) \text{ as } k \rightarrow \infty \text{ for a.e. } t \in [0, T].$$

As $|u'_{n_{j_k}}(t)| \leq c$ for a.e. $t \in [0, T]$, then also

$$|\gamma'(t)| \leq c \text{ for a.e. } t \in [0, T].$$

Thus $\gamma \in U_{ad}$. □

Proposition 2.2. *The control-to-state mapping $G : \gamma \in U_{ad} \mapsto y(\gamma) \in W(0, T)$ is weakly sequentially continuous.*

Proof. Let $(\gamma_n)_{n \in \mathbb{N}} \subset U_{ad}$ be a weakly convergent sequence in $H^1(0, T; \mathbb{R}^2)$ and let $\gamma \in U_{ad}$ its weak limit. From Theorem 3.13 in [18] it follows that the sequence $(y(\gamma_n))_{n \in \mathbb{N}}$ is a bounded sequence in the space $W(0, T)$. Consequently, it possesses a weakly convergent subsequence $(y(\gamma_{n_j}))_{j \in \mathbb{N}}$ in the space $W(0, T)$. Let y be the weak limit of $(y(\gamma_{n_j}))_{j \in \mathbb{N}}$.

For $\frac{1}{2} < \varepsilon < 1$, the embedding from $W(0, T)$ into $L^2(0, T; H^\varepsilon(\Omega))$ is a linear continuous compact mapping by the Compactity Lemma [12]. The trace mapping

$$\begin{array}{ccc} L^2(0, T; H^\varepsilon(\Omega)) & \longrightarrow & L^2(0, T; H^{\varepsilon-1/2}(\Gamma)) \\ y & \longmapsto & y|_\Sigma. \end{array}$$

is linear and continuous [13], and thus the trace mapping from $L^2(0, T; H^\varepsilon(\Omega))$ into $L^2(\Sigma) = L^2(0, T; L^2(\Gamma))$ is also linear and continuous. This implies that the traces on Σ of $(y(\gamma_{n_j}))_{j \in \mathbb{N}}$ strongly converge to u in $L^2(\Sigma)$.

Now we recall that each $y(\gamma_{n_j})$ satisfies the equivalent weak formulation of problem (1.1), namely

$$\begin{aligned} & \rho c \int_0^T \langle \frac{dy}{dt}(\gamma_{n_j})(\cdot, t), v \rangle_{(H_{\Gamma_3}^1(\Omega))^*, H_{\Gamma_3}^1(\Omega)} \varphi(t) dt + \kappa \int_0^T \int_\Omega \nabla y(\gamma_{n_j})(x, t) \cdot \nabla v(x) \varphi(t) dx dt \\ & + h \int_0^T \int_{\Gamma_1} y(\gamma_{n_j})(x, t) v(x) \varphi(t) dS(x) dt + h \int_0^T \int_{\Gamma_2} y(\gamma_{n_j})(x, t) v(x) \varphi(t) dS(x) dt \\ & - \alpha \frac{2P}{\pi R^2} \int_0^T \int_{\Gamma_1} \exp\left(-2 \frac{|x - \gamma_{n_j}(t)|^2}{R^2}\right) v(x) \varphi(t) dS(x) dt = 0, \end{aligned}$$

$$\forall v \in H_{\Gamma_3}^1(\Omega), \forall \varphi \in L^2(0, T).$$

By the Lebesgue convergence theorem we have,

$$\begin{aligned} \int_0^T \int_{\Gamma_1} \exp\left(-2 \frac{|x - \gamma_{n_j}(t)|^2}{R^2}\right) v(x) \varphi(t) dS(x) dt & \longrightarrow \\ \int_0^T \int_{\Gamma_1} \exp\left(-2 \frac{|x - \gamma(t)|^2}{R^2}\right) v(x) \varphi(t) dS(x) dt & \end{aligned}$$

$$\text{as } j \rightarrow \infty, \forall v \in H_{\Gamma_3}^1(\Omega) \text{ and } \forall \varphi \in L^2(0, T).$$

Using all the previous convergence properties to pass to the limit in the above equation as $j \rightarrow \infty$, we obtain $y = y(\gamma)$. Thus $(y(\gamma_{n_j}))_{j \in \mathbb{N}}$ weakly converges to $y(\gamma)$ in $W(0, T)$. Therefore any subsequence of $(y(\gamma_n))_{n \in \mathbb{N}}$ contains a further subsequence which converges weakly to $y(\gamma)$ in $W(0, T)$. This implies that the whole sequence itself $(y(\gamma_n))_{n \in \mathbb{N}}$ converges weakly to $y(\gamma)$ in $W(0, T)$. This proves the proposition. \square

Definition 2.3. *The reduced cost functional is defined by*

$$\begin{aligned} \hat{J} : U_{ad} &\longrightarrow L^2(0, T; H_{\Gamma_3}^1(\Omega)) \\ \gamma &\longmapsto J(G(\gamma), \gamma). \end{aligned}$$

We are ready to prove our main result, namely the existence of at least one optimal control.

Theorem 2.4 (Existence of an optimal control). *Supposing $U_{ad} \neq \emptyset$, then the optimal control problem (OCP) admits at least one optimal control $\bar{\gamma} \in U_{ad}$.*

Proof. Since $\hat{J}(\gamma) \geq 0$, the infimum

$$L := \inf_{\gamma \in U_{ad}} \hat{J}(\gamma),$$

exists and there is a sequence $(\gamma_n)_{n \in \mathbb{N}} \subset U_{ad}$ such that $\hat{J}(\gamma_n) \rightarrow L$ as $n \rightarrow +\infty$.

The sequence $(\gamma_n)_{n \in \mathbb{N}} \subset U_{ad}$ is bounded in $H^1(0, T; \mathbb{R}^2)$, because $\|\gamma_n\|_{H^1(0, T; \mathbb{R}^2)}^2 \leq \frac{2}{\lambda_\gamma} \hat{J}(\gamma_n)$ for all $n \in \mathbb{N}$. Hence, it possesses a subsequence $(\gamma_{n_j})_{j \in \mathbb{N}}$ weakly convergent to some element $\bar{\gamma} \in U_{ad}$. This implies that

$$\|\bar{\gamma}\|_{H^1(0, T; \mathbb{R}^2)} \leq \liminf_{j \rightarrow \infty} \|\gamma_{n_j}\|_{H^1(0, T; \mathbb{R}^2)} \leq \sqrt{\frac{2L}{\lambda_\gamma}}. \quad (2.7)$$

By the previous proposition $G(\gamma_{n_j}) \rightharpoonup G(\bar{\gamma})$ in $W(0, T)$ which implies that $G(\gamma_{n_j}) \rightharpoonup G(\bar{\gamma})$ in $L^2(0, T; H_{\Gamma_3}^1(\Omega))$, and thus

$$\|G(\bar{\gamma})\|_{L^2(0, T; H_{\Gamma_3}^1(\Omega))} \leq \liminf_{j \rightarrow \infty} \|G(\gamma_{n_j})\|_{L^2(0, T; H_{\Gamma_3}^1(\Omega))}. \quad (2.8)$$

The embedding from $W(0, T)$ into $L^2(0, T; L^2(\Omega))$ being compact [12], the sequence $G(\gamma_{n_j})$ also strongly converges to $G(\bar{\gamma})$ in $L^2(0, T; L^2(\Omega))$.

Using all the previous convergence properties and formula (2.2) we have

$$\begin{aligned} L &\geq \frac{1}{2} \liminf_{j \rightarrow \infty} \|G(\gamma_{n_j})\|_{L^2(0, T; H_{\Gamma_3}^1(\Omega))}^2 + \frac{\lambda_Q}{2} \liminf_{j \rightarrow \infty} \|G(\gamma_{n_j}) - y_Q\|_{L^2(Q)}^2 \\ &\quad + \frac{\lambda_\gamma}{2} \liminf_{j \rightarrow \infty} \|\gamma_{n_j}\|_{H^1(0, T; \mathbb{R}^2)}^2 \geq \hat{J}(\bar{\gamma}). \end{aligned}$$

By the definition of L we have also that $L \leq \hat{J}(\bar{\gamma})$. Thus $L = \hat{J}(\bar{\gamma})$. □

3 Differentiability of the control-to-state mapping.

Our aim is to derive necessary optimality conditions for an admissible control to be optimal. We first have to discuss the differentiability of the control-to-state mapping.

Lemma 3.1. *The mapping*

$$\begin{aligned} G : U_{ad} &\longrightarrow L^2(0, T; H_{\Gamma_3}^1(\Omega)) \\ \gamma &\longmapsto y(\gamma) \end{aligned}$$

is Fréchet differentiable.

Proof. We can write G as the restriction to U_{ad} of the composition of the Fréchet differentiable mappings w , g and q (see [11, p.262]), where w , g and q are defined as follows:

$$\begin{aligned} w : H^1(0, T; \mathbb{R}^2) &\longrightarrow C(\bar{\Gamma}_1 \times [0, T]) \\ \gamma &\longmapsto -c_R |\tilde{\gamma}(\gamma)|^2 \end{aligned} \quad (3.1)$$

where $c_R = \frac{2}{R^2}$ and $\tilde{\gamma}(\gamma)(x, t) := x - \gamma(t)$, $\forall (x, t) \in \bar{\Gamma}_1 \times [0, T]$,

$$\begin{aligned} g : C(\bar{\Gamma}_1 \times [0, T]) &\longrightarrow L^2(\Sigma_1) \\ u &\longmapsto a \exp(u) \end{aligned} \quad (3.2)$$

where $a = \alpha \frac{2P}{\pi R^2}$, and

$$\begin{aligned} q : L^2(\Sigma_1) &\longrightarrow L^2(0, T; H_{\Gamma_3}^1(\Omega)) \\ g &\longmapsto y \end{aligned} \quad (3.3)$$

where y denotes the weak solution of the initial boundary value problem:

$$\begin{cases} \rho c \partial_t y - \kappa \Delta y = 0 & \text{in } Q, \\ -\kappa \frac{\partial y}{\partial \nu} = h y - g & \text{on } \Sigma_1, \\ -\kappa \frac{\partial y}{\partial \nu} = h y & \text{on } \Sigma_2, \\ y = 0 & \text{on } \Sigma_3, \\ y(x, 0) = y_0(x) & \text{for } x \in \Omega, \end{cases} \quad (3.4)$$

$y_0 \in L^2(\Omega)$ denoting a fixed initial condition.

• We start by proving that w is Fréchet differentiable, when $C(\bar{\Gamma}_1 \times [0, T])$ is endowed with its natural norm $\|\gamma\|_\infty := \sup_{(x,t) \in \bar{\Gamma}_1 \times [0, T]} |\gamma(x, t)|$. For all $\delta\gamma \in H^1(0, T; \mathbb{R}^2)$ with sufficiently small norm we have $\gamma + \delta\gamma \in \text{Dom}(w)$:

$$\begin{aligned} [w(\gamma + \delta\gamma) - w(\gamma)](x, t) &= -c_R [|x - (\gamma + \delta\gamma)(t)|^2 - |x - \gamma(t)|^2] \\ &= c_R \delta\gamma(t) \cdot (2x - 2\gamma(t) - \delta\gamma(t)) \\ &= 2c_R (x - \gamma(t)) \cdot \delta\gamma(t) - c_R |\delta\gamma(t)|^2, \quad \forall (x, t) \in \bar{\Gamma}_1 \times [0, T], \end{aligned}$$

where \cdot denotes here the inner product in \mathbb{R}^2 . Using the fact that $H^1(0, T; \mathbb{R}^2) \hookrightarrow C([0, T]; \mathbb{R}^2)$, there exists a constant $\eta \geq 0$ such that

$$\frac{\| |\delta\gamma|^2 \|_\infty}{\|\delta\gamma\|_{H^1(0, T; \mathbb{R}^2)}} \leq \eta \frac{\|\delta\gamma\|_{H^1(0, T; \mathbb{R}^2)}^2}{\|\delta\gamma\|_{H^1(0, T; \mathbb{R}^2)}} = \eta \|\delta\gamma\|_{H^1(0, T; \mathbb{R}^2)} \rightarrow 0 \text{ as } \|\delta\gamma\|_{H^1(0, T; \mathbb{R}^2)} \rightarrow 0.$$

Hence, w is Fréchet differentiable with Fréchet derivative

$$Dw(\gamma) \cdot \delta\gamma = 2c_R \tilde{\gamma}(\gamma) \cdot \delta\gamma, \text{ for all } \delta\gamma \in H^1(0, T; \mathbb{R}^2).$$

• Now we prove that g is Fréchet differentiable. For all $\delta u \in C(\bar{\Gamma}_1 \times [0, T])$, we have

$$\begin{aligned} \exp(u + \delta u) - \exp(u) &= \exp(u)(\exp(\delta u) - 1) \\ &= \exp(u)(\delta u + \exp(\delta u) - \delta u - 1) \\ &= \exp(u)\delta u + \exp(u)(\exp(\delta u) - \delta u - 1). \end{aligned}$$

Let us set $r := \exp(u)(\exp(\delta u) - \delta u - 1)$. We see that

$$\| r \|_{L^2(\Sigma_1)} \leq \| \exp(u) \|_\infty \| \exp(\delta u) - \delta u - 1 \|_{L^2(\Sigma_1)}.$$

Furthermore,

$$| \exp(\delta u) - 1 - \delta u | = \left| \int_0^{\delta u} (\delta u - s) \exp(s) ds \right|.$$

If we perform the simple change of variable $s = \xi \delta u$ we have

$$\begin{aligned} \int_0^{\delta u} (\delta u - s) \exp(s) ds &= \int_0^1 \delta u^2 \exp(\xi \delta u) (1 - \xi) d\xi \\ &\leq | \delta u |^2 \exp(\| \delta u \|_\infty). \end{aligned}$$

This implies that

$$\begin{aligned} \| \exp(\delta u) - 1 - \delta u \|_{L^2(\Sigma_1)} &\leq | \Sigma_1 |^{\frac{1}{2}} \| \exp(\delta u) - 1 - \delta u \|_\infty \\ &\leq | \Sigma_1 |^{\frac{1}{2}} \| \delta u \|_\infty^2 \exp(\| \delta u \|_\infty). \end{aligned}$$

Thus,

$$\frac{\| r \|_{L^2(\Sigma_1)}}{\| \delta u \|_\infty} \rightarrow 0 \text{ as } \| \delta u \|_\infty \rightarrow 0.$$

Hence, g is Fréchet differentiable with Fréchet derivative

$$Dg(u) \cdot \delta u = a \exp(u) \delta u, \text{ for all } \delta u \in C(\bar{\Gamma}_1 \times [0, T]).$$

• It remains to prove that q is Fréchet differentiable. Using the weak formulation of (3.4) we obtain

$$\begin{aligned} \rho c \int_\Omega \partial_t y(x, t) y(x, t) dx &= \kappa \int_{\Gamma_1} \frac{\partial y}{\partial \nu}(x, t) y(x, t) dS(x) + \kappa \int_{\Gamma_2} \frac{\partial y}{\partial \nu}(x, t) y(x, t) dS(x) \\ &\quad - \kappa \int_\Omega | \nabla y(x, t) |^2 dx \\ &= -h \int_{\Gamma_1} | y(x, t) |^2 dS(x) - h \int_{\Gamma_2} | y(x, t) |^2 dS(x) \\ &\quad + \int_{\Gamma_1} g(x, t) y(x, t) dS(x) - \kappa \int_\Omega | \nabla y(x, t) |^2 dx. \end{aligned}$$

If we integrate the above identity from 0 to T we obtain

$$\begin{aligned} \kappa \int_Q | \nabla y(x, t) |^2 dx dt &= -h \int_{\Sigma_1} | y(x, t) |^2 dS(x) dt - h \int_{\Sigma_2} | y(x, t) |^2 dS(x) dt \\ &\quad + \int_{\Sigma_1} g(x, t) y(x, t) dS(x) dt - \frac{\rho c}{2} \| y(\cdot, T) \|_{L^2(\Omega)}^2 + \frac{\rho c}{2} \| y_0 \|_{L^2(\Omega)}^2. \end{aligned}$$

Then by Cauchy-Schwarz's and Young's inequalities, we get

$$\begin{aligned}
& \kappa \int_Q |\nabla y(x, t)|^2 dxdt + h \int_{\Sigma_1} |y(x, t)|^2 dS(x)dt + h \int_{\Sigma_2} |y(x, t)|^2 dS(x)dt + \frac{\rho c}{2} \|y(\cdot, T)\|_{L^2(\Omega)}^2 \\
&= \frac{\rho c}{2} \|y_0\|_{L^2(\Omega)}^2 + \int_{\Sigma_1} g(x, t)y(x, t)dS(x)dt \\
&\leq \frac{\rho c}{2} \|y_0\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon^2} \|g\|_{L^2(\Sigma_1)}^2 + \epsilon^2 \|y\|_{L^2(\Sigma_1)}^2, \text{ for all } \epsilon > 0.
\end{aligned}$$

For $\epsilon = \sqrt{\frac{h}{2}}$, we have $h - \epsilon^2 > 0$, so that we obtain for some constant $C > 0$:

$$\|y\|_{L^2(0, T; H_{\Gamma_3}^1(\Omega))}^2 \leq C \left[\|y_0\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\Sigma_1)}^2 \right]. \quad (3.5)$$

Let $y = y_1 + y_2$ where y_1 is solution of

$$\begin{cases} \rho c \partial_t y_1 - \kappa \Delta y_1 = 0 & \text{in } Q, \\ -\kappa \frac{\partial y_1}{\partial \nu} = h y_1 & \text{on } \Sigma_1, \\ -\kappa \frac{\partial y_1}{\partial \nu} = h y_1 & \text{on } \Sigma_2, \\ y_1 = 0 & \text{on } \Sigma_3, \\ y_1(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases} \quad (3.6)$$

and y_2 is solution of

$$\begin{cases} \rho c \partial_t y_2 - \kappa \Delta y_2 = 0 & \text{in } Q, \\ -\kappa \frac{\partial y_2}{\partial \nu} = h y_2 - g & \text{on } \Sigma_1, \\ -\kappa \frac{\partial y_2}{\partial \nu} = h y_2 & \text{on } \Sigma_2, \\ y_2 = 0 & \text{on } \Sigma_3, \\ y_2(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \quad (3.7)$$

Then by the principle of superposition of linear PDEs and (3.5) the map

$$\begin{aligned}
q: L^2(\Sigma_1) &\longrightarrow L^2(0, T; H_{\Gamma_3}^1(\Omega)) \\
g &\longmapsto y
\end{aligned}$$

is affine and continuous. Hence its derivative at any point of $L^2(\Sigma_1)$ is the linear continuous mapping:

$$\begin{aligned}
\tau: L^2(\Sigma_1) &\longrightarrow L^2(0, T; H_{\Gamma_3}^1(\Omega)) \\
g &\longmapsto \tau(g) := y_2.
\end{aligned}$$

In conclusion, G is Fréchet differentiable with Fréchet derivative

$$\begin{aligned}
DG(\gamma) \cdot \delta\gamma &= D(q \circ g \circ w)(\gamma) \cdot \delta\gamma \\
&= Dq(g(w(\gamma))) \cdot (D(g \circ w)(\gamma) \cdot \delta\gamma) \\
&= (Dq(g(w(\gamma))) \circ Dg(w(\gamma))) \cdot (Dw(\gamma) \cdot \delta\gamma) \\
&= \tau((Dg(w(\gamma)) \circ Dw(\gamma)) \cdot \delta\gamma) \\
&= 2ac_R \tau(\exp(w(\gamma)) \tilde{\gamma}(\gamma) \cdot \delta\gamma), \text{ for all } \delta\gamma \in H^1(0, T; \mathbb{R}^2).
\end{aligned}$$

□

From the previous lemma and by composition we deduce that the reduced cost functional $\gamma \in H^1(0, T; \mathbb{R}^2) \mapsto J(G(\gamma), \gamma) \in \mathbb{R}$ is Fréchet differentiable. Let us denote by

$$v(\gamma, \delta\gamma) := 2ac_R \tau(\exp(w(\gamma))) \tilde{\gamma}(\gamma) \cdot \delta\gamma \quad (3.8)$$

the Fréchet derivative of the control-to-state mapping $\gamma \in U_{ad} \mapsto G(\gamma) \in L^2(0, T; H_{\Gamma_3}^1(\Omega))$. Using the previous result, we obtain:

$$\begin{aligned} D\hat{J}(\gamma) \cdot \delta\gamma &= (G(\gamma), v(\gamma, \delta\gamma))_{L^2(0, T; H_{\Gamma_3}^1(\Omega))} + \lambda_Q (G(\gamma) - y_Q, v(\gamma, \delta\gamma))_{L^2(Q)} \\ &+ \lambda_\gamma (\gamma, \delta\gamma)_{H^1(0, T; \mathbb{R}^2)} = \int_0^T \int_\Omega \nabla G(\gamma)(x, t) \cdot \nabla v(\gamma, \delta\gamma)(x, t) \, dx dt \\ &+ \lambda_Q \int_0^T \int_\Omega G(\gamma)(x, t) v(\gamma, \delta\gamma)(x, t) \, dx dt - \lambda_Q \int_0^T \int_\Omega y_Q(x, t) v(\gamma, \delta\gamma)(x, t) \, dx dt \\ &+ \lambda_\gamma \int_0^T \gamma(t) \cdot \delta\gamma(t) \, dt + \lambda_\gamma \int_0^T \gamma'(t) \cdot \delta\gamma'(t) \, dt, \text{ for all } \delta\gamma \in H^1(0, T; \mathbb{R}^2). \end{aligned} \quad (3.9)$$

4 Adjoint equation and necessary optimality conditions

It is well known that an optimal control $\bar{\gamma}$ minimizing \hat{J} in U_{ad} has to obey the variational inequality

$$D\hat{J}(\bar{\gamma})(\gamma - \bar{\gamma}) \geq 0 \text{ for all } \gamma \in U_{ad}, \quad (4.1)$$

provided that \hat{J} is Gâteaux differentiable at $\bar{\gamma}$ and U_{ad} convex. In our case \hat{J} is Fréchet differentiable but U_{ad} is not convex, thus (4.1) is no more true. Therefore, we introduce at any point $\gamma \in U_{ad}$ the cone of feasible directions and we use the Kuhn-Tucker conditions. More precisely, we will recall from [5] the following definition and result.

Definition 4.1. *Let V a normed vector space and U_{ad} a non-empty subset of V . For every $\gamma \in U_{ad}$, the cone of admissible directions is*

$$\begin{aligned} C(\gamma) := \{0\} \cup \{w \in V; (\gamma_k)_{k \geq 0} \subset U_{ad}, \gamma_k \neq \gamma \, \forall k \geq 0 \text{ s.t. } \lim_{k \rightarrow \infty} \gamma_k = \gamma \text{ and} \\ \lim_{k \rightarrow \infty} \frac{\gamma_k - \gamma}{\|\gamma_k - \gamma\|} = \frac{w}{\|w\|}, w \neq 0\}. \end{aligned} \quad (4.2)$$

Theorem 4.2 (Kuhn-Tucker). *Let $J : O \subset V \rightarrow \mathbb{R}$, O an open set of V such that $U_{ad} \subset O$. If J has at $\gamma \in U_{ad}$ a relative minimum compared to the subset U_{ad} , and if J is Fréchet differentiable at γ then*

$$DJ(\gamma) \cdot (v - \gamma) \geq 0 \text{ for every } v \in \{\gamma + C(\gamma)\}. \quad (4.3)$$

Since \hat{J} is Fréchet differentiable Theorem 4.2 allows us to derive a necessary condition for an admissible control to be optimal. Let us first introduce the adjoint state. We claim that the adjoint

system of our problem is the following linear backward boundary value problem

$$\begin{cases} \rho c \partial_t p + \kappa \Delta p = \Delta G(\bar{\gamma}) - \lambda_d(G(\bar{\gamma}) - y_Q) & \text{in } Q, \\ \frac{\partial G(\bar{\gamma})}{\partial \nu} - \kappa \frac{\partial p}{\partial \nu} = h p & \text{on } \Sigma_1, \\ \frac{\partial G(\bar{\gamma})}{\partial \nu} - \kappa \frac{\partial p}{\partial \nu} = h p & \text{on } \Sigma_2, \\ p = 0 & \text{on } \Sigma_3, \\ p(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \quad (4.4)$$

Definition 4.3. Let $\bar{\gamma}$ be an optimal control of (OCP) with associated state $G(\bar{\gamma})$. A function $p \in W(0, T)$ is said to be a weak solution to (4.4) if $p(\cdot, T) = 0$ in Ω and

$$\begin{aligned} & -\rho c \int_0^T \int_{\Omega} \langle \partial_t p(\cdot, t) \varphi(\cdot, t) \rangle_{(H_{\Gamma_3}^1(\Omega))^*, H_{\Gamma_3}^1(\Omega)} dt + \kappa \int_0^T \int_{\Omega} \nabla p(x, t) \cdot \nabla \varphi(x, t) dx dt \\ & + h \int_0^T \int_{\Gamma_1 \cup \Gamma_2} p(x, t) \varphi(x, t) dS(x) dt = \int_0^T \int_{\Omega} \nabla G(\bar{\gamma})(x, t) \nabla \varphi(x, t) dx dt \\ & + \lambda_Q \int_0^T \int_{\Omega} (G(\bar{\gamma}) - y_Q)(x, t) \varphi(x, t) dx dt \end{aligned} \quad (4.5)$$

for every $\varphi \in L^2(0, T; H_{\Gamma_3}^1(\Omega))$.

Let us notice that (4.4) admits a unique weak solution in $W(0, T)$, see [7] for instance.

Theorem 4.4. If $\bar{\gamma} \in U_{ad}$ is an optimal control of (OCP) with associated state $G(\bar{\gamma})$, and $p \in W(0, T)$ the corresponding adjoint state that solves (4.4), then the variational inequality

$$\begin{aligned} & \lambda_{\gamma} \int_0^T \bar{\gamma}(t) \cdot (\gamma - \bar{\gamma})(t) dt + \lambda_{\gamma} \int_0^T \bar{\gamma}'(t) \cdot (\gamma - \bar{\gamma})'(t) dt \\ & + 2ac_R \int \int_{\Sigma_1} \exp(w(\bar{\gamma})(x, t)) \bar{\gamma}(\bar{\gamma})(x, t) \cdot (\gamma - \bar{\gamma})(t) p(x, t) dS(x) dt \geq 0 \end{aligned} \quad (4.6)$$

holds for all $\gamma \in \{\bar{\gamma} + C(\bar{\gamma})\}$.

Proof. If $\bar{\gamma}$ is an optimal control for the problem (OCP) then by Theorem 4.2

$$D\hat{J}(\bar{\gamma}) \cdot (\gamma - \bar{\gamma}) \geq 0 \text{ for all } \gamma \in \{\bar{\gamma} + C(\bar{\gamma})\}.$$

By (3.9) and (4.5) we have

$$\begin{aligned} D\hat{J}(\bar{\gamma}) \cdot (\gamma - \bar{\gamma}) &= -\rho c \int_0^T \int_{\Omega} \partial_t p(x, t) v(\gamma, \gamma - \bar{\gamma})(x, t) dx dt \\ &+ \kappa \int_0^T \int_{\Omega} \nabla p(x, t) \cdot \nabla v(\gamma, \gamma - \bar{\gamma})(x, t) dx dt + h \int_0^T \int_{\Gamma_1 \cup \Gamma_2} p(x, t) v(\gamma, \gamma - \bar{\gamma})(x, t) dS(x) dt \\ &+ \lambda_{\bar{\gamma}} \int_0^T \bar{\gamma}(t) \cdot (\gamma - \bar{\gamma})(t) dt + \lambda_{\bar{\gamma}} \int_0^T \bar{\gamma}'(t) \cdot (\gamma - \bar{\gamma})'(t) dt \end{aligned} \quad (4.7)$$

where we recall that $v(\bar{\gamma}, \gamma - \bar{\gamma})$ given by (3.8) is the weak solution of

$$\begin{cases} \rho c \partial_t v(\bar{\gamma}, \gamma - \bar{\gamma}) - \kappa \Delta v(\bar{\gamma}, \gamma - \bar{\gamma}) = 0 & \text{in } Q, \\ -\kappa \frac{\partial v(\bar{\gamma}, \gamma - \bar{\gamma})}{\partial \nu} = h v(\bar{\gamma}, \gamma - \bar{\gamma}) - 2ac_R \exp(w(\bar{\gamma})) \bar{\gamma}(\bar{\gamma}) \cdot (\gamma - \bar{\gamma}) & \text{on } \Sigma_1, \\ -\kappa \frac{\partial v}{\partial \nu} = h v(\bar{\gamma}, \gamma - \bar{\gamma}) & \text{on } \Sigma_2, \\ v(\bar{\gamma}, \gamma - \bar{\gamma}) = 0 & \text{on } \Sigma_3, \\ v(\bar{\gamma}, \gamma - \bar{\gamma})(x, 0) = 0 & \text{for } x \in \Omega, \end{cases} \quad (4.8)$$

and p is the weak solution of (4.4). Using the fact that v is the weak solution of (4.8) and $p(\cdot, T) = 0$, $v(\cdot, 0) = 0$ we obtain

$$\begin{aligned} 2ac_R \int_0^T \int_{\Gamma_1} \exp(w(\bar{\gamma})(x, t)) \bar{\gamma}(x, t) \cdot (\gamma - \bar{\gamma})(t) p(x, t) dS(x) dt &= -\rho c \int_0^T \int_{\Omega} \partial_t p(x, t) v(\gamma, \gamma - \bar{\gamma})(x, t) dx dt \\ + \kappa \int_0^T \int_{\Omega} \nabla p(x, t) \cdot \nabla v(\gamma, \gamma - \bar{\gamma})(x, t) dx dt &+ h \int_0^T \int_{\Gamma_1 \cup \Gamma_2} p(x, t) v(\gamma, \gamma - \bar{\gamma})(x, t) dS(x) dt. \end{aligned} \quad (4.9)$$

By replacing this last identity in (4.7) we get

$$\begin{aligned} D\hat{J}(\bar{\gamma}) \cdot (\gamma - \bar{\gamma}) &= 2ac_R \int \int_{\Sigma_1} \exp(w(\bar{\gamma})(x, t)) \bar{\gamma}(x, t) \cdot (\gamma - \bar{\gamma})(t) p(x, t) dS(x) dt \\ + \lambda_\gamma \int_0^T \bar{\gamma}(t) \cdot (\gamma - \bar{\gamma})(t) dt &+ \lambda_\gamma \int_0^T \bar{\gamma}'(t) \cdot (\gamma - \bar{\gamma})'(t) dt \geq 0. \end{aligned} \quad (4.10)$$

This concludes the proof of the Theorem. □

5 Perspectives

In Theorem 4.6 the variational inequality is only valid in a tangent cone (not necessarily convex) [5] which complicates the discretization of the problem. In a further work, we want to analyse numerically the solution of the optimal control problem by applying the projected gradient method, that requires a projection formula on a convex set (see for example chapter 2 in [8]). To solve this issue, we intend to relax the non convex constraints by adding a penalization term to the cost functional (2.2) while respecting the specificity of our control. More precisely, $\delta > 0$ being a penalized parameter, we want to add to $\hat{J}(\gamma)$, the term

$$\frac{1}{\delta^2} \left(2R \int_0^T \sqrt{|\gamma'(t)|^2 + \delta^2} dt - |\Gamma_1| \right)^2.$$

Formally as δ is close to zero, this will force the control to satisfy

$$2R \int_0^T |\gamma'(t)| dt \simeq |\Gamma_1|, \quad (5.1)$$

which means that the area covered by the laser is close to the area of Γ_1 . This allows to hope that the laser path would have covered the whole Γ_1 avoiding self-intersection because, in the same

time, we also minimize the temperature gradient. This will be properly investigated analytically and numerically in a forthcoming paper.

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