



# EXTENDING POTOČNIK AND ŠAJNA'S CONDITIONS ON VERTEX-TRANSITIVE SELF-COMPLEMENTARY k-HYPERGRAPHS

Linda Lesniak, Henri Thuillier, Adam Pawel Wojda

## ► To cite this version:

Linda Lesniak, Henri Thuillier, Adam Pawel Wojda. EXTENDING POTOČNIK AND ŠAJNA'S CONDITIONS ON VERTEX-TRANSITIVE SELF-COMPLEMENTARY k-HYPERGRAPHS. 2019. <hal-02302371>

**HAL Id: hal-02302371**

**<https://hal.science/hal-02302371v1>**

Preprint submitted on 1 Oct 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# EXTENDING POTOČNIK AND ŠAJNA'S CONDITIONS ON VERTEX-TRANSITIVE SELF-COMPLEMENTARY $k$ -HYPERGRAPHS

L. LESNIAK  
WESTERN MICHIGAN UNIVERSITY, KALAMAZOO, MI USA  
H. THUILLIER  
UNIVERSITÉ D'ORLÉANS, FRANCE  
A.P. WOJDA  
AGH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KRAKÓW, POLAND

ABSTRACT. Let  $l$  be a positive integer,  $k = 2^l$  or  $k = 2^l + 1$  and let  $n$  be a positive integer with  $n \equiv 1 \pmod{2^{l+1}}$ . Potočník and Šajna showed that if there exists a vertex-transitive self-complementary  $k$ -hypergraph of order  $n$ , then for every prime  $p$  we have  $p^{n(p)} \equiv 1 \pmod{2^{l+1}}$ . Here we extend their result to any integer  $k$  and a larger class of integers  $n$ .

August 3, 2019

## 1. INTRODUCTION

In 1985 Rao [6] determined a sufficient condition on the order  $n$  of a vertex-transitive self-complementary graph. Following many partial results, Muzychuk [2] showed in 1999, in an elegant proof, that Rao's sufficient condition was, indeed, also necessary.

For a prime  $p$  and a positive integer  $n$ , let  $n_{(p)}$  denote the largest integer  $i$  for which  $p^i$  divides  $n$ . Using this notation, we combine the theorems of Rao and Muzychuk as follows.

**Theorem 1.1** (Rao/Muzychuk). *For a positive integer  $n$ , there exists a vertex-transitive self-complementary graph of order  $n$  if and only if  $p^{n_{(p)}} \equiv 1 \pmod{4}$  for every prime  $p$ .*

For an interesting discussion of the history of the vertex-transitive self-complementary graph problem, see [1].

For every integer  $k \geq 2$ , a  **$k$ -uniform hypergraph**, or  **$k$ -hypergraph**, for short, is a pair  $(V; E)$  consisting of a vertex set  $V$  and edge set  $E \subseteq \binom{V}{k}$ , where  $\binom{V}{k}$  denotes the set of all  $k$ -subsets of  $V$ . Clearly a 2-hypergraph is simply a graph. A hypergraph  $H$  is called **vertex-transitive** if for every two vertices  $u, v$  of  $H$  there is an automorphism  $\phi$  of  $H$  for which  $u = \phi(v)$ . In 2009, Potočník and Šajna [5] proposed studying the problem analogous to the previous theorem for  $k$ -hypergraphs. In particular, they extended Muzychuk's necessary condition to

---

*Date:* August 3, 2019.

*2010 Mathematics Subject Classification.* 05C65.

*Key words and phrases.* vertex-transitive hypergraphs, self-complementary hypergraphs.

$k$ -hypergraphs when  $k = 2^l$  or  $k = 2^l + 1$  for some positive integer  $l$ . Shortly after, Gosselin [3] established the sufficiency of the Potočnik and Šajna result.

**Theorem 1.2** (Potočnik-Šajna/Gosselin). *Let  $l$  be a positive integer,  $k = 2^l$  or  $k = 2^l + 1$  and let  $n$  be a positive integer with  $n \equiv 1 \pmod{2^{l+1}}$ . Then there exists a vertex-transitive self-complementary  $k$ -hypergraph of order  $n$  if and only if for every prime  $p$  we have  $p^{n(p)} \equiv 1 \pmod{2^{l+1}}$ .*

In Theorem 1.2, the only considered values of  $k$  are of the form  $k = 2^l$  or  $k = 2^l + 1$ , for some positive integer  $l$ . We now consider any integer  $k \geq 2$  and look at the binary expansion of  $k$ . Then there is a minimum positive integer  $l$  such that  $k = \sum_{i>l} k_i 2^i + 2^l$  or  $k = \sum_{i>l} k_i 2^i + 2^l + 1$ , where  $k_i \in \{0, 1\}$ , for every  $i$ . In Theorem 1.2, each such  $k_i = 0$ . Furthermore, in Theorem 1.2,  $n \equiv 1 \pmod{2^{l+1}}$  or, equivalently,  $n = \sum_{i>l} n_i 2^i + 1$ , with  $n_i \in \{0, 1\}$  for  $i > l$ . This suggests our next theorem which extends the necessary condition of Potočnik and Šajna to any integer  $k$  and a larger class of integers  $n$ .

**Theorem 1.3.** *For any positive integer  $k \geq 2$ , let  $l$  be a positive integer for which  $k = \sum_{i>l} k_i 2^i + 2^l$  or  $k = \sum_{i>l} k_i 2^i + 2^l + 1$ . Assume  $n = \sum_{i>l} n_i 2^i + 1$ , where  $n_i, k_i \in \{0, 1\}$  for every  $i$ , and for every  $i > l$ ,*

$$n_i \geq k_i.$$

*If there exists a vertex-transitive self-complementary  $k$ -hypergraph of order  $n$ , then for every prime  $p$  we have  $p^{n(p)} \equiv 1 \pmod{2^{l+1}}$ .*

## 2. PROOF OF THEOREM 1.3

If  $H = (V; E)$  is a self-complementary  $k$ -hypergraph, then there is a permutation  $\sigma$  of the set  $V$  such that  $\sigma(e)$  is an edge of  $\bar{H}$  if and only if  $e \in E$ . Such a permutation is called a **self-complementing permutation**. The set of all self-complementing permutations of a self-complementary hypergraph  $H$  will be denoted by  $C(H)$ . In [7] the following characterization of self-complementing permutations for  $k$ -hypergraphs was given.

**Theorem 2.1.** *Let  $n$  and  $k$  be positive integers,  $2 \leq k < n$ , with  $n = \sum_{i \geq 0} n_i 2^i$  and  $k = \sum_{i \geq 0} k_i 2^i$ , where  $n_i, k_i \in \{0, 1\}$ . A permutation  $\sigma$  on the set  $V = \{1, 2, \dots, n\}$  is a self-complementing permutation of a  $k$ -hypergraph with vertex set  $V$  if and only if there exist integers  $l_0, l_1$ ;  $l_0 \leq l_1$ , such that  $n_{l_1} = k_{l_1}, n_{l_1-1} = k_{l_1-1}, \dots, n_{l_0+1} = k_{l_0+1}, k_{l_0} = 1, n_{l_0} = 0$  and  $\sum_{i>l_1} n_i 2^i$  vertices of  $V$  are contained in cycles of  $\sigma$  of lengths divisible by  $2^{l_1+1}$ .*

Theorem 2.1 easily implies the following corollary.

**Corollary 2.2.** *Let  $n$  and  $k$ ,  $n > k$ , be positive integers for which there is a positive integer  $l$  such that  $n = \sum_{i>l} n_i 2^i + 1$  and  $k = \sum_{i>l} k_i 2^i + 2^l$  or  $k = \sum_{i>l} k_i 2^i + 2^l + 1$ , where  $n_i, k_i \in \{0, 1\}$  for all  $i$ , and*

$$n_i \geq k_i \text{ for } i > l.$$

*Then there exists a self-complementary  $k$ -hypergraph of order  $n$  and every self-complementing permutation of any self-complementary  $k$ -hypergraph of order  $n$  has all cycles of lengths divisible by  $2^{l+1}$ , except one cycle of length 1 (fixpoint).*

The proof of Theorem 1.3 uses the technique of Muzychuk [2]. The proof also depends on the second Sylow theorem (see [4], for example.) Finally, one may use the well-known Orbit Decomposition Theorem to prove the following.

**Theorem 2.3.** *If  $P$  is  $p$ -Sylow subgroup of the normalizer of  $G$  then  $P$  is a  $p$ -Sylow subgroup of  $G$ .*

### Proof of Theorem 1.3

Suppose that  $H = (V; E)$  is a self-complementary and vertex-transitive  $k$ -hypergraph of order  $n$ , where  $k$  and  $n$  satisfy the conditions of our theorem. Let  $p$  be a prime; if  $n_{(p)} = 0$ , then the result is clear. Thus assume that  $n_{(p)} > 0$ . Here we will find a self-complementary vertex transitive  $k$ -subhypergraph  $H'$  of  $H$  of order  $p^{n_{(p)}}$  such that the cycles of a self-complementing permutation of  $H'$  are cycles of a self-complementing permutation  $\sigma$  of  $H$  and the fixpoint of  $\sigma$  is one of the vertices of  $H'$ . Then Theorem 1.3 follows by Corollary 2.2.

Let  $M = \text{Aut}(H)$  be the automorphism group of  $H$ . For a subgroup  $G$  of  $M$  and a vertex  $v$  of  $H$ , denote its stabilizer in  $G$  by  $G_v = \{\sigma \in G : \sigma(v) = v\}$  and the set of the Sylow  $p$ -subgroups of  $G$  by  $\text{Syl}_p(G)$ .

Note that for every  $\sigma \in C(H)$  we have  $\sigma^2 \in \text{Aut}(H)$ . More precisely, a composition of a number of automorphisms and self-complementing permutations is an automorphism of  $H$  if the number of self-complementing permutations is even; otherwise, the product is a self-complementing permutation of  $H$ . The set  $G = \text{Aut}(H) \cup C(H)$  is a group which is generated by  $\text{Aut}(H) \cup \{\sigma\}$ , where  $\sigma$  is an arbitrary element of  $C(H)$ .

Define  $\mathcal{P}$  to be the set of  $p$ -subgroups  $P$  of  $M$  with the following properties: There is a vertex  $v$  of  $H$  and  $\tau \in C(H)$  such that

- (1)  $\tau(v) = v$ ;
- (2)  $\tau P \tau^{-1} = P$  ( $\tau$  normalizes  $P$ );
- (3)  $P_v \in \text{Syl}_p(M_v)$ .

We will show that  $\mathcal{P}$  is not empty and any maximal element of  $\mathcal{P}$  is, in fact, a Sylow  $p$ -subgroup of  $M$ .

Choose any  $v \in V$ , any  $P \in \text{Syl}_p(M_v)$  and any  $\sigma \in C(H)$ . Since  $H$  is self-complementary,  $C(H)$  is not empty. Note that if  $p$  does not divide  $|M_v|$ , then  $P$  is trivial. Since  $H$  is vertex transitive, there exists  $h \in M$  with  $h(v) = \sigma(v)$ . Thus  $\tilde{\sigma} := \sigma^{-1}h$  is an element of  $C(H)$  that fixes  $v$ .

By the second Sylow Theorem, there exists  $g \in M_v$  such that  $\tilde{\sigma} P \tilde{\sigma}^{-1} = g P g^{-1}$ ; hence  $(g^{-1} \tilde{\sigma}) P (g^{-1} \tilde{\sigma})^{-1} = P$ . Set  $\tau := g^{-1} \tilde{\sigma}$ . Then  $\tau \in C(H)$ ,  $\tau(v) = g^{-1} \tilde{\sigma}(v) = g^{-1}(v) = v$  and  $P_v = P$  (since  $P \in \text{Syl}_p(M_v)$ ). Thus  $\mathcal{P} \neq \emptyset$ .

From now on we shall assume that

- $P \in \mathcal{P}$  is a maximal element of  $\mathcal{P}$ ,
- $N$  is the normalizer of  $P$  in  $M$ ,
- $Q$  is a Sylow  $p$ -subgroup of  $N$  containing  $P$  ( $Q$  exists by the second Sylow Theorem).

We shall prove that  $P$  is a Sylow  $p$ -subgroup of  $M$  (we shall do it by proving that  $Q \in \mathcal{P}$  – then  $Q = P$  by maximality of  $P$ ). It will mean that  $P$  is a Sylow  $p$ -subgroup of its normalizer. Hence, by the Orbit Decomposition Theorem,  $P$  is

Sylow  $p$ -subgroup of  $M$ .

Since  $P \in \mathcal{P}$ , there are  $\tau \in C(H)$  and a vertex  $v$  such that  $\tau(v) = v$ ,  $\tau P \tau^{-1} = P$  and  $P_v \in \text{Syl}_p(M_v)$ .

It is straightforward to show that  $\tau$  normalizes  $N$ , that is,  $\tau N \tau^{-1} = N$ . Thus,  $\tau N = N \tau$ .

We next show that  $\tau Q \tau^{-1}$  is a Sylow  $p$ -subgroup of  $N$ .

$$(1) \quad \tau Q \tau^{-1} \subseteq_{Q \leq N} \tau N \tau^{-1} = N$$

Hence  $\tau Q \tau^{-1}$  is a subgroup of  $N$

$$(2) \quad |\tau Q \tau^{-1}| = |Q|.$$

Therefore  $\tau Q \tau^{-1}$  is a Sylow  $p$ -subgroup of  $N$ .

Let  $U := N(v) (= \{h(v) : h \in N\})$ , where  $v$  is a fix point of  $\tau$ . Then we have  $\tau(U) = \tau(N(v)) = (\tau N)(v) = (N \tau)(v)$ , since  $\tau N = N \tau$  by our previous argument. This implies that  $\tau(U) = N(\tau(v)) = N(v) = U$ .

Recall that, by Corollary 2.2, every cycle  $c$  of the antimorphism  $\tau$  has length divisible by  $2^{l+1}$ . Since  $\tau(U) = U$ , for every cycle  $c$  of the permutation  $\tau$  we know that either all the vertices of  $c$  are in  $U$  or else, the set of vertices of  $c$  is disjoint with  $U$ . Then,  $U$  is a set of vertices of a self-complementary vertex-transitive  $k$ -uniform hypergraph with self-complementing permutation  $\tau$  (restricted to  $U$ ) and vertex-transitive group of automorphisms containing  $N$ . Moreover, the fix point of  $\tau$  (the vertex  $v$ ) is in  $U$ . Hence we have

$$|U| \equiv 1 \pmod{2^{l+1}}.$$

Since  $\tau Q \tau^{-1}$  and  $Q$  are two Sylow  $p$ -subgroups of the group  $N$ , there is  $g \in N$  such that  $\tau Q \tau^{-1} = g Q g^{-1}$  (second Sylow Theorem). Hence  $(g^{-1} \tau) Q (g^{-1} \tau)^{-1} = Q$ .

Write  $\sigma = \tau^{-1} g$  (we have  $\sigma Q \sigma^{-1} = Q$  and  $\sigma \in C(H)$ ). By the definition of  $U$  and since  $g \in N$  we have  $g(U) = U$ , and hence,  $\sigma(U) = U$ .

Let  $u$  be a vertex of  $H$  such that  $\sigma(u) = u$  (the vertex  $u$  exists by Corollary 2.2).

Since the group  $M$  is vertex-transitive, there is  $h \in M$  such that  $h(v) = u$ .

Thus the subgroups  $M_v$  and  $M_u$  and the subgroups  $Q_u$  and  $Q_v$  are conjugated by  $h$ , that is,

- $M_u = h M_v h^{-1}$  and
- $Q_u = h Q_v h^{-1}$ .

Hence  $|M_u| = |M_v|$  and  $|Q_u| = |Q_v|$ . Consequently,  $Q_u \in \text{Syl}_p(M_u)$  and  $Q \in \mathcal{P}$ . By the maximality of  $P$  we have  $Q = P$  and  $Q \in \text{Syl}_p(M)$ .

Now we shall show that the orbit  $P(v)$  induces a self-complementary and vertex-transitive  $k$ -hypergraph of order  $p^r$ . Note first that since  $\tau P = P \tau$  and  $\tau(v) = v$  we have

$$\tau(P(v)) = P(\tau(v)) = P(v)$$

and therefore the  $k$ -hypergraph induced in  $H$  by  $P(v)$  is self-complementary and vertex transitive.

Write  $|M| = p^d q$ , where  $q$  and  $p$  are relatively prime. Then  $|P| = p^d$ . Since  $M$  acts vertex transitively on  $V$  we have

$$|M_v| = \frac{|M|}{|M(v)|} = \frac{p^d q}{p^r m} = p^{d-r} r,$$

where  $r$  and  $p$  are relatively prime.

Hence  $|P_v| = p^{d-r}$ . On the other hand, since  $P \in \text{Syl}_p(M)$  and  $P_v \in \text{Syl}_p(M_v)$  we have

$$p^{d-r} = |P_v| = \frac{|P|}{|P(v)|} = \frac{p^d}{|P(v)|}$$

This implies  $|P(v)| = p^r$  and, by Corollary 2.2,  $p^r \equiv 1 \pmod{2^{l+1}}$ , and the proof is complete.

#### REFERENCES

- [1] R.A. Beezer, Sylow subgraphs in self-complementary vertex transitive graphs, *Expo. Math.* **24** (2006) 185-194.
- [2] M. Muzychuk, On Sylow subgraphs of vertex-transitive self-complementary graphs, *Bull. London Math. Soc.* **31** (1999) 531-533.
- [3] S.Gosselin, Vertex-transitive self-complementary uniform hypergraphs of prime order, *Discrete Math.* **310** (2010) 671-680.
- [4] W.K. Nicholson, Introduction to Abstract Algebra, *Wiley-Interscience* 2007.
- [5] P. Potočnik and M. Šajna, Vertex-transitive self-complementary uniform hypergraphs, *European J. Combin.* **28** (2009) 227-237.
- [6] S.B. Rao, On regular and strongly-regular self-complementary graphs, *Discrete Math.* **54** (1985) 73-82.
- [7] A. Szymański & A.P. Wojda, Self-complementing permutations of  $k$ -uniform hypergraphs, *DMTCS* **11:1** (2009) 117-134.