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SHARP REACHABILITY RESULTS FOR THE HEAT EQUATION IN ONE SPACE DIMENSION

KARIM KELLAY

Institut de Mathématiques de Bordeaux
Université de Bordeaux
351, Cours de la Libération - F 33 405 TALENCE, France

THOMAS NORMAND

Institut de Mathématiques de Bordeaux
Université de Bordeaux
351, Cours de la Libération - F 33 405 TALENCE, France

MARIUS TUCSNAK*

Institut de Mathématiques de Bordeaux
Université de Bordeaux
351, Cours de la Libération - F 33 405 TALENCE, France

ABSTRACT. This paper gives a complete characterization of the reachable space for a system described by the 1D heat equation with L^2 (with respect to time) Dirichlet boundary controls at both ends. More precisely, we prove that this space coincides with the sum of two spaces of analytic functions (of Bergman type). These results are then applied to give a complete description of the reachable space via inputs which are n -times differentiable functions of time. Moreover, we establish a connection between the norm in the obtained sum of Bergman spaces and the cost of null controllability in small time. Finally we show that our methods yield new complex analytic results on the sums of Bergman spaces in infinite sectors.

1. INTRODUCTION

Determining the reachable space of a controlled dynamical system is a major question in control theory. The knowledge of this set gives important information on our capability of acting on the state of a system and for safety verifications. This fundamental question is well understood for linear finite dimensional systems (see Section 2 below for some background material) but much less is known for time invariant linear infinite dimensional systems (namely those governed by partial differential equations). Most of the known results in this context concern the case when the system is exactly controllable, which means, as reminded below, that the reachable state coincides with the state space of the system. When the reachable space is a strict subspace of the state space, its description is generally far of being complete. In this work we focus on a case which might look very elementary but

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* Corresponding author.

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which englobes a rich structure: a system described by the heat equation in one space dimension with Dirichlet boundary control. The first results on this problem go back to the seminal paper of Fattorini and Russell [5] but more refined estimates have been obtained only during the last 3 years, see Martin, Rosier and Rouchon [18], Dardé and Ervedoza [2] and Hartmann, Kellay and Tucsnak [11]. The results in the papers quoted above reveal surprising and deep connections between controllability and reachability theory for the heat equation and spaces of analytic or Gevrey type functions and open the way towards new applications, namely for the control of nonlinear parabolic equations and for time optimal control problems (with point target) for the heat equation. The main contribution brought in by the present work consists in providing a complete characterization of the reachable space of the system described by the heat equation in one space dimension with Dirichlet boundary control. Our results, presented in terms of sums of classical Hilbert spaces of analytic functions are sharp, in the following sense: unlike the existing results quoted above, which assert that the reachable space is sandwiched between two spaces of analytic functions, we prove that this space *coincides* with the sum of two spaces of analytic functions. This main result is further applied in obtaining a complete characterization of the space of functions which can be reached by smooth (in a Sobolev scale) inputs and then in deriving an estimate for the cost of null controllability in small time. Note that very recently Orsoni [20] gave an apparently different characterization of the reachable space for the same system. This result motivated the discussion in Section 7 below, where we obtain new connections, which might be of independent interest, concerning sums of possibly weighted Bergman spaces.

In the remaining part of this introduction we give an overview of the existing theory and we state the main results which will be proved in the following sections.

We consider the system

$$(1.1) \quad \begin{cases} \frac{\partial w}{\partial t}(t, x) = \frac{\partial^2 w}{\partial x^2}(t, x) & t \geq 0, x \in (0, \pi), \\ w(t, 0) = u_0(t), \quad w(t, \pi) = u_\pi(t) & t \in [0, \infty), \\ w(0, x) = 0 & x \in (0, \pi), \end{cases}$$

which models the heat propagation in a rod of length π , controlled by prescribing the temperature at both ends. It is well known that for every $u_0, u_\pi \in L^2[0, \infty)$ the problem (1.1) admits a unique solution w and that the restriction of this function to $(0, \infty) \times (0, \pi)$ is an analytic function. The *input-to-state maps* (briefly, input maps) $(\Phi_\tau)_{\tau \geq 0}$ are defined by

$$\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = w(\tau, \cdot) \quad (\tau \geq 0, u_0, u_\pi \in L^2[0, \tau]).$$

Determining the reachable space at instant τ of the system determined by the 1D heat equation with boundary control consists in determining $\text{Ran } \Phi_\tau$. This question was considered long time as elusive so that the efforts went first towards determining the largest possible spaces of $\text{Ran } \Phi_\tau$. As mentioned above, the beginnings of the research in this direction go back to [5], where it is shown that $\text{Ran } \Phi_\tau$ contains the space of continuous functions which are 2π -periodic on \mathbb{R} , which extend holomorphically on the strip $|\text{Im } z| < \frac{\pi}{2}$ and with all the derivatives of even order vanishing at $x = 0$ and $x = \pi$. This last condition is quite restrictive since it does

not provide information on the reachability of very smooth functions (like polynomials) which are not vanishing at $x = 0$ and $x = \pi$. This lack of information has been partially filled in Schmidt [22] (see also references therein), where it has been proved that particular types of smooth functions (in particular polynomials) are in the reachable space, independently of their boundary values.

The interest on this fascinating question has been revealed by the recent work [18]. To give a precise statement of the main recent contributions to this problem we need some notation which will be used in the remaining part of this work.

Given an open set Ω containing $(0, \pi)$, we denote by $\text{Hol}(\Omega)$ the space of continuous functions on $(0, \pi)$ admitting a holomorphic extension to Ω . Moreover we identify these functions with their holomorphic extensions to Ω . The article [18] is, to our knowledge, the first work proving that $\text{Ran } \Phi_\tau$ can be sandwiched between two spaces of analytic functions. More precisely, the main result in [18] asserts that

$$\text{Hol}(\tilde{D}) \subset \text{Ran } \Phi_\tau \subset \text{Hol}(D),$$

where \tilde{D} is the disk centered in $\frac{\pi}{2}$ and of diameter $\pi e^{(2e)^{-1}}$ and

$$(1.2) \quad D = \{s = x + iy \in \mathbb{C} \mid |y| < x \text{ and } |y| < \pi - x\},$$

This result has been further improved in [2], where it has been shown that for every $\varepsilon > 0$ we have

$$\text{Hol}(D_\varepsilon) \subset \text{Ran } \Phi_\tau \subset \text{Hol}(D),$$

where D_ε is the set of those $s \in \mathbb{C}$ such that $\text{dist}(s, D) < \varepsilon$. A significant advancement towards a characterization, in terms of Banach spaces of analytic functions, of $\text{Ran } \Phi_\tau$ has been obtained in [11]. In this work it has been proved that

$$E^2(D) \subset \text{Ran } \Phi_\tau \subset A^2(D),$$

where $E^2(D)$ and $A^2(D)$ are the Hardy-Smirnov and Bergman spaces on D , respectively. For reader's convenience we remind simplified definitions of these spaces, which are

$$(1.3) \quad E^2(D) = \left\{ f \in \text{Hol}(D) \cap L^2(\partial D) \mid \int_{\partial D} f(\zeta) \zeta^n dz = 0 \text{ for all } n \geq 1 \right\},$$

$$(1.4) \quad A^2(D) = \text{Hol}(D) \cap L^2(D).$$

When these spaces are endowed, respectively, with norms

$$\|f\|_{E^2(D)}^2 = \int_{\partial D} |f(\zeta)|^2 |d\zeta| \quad \text{and} \quad \|f\|_{A^2(D)}^2 = \int_D |f(x + iy)|^2 dx dy,$$

they become Hilbert spaces. The main new result in this paper is a complete characterization of $\text{Ran } \Phi_\tau$ in terms of the sum of two weighted Bergman spaces. A surprising consequence of this result is that although each one of these spaces depends on a parameter $\delta > 0$, their sum is independent of this parameter (firstly for δ small enough and then for all $\delta > 0$, see Theorem 1.1 and Proposition 1.1 below). To state this result we introduce the sets

$$(1.5) \quad \Delta = \left\{ s \in \mathbb{C} \mid -\frac{\pi}{4} < \arg s < \frac{\pi}{4} \right\}, \quad \tilde{\Delta} = \pi - \Delta,$$

the weight functions

$$(1.6) \quad \omega_{0,\delta}(s) = \frac{e^{\frac{\text{Re}(s^2)}{2\delta}}}{\delta} \quad (\delta > 0, s \in \Delta),$$

$$(1.7) \quad \omega_{\pi,\delta}(\tilde{s}) = \frac{e^{\frac{\operatorname{Re}[(\pi-\tilde{s})^2]}{2\delta}}}{\delta} \quad (\delta > 0, \tilde{s} \in \tilde{\Delta}),$$

and the weighted Bergman spaces $A^2(\Delta, \omega_{0,\delta})$ and $A^2(\tilde{\Delta}, \omega_{\pi,\delta})$:

$$\|f\|_{A^2(\Delta, \omega_{0,\delta})}^2 = \left\{ f \in \operatorname{Hol}(\Delta) \mid \int_{\Delta} |f(x+iy)|^2 \omega_{0,\delta}(x+iy) dx dy < \infty \right\},$$

$$A^2(\tilde{\Delta}, \omega_{\pi,\delta}) = \left\{ f \in \operatorname{Hol}(\tilde{\Delta}) \mid \int_{\tilde{\Delta}} |f(x+iy)|^2 \omega_{\pi,\delta}(x+iy) dx dy < \infty \right\}.$$

When endowed with the norms

$$\|f\|_{A^2(\Delta, \omega_{0,\delta})}^2 = \int_{\Delta} |f(x+iy)|^2 \omega_{0,\delta}(x+iy) dx dy \quad (f \in A^2(\Delta, \omega_{0,\delta})),$$

$$\|\tilde{f}\|_{A^2(\tilde{\Delta}, \omega_{\pi,\delta})}^2 = \int_{\tilde{\Delta}} |\tilde{f}(x+iy)|^2 \omega_{\pi,\delta}(x+iy) dx dy \quad (\tilde{f} \in A^2(\tilde{\Delta}, \omega_{\pi,\delta})),$$

$A^2(\Delta, \omega_{0,\delta})$ and $A^2(\tilde{\Delta}, \omega_{\pi,\delta})$ become Hilbert spaces. An important role in the remaining part of this work will be played by the sum of the two spaces above, i.e., the space X_δ defined for every $\delta > 0$ by

$$(1.8) \quad X_\delta = \left\{ \psi \in C(0, \pi) \mid \begin{array}{l} \exists \varphi_0 \in A^2(\Delta, \omega_{0,\delta}) \\ \exists \varphi_\pi \in A^2(\tilde{\Delta}, \omega_{\pi,\delta}) \end{array}, \psi = \varphi_0 + \varphi_\pi \text{ on } (0, \pi) \right\},$$

which is endowed with the norm

$$(1.9) \quad \|\varphi\|_\delta = \inf \left\{ \|\varphi_0\|_{A^2(\Delta, \omega_{0,\delta})} + \|\varphi_\pi\|_{A^2(\tilde{\Delta}, \omega_{\pi,\delta})} \mid \begin{array}{l} \varphi_0 + \varphi_\pi = \varphi \\ \varphi_0 \in A^2(\Delta, \omega_{0,\delta}) \\ \varphi_\pi \in A^2(\tilde{\Delta}, \omega_{\pi,\delta}) \end{array} \right\}.$$

Theorem 1.1. *For every $\tau > 0$ we have*

$$(1.10) \quad \Phi_\tau \in \mathcal{L}(L^2([0, \tau]; \mathbb{C}^2); X_\tau).$$

Moreover, there exists $\delta^ > 0$ such that for every $\tau > 0$ and every $\delta \in (0, \delta^*)$ we have*

$$(1.11) \quad \operatorname{Ran} \Phi_\tau = X_\delta \quad (\tau > 0).$$

According to a general property of control LTI systems which are null controllable in any time (see Proposition 3.1 below), it is known that $\operatorname{Ran} \Phi_\tau$ is independent of $\tau > 0$. This fact is one of the ingredients of the proof of Theorem 1.1, as shown in Section 4. The fact that the sum of Bergman spaces in the right hand side of (1.8) is independent of $\delta \in (0, \delta^*)$ seems a new result, which is strengthened by the following proposition, which will be proved in Section 7.

Proposition 1.1. *For each $\delta > 0$ let X_δ is the space defined in (1.8). Then*

$$X_t = X_\tau \quad (t, \tau > 0).$$

Putting together Theorem 1.1 and Proposition 1.1 we obtain

Corollary 1.1. *With the notation in Theorem 1.1 we have*

$$\operatorname{Ran} \Phi_\tau = X_\delta \quad (\tau, \delta > 0).$$

The remaining part of this work is organized as follows. In Section 2 we remind, in order to introduce the appropriate vocabulary, some necessary concepts from linear finite dimensional systems theory. Section 3 is devoted to some background on infinite dimensional well-posed linear systems, with emphasis on the concept of reachable space and on its general properties. Moreover, we end this section by showing that the system determined by the boundary controlled heat equation fits the introduced abstract framework and we describe the corresponding output maps. Section 4 is devoted to the proof of the main result, which gives a complete characterization of the reachable space. In Section 5 this result is applied to give a complete characterization of the reachable space obtained via controls which are n times differentiable and with derivatives up to order $n - 1$ vanishing at the initial time. Section 6 provides a link, of possible interest for improving the existing estimates on the control cost in small time, between the natural norm on the reachable space and the above mentioned control cost. Finally, in Section 7 we discuss the implications of our results on sums of Bergman spaces on infinite sectors and we obtain a new proof of a different characterization of the reachable space, recently obtained in Orsoni [20].

Notation.

Throughout this paper, the notation \mathbb{N} , \mathbb{Z} stands for the sets of natural numbers (starting with 1) and integer numbers, respectively. We denote $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$.

2. SOME BACKGROUND ON FINITE DIMENSIONAL LINEAR TIME INVARIANT SYSTEMS

In this section, in order to introduce several concepts and operators in a simple motivating case, we briefly remind some well-known facts for linear time invariant systems (LTIs) in the finite dimensional case. For more details on this subject we refer to good introductory chapters on this classical subject, such as D'Azzo and Houpis [4], Friedland [8], Ionescu, Oară and Weiss [12], Kwakernaak and Sivan [13], Maciejowski [17], Rugh [21] and Wonham [31].

Let U (the input space) and X (the state space) be finite-dimensional inner product spaces, with $\dim X = n$. A finite-dimensional linear time-invariant control system with input space U and state space X is traditionally described by the equations

$$(2.1) \quad \dot{z}(t) = Az(t) + Bu(t) \quad (t \geq 0).$$

In the above equations $u \in L^2([0, \infty); U)$ is the *input function* and $z \in C([0, \infty); X)$ is the *state trajectory*, whereas A, B are linear operators such that $A : X \rightarrow X$, and $B : U \rightarrow X$. By the variation of constants formula (sometimes called Duhamel's formula) equation (2.1) yields

$$(2.2) \quad z(t) = e^{tA}z(0) + \int_0^t e^{(t-\sigma)A}Bu(\sigma) d\sigma \quad (t \geq 0).$$

In the above formula we can notice the appearance of two families of operators $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ (the C^0 semigroup on X generated by A) and $\Phi = (\Phi_t)_{t \geq 0}$ (the input to state maps) defined by

$$(2.3) \quad \mathbb{T}_t\varphi = e^{tA}\varphi \quad (t \geq 0, \varphi \in X),$$

$$(2.4) \quad \Phi_t u = \int_0^t e^{(t-\sigma)A} B u(\sigma) d\sigma \quad (t \geq 0, u \in L^2([0, \infty); U)).$$

The two operator families above have important properties (which will be used below to define general well-posed control systems). More precisely:

- $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ is a C^0 semigroup of operators (briefly an *operator semigroup*) on X , which means that $\mathbb{T}_t \in \mathcal{L}(X)$ for every $t \geq 0$ and

$$(2.5) \quad \mathbb{T}_0 \varphi = \varphi \quad (\varphi \in X),$$

$$(2.6) \quad \mathbb{T}_{t+\tau} = \mathbb{T}_t \mathbb{T}_\tau \quad (t, \tau \geq 0),$$

$$(2.7) \quad \lim_{t \rightarrow 0^+} \mathbb{T}_t \varphi = \varphi \quad (\varphi \in X).$$

- For every $t \geq 0$ we have $\Phi_t \in \mathcal{L}(L^2([0, \infty); U), X)$ and

$$(2.8) \quad \Phi_{\tau+t}(u \underset{\tau}{\diamond} v) = \mathbb{T}_t \Phi_\tau u + \Phi_t v \quad (t, \tau \geq 0),$$

where the τ -concatenation of two signals u and v , denoted $u \underset{\tau}{\diamond} v$, is the function

$$(2.9) \quad u \underset{\tau}{\diamond} v = \begin{cases} u(t) & \text{for } t \in [0, \tau), \\ v(t - \tau) & \text{for } t \geq \tau. \end{cases}$$

Let us also note that, with the above notation, formula (2.2) can be rewritten

$$z(t) = \mathbb{T}_t z(0) + \Phi_t u \quad (t \geq 0).$$

Definition 2.1. Given a finite dimensional LTI control system described by (2.1) and $\tau > 0$, the reachable space of this system at time $\tau > 0$ is the range $\text{Ran } \Phi_\tau$ of the operator Φ_τ defined in (2.4). The system is said controllable in time τ if $\text{Ran } \Phi_\tau = X$.

The result below shows that, within the very simple framework considered in this section, the reachable space and the controllability property do not depend on the time $\tau > 0$. More precisely, the following result, known as the *Kalman rank condition* for controllability holds:

Theorem 2.1. *We have, for every $\tau > 0$,*

$$(2.10) \quad \text{Ran } \Phi_\tau = \text{Ran} [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B].$$

Moreover, the pair (A, B) is controllable if and only if

$$\text{rank} [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] = n.$$

For exactly controllable systems there exist controls steering the state trajectory z of (2.1) from any initial state z_0 to any final state z_1 . Among these controls there is one of “minimal energy”. To state these facts in a precise manner we need the concept of controllability Gramian R_τ of the pair (A, B) , which is defined by

$$R_\tau = \Phi_\tau \Phi_\tau^* \quad (\tau \geq 0).$$

It is easily checked that $R_\tau \in \mathcal{L}(X)$ can alternatively be written as

$$R_\tau = \int_0^\tau e^{tA} B B^* e^{tA^*} dt,$$

and that (A, B) is controllable iff R_τ is a strictly positive operator. Moreover, we have

Proposition 2.1. *Suppose that (A, B) is controllable and let $z_0, z_1 \in X, \tau > 0$. If*

$$u = \Phi_\tau^* R_\tau^{-1} (z_1 - \mathbb{T}_\tau z_0),$$

then the corresponding state trajectory z of (2.1) satisfies $z(0) = z_0$ and $z(\tau) = z_1$. Moreover, among all the inputs $v \in L^2([0, \tau]; U)$ for which $z(0) = z_0$ and $z(\tau) = z_1$, u is the unique one that has minimal $L^2([0, \tau]; U)$ norm.

Remark 2.1. From (2.10) it follows that $\text{Ran } \Phi_\tau$ contains $\text{Ran } B$ and it is invariant under A , thus under \mathbb{T}_t , for every $t \geq 0$. Denoting by \tilde{A} and $\tilde{\mathbb{T}}$ the restrictions of A and of \mathbb{T} to $\text{Ran } \Phi_\tau$, the above facts imply that the input maps of (\tilde{A}, B) coincide with those of (A, B) . We thus have that $(\Phi, \tilde{\mathbb{T}})$ (alternatively described by (\tilde{A}, B)) is an exactly controllable LTI system with state space $\text{Ran } \Phi_\tau$ and input space U .

Remark 2.2. Noting that, given $\tau > 0$, the space $L^\infty([0, \tau]; U)$ can be seen as a subspace of $L^2([0, \infty); U)$ and having in mind that controls which can be effectively applied are generally bounded functions on time, a natural question is the characterization of the set $\text{Ran } \Phi_\tau^\infty$, where Φ_τ^∞ is the restriction of Φ_τ to $L^\infty([0, \tau]; U)$. It turns out that, in the simple case considered in this section, $\text{Ran } \Phi_\tau^\infty$ coincides with $\text{Ran } \Phi_\tau$. Indeed, let $\eta \in \text{Ran } \Phi_\tau$. As seen in Remark 2.1, the system $(\Phi, \tilde{\mathbb{T}})$ is exactly controllable. Thus, the minimal $L^2([0, \tau]; U)$ control \tilde{u} steering its state trajectory from 0 at $t = 0$ to η at $t = \tau$ is given, according to Proposition 2.1, by

$$\tilde{u} = \Phi_\tau^* \tilde{R}_\tau^{-1} \eta,$$

where \tilde{R}_τ is the controllability Gramian in time τ of the system (\tilde{A}, B) . The control \tilde{u} above clearly steers the state trajectory of the original system (A, B) from the null state at $t = 0$ to the state η at $t = \tau$ and \tilde{u} obviously extends to an analytic function from \mathbb{C} to U . We have thus shown the stronger property that $\text{Ran } \Phi_\tau$ coincides with the range of the restriction of Φ_τ to signals which can be extended to analytic functions from \mathbb{C} to U .

3. REACHABLE SPACE FOR WELL-POSED LINEAR CONTROL SYSTEMS

We begin this section by reminding some basic definitions and properties (mostly without proofs but with appropriate references) of well-posed linear control time invariant systems, with emphasis on the concept of reachability and on properties of the reachable space. These systems provide a framework to generalize some of the concepts in classical linear control theory to infinite dimensional systems. In particular, all finite dimensional control LTI systems are well-posed in the sense of the definition below, which are general enough to include some basic examples of systems governed by partial differential equations (such as the heat, Schrödinger and wave equations) with boundary control. We generally do not give proofs and we refer to Weiss [30] (where these systems have been introduced under the name of *abstract control systems*), Tucsnak and Weiss [27, Chapters 2,3,4], [28] and references therein for more information, including detailed proofs. In the last part of this section we also provide proofs for two results which are quite simple but which play an important role in this paper.

Let U (the input space) and X (the state space) be Hilbert spaces (possibly infinite-dimensional). From a system theoretic view-point the simplest way to define a linear well-posed time invariant systems in a possibly infinite dimensional setting is to introduce families of operators, inspired by those in (2.3), (2.4) and sharing a part of their properties.

Definition 3.1. Let U and Y be Hilbert spaces. A *well-posed linear control system* is a couple (\mathbb{T}, Φ) of families of operators such that

- (1) $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ is an operator semigroup on X , i.e., it satisfies conditions (2.5)-(2.7);
- (2) $\Phi = (\Phi_t)_{t \geq 0}$ is a family of bounded linear operators from $L^2([0, \infty); U)$ to X such that (2.8) holds for every $u, v \in L^2([0, \infty); U)$ and all $\tau, t \geq 0$.

It follows from the above definition that Φ is *causal*, i.e., the state does not depend on the future input: $\Phi_\tau \Pi_\tau = \Phi_\tau$ for all $\tau \geq 0$, where Π_τ stands for the orthogonal projection from $L^2([0, \infty); U)$ onto $L^2[0, \tau]; U)$. Moreover, it can be shown that the above properties imply that the map

$$(t, u) \longmapsto \Phi_t u,$$

is continuous from $[0, \infty) \times L^2([0, \infty); U)$ to X .

From a PDEs viewpoint, the above definition is, in general, not easy to use. In most of the cases encountered in applications, an infinite dimensional system is described by evolution partial differential equations with appropriate boundary conditions (some of them being the boundary controls), thus by partial differential and trace operators. To describe such a system in the terms of Definition 3.1 one needs to define a notion of solution of the considered PDE system and to prove appropriate existence and uniqueness results for these solutions (including the “correct” choices for X and U) and allowing to define the families of operators (\mathbb{T}_t) and (Φ_t) . In general there are no explicit formulas for these families of operators. However, as shown at the end of this section, such formulas are available for the system described by the first two equations in (1.1), so this view point is quite convenient in our case.

As in the finite dimensional case, the *reachable space* of a well-posed control system at time $\tau > 0$ is defined as $\text{Ran } \Phi_\tau$. Unlike the finite dimensional case, in this more general framework there is no simple characterization of the reachable space in term of the operators A and B . Moreover, this space depends in general on τ and for most systems described by partial differential equations we have only a small amount of information on the reachable space. Another difference with respect to the finite dimensional case is that the range $\text{Ran } \Phi_\tau^\infty$ of the restriction of Φ_τ to $L^\infty([0, \tau]; U)$ is in general a strict subset of $\text{Ran } \Phi_\tau$. We also note that, given $\tau > 0$, the reachable space $\text{Ran } \Phi_\tau$ can be endowed with the norm induced from $L^2([0, \tau]; U)$ which is

$$(3.1) \quad \|\eta\|_{\text{Ran } \Phi_\tau} = \inf_{\substack{u \in L^2([0, \tau]; U) \\ \Phi_\tau u = \psi}} \|u\|_{L^2([0, \tau]; U)} \quad (\eta \in \text{Ran } \Phi_\tau).$$

As mentioned in the Introduction, the concept of reachable set plays an essential role in control theory. It appears, in particular, in the definition of the main three controllability concepts used in infinite dimensional system theory.

Definition 3.2. Let $\tau > 0$ and let the pair (\mathbb{T}, Φ) define a well-posed control LTI system.

- The pair (\mathbb{T}, Φ) is *exactly controllable in time τ* if $\text{Ran } \Phi_\tau = X$.
- (\mathbb{T}, Φ) is *approximately controllable in time τ* if $\text{Ran } \Phi_\tau$ is dense in X .
- The pair (\mathbb{T}, Φ) is *null-controllable in time τ* if $\text{Ran } \Phi_\tau \supset \text{Ran } \mathbb{T}_\tau$.

Remark 3.1. Let (\mathbb{T}, Φ) be a well-posed linear LTI control system which is approximately controllable in time τ . It is not difficult to check for every $\eta \in \text{Ran } \Phi_\tau$ there exists a unique $\psi \in X$ such that $\eta = \Phi_\tau \Phi_\tau^* \psi$. Moreover, we have

$$\|\eta\|_{\text{Ran } \Phi_\tau} = \|\Phi_\tau^* \psi\|_{L^2([0, \tau]; U)}.$$

The above facts imply that $\text{Ran } \Phi_\tau$ endowed with the norm (3.1) is a Banach space. Indeed, let $(\eta_k)_{k \in \mathbb{N}} \subset \text{Ran } \Phi_\tau$ be a Cauchy sequence with respect to the norm (3.1). For each $k \in \mathbb{N}$ let ψ_k be the unique vector in X such that $\eta_k = \Phi_\tau \Phi_\tau^* \psi_k$. Then for every $k, l \in \mathbb{N}$ we have

$$\eta_k - \eta_l = \Phi_\tau \Phi_\tau^* (\psi_k - \psi_l), \quad \|\eta_k - \eta_l\|_{\text{Ran } \Phi_\tau} = \|\Phi_\tau^* (\psi_k - \psi_l)\|_{L^2([0, \tau]; U)}.$$

It follows that $(\Phi_\tau^* \psi_k)$ is a Cauchy sequence in $L^2([0, \tau]; U)$ so that there

$$\lim_{k \rightarrow \infty} \|\Phi_\tau^* \psi_k - v\|_{L^2([0, \tau]; U)} = 0$$

for some $v \in L^2([0, \tau]; U)$. Setting $\eta = \Phi_\tau v$ we see that $\|\eta_k - \eta\|_{\text{Ran } \Phi_\tau} \rightarrow 0$, thus we obtain the desired conclusion.

We continue this section with two results on well-posed control LTI systems which are null-controllable in any time $\tau > 0$. Although the first one is classical, see Fattorini [7], we give a very short proof below, following essentially Seidman [23].

Proposition 3.1. *Assume that the well-posed control LTI system (\mathbb{T}, Φ) is null controllable in any positive time. Then $\text{Ran } \Phi_\tau$ does not depend on $\tau > 0$.*

Proof. Let $0 < \tau < t$. The inclusion $\text{Ran } \Phi_\tau \subset \text{Ran } \Phi_t$ is easy to establish. Indeed, let $\eta \in \text{Ran } \Phi_\tau$ and \tilde{u} be a control such that $\Phi_\tau \tilde{u} = \eta$. Let $u = 0 \underset{t-\tau}{\diamond} \tilde{u}$ (for the notation $\underset{t-\tau}{\diamond}$ we remind (2.9)). Then, according to (2.8)

$$\Phi_t u = \Phi_{(t-\tau)+\tau} u = \mathbb{T}_\tau \Phi_{t-\tau} 0 + \Phi_\tau \tilde{u} = \eta,$$

thus we have shown that $\text{Ran } \Phi_\tau \subset \text{Ran } \Phi_t$.

To establish the inclusion $\text{Ran } \Phi_t \subset \text{Ran } \Phi_\tau$, take $\eta \in \text{Ran } \Phi_t$ and $u \in L^2([0, \tau]; U)$ such that $\Phi_t u = \eta$. Denoting

$$\tilde{u}(\sigma) = u(\sigma + t - \tau) \quad (\sigma \in [0, \tau]),$$

we remark that $u = u \underset{t-\tau}{\diamond} \tilde{u}$. Consequently, applying again (2.8), it follows that

$$\eta = \Phi_t u = \Phi_{(t-\tau)+\tau} (u \underset{t-\tau}{\diamond} \tilde{u}) = \mathbb{T}_\tau \Phi_{t-\tau} u + \Phi_\tau \tilde{u}.$$

Since the system is null controllable in any time, we have $\text{Ran } \Phi_\tau \supset \text{Ran } \mathbb{T}_\tau$. This, combined with the above formula, implies that $\eta \in \text{Ran } \Phi_\tau$, so that indeed we have $\text{Ran } \Phi_t \subset \text{Ran } \Phi_\tau$, which ends the proof. \square

The following result in this section, although simple, does not seem to have been explicitly stated in the literature.

Proposition 3.2. *Let (\mathbb{T}, Φ) be a well-posed control LTI system which is null controllable in any positive time and let $\tau, \alpha > 0$. Denote*

$$(3.2) \quad \mathcal{U}_{\tau, \alpha} = \{u \in L^2([0, \tau]; U) \mid (t \mapsto t^{-\alpha} u(t)) \in L^2([0, \tau]; U)\}.$$

Then for every $\alpha > 0, \tau > 0$ we have $\Phi_\tau(\mathcal{U}_{\tau, \alpha}) = \text{Ran } \Phi_\tau$.

Proof. The inclusion $\Phi_\tau(\mathcal{U}_{\tau,\alpha}) \subset \text{Ran } \Phi_\tau$ being an obvious one we only need to check that $\text{Ran } \Phi_\tau \subset \Phi_\tau(\mathcal{U}_{\tau,\alpha})$. To this aim we note that according to Proposition 3.1 we have $\text{Ran } \Phi_{\frac{\tau}{2}} = \text{Ran } \Phi_\tau$, thus for every $\eta \in \text{Ran } \Phi_\tau$ there exists $u \in L^2([0, \infty); U)$ such that $\Phi_{\frac{\tau}{2}} u = \eta$. Setting $\tilde{u} = 0 \diamond_{\frac{\tau}{2}} u$ and applying (2.8) it follows that

$$\Phi_\tau \tilde{u} = \Phi_{\frac{\tau}{2} + \frac{\tau}{2}}(0 \diamond_{\frac{\tau}{2}} u) = \mathbb{T}_{\frac{\tau}{2}} \Phi_{\frac{\tau}{2}} 0 + \Phi_{\frac{\tau}{2}} u = \eta.$$

Moreover, since $\tilde{u} = 0$ on $[0, \frac{\tau}{2}]$ we have that $\tilde{u} \in \mathcal{U}_{\tau,\alpha}$ for every $\alpha > 0$, so that $\eta \in \Phi_\tau(\mathcal{U}_{\tau,\alpha})$, which ends the proof. \square

We end this section by reminding the known fact (this goes back to [6], see also [27, Proposition 10.7.1]) that the two first equations of (1.1) determine a well-posed control LTI system, with appropriate choices for X and U , which is null controllable in any positive time. Moreover, we remind an expression of the output maps which has already been used in [11].

Proposition 3.3. *The first two equations in (1.1) determine a well posed control LTI system with state space $X = W^{-1,2}(0, \pi)$ and input space $U = \mathbb{C}^2$. Moreover the corresponding family Φ of input maps is given by*

$$(3.3) \quad \left(\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} \right) (x) = \int_0^\tau \frac{\partial K_0}{\partial x}(\tau - \sigma, x) u_0(\sigma) d\sigma \\ + \int_0^\tau \frac{\partial K_\pi}{\partial x}(\tau - \sigma, x) u_\pi(\sigma) d\sigma \quad (\tau > 0, u_0, u_\pi \in L^2[0, \tau], x \in (0, \pi)),$$

where

$$(3.4) \quad K_0(\sigma, x) = -\sqrt{\frac{1}{\pi\sigma}} \sum_{m \in \mathbb{Z}} e^{-\frac{(x+2m\pi)^2}{4\sigma}} \quad (\sigma > 0, x \in [0, \pi]),$$

$$(3.5) \quad K_\pi(\sigma, x) = \sqrt{\frac{1}{\pi\sigma}} \sum_{m \in \mathbb{Z}} e^{-\frac{(x+(2m-1)\pi)^2}{4\sigma}} \quad (\sigma > 0, x \in [0, \pi]).$$

Finally, the considered system is null-controllable in any time $\tau > 0$.

4. PROOF OF THEOREM 1.1

As already mentioned, the proof of Theorem 1.1 uses in an essential manner results from [11]. More precisely, the main ingredient of this proof is a result which is not explicitly stated in [11], but which is implicitly proved in this reference. To make this clear, we give its precise statement and we describe the main steps of the proof.

Proposition 4.1. *For $\tau > 0$ let $\mathcal{U}_{\tau, \frac{1}{2}}$ be the set defined in (3.2) (with $\alpha = \frac{1}{2}$). Then there exists $\delta^* > 0$ such that*

$$(4.1) \quad \Phi_\tau(\mathcal{U}_{\tau, \frac{1}{2}}) = A^2(\Delta, \omega_{0,\tau}) + A^2(\tilde{\Delta}, \omega_{\pi,\tau}) \quad (\tau \in (0, \delta^*)),$$

where the weights $\omega_{0,\tau}$ and $\omega_{\pi,\tau}$ have been introduced in (1.6) and (1.7), respectively.

To prove the above result we introduce some notation and remind some results from [11]. We first introduce the families of operators

$$(4.2) \quad (P_\tau f)(s) = \int_0^\tau \frac{se^{-\frac{s^2}{4(\tau-\sigma)}}}{2\sqrt{\pi}(\tau-\sigma)^{\frac{3}{2}}} f(\sigma) \sqrt{\sigma} d\sigma \quad (\tau \geq 0, f \in L^2[0, \tau], s \in \Delta),$$

(4.3)

$$(Q_\tau g)(s) = \int_0^\tau \frac{(\pi-s)e^{-\frac{(\pi-s)^2}{4(\tau-\sigma)}}}{2\sqrt{\pi}(\tau-\sigma)^{\frac{3}{2}}} g(\sigma)\sqrt{\sigma} d\sigma \quad (\tau \geq 0, g \in L^2[0, \tau], s \in \tilde{\Delta}),$$

where the sets Δ and $\tilde{\Delta}$ have been introduced in (1.5). Each of the two operators above can be seen as the input maps of a system governed by the boundary controlled heat equation on a half-line. Looking, for instance, to $(P_\tau)_{\tau \geq 0}$ and setting

$$w_l(t, x) = (P_t f)(x) \quad (t \geq 0, x \geq 0),$$

we have that

$$\begin{cases} \frac{\partial w_l}{\partial t}(t, x) = \frac{\partial^2 w_l}{\partial x^2}(t, x) & (t \geq 0, x \geq 0), \\ w_l(t, 0) = \sqrt{t}f(t), & t \in [0, \infty), \\ w_l(0, x) = 0 & x \geq 0. \end{cases}$$

Using results from Aikawa, Hayashi and Saitoh [1] it has been shown in [11, Theorem 2.2 and Corollary 2.3] that the following result holds:

Lemma 4.1. *For every $\tau > 0$ the operator $\begin{bmatrix} P_\tau & 0 \\ 0 & Q_\tau \end{bmatrix}$ is bounded and invertible from $L^2([0, \pi])^2$ onto $A^2(\Delta, \omega_{0, \tau}) \times A^2(\tilde{\Delta}, \omega_{\pi, \tau})$. Moreover,*

$$\left\| \begin{bmatrix} P_\tau & 0 \\ 0 & Q_\tau \end{bmatrix} \right\|_{\mathcal{L}((L^2([0, \pi]))^2, A^2(\Delta, \omega_{0, \tau}) \times A^2(\tilde{\Delta}, \omega_{\pi, \tau}))} = 1 \quad (\tau > 0).$$

For $\tau > 0$ we introduce the family of operators $(M_\tau)_{\tau > 0}$ defined by

$$(4.4) \quad M_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \begin{bmatrix} g_0(u_0) \\ g_\pi(u_\pi) \end{bmatrix} \quad (u_0, u_\pi \in L^2[0, \tau]),$$

where

$$g_0(u_0)(s) = \int_0^\tau \frac{\partial K_0}{\partial s}(\sigma, s)\sqrt{\sigma} u_0(\sigma) d\sigma \quad (u_0 \in L^2[0, \tau], s \in \Delta),$$

$$g_\pi(u_\pi)(s) = \int_0^\tau \frac{\partial K_\pi}{\partial s}(\sigma, s)\sqrt{\sigma} u_\pi(\sigma) d\sigma \quad (u_\pi \in L^2[0, \tau], s \in \tilde{\Delta}),$$

and the kernels K_0 and K_π have been introduced in (3.4) and (3.5), respectively. Comparing the above formulas with (3.3) we see that

$$(4.5) \quad g_0(u_0)(s) + g_\pi(u_\pi)(s) = \Phi_\tau \begin{bmatrix} v_0 \\ v_\pi \end{bmatrix} (s) \quad (s \in D),$$

where

$$v_0(t) = \sqrt{t} u_0(t), \quad v_\pi(t) = \sqrt{t} u_\pi(t) \quad (t \in [0, \tau]),$$

and D has been defined in (1.2).

Another important estimate proved in [11] is

Lemma 4.2. *For every $\tau > 0$ the operator M_τ defined in (4.4) is bounded from $L^2([0, \pi])^2$ to $A^2(\Delta, \omega_{0, \tau}) \times A^2(\tilde{\Delta}, \omega_{\pi, \tau})$. Moreover,*

$$\lim_{\tau \rightarrow 0^+} \left\| M_\tau - \begin{bmatrix} P_\tau & 0 \\ 0 & Q_\tau \end{bmatrix} \right\|_{\mathcal{L}((L^2([0, \pi]))^2, A^2(\Delta, \omega_{0, \tau}) \times A^2(\tilde{\Delta}, \omega_{\pi, \tau}))} = 0.$$

We are now in a position to prove Proposition 4.1.

Proof of Proposition 4.1. Let $\varphi \in A^2(\Delta, \omega_0) + A^2(\tilde{\Delta}, \omega_\pi)$, so that there exist $\varphi_0 \in A^2(\Delta, \omega_0)$ and $\varphi_\pi \in A^2(\tilde{\Delta}, \omega_\pi)$ such that $\varphi = \varphi_0 + \varphi_\pi$. By combining Lemma 4.1 and Lemma 4.2 it follows that there exists $\delta^* > 0$ such that the operator M_τ is bounded and invertible from $L^2([0, \pi])^2$ onto $A^2(\Delta, \omega_0) \times A^2(\tilde{\Delta}, \omega_\pi)$ for every $\tau \in (0, \delta^*)$. According to the definition (4.4) it follows that for every $\tau \in (0, \delta^*)$ there exist $\tilde{u}_0, \tilde{u}_\pi \in L^2[0, \tau]$ such that

$$\begin{aligned} \int_0^\tau \frac{\partial K_0}{\partial s}(\sigma, s) \sqrt{\sigma} \tilde{u}_0(\sigma) d\sigma &= \varphi_0(s) & (\tau \in (0, \delta^*), s \in \Delta), \\ \int_0^\tau \frac{\partial K_\pi}{\partial s}(\sigma, s) \sqrt{\sigma} \tilde{u}_\pi(\sigma) d\sigma &= \varphi_\pi(s) & (\tau \in (0, \delta^*), s \in \tilde{\Delta}). \end{aligned}$$

The last two formulas, combined with (3.3) imply that

$$\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \varphi_0 + \varphi_\pi = \varphi \quad (\tau \in (0, \delta^*)),$$

where $u_0(t) = \sqrt{t} \tilde{u}_0(t)$ and $u_\pi(t) = \sqrt{t} \tilde{u}_\pi(t)$ for $t \in [0, \tau]$. The conclusion (4.1) follows now from the obvious fact that u_0, u_π lie in $\mathcal{U}_{\tau, \frac{1}{2}}$. \square

Finally, we give below the proof of Theorem 1.1.

Proof. The fact that Φ_τ is bounded from $(L^2[0, \tau])^2$ to $A^2(\Delta, \omega_{0,\tau}) + A^2(\tilde{\Delta}, \omega_{\pi,\tau})$ is shown in the proof of Proposition 2.1 from [11], but, for the sake of completeness, we make this clear below.

For $u_0, u_\pi \in L^2[0, \tau]$ we note that from (3.3) it follows that

$$\begin{aligned} \left(\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} \right) (x) &= \int_0^{\frac{\tau}{2}} \frac{\partial K_0}{\partial x}(\tau - \sigma, x) u_0(\sigma) d\sigma + \int_0^{\frac{\tau}{2}} \frac{\partial K_\pi}{\partial x}(\tau - \sigma, x) u_\pi(\sigma) d\sigma \\ &\quad + \int_0^\tau \frac{\partial K_0}{\partial x}(\tau - \sigma, x) \tilde{u}_0(\sigma) \sqrt{\sigma} d\sigma + \int_0^\tau \frac{\partial K_\pi}{\partial x}(\tau - \sigma, x) \tilde{u}_\pi(\sigma) \sqrt{\sigma} d\sigma, \end{aligned}$$

where, for $\gamma \in \{0, \pi\}$ we define

$$(4.6) \quad \tilde{u}_\gamma(\sigma) := \begin{cases} 0 & \text{if } \sigma \in [0, \tau/2], \\ \frac{u_\gamma(\sigma)}{\sqrt{\sigma}} & \text{if } \sigma \in [\tau/2, \tau]. \end{cases}$$

It can be checked by direct calculations that $\frac{\partial K_\gamma}{\partial s}(\tau - \sigma, \cdot) \in L^2(\partial D)$ for $\sigma \in [0, \tau/2]$ where ∂D is the boundary of the open set D defined in (1.2). Hence, by Cauchy-Schwarz inequality, the operator $\Phi_{\tau,1}$ defined by

$$(4.7) \quad \Phi_{\tau,1} \left(\begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} \right) (s) = \int_0^{\frac{\tau}{2}} \frac{\partial K_0}{\partial s}(\tau - \sigma, s) u_0(\sigma) d\sigma + \int_0^{\frac{\tau}{2}} \frac{\partial K_\pi}{\partial s}(\tau - \sigma, s) u_\pi(\sigma) d\sigma,$$

is linear and bounded from $(L^2[0, \tau])^2$ to the Hardy-Smirnov space $E^2(D)$ defined in (1.3). On the other hand it has been shown in [11] that we have $E^2(D) \subset X_\tau$, where

$$X_\tau = A^2(\Delta, \omega_{0,\tau}) + A^2(\tilde{\Delta}, \omega_{\pi,\tau})$$

with continuous embeddedding so that the operator defined in (4.7) is linear and bounded from $(L^2[0, \tau])^2$ to X_τ . The fact that the operator $\Phi_{\tau,2}$ defined by

$$\begin{aligned} \Phi_{\tau,2} \left(\begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} \right) (s) &= \int_0^\tau \frac{\partial K_0}{\partial s}(\tau - \sigma, s) \tilde{u}_0(\sigma) \sqrt{\sigma} \, d\sigma \\ &\quad + \int_0^\tau \frac{\partial K_\pi}{\partial s}(\tau - \sigma, s) \tilde{u}_\pi(\sigma) \sqrt{\sigma} \, d\sigma, \end{aligned}$$

with \tilde{u}_0 and \tilde{u}_π defined in (4.6), is linear and bounded from $(L^2[0, \tau])^2$ to X_τ follows easily from Lemma 4.2. Putting together the above estimates and the fact that

$$\Phi_\tau = \Phi_{\tau,1} + \Phi_{\tau,2},$$

the conclusion (1.10).

The main assertion of Theorem 1.1 which says that the operator Φ_τ is onto can now be obtained by putting together several of our previous results. Indeed, let δ^* be the constant in Proposition 4.1. By combining Proposition 3.2 and Proposition 4.1 it follows that

$$\text{Ran } \Phi_\delta = A^2(\Delta, \omega_{0,\delta}) + A^2(\tilde{\Delta}, \omega_{\pi,\delta}) \quad (\delta \in (0, \delta^*)).$$

On the other hand, we know from Proposition 3.1 and Proposition 3.3 that $\text{Ran } \Phi_\tau = \text{Ran } \Phi_\delta$ for every $\tau, \delta > 0$, thus we obtain the conclusion (1.11). \square

5. REACHABLE SPACE WITH SMOOTH INPUTS

It can be seen as an obvious consequence of Theorem 1.1 (but it can also be checked in a more elementary manner) that the well-posed control LTI system determined by (1.1), with state space $X = W^{-1,2}(0, \pi)$ and input space $U = L^2[0, \pi]$, is not exactly controllable in any time $\tau > 0$. A natural question is: can this system be seen as an exactly controllable one by choosing a different state space (and possibly a class of smoother input functions)? For the system introduced in Theorem 1.1 this view point has been developed in Laurent and Rosier [15], where such "exact controllability" properties have been exploited to obtain reachability and controllability results for some systems governed by nonlinear parabolic equations. By analogy with Remark 2.1 (valid for finite dimensional LTI systems) and based on our main result in Theorem 1.1, a candidate for the new state space is $X_\delta = \text{Ran } \Phi_\tau$. Indeed, in our case it is easily checked that $\mathbb{T}_t(\text{Ran } \Phi_\tau) \subset \text{Ran } \Phi_\tau$ for every $t \geq 0$ and $\tau > 0$ and that the family $\tilde{\mathbb{T}} = \mathbb{T}|_{\text{Ran } \Phi_\tau}$ satisfies the semigroup properties (2.5) and (2.6). Moreover, it is clear that the couple $(\tilde{\mathbb{T}}, \Phi)$ satisfies (2.8). It follows that the only condition still to be checked in order to prove that $(\tilde{\mathbb{T}}, \Phi)$ is a well-posed exactly controllable system, with control space X_δ and input space U , is the strong continuity property of the semigroup $\tilde{\mathbb{T}}$ on X_δ . This seems a difficult question. The recently developed theory on C^0 -semigroups on spaces of analytic functions (see, for instance, Gal and Gal [9] and references therein) could provide a good track in exploring this question.

Motivated by applications to nonlinear problems, the articles [18] and [15] study a controllability concept for the heat equation which is quite different of the exact controllability introduced in Definition 3.2. More precisely, given $\tau > 0$, the main results in [18] and [15] assert that there exists controls u_0, u_π having a Gevrey type regularity on $[0, \tau]$ which steer the solution of (1.1) to any state which can be

holomorphically extended to a ball in \mathbb{C} which is centered at $\frac{\pi}{2}$ and of diameter large enough. We give below a result in the same direction, with less regularity for both the target states and the input signals. More precisely, our result below gives a complete characterization of the states which can be reached by inputs lying in

$$(5.1) \quad W_L^{n,2}(0, \tau) := \left\{ v \in W^{n,2}(0, \tau) \mid v(0) = \dots = \frac{d^{n-1}v}{dt^{n-1}}(0) = 0 \right\},$$

for some $n \in \mathbb{N}$ and $\tau > 0$. Moreover, we set $W_L^{0,2}(0, \tau) := L^2[0, \tau]$.

To state the main result in this section we introduce, for each $n \in \mathbb{Z}_+$ and $\tau > 0$, the space $X_{n,\tau}$ defined by open set $\Omega \subset \mathbb{C}$ we denote $E^{0,2}(\Omega) := E^2(\Omega)$ and

$$(5.2) \quad X_{n,\tau} := \left\{ \psi \in X_\tau \mid \frac{d^{2k}\psi}{ds^{2k}} \in X_\tau \text{ for } k = 0, 1, \dots, n \right\},$$

where the family of Banach spaces $(X_\delta)_{\delta>0}$ has been defined in (1.8) and (1.9). Note that $W_L^{n,2}(0, \tau)$ and $X_{n,\tau}$ are Banach spaces when endowed with the norms

$$\|v\|_{W_L^{n,2}(0,\tau)} = \left\| \frac{d^n v}{dt^n} \right\|_{L^2[0,\tau]} \quad (v \in W_L^{n,2}(0, \tau)),$$

$$\|\psi\|_{X_{n,\tau}}^2 = \sum_{k=1}^n \left\| \frac{d^k \psi}{ds^{2k}} \right\|_{X_{n,\tau}}^2 \quad (\psi \in X_{n,\tau}).$$

Proposition 5.1. *Let $n \in \mathbb{Z}_+$. Then for every $\tau > 0$ the restriction of Φ_τ to the space $W_L^{n,2}(0, \tau)$ introduced in (5.1) is a linear bounded operator from $W_L^{n,2}(0, \tau)$ onto $X_{n,\tau}$, where the Banach space $X_{n,\tau}$ has been defined in (5.2).*

Proof. The fact that for every $\tau > 0$ we have $\Phi_\tau \in \mathcal{L}(W_L^{0,2}(0, \tau); X_{0,\tau})$ has been proven in Theorem 1.1. For every $n \in \mathbb{N}$ and $u_0, u_\pi \in W_L^{n,2}(0, \tau)$ we denote $z(t, \cdot) := \Phi_t \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix}$, with $t \in [0, \tau]$. By applying Lemma 2.1 from Tucsnak and Weiss [29] it follows that $z \in C^n([0, \tau]; W^{-1,2}(0, \pi))$ and

$$\frac{\partial^{2k} z}{\partial x^{2k}}(\tau, x) = \frac{\partial^k z}{\partial t^k}(\tau, x) \quad (k \in \{1, \dots, n\}, x \in (0, \pi)),$$

$$\frac{\partial^k z}{\partial t^k}(\tau, x) = \phi_\tau \begin{bmatrix} \frac{d^k u_0}{dt^k} \\ \frac{d^k u_\pi}{dt^k} \end{bmatrix} (x) \quad (k \in \{1, \dots, n\}, x \in (0, \pi)).$$

The two relations above combined with the fact, following from Theorem 1.1, that the maps

$$\begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} \mapsto \phi_\tau \begin{bmatrix} \frac{d^k u_0}{dt^k} \\ \frac{d^k u_\pi}{dt^k} \end{bmatrix} \quad (k \in \{0, \dots, n\}, u_0, u_\pi \in W_L^{n,2}(0, \tau)),$$

are bounded from $W_L^{n,2}(0, \tau)$ into X_τ yield that indeed $\Phi_\tau \in \mathcal{L}(W_L^{n,2}(0, \tau); X_{n,\tau})$.

To show that Φ_τ maps $W_L^{n,2}(0, \tau)$ onto $X_{n,\tau}$ we begin by noticing that, according to Theorem 1.1, this holds for $n = 0$. For $n \in \mathbb{N}$ we remark that, by using again Theorem 1.1, for every $\psi \in X_{n,\tau}$ there exist $v_0, v_\pi \in L^2[0, \tau]$ with the solution w

of

$$(5.3) \quad \begin{cases} \frac{\partial w}{\partial t}(t, x) = \frac{\partial^2 w}{\partial x^2}(t, x) & (t \geq 0, x \in (0, \pi)), \\ w(t, 0) = v_0(t), \quad w(t, \pi) = v_\pi(t) & (t \in [0, \infty)), \\ w(0, x) = 0 & (x \in (0, \pi)), \end{cases}$$

satisfies

$$(5.4) \quad w(\tau, x) = \frac{d^{2n}\psi}{dx^{2n}}(x) \quad (x \in (0, \pi)).$$

Consider the functions \tilde{v}_0 and \tilde{v}_π defined by

$$\tilde{v}_0(t) = \int_0^t v_0(\sigma) d\sigma, \quad \tilde{v}_\pi(t) = \int_0^t v_\pi(\sigma) d\sigma \quad (t \in [0, \tau]),$$

for $n = 1$ and

$$\tilde{v}_0(t) = \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} \cdots \int_0^{\sigma_{n-1}} v_{0,n}(\sigma_n) d\sigma_n d\sigma_{n-1} \cdots d\sigma_1, \quad (t \in [0, \tau]),$$

$$\tilde{v}_\pi(t) = \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} \cdots \int_0^{\sigma_{n-1}} v_\pi(\sigma_n) d\sigma_n d\sigma_{n-1} \cdots d\sigma_1, \quad (t \in [0, \tau]),$$

for $n \geq 2$. We clearly have

$$(5.5) \quad \tilde{v}_0, \tilde{v}_\pi \in W_L^{n,2}(0, \tau),$$

and

$$(5.6) \quad \frac{d^n \tilde{v}_0}{dt^n} = v_0, \quad \frac{d^n \tilde{v}_\pi}{dt^n} = v_\pi.$$

Let \tilde{w} be the solution of the initial and boundary value problem

$$(5.7) \quad \begin{cases} \frac{\partial \tilde{w}}{\partial t}(t, x) = \frac{\partial^2 \tilde{w}}{\partial x^2}(t, x) & (t \in [0, \tau], x \in (0, \pi)), \\ \tilde{w}(t, 0) = \tilde{v}_0(t), \quad \tilde{w}(t, \pi) = \tilde{v}_\pi(t) & (t \in [0, \tau]), \\ \tilde{w}(0, x) = 0 & (x \in (0, \pi)). \end{cases}$$

Thanks to (5.5) and (5.6) and using again Lemma 2.1 from [29] we have

$$\tilde{w} \in C([0, \tau]; W^{2n-1,2}(0, \pi)) \cap W^{n,2}([0, \tau]; W^{-1,2}(0, \pi)),$$

and $\frac{\partial^n \tilde{w}}{\partial t^n} = w$, where w is the solution of (5.3)-(5.4). It follows that

$$\frac{\partial^n \tilde{w}}{\partial t^n}(\tau, x) = \frac{d^{2n}\psi}{dx^{2n}}(x) \quad (x \in (0, \pi)).$$

The above relation and the first equation in (5.7) imply that

$$\frac{\partial^{2n} \tilde{w}}{\partial x^{2n}}(\tau, x) = \frac{d^{2n}\psi}{dx^{2n}}(x) \quad (x \in (0, \pi)).$$

It follows that

$$\tilde{w}(\tau, x) = \psi(x) + P(x) \quad (x \in (0, \pi)),$$

where P is a polynomial of degree $2n - 1$. The last formula can alternatively be written

$$(5.8) \quad \Phi_\tau \begin{bmatrix} \tilde{v}_0 \\ \tilde{v}_\pi \end{bmatrix} = \psi + P.$$

On the other hand, according to Theorem 2 in Laroche, Martin, Rouchon [14] (see also [18]) there exist $\xi_0, \xi_\pi \in C^\infty([0, \tau])$ such that

$$(5.9) \quad \Phi_\tau \begin{bmatrix} \xi_0 \\ \xi_\pi \end{bmatrix} = P,$$

$$\frac{d^k \xi_0}{dt^k}(0) = \frac{d^k \xi_\pi}{dt^k}(0) = 0 \quad (k \in \mathbb{Z}_+).$$

Finally, setting

$$u_0 = \tilde{v}_0 - \xi_0, \quad u_\pi = \tilde{v}_\pi - \xi_\pi,$$

and using (5.8) and (5.9) it follows that $u_0, u_\pi \in W_L^{n,2}(0, \tau)$ and they satisfy

$$\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \psi,$$

which ends the proof. \square

Let us now introduce, for each $n \in \mathbb{Z}_+$ and open set $\Omega \subset \mathbb{C}$ the spaces

$$A^{0,2}(\Omega) := A^2(\Omega), \quad E^{0,2}(\Omega) := E^2(\Omega)$$

and

$$(5.10) \quad A^{n,2}(\Omega) := \left\{ \psi \in A^2(\Omega) \mid \frac{d^{2k} \psi}{ds^{2k}} \in A^2(\Omega) \text{ for } k = 1, \dots, n \right\} \quad (n \geq 1),$$

$$E^{n,2}(\Omega) := \left\{ \psi \in E^2(\Omega) \mid \frac{d^{2k} \psi}{ds^{2k}} \in E^2(\Omega) \text{ for } k = 1, \dots, n \right\} \quad (n \geq 1).$$

where the Bergman and Hardy-Smirnov spaces $A^2(\Omega)$ and $E^2(\Omega)$ have been introduced in (1.4) and (1.3), respectively.

By combining Proposition 5.1 above with Proposition 1.1 and Theorem 1.3 in [11] it follows that

Corollary 5.1. *Given $n \in \mathbb{Z}_+$ we have*

$$E^{n,2}(D) \subset \Phi_\tau \left(W_L^{n,2}(0, \tau) \right) \subset A^{n,2}(D) \quad (\tau > 0),$$

where D is the open set defined in (1.2).

Remark 5.1. As already mentioned, the main result in [18] asserts that functions which are analytic in a ball centered at $\frac{\pi}{2}$ and of a radius large enough are reachable by controls lying in the Gevrey class $G^2([0, \tau])$. We remind that the Gevrey class $G^\alpha([0, \tau])$ of order $\alpha > 1$ is defined as the set of all functions $g \in C^\infty([0, \tau])$ such that for every $n \in \mathbb{Z}_+$ we have $\|f^{(n)}\|_\infty \leq A_f R_f^n (n!)^\alpha$ for some positive constants A_f, R_f . We conjecture that this result can be strengthened to the following ‘‘analytic’’ version of Proposition 5.1: for every $\tau > 0$ and $\psi \in \text{Hol}(\tilde{D})$, where $\tilde{D} \subset \mathbb{C}$ is an open set containing \bar{D} , there exist $u_0, u_\pi \in G^2([0, \tau])$ such that $\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \psi$.

A possible approach in proving this conjecture could consist in applying Proposition 5.1 with $n \rightarrow \infty$. This approach would require appropriate estimates of the

derivatives, up to order $n - 1$, of the controls u_0 , u_π constructed in Proposition 5.1. Obtaining such estimates is for now an open question.

6. REACHABLE SPACE AND THE COST OF NULL-CONTROLLABILITY

In this section we describe an application of the results and methods developed above in order to obtain estimates for the cost of null controllability in small time for the system determined by the two first equations in (1.1). We begin by reminding, in the general context introduced in Section 3 the definition of the cost of null controllability. To this aim, let X and U be Hilbert spaces and let (\mathbb{T}, Φ) be a well-posed control LTI system with state space X and input space U (in the sense of Definition 3.1). Assuming that the system (\mathbb{T}, Φ) is null controllable in some time $\tau > 0$ (According to Definition 3.2 this means that $\text{Ran } \Phi_\tau \supset \text{Ran } \mathbb{T}_\tau$), the cost of null controllability in time τ is the number c_τ defined by

$$(6.1) \quad c_\tau = \sup_{\|\psi\|_X \leq 1} \|\mathbb{T}_\tau \psi\|_{\text{Ran } \Phi_\tau},$$

where the norm $\|\cdot\|_{\text{Ran } \Phi_\tau}$ has been defined in (3.1).

For systems which are null controllable in every time $\tau > 0$ we clearly have that $\limsup_{\tau \rightarrow 0+} c_\tau = +\infty$. A question of interest in this case is to estimate the blow-up rate of c_τ when $\tau \rightarrow 0$. For finite dimensional LTI systems the question has been first investigated in Seidman [24] and Seidman–Yong [25], where it has been showed that, as τ tends to zero, c_τ behaves like $1/\tau^{k+1/2}$, where $k \in \mathbb{Z}_+$ is the smallest integer such that

$$\text{Ran} [B \quad AB \quad A^2B \quad \dots \quad A^k B] = X.$$

In the case of the system determined by the two first equations in (1.1), which is null controllable in any positive time (see Proposition 3.3), the study of the behavior of the cost of null controllability when $\tau \rightarrow 0+$ began with the classical work [5] and continued with a series of papers including, with successive improvements, Güichal [10], Miller [19], Tenenbaum and Tucsnak [26], Lissy [16], Dardé and Ervedoza [3]. As far as we know, the most precise available results on the behavior of the c_τ defined in (6.1) when $\tau \rightarrow 0+$ are

$$\liminf_{\tau \rightarrow 0+} \tau \log c_\tau \geq \frac{\pi^2}{8},$$

which has been proved [16], and

$$(6.2) \quad \limsup_{\tau \rightarrow 0+} \tau \log c_\tau \leq \kappa_0 \frac{\pi^2}{4},$$

where κ_0 is a constant approximately equal to 0.6966, which has been proved in [3].

In this section we prove that for small τ the constant c_τ is smaller than a constant d_τ which is simply defined in terms of the semigroup \mathbb{T} and of the norm of the space X_τ defined in (1.8). Whether this estimate can lead to an improvement of the κ_0 in (6.2) is an open question, to be treated in a forthcoming work.

To give a precise statement of the main result in this section we note that for every $\tau > 0$ and $\psi \in W^{-1,2}(0, \pi)$, the function $x \mapsto (\mathbb{T}_\tau \psi)(x)$ clearly extends to a function which is holomorphic on \mathbb{C} , so that, according to Corollary 3.6 in [11] (or just using the null controllability of (\mathbb{T}, Φ) combined with Theorem 1.1) we have

that $\mathbb{T}_\tau \psi \in X_\tau$. We can thus define, for each $\tau > 0$, the constant

$$(6.3) \quad d_\tau = \sup_{\|\psi\|_{W^{-1,2}(0,\pi)} \leq 1} \|\mathbb{T}_\tau \psi\|_\tau,$$

where \mathbb{T} is the heat semigroup and the norm $\|\cdot\|_\tau$ has been defined in (1.9).

Proposition 6.1. *With the above notation we have*

$$(6.4) \quad \limsup_{\tau \rightarrow 0^+} \frac{c_\tau}{d_\tau} \leq 1.$$

Proof. We have seen above that $\mathbb{T}_\tau \psi$ lies in X_τ for every $\psi \in W^{-1,2}(0,\pi)$ and $\tau > 0$ so that we have

$$(6.5) \quad (\mathbb{T}_\tau \psi)(x) = \varphi_0(x) + \varphi_\pi(x) \quad (x \in (0,\pi)),$$

where $\varphi_0 \in A^2(\Delta, \omega_{0,\tau})$ and $\varphi_\pi \in A^2(\tilde{\Delta}, \omega_{\pi,\tau})$ depend on both ψ and τ .

On the other hand, remind the operators P_τ , Q_τ and M_τ defined in (4.2), (4.3) and (4.4), respectively. Using Lemmas 4.1 and 4.2 follows that there exists $\delta^* > 0$ such that M_τ is invertible for every $\tau \in (0, \delta^*)$ and

$$\|M_\tau^{-1}\|_{\mathcal{L}(A^2(\Delta, \omega_{0,\tau}) \times A^2(\tilde{\Delta}, \omega_{\pi,\tau}), (L^2([0,\tau]))^2)} \leq \frac{1}{1 - \gamma_\tau},$$

where for every $\tau > 0$ we have set

$$(6.6) \quad \gamma_\tau = \left\| M_\tau - \begin{bmatrix} P_\tau & 0 \\ 0 & Q_\tau \end{bmatrix} \right\|_{\mathcal{L}((L^2([0,\pi]))^2, A^2(\Delta, \omega_{0,\tau}) \times A^2(\tilde{\Delta}, \omega_{\pi,\tau}))}.$$

Consequently, for each $\tau \in (0, \delta^*)$ and φ_0 and φ_π as above there exist $v_0, v_\pi \in L^2[0, \tau]$ such that

$$M_\tau \begin{bmatrix} v_0 \\ v_\pi \end{bmatrix} = \begin{bmatrix} \varphi_0 \\ \varphi_\pi \end{bmatrix},$$

$$(6.7) \quad \left\| \begin{bmatrix} v_0 \\ v_\pi \end{bmatrix} \right\|_{(L^2[0,\tau])^2} \leq \frac{1}{1 - \gamma_\tau} \left\| \begin{bmatrix} \varphi_0 \\ \varphi_\pi \end{bmatrix} \right\|_{A^2(\Delta, \omega_{0,\tau}) \times A^2(\tilde{\Delta}, \omega_{\pi,\tau})}$$

We can thus conclude, reminding (4.4) and (4.5), that for every $\psi \in W^{-1,2}(0,\pi)$, $\tau \in (0, \delta^*)$ and $\varphi_0 \in A^2(\Delta, \omega_{0,\tau})$, $\varphi_\pi \in A^2(\tilde{\Delta}, \omega_{\pi,\tau})$ satisfying (6.5) there exist $v_0, v_\pi \in L^2[0, \tau]$ satisfying (6.7) such that

$$(6.8) \quad \Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \mathbb{T}_\tau \psi,$$

where we have denoted

$$u_0(t) = \sqrt{t} v_0(t), \quad u_\pi(t) = \sqrt{t} v_\pi(t) \quad (t \in [0, \tau]).$$

With no loss of generality we can assume that $\delta^* < 1$, so that

$$\|u_0\|_{L^2[0,\tau]} \leq \|v_0\|_{L^2[0,\tau]}, \quad \|u_\pi\|_{L^2[0,\tau]} \leq \|v_\pi\|_{L^2[0,\tau]}.$$

We have shown that for every $\psi \in W^{-1,2}(0,\pi)$, $\tau \in (0, \delta^*)$ and $\varphi_0 \in A^2(\Delta, \omega_{0,\tau})$, $\varphi_\pi \in A^2(\tilde{\Delta}, \omega_{\pi,\tau})$ satisfying (6.5) there exist $u_0, u_\pi \in L^2[0, \tau]$ satisfying (6.8), together with

$$(6.9) \quad \left\| \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} \right\|_{(L^2[0,\tau])^2} \leq \frac{1}{1 - \gamma_\tau} \left\| \begin{bmatrix} \varphi_0 \\ \varphi_\pi \end{bmatrix} \right\|_{A^2(\Delta, \omega_{0,\tau}) \times A^2(\tilde{\Delta}, \omega_{\pi,\tau})},$$

where γ_τ has been defined in (6.6). Since (6.9) holds for every $\varphi_0 \in A^2(\Delta, \omega_{0,\tau})$ and $\varphi_\pi \in A^2(\tilde{\Delta}, \omega_{\pi,\tau})$ satisfying (6.5), using (1.9) and (3.1) it follows that

$$(6.10) \quad \|\mathbb{T}_\tau \psi\|_{\text{Ran } \Phi_\tau} \leq \frac{1}{1 - \gamma_\tau} \|\mathbb{T}_\tau \psi\|_\tau \quad (\psi \in W^{-1,2}(0, \pi), \tau \in (0, \delta^*)).$$

Since by Lemma 4.2 we have that $\lim_{\tau \rightarrow 0^+} \gamma_\tau = 0$, the conclusion (6.4) follows from (6.1) and (6.3). \square

Remark 6.1. Analyzing the proof of Proposition 6.1 it is easily seen that (6.10) holds with an arbitrary $\eta \in \text{Ran } \Phi_\tau$ instead of $\mathbb{T}_\tau \psi$. We thus have that

$$\limsup_{\tau \rightarrow 0^+} \frac{\|\eta\|_{\text{Ran } \Phi_\tau}}{\|\eta\|_{X_\tau}} \leq 1 \quad (\eta \in \text{Ran } \Phi_\tau \setminus \{0\}),$$

and the existence of a constant $K^* > 0$ such that

$$\|\eta\|_{\text{Ran } \Phi_\tau} \leq K^* \|\eta\|_\tau \quad (\tau \in (0, \delta^*), \eta \in \text{Ran } \Phi_\tau).$$

By the closed graph theorem, it follows that for every $\tau \in (0, \delta^*)$ the norms $\|\cdot\|_{\text{Ran } \Phi_\tau}$ and $\|\cdot\|_\tau$ are equivalent.

7. SUMS OF BERGMAN SPACES ON SYMMETRIC SECTORS

The aim of this section is twofold. We first prove Proposition 1.1 thus, consequently, Corollary 1.1. We next connect our results to those obtained recently, with a different methodology, in [20], where an apparently different characterization of the reachable space has been given. More precisely, the main result in [20] asserts that

$$(7.1) \quad \text{Ran } \Phi_\tau = A^2(\Delta) + A^2(\tilde{\Delta}) \quad (\tau > 0).$$

Putting together (7.1) and Corollary 1.1 it follows that

Proposition 7.1. *For every $\delta > 0$ we have*

$$(7.2) \quad X_\delta = A^2(\Delta) + A^2(\tilde{\Delta}),$$

where we remind that the spaces $(X_\delta)_{\delta > 0}$ have been defined in (7.2).

The second main aim of this section is to show that Proposition 7.1 follows from Proposition 1.1 and thus providing a new proof of (7.1).

An essential step in proving Proposition 1.1 is the construction of a family of entire functions $(\Theta_{\tau,t})$ having the property that if $0 < \tau < t$ and $t - \tau$ is small enough then the multiplication by $\Theta_{\tau,t}$ defines a bounded linear operator from X_τ to X_t , provided that τ and t are close enough. To this aim, we need several lemmas involving the families of functions

$$(7.3) \quad \theta_{\tau,t}(s) := e^{\frac{s^2(t-\tau)}{4t\tau}} \quad (\tau, t \geq 0, s \in \Delta \cup \tilde{\Delta}),$$

$$(7.4) \quad \tilde{\theta}_{\tau,t}(s) := e^{\frac{(\pi-s)^2(t-\tau)}{4t\tau}} \quad (\tau, t \geq 0, s \in \Delta \cup \tilde{\Delta}),$$

$$(7.5) \quad \Theta_{\tau,t}(s) := \theta_{\tau,t}(s) + \tilde{\theta}_{\tau,t}(s) \quad (\tau, t \geq 0, s \in \Delta \cup \tilde{\Delta}).$$

The three functions defined above are, for every $\tau, t > 0$, holomorphic on \mathbb{C} and for every $s \in \mathbb{C}$ we have

$$(7.6) \quad \tilde{\theta}_{\tau,t}(s) = \theta_{\tau,t}(\pi - s), \quad \Theta_{\tau,t}(s) = \Theta_{\tau,t}(\pi - s).$$

Moreover, we have:

Lemma 7.1. *Assume $t, \tau, > 0$ are such that*

$$(7.7) \quad \frac{t\tau}{t-\tau} > \frac{\pi}{4}.$$

Then the function $\Theta_{\tau,t}$ defined in (7.5) has no zeros on $\overline{\Delta \cup \widetilde{\Delta}}$. Moreover, there exist $\alpha, \beta > 0$ (possibly depending on τ and t) such that the functions $\theta_{\tau,t}$ and $\tilde{\theta}_{\tau,t}$ defined in (7.3) and (7.4), respectively, satisfy

$$(7.8) \quad \alpha \leq \left| 1 + \frac{\tilde{\theta}_{\tau,t}(s)}{\theta_{\tau,t}(s)} \right| \leq \beta \quad (s \in \Delta).$$

Proof. Using the fact that

$$(7.9) \quad \Theta_{\tau,t}(s) = \theta_{\tau,t}(s) \left(1 + e^{\frac{(\pi^2 - 2\pi s)(t-\tau)}{4t\tau}} \right) \quad (\tau, t \geq 0, s \in \mathbb{C}),$$

it can be easily checked that $\Theta_{\tau,t}$ vanishes for some $s \in \mathbb{C}$ iff

$$(7.10) \quad \operatorname{Re} s = \pi/2, \quad \operatorname{Im} s \in \frac{-2t\tau}{t-\tau} + \frac{4t\tau}{t-\tau}\mathbb{Z}.$$

On the other hand, for every τ, t satisfying (7.7) we have

$$\left(\frac{-2t\tau}{t-\tau} + \frac{4t\tau}{t-\tau}\mathbb{Z} \right) \cap [-\pi/2, \pi/2] = \emptyset,$$

which, together with (7.10), implies that indeed $\Theta_{\tau,t}$ has no zeros in $\overline{\Delta \cup \widetilde{\Delta}}$.

In order to prove (7.8) we introduce the compact set

$$K := \{s \in \overline{\Delta} \mid \operatorname{Re} s \in [0, \pi]\}.$$

The function

$$s \mapsto \left| 1 + e^{\frac{(\pi^2 - 2\pi s)(t-\tau)}{4t\tau}} \right| \quad (s \in \mathbb{C}),$$

is continuous on K and we have shown above that it is non vanishing on K . Consequently, there exists $\alpha_1 > 0$ such that

$$\left| 1 + e^{\frac{(\pi^2 - 2\pi s)(t-\tau)}{4t\tau}} \right| \geq \alpha_1 \quad (s \in K).$$

For $s \in \overline{\Delta} \setminus K$ we have $\operatorname{Re} s > \pi$ thus

$$\left| e^{\frac{(\pi^2 - 2\pi s)(t-\tau)}{4t\tau}} \right| \leq e^{\frac{-\pi^2(t-\tau)}{4t\tau}} < 1.$$

Hence, there exists $\alpha_2 > 0$ such that for every $s \in \overline{\Delta} \setminus K$ we have

$$\left| 1 + e^{\frac{(\pi^2 - 2\pi s)(t-\tau)}{4t\tau}} \right| \geq 1 - e^{\frac{-\pi^2(t-\tau)}{4t\tau}} \geq \alpha_2 > 0.$$

Setting $\alpha := \min(\alpha_1, \alpha_2) > 0$, we obtain the first inequality in (7.8).

Finally, using the fact that $\operatorname{Re} s \geq 0$ for every $s \in \overline{\Delta}$, it follows that

$$\left| 1 + e^{\frac{(\pi^2 - 2\pi s)(t-\tau)}{4t\tau}} \right| \leq 1 + e^{\frac{\pi^2(t-\tau)}{4t\tau}} \quad (s \in \overline{\Delta}),$$

which implies the second inequality in (7.8). \square

Lemma 7.2. *Let τ, t satisfy the assumptions in Lemma 7.1 and let $\Theta_{\tau,t}$ be the function defined in (7.9). Then for every $f \in A^2(\Delta; \omega_{0,t})$ and every $\tilde{f} \in A^2(\tilde{\Delta}; \omega_{\pi,t})$ we have*

$$\frac{f}{\Theta_{\tau,t}} \in A^2(\Delta; \omega_{0,\tau}), \quad \frac{\tilde{f}}{\Theta_{\tau,t}} \in A^2(\tilde{\Delta}; \omega_{\pi,\tau}).$$

Proof. Let $f \in A^2(\Delta; \omega_{0,t})$. We know from Lemma 7.1 that $\frac{f}{\Theta_{\tau,t}}$ is holomorphic on Δ . Moreover, by combining (7.9) and Lemma 7.1, it follows that for every $s \in \Delta$ we have

$$\begin{aligned} \frac{|f(s)|^2}{|\Theta_{\tau,t}(s)|^2} \omega_{0,\tau}(s) &= \frac{|f(s)|^2}{\left| \theta_{\tau,t}(s) \right|^2 \left| 1 + e^{\frac{(\pi^2 - 2\pi s)(t-\tau)}{4t\tau}} \right|^2} \omega_{0,\tau}(s) \\ &\leq \frac{1}{\alpha^2} \frac{|f(s)|^2}{|\theta_{\tau,t}(s)|^2} \omega_{0,\tau}(s) = \frac{t}{\tau \alpha^2} |f(s)|^2 \omega_{0,t}(s). \end{aligned}$$

Using our assumptions on f it follows that for every $f \in A^2(\Delta, \omega_{0,t})$ we have

$$\frac{f}{\Theta_{\tau,t}} \in A^2(\Delta; \omega_{0,\tau}).$$

Using this fact and (7.6), the corresponding result for $\tilde{f} \in A^2(\tilde{\Delta}; \omega_{\pi,t})$ readily follows. \square

Lemma 7.3. *Let τ, t satisfy the assumptions in Lemma 7.1 and let $\Theta_{\tau,t}$ be the function defined in (7.9). Let $\gamma, \gamma' > 0$ be such that*

$$(7.11) \quad \frac{t-\tau}{t\tau} + \frac{1}{\gamma'} \leq \frac{1}{\gamma}.$$

Then for every $f \in A^2(\Delta; \omega_{0,\gamma})$ and every $\tilde{f} \in A^2(\tilde{\Delta}; \omega_{\pi,\gamma})$ we have

$$f\Theta_{\tau,t} \in A^2(\Delta; \omega_{0,\gamma'}), \quad \tilde{f}\Theta_{\tau,t} \in A^2(\tilde{\Delta}; \omega_{\pi,\gamma'}).$$

Proof. Let $f \in A^2(\Delta; \omega_{0,\gamma})$. Using Lemma 7.1 it follows that for every $s \in \Delta$ we have

$$\begin{aligned} |f(s)\Theta_{\tau,t}(s)|^2 \omega_{0,\gamma'}(s) &= |f(s)|^2 |\theta_{t,\tau}(s)|^2 \left| 1 + \frac{\tilde{\theta}_{\tau,t}(s)}{\theta_{\tau,t}(s)} \right|^2 \omega_{0,\gamma'}(s) \\ &\leq \frac{\beta^2}{\gamma'} |f(s)|^2 e^{\operatorname{Re}(s^2) \left(\frac{t-\tau}{2t\tau} + \frac{1}{2\gamma'} \right)} \leq \frac{\beta^2 \gamma}{\gamma'} |f(s)|^2 \omega_{0,\gamma}(s) \end{aligned}$$

which shows that indeed $f\Theta_{\tau,t} \in A^2(\Delta; \omega_{0,\gamma'})$. As above, the corresponding result for $A^2(\tilde{\Delta}; \omega_{\pi,\gamma})$ follows by symmetry, see (7.6). \square

Lemma 7.4. *Let τ, t satisfy the assumptions in Lemma 7.1 such that $\varepsilon := t-\tau < \tau$, and let $\Theta_{\tau,t}$ be the function defined in (7.9). Moreover, assume that $\tau > 0$ is such that*

$$(7.12) \quad X_\delta = X_\tau \quad (0 < \delta \leq \tau).$$

(This holds, in particular, for $\tau \in (0, \delta^)$, where δ^* is the constant in Theorem 1.1.). Then $X_t = X_\tau$.*

Proof. As already mentioned, it is obvious that $X_\tau \subset X_t$. To prove that $X_t \subset X_\tau$, let $\varphi \in A^2(\Delta; \omega_{0,t})$ and $\tilde{\varphi} \in A^2(\tilde{\Delta}; \omega_{\pi,t})$. According to Lemma 7.2 and (7.12) we have

$$\frac{\varphi}{\Theta_{\tau,t}} + \frac{\tilde{\varphi}}{\Theta_{\tau,t}} \in X_\tau = X_{\tau-\varepsilon},$$

here $\varepsilon = t - \tau < \tau$. It follows that there exist $f \in A^2(\Delta; \omega_{0,\tau-\varepsilon})$ et $\tilde{f} \in A^2(\tilde{\Delta}; \omega_{\pi,\tau-\varepsilon})$ such that

$$\frac{\varphi}{\Theta_{\tau,t}} + \frac{\tilde{\varphi}}{\Theta_{\tau,t}} = f + \tilde{f}$$

Let $\gamma = \tau - \varepsilon$ and $\gamma' = \tau$. Since $t \geq \tau - \varepsilon$ we have

$$\frac{t - \tau}{t\tau} + \frac{1}{\gamma'} = \frac{\varepsilon}{t\tau} + \frac{1}{\tau} \leq \frac{\varepsilon}{(\tau - \varepsilon)\tau} + \frac{1}{\tau} = \frac{1}{\tau - \varepsilon} = \frac{1}{\gamma},$$

so that inequality (7.11) holds. Consequently, using Lemma 7.3 it follows that

$$\varphi + \tilde{\varphi} = f\Theta_{\tau,t} + \tilde{f}\Theta_{\tau,t} \in A^2(\Delta; \omega_{0,\tau}) + A^2(\tilde{\Delta}; \omega_{\pi,\tau}) = X_\tau,$$

which ends the proof. \square

We are now in a position to prove Proposition 1.1.

Proof of Proposition 1.1. Let

$$\mathcal{I} = \{\tau > 0 \mid X_\delta = X_\tau \text{ for all } \delta \in (0, \tau]\}.$$

We clearly that if $\tau \in \mathcal{I}$ then $(0, \tau] \subset \mathcal{I}$. From Theorem 1.1 we know that $\mathcal{I} \supset (0, \frac{\delta^*}{2}]$, where δ^* is the constant in Theorem 1.1. Let

$$\tau_0 := \min\left(\pi/8, \frac{\delta^*}{2}\right), \quad m := \frac{\pi\tau_0}{\pi - 4\tau_0}, \quad \varepsilon_0 := (m - \tau_0)/2.$$

We clearly have that $\tau_0 \in \mathcal{I}$, $\varepsilon_0 < \tau_0$ and

$$\frac{(\tau_0 + \varepsilon_0)\tau_0}{\varepsilon_0} > \frac{\pi}{4}.$$

Since the function $t \mapsto \frac{(t+\varepsilon_0)t}{\varepsilon_0}$ is clearly increasing on $(0, \infty)$, it follows that

$$(7.13) \quad \frac{(\tau + \varepsilon_0)\tau}{\varepsilon_0} > \frac{\pi}{4} \quad (\tau > \tau_0).$$

Consider now the sequence $(t_n)_{n \in \mathbb{Z}_+}$ defined by $t_0 = \tau_0$ and

$$t_{n+1} = t_n + \varepsilon_0 \quad (n \in \mathbb{Z}_+),$$

which obviously satisfies $\lim_{n \rightarrow \infty} t_n = +\infty$. Then (7.13) enables us to recursively apply Lemma 7.4 and obtain that for every $n \in \mathbb{N}$ we have $t_n \in \mathcal{I}$ and, consequently, $X_{t_n} = X_{\tau_0}$. This shows that $\mathcal{I} = (0, \infty)$, which ends the proof. \square

As mentioned in the preamble of this section, our second aim here is to give a direct proof of Proposition 7.1, which provides an alternative proof of (7.1). To this aim, we need the following result:

Lemma 7.5. *Let $\varphi \in A^2(\Delta)$ and $\tilde{\varphi} \in A^2(\tilde{\Delta})$. Then*

$$\frac{\varphi}{\Theta_{\frac{\pi}{4}, \frac{\pi}{2}}} \in A^2(\Delta; \omega_{0,\pi/2}), \quad \frac{\tilde{\varphi}}{\Theta_{\frac{\pi}{4}, \frac{\pi}{2}}} \in A^2(\tilde{\Delta}; \omega_{\pi,\pi/2}).$$

Proof. Let $\varphi \in A^2(\Delta)$. We know from Lemma 7.1 that the function $\Theta_{\frac{\pi}{4}, \frac{\pi}{2}}$ has no zeros in $\overline{\Delta \cup \tilde{\Delta}}$, so that the function $s \mapsto \frac{\varphi}{\Theta_{\frac{\pi}{4}, \frac{\pi}{2}}}$ is holomorphic on Δ . Moreover, using (7.9) and again Lemma 7.1 it follows that for every $s \in \Delta$ we have

$$\begin{aligned} \frac{|\varphi(s)|^2}{|\Theta_{\frac{\pi}{4}, \frac{\pi}{2}}(s)|^2} \omega_{0, \pi/2}(s) &= \frac{|\varphi(s)|^2}{\left| \theta_{\frac{\pi}{4}, \frac{\pi}{2}}(s) \right|^2 \left| 1 + e^{\frac{(\pi^2 - 2\pi s)}{2\pi}} \right|^2} \omega_{0, \pi/2}(s) \\ &\leq \frac{1}{\alpha^2} \frac{|\varphi(s)|^2}{|\theta_{\frac{\pi}{4}, \frac{\pi}{2}}(s)|^2} \omega_{0, \pi/2}(s) = \frac{2}{\pi \alpha^2} |\varphi(s)|^2, \end{aligned}$$

which implies that $\frac{\varphi}{\Theta_{\frac{\pi}{4}, \frac{\pi}{2}}} \in A^2(\Delta; \omega_{0, \pi/2})$. The fact that $\frac{\tilde{\varphi}}{\Theta_{\frac{\pi}{4}, \frac{\pi}{2}}} \in A^2(\tilde{\Delta}; \omega_{\pi, \pi/2})$ is obtained by symmetry using (7.6). \square

We are now in a position to prove Proposition 7.1.

Proof of Proposition 7.1. Let $\varphi \in A^2(\Delta)$ and $\tilde{\varphi} \in A^2(\tilde{\Delta})$ and let $g = \varphi + \tilde{\varphi}$. According to Lemma 7.5 and Proposition 1.1 there exist $f \in A^2(\Delta; \omega_{0, \pi/4})$ and $\tilde{f} \in A^2(\tilde{\Delta}; \omega_{\pi, \pi/4})$ such that

$$\frac{g}{\Theta_{\frac{\pi}{4}, \frac{\pi}{2}}} = f + \tilde{f}.$$

Using next Lemma 7.3 with $t = \frac{\pi}{2}$, $\tau = \frac{\pi}{4}$, $\gamma = \frac{\pi}{4}$ and $\gamma' = \frac{\pi}{2}$, it follows that

$$g = f \Theta_{\frac{\pi}{4}, \frac{\pi}{2}} + \tilde{f} \Theta_{\frac{\pi}{4}, \frac{\pi}{2}} \in A^2(\Delta; \omega_{0, \pi/2}) + A^2(\tilde{\Delta}; \omega_{\pi, \pi/2}) = X_{\frac{\pi}{2}}.$$

We have thus shown that

$$A^2(\Delta) + A^2(\tilde{\Delta}) \subset X_{\frac{\pi}{2}}.$$

Since the inclusion $X_{\frac{\pi}{2}} \subset A^2(\Delta) + A^2(\tilde{\Delta})$ is an obvious one, we have

$$A^2(\Delta) + A^2(\tilde{\Delta}) = X_{\frac{\pi}{2}}.$$

The conclusion follow now by using again Proposition 1.1. \square

Finally, we remark that by combining Proposition 5.1 with Proposition 1.1 and Proposition 7.1 we obtain

Corollary 7.1. *Given $n \in \mathbb{Z}_+$ and $\tau > 0$ we have*

$$X_{n, \tau} = A^{n, 2}(\Delta) + A^{n, 2}(\tilde{\Delta}),$$

where Banach spaces $X_{n, \tau}$ and $A^{n, 2}(\Delta)$ have been defined in (5.2) and (5.10), respectively.

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Email address: kkellay@math.u-bordeaux.fr

Email address: thomas.normand@etu.u-bordeaux.fr

Email address: marius.tucsnak@u-bordeaux.fr