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**EXTENDING POTOČNIK AND ŠAJNA'S CONDITIONS ON
VERTEX-TRANSITIVE SELF-COMPLEMENTARY
 k -HYPERGRAPHS**

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ABSTRACT. Let l be a positive integer, $k = 2^l$ or $k = 2^l + 1$ and let n be a positive integer with $n \equiv 1 \pmod{2^{l+1}}$. Potočnik and Šajna showed that if there exists a vertex-transitive self-complementary k -hypergraph of order n , then for every prime p we have $p^{n(p)} \equiv 1 \pmod{2^{l+1}}$. Here we extend their result to any integer k and a larger class of integers n .

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1. INTRODUCTION

In 1985 Rao [6] determined a sufficient condition on the order n of a vertex-transitive self-complementary graph. Following many partial results, Muzychuk [2] showed in 1999, in an elegant proof, that Rao's sufficient condition was, indeed, also necessary.

For a prime p and a positive integer n , let $n_{(p)}$ denote the largest integer i for which p^i divides n . Using this notation, we combine the theorems of Rao and Muzychuk as follows.

Theorem 1.1 (Rao/Muzychuk). *For a positive integer n , there exists a vertex-transitive self-complementary graph of order n if and only if $p^{n(p)} \equiv 1 \pmod{4}$ for every prime p .*

For an interesting discussion of the history of the vertex-transitive self-complementary graph problem, see [1].

For every integer $k \geq 2$, a **k -uniform hypergraph**, or **k -hypergraph**, for short, is a pair $(V; E)$ consisting of a vertex set V and edge set $E \subseteq \binom{V}{k}$, where $\binom{V}{k}$ denotes the set of all k -subsets of V . Clearly a 2-hypergraph is simply a graph. A hypergraph H is called **vertex-transitive** if for every two vertices u, v of H there is an automorphism ϕ of H for which $u = \phi(v)$. In 2009, Potočnik and Šajna [5] proposed studying the problem analogous to the previous theorem for k -hypergraphs. In particular, they extended Muzychuk's necessary condition to

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k -hypergraphs when $k = 2^l$ or $k = 2^l + 1$ for some positive integer l . Shortly after, Gosselin [3] established the sufficiency of the Potočnik and Šajna result.

Theorem 1.2 (Potočnik-Šajna/Gosselin). *Let l be a positive integer, $k = 2^l$ or $k = 2^l + 1$ and let n be a positive integer with $n \equiv 1 \pmod{2^{l+1}}$. Then there exists a vertex-transitive self-complementary k -hypergraph of order n if and only if for every prime p we have $p^{n(p)} \equiv 1 \pmod{2^{l+1}}$.*

In Theorem 1.2, the only considered values of k are of the form $k = 2^l$ or $k = 2^l + 1$, for some positive integer l . We now consider any integer $k \geq 2$ and look at the binary expansion of k . Then there is a minimum positive integer l such that $k = \sum_{i>l} k_i 2^i + 2^l$ or $k = \sum_{i>l} k_i 2^i + 2^l + 1$, where $k_i \in \{0, 1\}$, for every i . In Theorem 1.2, each such $k_i = 0$. Furthermore, in Theorem 1.2, $n \equiv 1 \pmod{2^{l+1}}$ or, equivalently, $n = \sum_{i>l} n_i 2^i + 1$, with $n_i \in \{0, 1\}$ for $i > l$. This suggests our next theorem which extends the necessary condition of Potočnik and Šajna to any integer k and a larger class of integers n .

Theorem 1.3. *For any positive integer $k \geq 2$, let l be a positive integer for which $k = \sum_{i>l} k_i 2^i + 2^l$ or $k = \sum_{i>l} k_i 2^i + 2^l + 1$. Assume $n = \sum_{i>l} n_i 2^i + 1$, where $n_i, k_i \in \{0, 1\}$ for every i , and for every $i > l$,*

$$n_i \geq k_i.$$

If there exists a vertex-transitive self-complementary k -hypergraph of order n , then for every prime p we have $p^{n(p)} \equiv 1 \pmod{2^{l+1}}$.

2. PROOF OF THEOREM 1.3

If $H = (V; E)$ is a self-complementary k -hypergraph, then there is a permutation σ of the set V such that $\sigma(e)$ is an edge of \overline{H} if and only if $e \in E$. Such a permutation is called a **self-complementing permutation**. The set of all self-complementing permutations of a self-complementary hypergraph H will be denoted by $C(H)$. In [7] the following characterization of self-complementing permutations for k -hypergraphs was given.

Theorem 2.1. *Let n and k be positive integers, $2 \leq k < n$, with $n = \sum_{i \geq 0} n_i 2^i$ and $k = \sum_{i \geq 0} k_i 2^i$, where $n_i, k_i \in \{0, 1\}$. A permutation σ on the set $V = \{1, 2, \dots, n\}$ is a self-complementing permutation of a k -hypergraph with vertex set V if and only if there exist integers l_0, l_1 ; $l_0 \leq l_1$, such that $n_{l_1} = k_{l_1}, n_{l_1-1} = k_{l_1-1}, \dots, n_{l_0+1} = k_{l_0+1}, k_{l_0} = 1, n_{l_0} = 0$ and $\sum_{i>l_1} n_i 2^i$ vertices of V are contained in cycles of σ of lengths divisible by 2^{l_1+1} .*

Theorem 2.1 easily implies the following corollary.

Corollary 2.2. *Let n and k , $n > k$, be positive integers for which there is a positive integer l such that $n = \sum_{i>l} n_i 2^i + 1$ and $k = \sum_{i>l} k_i 2^i + 2^l$ or $k = \sum_{i>l} k_i 2^i + 2^l + 1$, where $n_i, k_i \in \{0, 1\}$ for all i , and*

$$n_i \geq k_i \text{ for } i > l.$$

Then there exists a self-complementary k -hypergraph of order n and every self-complementing permutation of any self-complementary k -hypergraph of order n has all cycles of lengths divisible by 2^{l+1} , except one cycle of length 1 (fixpoint).

The proof of Theorem 1.3 uses the technique of Muzychuk [2]. The proof also depends on the second Sylow theorem (see [4], for example.) Finally, one may use the well-known Orbit Decomposition Theorem to prove the following.

Theorem 2.3. *If P is p -Sylow subgroup of the normalizer of G then P is a p -Sylow subgroup of G .*

Proof of Theorem 1.3

Suppose that $H = (V; E)$ is a self-complementary and vertex-transitive k -hypergraph of order n , where k and n satisfy the conditions of our theorem. Let p be a prime; if $n_{(p)} = 0$, then the result is clear. Thus assume that $n_{(p)} > 0$. Here we will find a self-complementary vertex transitive k -subhypergraph H' of H of order $p^{n_{(p)}}$ such that the cycles of a self-complementing permutation of H' are cycles of a self-complementing permutation σ of H and the fixpoint of σ is one of the vertices of H' . Then Theorem 1.3 follows by Corollary 2.2.

Let $M = \text{Aut}(H)$ be the automorphism group of H . For a subgroup G of M and a vertex v of H , denote its stabilizer in G by $G_v = \{\sigma \in G : \sigma(v) = v\}$ and the set of the Sylow p -subgroups of G by $\text{Syl}_p(G)$.

Note that for every $\sigma \in C(H)$ we have $\sigma^2 \in \text{Aut}(H)$. More precisely, a composition of a number of automorphisms and self-complementing permutations is an automorphism of H if the number of self-complementing permutations is even; otherwise, the product is a self-complementing permutation of H . The set $G = \text{Aut}(H) \cup C(H)$ is a group which is generated by $\text{Aut}(H) \cup \{\sigma\}$, where σ is an arbitrary element of $C(H)$.

Define \mathcal{P} to be the set of p -subgroups P of M with the following properties: There is a vertex v of H and $\tau \in C(H)$ such that

- (1) $\tau(v) = v$;
- (2) $\tau P \tau^{-1} = P$ (τ normalizes P);
- (3) $P_v \in \text{Syl}_p(M_v)$.

We will show that \mathcal{P} is not empty and any maximal element of \mathcal{P} is, in fact, a Sylow p -subgroup of M .

Choose any $v \in V$, any $P \in \text{Syl}_p(M_v)$ and any $\sigma \in C(H)$. Since H is self-complementary, $C(H)$ is not empty. Note that if p does not divide $|M_v|$, then P is trivial. Since H is vertex transitive, there exists $h \in M$ with $h(v) = \sigma(v)$. Thus $\tilde{\sigma} := \sigma^{-1}h$ is an element of $C(H)$ that fixes v .

By the second Sylow Theorem, there exists $g \in M_v$ such that $\tilde{\sigma} P \tilde{\sigma}^{-1} = g P g^{-1}$; hence $(g^{-1} \tilde{\sigma}) P (g^{-1} \tilde{\sigma})^{-1} = P$. Set $\tau := g^{-1} \tilde{\sigma}$. Then $\tau \in C(H)$, $\tau(v) = g^{-1} \tilde{\sigma}(v) = g^{-1}(v) = v$ and $P_v = P$ (since $P \in \text{Syl}_p(M_v)$). Thus $\mathcal{P} \neq \emptyset$.

From now on we shall assume that

- $P \in \mathcal{P}$ is a maximal element of \mathcal{P} ,
- N is the normalizer of P in M ,
- Q is a Sylow p -subgroup of N containing P (Q exists by the second Sylow Theorem).

We shall prove that P is a Sylow p -subgroup of M (we shall do it by proving that $Q \in \mathcal{P}$ – then $Q = P$ by maximality of P). It will mean that P is a Sylow p -subgroup of its normalizer. Hence, by the Orbit Decomposition Theorem, P is

Sylow p -subgroup of M .

Since $P \in \mathcal{P}$, there are $\tau \in C(H)$ and a vertex v such that $\tau(v) = v$, $\tau P \tau^{-1} = P$ and $P_v \in \text{Syl}_p(M_v)$.

It is straightforward to show that τ normalizes N , that is, $\tau N \tau^{-1} = N$. Thus, $\tau N = N \tau$.

We next show that $\tau Q \tau^{-1}$ is a Sylow p -subgroup of N .

$$(1) \quad \tau Q \tau^{-1} \underset{Q \leq N}{\subset} \tau N \tau^{-1} = N$$

Hence $\tau Q \tau^{-1}$ is a subgroup of N

$$(2) \quad |\tau Q \tau^{-1}| = |Q|.$$

Therefore $\tau Q \tau^{-1}$ is a Sylow p -subgroup of N .

Let $U := N(v) (= \{h(v) : h \in N\})$, where v is a fix point of τ . Then we have $\tau(U) = \tau(N(v)) = (\tau N)(v) = (N \tau)(v)$, since $\tau N = N \tau$ by our previous argument. This implies that $\tau(U) = N(\tau(v)) = N(v) = U$.

Recall that, by Corollary 2.2, every cycle c of the antimorphism τ has length divisible by 2^{l+1} . Since $\tau(U) = U$, for every cycle c of the permutation τ we know that either all the vertices of c are in U or else, the set of vertices of c is disjoint with U . Then, U is a set of vertices of a self-complementary vertex-transitive k -uniform hypergraph with self-complementing permutation τ (restricted to U) and vertex-transitive group of automorphisms containing N . Moreover, the fix point of τ (the vertex v) is in U . Hence we have

$$|U| \equiv 1 \pmod{2^{l+1}}.$$

Since $\tau Q \tau^{-1}$ and Q are two Sylow p -subgroups of the group N , there is $g \in N$ such that $\tau Q \tau^{-1} = g Q g^{-1}$ (second Sylow Theorem). Hence $(g^{-1} \tau) Q (g^{-1} \tau)^{-1} = Q$.

Write $\sigma = \tau^{-1} g$ (we have $\sigma Q \sigma^{-1} = Q$ and $\sigma \in C(H)$). By the definition of U and since $g \in N$ we have $g(U) = U$, and hence, $\sigma(U) = U$.

Let u be a vertex of H such that $\sigma(u) = u$ (the vertex u exists by Corollary 2.2).

Since the group M is vertex-transitive, there is $h \in M$ such that $h(v) = u$.

Thus the subgroups M_v and M_u and the subgroups Q_u and Q_v are conjugated by h , that is,

- $M_u = h M_v h^{-1}$ and
- $Q_u = h Q_v h^{-1}$.

Hence $|M_u| = |M_v|$ and $|Q_u| = |Q_v|$. Consequently, $Q_u \in \text{Syl}_p(M_u)$ and $Q \in \mathcal{P}$. By the maximality of P we have $Q = P$ and $Q \in \text{Syl}_p(M)$.

Now we shall show that the orbit $P(v)$ induces a self-complementary and vertex-transitive k -hypergraph of order p^r . Note first that since $\tau P = P \tau$ and $\tau(v) = v$ we have

$$\tau(P(v)) = P(\tau(v)) = P(v)$$

and therefore the k -hypergraph induced in H by $P(v)$ is self-complementary and vertex transitive.

Write $|M| = p^d q$, where q and p are relatively prime. Then $|P| = p^d$. Since M acts vertex transitively on V we have

$$|M_v| = \frac{|M|}{|M(v)|} = \frac{p^d q}{p^r m} = p^{d-r} r,$$

where r and p are relatively prime.

Hence $|P_v| = p^{d-r}$. On the other hand, since $P \in \text{Syl}_p(M)$ and $P_v \in \text{Syl}_p(M_v)$ we have

$$p^{d-r} = |P_v| = \frac{|P|}{|P(v)|} = \frac{p^d}{|P(v)|}$$

This implies $|P(v)| = p^r$ and, by Corollary 2.2, $p^r \equiv 1 \pmod{2^{l+1}}$, and the proof is complete.

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