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A gentle introduction to unification in modal logics

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Abstract

Unification in propositional logics is an active research area. In this paper, we introduce the results we have obtained within the context of modal logics and epistemic logics and we present some of the open problems whose solution will have an important impact on the future of the area.

Résumé

L'unification dans les logiques propositionnelles est un domaine de recherche actif. Dans cet article, nous présentons les résultats que nous avons obtenus dans le cadre des logiques modales et des logiques épistémiques et nous introduisons quelques uns des problèmes ouverts dont la résolution aura un impact important sur l'avenir du domaine.

1 Introduction

The problem of solving equations was at the heart of the algebra of logic created by Boole. In modern terms, owing to the fact that given $n \in \mathbb{N}$ and pairs $(\varphi_1, \psi_1), \dots, (\varphi_n, \psi_n)$ of Boolean formulas, the system consisting of the n equivalences $\varphi_1 \leftrightarrow \psi_1, \dots, \varphi_n \leftrightarrow \psi_n$ can be readily transformed into an equivalent system

$\chi \leftrightarrow \top$ consisting of only one equivalence, the problem of solving equations can be interpreted as a satisfiability problem, or as a unification problem. The satisfiability problem asks whether the variables of χ can be uniformly replaced by constant formulas (i.e. the formulas \perp and \top) in such a way that the resulting formula evaluates to 1 in the two-element Boolean algebra. The unification problem asks whether the variables of χ can be uniformly replaced by formulas (i.e. arbitrary formulas) in such a way that the resulting formula evaluates to 1 under all truth assignments in the two-element Boolean algebra.

Of relevance in many applications of Artificial Intelligence, the satisfiability problem has given rise to a significant corpus of results in automated reasoning. This is borne out by the numerous problems having translations to the satisfiability problem and by the multifarious tools available to solve these translations. In the case of most applied logics, a similar remark also applies. Witness, the systematic interest falling on optimal procedures deciding the satisfiability problem of freshly designed applied logics rather than on elegant axiomatizations explaining the meaning of their logical connectives. In comparison, the unification problem has attracted less attention. The truth is that it has been examined from a different angle : a desirable output of the satisfiability problem is a model satisfying the given formula whereas a desirable output of the unification problem is a substitution making the

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given formula valid.

From a mathematical point of view, the unification problem is strongly related to the admissibility problem which asks with respect to some predetermined propositional logics \mathbf{L} whether given a rule $\frac{\varphi_1, \dots, \varphi_n}{\chi}$, every substitution turning $\varphi_1, \dots, \varphi_n$ into members of \mathbf{L} also turns χ into a member of \mathbf{L} . Firstly, since a given formula φ is unifiable if and only if, when \mathbf{L} is consistent, the associated rule $\frac{\varphi}{\perp}$ is non-admissible, we can turn any algorithm deciding the admissibility problem into an algorithm deciding the unification problem. Secondly, since a given rule $\frac{\varphi_1, \dots, \varphi_n}{\chi}$ is admissible if and only if, when \mathbf{L} is finitary¹, every maximal unifier of $\varphi_1, \dots, \varphi_n$ is also a unifier of χ , we can turn, when \mathbf{L} is decidable, any algorithm producing minimal complete sets of unifiers into an algorithm deciding the admissibility problem². See Ghilardi [18, 19] for illustrations within the context of intuitionistic logic and modal logics like $\mathbf{K4}$ and $\mathbf{S4}$. See also [17, 21].

From the point of view of Artificial Intelligence, unification in propositional logics is an active research area and several applications of the unification problem in the maintenance of knowledge bases have been considered. In this respect, within the context of a terminology of concepts, we may ask whether given $n \in \mathbb{N}$ and pairs $(C_1, D_1), \dots, (C_n, D_n)$ of concept descriptions, a substitution can make these pairs equivalent by replacing some of their variables by appropriate concept descriptions. Moreover, we may be interested to obtain, if possible, the most general substitution that can make the pairs equivalent. See the related unification algorithms presented in [3, 4] for the description languages \mathcal{EL} and \mathcal{FL}_0 .

The thing is that there is a wide variety of situations where the unification problem arises. Let us explain our motivation for considering them within the context of Public Announcement Logic³. Suppose the epistemic formula $\varphi(\bar{p})$ describes a given initial situation in terms of the knowledge of a group of agents about the list of parameter facts $\bar{p} = (p_1, \dots, p_m)$ and the epistemic formula $\langle\langle\chi(\bar{p}\bar{x})\rangle\rangle\psi(\bar{p})$ represents the knowledge of the group about \bar{p} in a desirable final situation (the ψ -part) after an executable public announcement concerning \bar{p} and the list of variable facts $\bar{x} = (x_1, \dots, x_n)$ (the χ -part) has been performed. It

1. See Section 2 for a definition.

2. It is well-known that owing to its structural completeness, Boolean logic has a decidable admissibility problem. It was only after the results of Rybakov [29, 30] that it has been known that intuitionistic logic and modal logics like $\mathbf{K4}$ and $\mathbf{S4}$ have a decidable admissibility problem too.

3. In a multi-agent system, public announcements are atomic actions performed by an outside observer consisting of publicly announcing a formula [12, Chapter 4]. Perceived by all agents, it is common knowledge that these public announcements are truthful.

may happen that $\varphi(\bar{p}) \rightarrow \langle\langle\chi(\bar{p}\bar{x})\rangle\rangle\psi(\bar{p})$ is not valid in Public Announcement Logic. Hence, we may ask whether there are tuples $\bar{\phi} = (\phi_1, \dots, \phi_n)$ of formulas such that $\varphi(\bar{p}) \rightarrow \langle\langle\chi(\bar{p}\bar{\phi})\rangle\rangle\psi(\bar{p})$ is valid. Moreover, we may be interested to obtain the most general tuple $\bar{\phi}$ of formulas such that $\varphi(\bar{p}) \rightarrow \langle\langle\chi(\bar{p}\bar{\phi})\rangle\rangle\psi(\bar{p})$ is valid. In Section 5, we will prove that some unifiable epistemic formulas as above have always most general unifiers.

In this paper, we introduce the results we have obtained within the context of modal logics and epistemic logics and we present some of the open problems whose solution will have an important impact on the future of the area.

2 Unification : preliminary definitions

Syntax We consider a propositional language with a countably infinite set \mathcal{PAR} of *propositional parameters* (p, q , etc), a countably infinite set \mathcal{VAR} of *propositional variables* (x, y , etc), a countable set O of *operators* and an *arity function* $\rho : O \rightarrow \mathbb{N}$. *Atoms* (α, β , etc) are parameters, or variables. We denote this language by \mathcal{PL} . Its *formulas* are defined by the rule $\varphi ::= \alpha \mid o(\varphi_1, \dots, \varphi_n)$ where o ranges over O and $n = \rho(o)$. For all formulas φ , its *set of variables* (in symbols, $\text{var}(\varphi)$) is defined as usual. Let (p_1, p_2, \dots) be an enumeration of \mathcal{PAR} without repetitions and (x_1, x_2, \dots) be an enumeration of \mathcal{VAR} without repetitions. We write $\varphi(\bar{p}\bar{x})$ to denote a formula whose parameters form a sublist of $\bar{p} = (p_1, \dots, p_m)$ and whose variables form a sublist of $\bar{x} = (x_1, \dots, x_n)$. We use the standard definition for the notion of *subformula*.

Semantics Formulas of \mathcal{PL} are interpreted in *algebraic structures*, or in *relational structures* : nondegenerate Boolean algebras in the case of Boolean logic, Kripke frames in the case of modal logics, etc. A set of *values* and a set of *designated values* are coming with each of these structures : A and $\{1_A\}$ in the case of the nondegenerate Boolean algebra $(A, 0_A, \star_A, +_A)$, $\mathcal{P}(W)$ and $\{W\}$ in the case of the Kripke frame (W, R) , etc. *Models* are pairs of the form (S, V) where S is a structure and V is a *truth assignment* associating each formula φ with a value $V(\varphi)$ in S . We shall say that a formula φ is *true in the model* (S, V) if $V(\varphi)$ is a designated value in S . Let \mathcal{C} be a class of structures. We shall say that a formula is \mathcal{C} -*valid* if it is true in all models based on a structure in \mathcal{C} . The *propositional logic* determined by \mathcal{C} is the set of all \mathcal{C} -valid formulas : \mathbf{BL} in the case of Boolean logic, $\mathbf{K4}$, $\mathbf{S4}$, etc in the case of modal logics, etc.

Substitutions A *substitution* is a homomorphism $\sigma : \mathcal{PL} \rightarrow \mathcal{PL}$ which moves at most finitely many va-

riables. It is *parameter-free* if for all variables x , $\sigma(x)$ is parameter-free. It is *ground* if for all variables x , $\sigma(x)$ is variable-free. The *composition* of the substitutions σ and τ is the substitution $\sigma \circ \tau$ such that for all variables x , $(\sigma \circ \tau)(x) = \tau(\sigma(x))$.

We shall say that a substitution σ is \mathcal{C} -*equivalent* to a substitution τ (in symbols $\sigma \simeq \tau$) if for all variables x , each model associate $\sigma(x)$ and $\tau(x)$ with the same value. We shall say that a substitution σ is *more general than* a substitution τ (in symbols $\sigma \preceq \tau$) if there exists a substitution ν such that $\sigma \circ \nu \simeq \tau$. Obviously, \preceq contains \simeq . Moreover, on the set of all substitutions, \simeq is reflexive, symmetric and transitive and \preceq is reflexive and transitive. We shall say that a set Σ of substitutions is *minimal* if for all $\sigma, \tau \in \Sigma$, if $\sigma \preceq \tau$ then $\sigma = \tau$.

Unifiers We shall say that a formula φ is \mathcal{C} -*unifiable* if there exists a substitution σ such that $\sigma(\varphi)$ is \mathcal{C} -valid. In that case, σ is a unifier of φ . We traditionally distinguish between *elementary unification* (*ELU*) and *unification with parameters* (*UWP*). *ELU* is the problem of asking whether a given parameter-free formula is \mathcal{C} -unifiable. *UWP* is the problem of asking whether a given formula is \mathcal{C} -unifiable. It goes without saying that the computability of the unification problem in a propositional logic may vary according to whether one considers *ELU*, or considers *UWP*.

We shall say that a set Σ of unifiers of a \mathcal{C} -unifiable formula φ is *complete* if for all unifiers σ of φ , there exists $\tau \in \Sigma$ such that $\tau \preceq \sigma$. It can be easily proved that if a \mathcal{C} -unifiable formula has minimal complete sets of unifiers then these sets have the same cardinality. Hence, an important question is the following [14] : when a formula is \mathcal{C} -unifiable, has it a minimal complete set of unifiers? When the answer is “yes”, how large is this set?

Let φ be a \mathcal{C} -unifiable formula. φ is said to be *nullary* if there exists no minimal complete set of unifiers of φ , *infinitary* if there exists a minimal complete set of unifiers of φ but there exists no finite one, *finitary* if there exists a finite minimal complete set of unifiers of φ but there exists no with cardinality 1 and *unitary* if there exists a minimal complete set of unifiers of φ with cardinality 1. Obviously, the types “nullary”, “infinitary”, “finitary” and “unitary” constitute a set of jointly exhaustive and pairwise distinct situations for each unifiable formula.

\mathcal{C} — or the propositional logic determined by \mathcal{C} — is said to be *nullary* if there exists a nullary unifiable formula, *infinitary* if every unifiable formula possesses a minimal complete set of unifiers and there is an infinitary unifiable formula, *finitary* if every unifiable formula possesses a finite minimal complete set of unifiers

and there is a finitary unifiable formula and *unitary* if every unifiable formula possesses a minimal complete set of unifiers with cardinality 1. Obviously, the types “nullary”, “infinitary”, “finitary” and “unitary” constitute a set of jointly exhaustive and pairwise distinct situations for each propositional logic. Let us remark that the type of a propositional logic may vary according to whether one considers *ELU*, or considers *UWP*.

3 Case of Boolean logic

Syntax and semantics In the case of Boolean logic, \mathcal{O} consists of the usual Boolean connectives \perp , \neg and \vee together with their usual arities. We denote the associated language by \mathcal{BPL} . The Boolean connectives \top , \wedge , \rightarrow and \leftrightarrow are defined by the usual abbreviations.

Formulas of \mathcal{BPL} are interpreted in nondegenerate Boolean algebras $(A, 0_A, \star_A, +_A)$ — with A as set of values and $\{1_A\}$ as set of designated values — by means of truth assignments V such that $V(\perp) = 0_A$, $V(\neg\varphi) = V(\varphi)^{\star_A}$ and $V(\varphi \vee \psi) = V(\varphi) +_A V(\psi)$.

Computability of unification The unification problem in Boolean logic has been firstly investigated by Boole, his method involving successive elimination of variables [28]. In Boolean logic, both *ELU* and *UWP* are decidable. To be more precise, *ELU* and the satisfiability problem are inter-reducible whereas *UWP* and the validity problem of $\forall\exists$ -**QBF**-formulas are inter-reducible. Hence, in Boolean logic, *ELU* is **NP**-complete and *UWP* is Π_2^P -complete. See [1] for a comprehensive analysis.

Unification types Löwenheim [24] has given a formula for the most general solution of a unification problem expressed in terms of a particular solution. Given a unifier σ of a unifiable formula φ , let ϵ_φ^σ be the substitution such that for all $x \in \mathcal{VAR}$, if $x \in \mathbf{var}(\varphi)$ then $\epsilon_\varphi^\sigma(x) = (\varphi \wedge x) \vee (\neg\varphi \wedge \sigma(x))$ else $\epsilon_\varphi^\sigma(x) = x$. By induction on the formula ψ , the reader may prove that if $\mathbf{var}(\psi) \subseteq \mathbf{var}(\varphi)$ then $\varphi \rightarrow (\epsilon_\varphi^\sigma(\psi) \leftrightarrow \psi)$ and $\neg\varphi \rightarrow (\epsilon_\varphi^\sigma(\psi) \leftrightarrow \sigma(\psi))$ are valid. Since σ is a unifier of φ , we obtain the validity of $\varphi \rightarrow \epsilon_\varphi^\sigma(\varphi)$ and $\neg\varphi \rightarrow \epsilon_\varphi^\sigma(\varphi)$. Moreover, for all unifiers τ of φ and for all variables x , $\tau(\epsilon_\varphi^\sigma(x)) \leftrightarrow \tau(x)$ is valid. Hence, ϵ_φ^σ is a unifier of φ and for all unifiers τ of φ , $\epsilon_\varphi^\sigma \preceq \tau$. Thus, ϵ_φ^σ is a most general unifier of φ : it constitutes by its own a complete set of unifiers of φ . Consequently, φ is unitary.

In the case of *ELU*, given a unifiable parameter-free formula φ , we can define a most general unifier of it by firstly considering a truth assignment V in the two-element Boolean algebra such that $V(\varphi) = 1$ and by

secondly defining the substitution ϵ' such that for all $x \in \mathcal{VAR}$, if $x \in \mathbf{var}(\varphi)$ then $\epsilon'(x) = \text{“if } V(x) = 0 \text{ then } \varphi \wedge x \text{ else } \varphi \rightarrow x\text{”}$ else $\epsilon'(x) = x$. Obviously, the substitution ϵ' is equivalent to the substitution $\epsilon_{\varphi}^{\sigma_V}$ defined in the previous paragraph when σ_V is the parameter-free ground substitution such that for all $x \in \mathcal{VAR}$, if $x \in \mathbf{var}(\varphi)$ then $\sigma_V(x) = \text{“if } V(x) = 0 \text{ then } \perp \text{ else } \top\text{”}$ else $\sigma_V(x) = x$.

Additional comments Restricting \mathcal{BPL} to a language (denoted $\mathcal{BPL}^{\rightarrow}$) where \rightarrow is the sole Boolean connective makes the satisfiability problem trivial seeing that every formula then becomes satisfiable. Nevertheless, the validity problem still remains coNP-hard. So, all in all, $\mathcal{BPL}^{\rightarrow}$ is at least easier than \mathcal{BPL} for what concerns the satisfiability problem, or the validity problem. Surprisingly, the effect of restricting \mathcal{BPL} to $\mathcal{BPL}^{\rightarrow}$ is opposite for what concerns the unification problem. This opposite effect is visible for example, in the formula $x \rightarrow (p \vee q)$ which is unifiable and non-unitary⁴. It is unifiable : $\mathcal{BPL}^{\rightarrow}$ -substitutions like $\sigma_p(x) = p \vee (q \wedge x)$ and $\sigma_q(x) = (p \wedge x) \vee q$ are unifiers of it⁵. However, it is non-unitary. Let us see why. Indeed, suppose it is unitary. Let τ be a $\mathcal{BPL}^{\rightarrow}$ -unifier of it such that $\tau \preceq \sigma_p$ and $\tau \preceq \sigma_q$. As can be proved by induction on $\varphi \in \mathcal{BPL}^{\rightarrow}$, if $\varphi \rightarrow (p \vee q)$ is valid then $p \rightarrow \varphi$ is valid, or $q \rightarrow \varphi$ is valid. Since τ is a unifier of $x \rightarrow (p \vee q)$, therefore $p \rightarrow \tau(x)$ is valid, or $q \rightarrow \tau(x)$ is valid. Without loss of generality, suppose $p \rightarrow \tau(x)$ is valid. Since $\tau \preceq \sigma_q$, therefore $\lambda(\tau(x)) \leftrightarrow \sigma_q(x)$ is valid for some substitution λ . Since $p \rightarrow \tau(x)$ is valid, therefore $p \rightarrow ((p \wedge x) \vee q)$ is valid : a contradiction.

This increased difficulty in the unification problem has to be related to the fact that in $\mathcal{BPL}^{\rightarrow}$, Boolean logic is losing its structural completeness. In order to understand why, let us consider the rule $\frac{x \rightarrow p}{p \rightarrow x}$. As the reader can prove, for all $\mathcal{BPL}^{\rightarrow}$ -formulas φ , if $\varphi \rightarrow p$ is valid then $p \rightarrow \varphi$ is valid. Hence, the considered rule is admissible. Nevertheless, it is not derivable. Let us see why. Indeed, suppose the rule is derivable. Thus, by using the derivation theorem which still holds in $\mathcal{BPL}^{\rightarrow}$ [13, Chapter 9], the formula $(x \rightarrow p) \rightarrow (p \rightarrow x)$ is valid : a contradiction.

The type of $\mathcal{BPL}^{\rightarrow}$ for UWP is unknown and we conjecture that it is finitary. See Balbiani and Mojtahedi [8] for preliminary results. For $\mathcal{BPL}^{\rightarrow}$ -ELU, it is relatively easy to prove that every formula is unifiable and unitary. In order to understand why, let $\varphi \in \mathcal{BPL}^{\rightarrow}$ be a parameter-free formula. Let $x \in \mathcal{VAR}$ and $\psi \in \mathcal{BPL}$ be such that x does

not occur in ψ and $\varphi \leftrightarrow (\psi \vee x)$ is valid⁶. Let ϵ be the $\mathcal{BPL}^{\rightarrow}$ -substitution such that for all $y \in \mathcal{VAR}$, if $y = x$ then $\epsilon(y) = \varphi \rightarrow x$ else $\epsilon(y) = y$. Since x does not occur in ψ and $\varphi \leftrightarrow (\psi \vee x)$ is valid, therefore $\epsilon(\varphi) \leftrightarrow (\psi \vee ((\varphi \rightarrow x) \rightarrow x))$ is valid. Moreover, for all $\mathcal{BPL}^{\rightarrow}$ -unifiers τ of φ and for all $y \in \mathcal{VAR}$, $\tau(\epsilon(y)) \leftrightarrow \tau(y)$ is valid. Hence, ϵ is a $\mathcal{BPL}^{\rightarrow}$ -unifier of φ and for all $\mathcal{BPL}^{\rightarrow}$ -unifiers τ of φ , $\epsilon \preceq \tau$. Thus, ϵ is a most general unifier of φ : it constitutes by its own a complete set of unifiers of φ . Consequently, φ is unitary.

4 Case of modal logics

Syntax and semantics In the case of modal logics, on top of the Boolean connectives considered in Section 3, \mathcal{O} contains the modality $\Box \dots$ (“necessarily, ...”) of arity 1. We denote the associated language by \mathcal{MPL} . The modality $\Diamond \dots$ (“possibly, ...”) of arity 1 is defined by the abbreviation : $\Diamond \varphi ::= \neg \Box \neg \varphi$. For all $n \geq 0$, we write $\varphi_1 \dots \varphi_n$ to mean $\varphi_1 \wedge \dots \wedge \varphi_n$. We write φ^0 to mean $\neg \varphi$ and we write φ^1 to mean φ . For all $d \geq 0$, we write $\Box^{<d} \varphi$ to mean \top when $d = 0$, $(\varphi \wedge \Box \Box^{<d-1} \varphi)$ otherwise and we write $\Box^d \varphi$ to mean φ when $d = 0$, $\Box \Box^{d-1} \varphi$ otherwise. For all formulas ψ , we write $[\psi] \varphi$ to mean $\Box(\psi \rightarrow \varphi)$. For all formulas ψ and for all $d \geq 0$, we write $[\psi]^{<d} \varphi$ to mean \top when $d = 0$, $(\varphi \wedge [\psi][\psi]^{<d-1} \varphi)$ otherwise and we write $[\psi]^d \varphi$ to mean φ when $d = 0$, $[\psi][\psi]^{d-1} \varphi$ otherwise. For all formulas φ , its *modal degree* (in symbols, $\text{deg}(\varphi)$) is defined as usual.

Formulas of \mathcal{MPL} are interpreted in Kripke frames (W, R) — with $\mathcal{P}(W)$ as set of values and $\{W\}$ as set of designated values — by means of truth assignments V such that $V(\perp) = \emptyset$, $V(\neg \varphi) = W \setminus V(\varphi)$, $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$ and $V(\Box \varphi) = \{s \in W : \text{for all } t \in W, \text{ if } R(s, t) \text{ then } t \in V(\varphi)\}$. In this section, we will consider the modal logics enumerated in Table 1.

Computability of unification In some popular modal logics, to solve the unification problem is not a mere formality. The truth is that contrary to the case of Boolean logic, there exists parameter-free formulas that are satisfiable without being unifiable. To see why, take the parameter-free formula $\Diamond x \wedge \Diamond \neg x$. In many modal logics such as **K4**, **S4**, etc, this formula is satisfiable. Nevertheless, it is known that if this formula is unifiable in a modal logic **L** then **L** is inconsistent [15, 27]. The thing is that, while attacking the unification problem, little, if anything, from the standard tools in modal logics (canonical models,

4. In $\mathcal{BPL}^{\rightarrow}$, the Boolean connective \vee is definable by $(\varphi \vee \psi) ::= ((\varphi \rightarrow \psi) \rightarrow \psi)$.

5. As the reader can prove, formulas like $p \vee (q \wedge x)$ and $(p \wedge x) \vee q$ are definable in $\mathcal{BPL}^{\rightarrow}$.

6. Given a parameter-free formula $\varphi \in \mathcal{BPL}^{\rightarrow}$, the existence of $x \in \mathcal{VAR}$ and $\psi \in \mathcal{BPL}$ such that x does not occur in ψ and $\varphi \leftrightarrow (\psi \vee x)$ is valid can be proved by induction on φ .

Modal logics	Class of Kripke frames
K	All
KD	Serial
KT	Reflexive
KB	Symmetric
KDB	Serial symmetric
KTB	Reflexive symmetric
KG	Church-Rosser
KDG	Serial Church-Rosser
KTG	Reflexive Church-Rosser
K4	Transitive
S4	Reflexive transitive
S5	Reflexive transitive Euclidean
Alt₁	Deterministic

TABLE 1 – Some modal logics together with the classes of Kripke frames that determine them.

filtrations, etc) is helpful — to such an extent that **the computability of the following problems remains open : ELU in \mathbf{K} and \mathbf{KB} and UWP in \mathbf{KD} , \mathbf{KT} , \mathbf{KDB} , \mathbf{KTB} and \mathbf{Alt}_1 .**

The case of \mathbf{Alt}_1 constitutes a good example of what are the difficulties of the unification problem in modal logics. Within the context of this modal logic, the modality \Box corresponds in Kripke frames to a deterministic binary relation similar to the binary relation corresponding to the “next” modality of linear temporal logics. It is well-known that \mathbf{Alt}_1 gives rise to an \mathbf{NP} -complete satisfiability problem. Recently, by reducing $\mathbf{Alt}_1\text{-}ELU$ to the problem of determining whether a graph contains an Hamiltonian path, Balbiani and Tinchev [9] have proved that $\mathbf{Alt}_1\text{-}ELU$ is in \mathbf{PSPACE} . Nevertheless, the argument they have put forward does not seem to be adaptable to the case of $\mathbf{Alt}_1\text{-}UWP$ and **the computability of $\mathbf{Alt}_1\text{-}UWP$ remains open.**

Luckily, in some other popular modal logics, to solve the unification problem is relatively easy. For instance, in modal logics \mathbf{L} containing \mathbf{KD} , ELU is in \mathbf{NP} . This can be proved by reducing $\mathbf{L}\text{-}ELU$ to the problem of determining whether, given a parameter-free formula $\varphi(\bar{x})$, there are tuples $\bar{\phi}$ of atom-free formulas uniformly replacing the variables in the tuple \bar{x} such that $\varphi(\bar{\phi})$ is in \mathbf{KD} . Owing to the fact that every atom-free formula is \mathbf{KD} -equivalent to \perp , or \mathbf{KD} -equivalent to \top , one readily observes that the atom-free formulas in $\bar{\phi}$ uniformly replacing the variables in \bar{x} can be restricted to the formulas \perp and \top . This gives rise to a nondeterministic algorithm able to solve $\mathbf{L}\text{-}ELU$ in polynomial time.

If one considers a modal logic not containing \mathbf{KD} , or one interests in UWP then the complexity of the uni-

fication problem may dramatically increase. Investigated as a subproblem of the non-admissibility problem, the unification problem has been proved by Rybakov [29] to be decidable in modal logics such as $\mathbf{K4}$, $\mathbf{S4}$, etc. Nevertheless, apart from the work of Jeřábek [22] who has studied the computability of the admissibility problem in modal logics such as $\mathbf{K4}$, $\mathbf{S4}$, etc, **the computability of ELU and UWP is still largely terra incognita in most modal logics.**

Unification types We have seen in Section 3 that in Boolean logic, every unifiable formulas can be proved to be unitary. In some modal logics, that is quite a different matter. Consider for example, the formula $\Box x \vee \Box \neg x$. It is \mathbf{K} -unifiable : substitutions like $\sigma_{\top}(x) = \top$ and $\sigma_{\perp}(x) = \perp$ are unifiers of it. However, it is finitary. Let us see why. Firstly, the above-mentioned unifiers σ_{\top} and σ_{\perp} constitute a complete set of unifiers of it. This is a consequence of the fact that \mathbf{K} satisfies the modal disjunction property : for all formulas φ, ψ , if $\Box \varphi \vee \Box \psi$ is in \mathbf{K} then φ is in \mathbf{K} , or ψ is in \mathbf{K} . Secondly, there exists no unifier τ of $\Box x \vee \Box \neg x$ such that $\tau \preceq \sigma_{\top}$ and $\tau \preceq \sigma_{\perp}$. Indeed, suppose τ is a unifier such that $\tau \preceq \sigma_{\top}$ and $\tau \preceq \sigma_{\perp}$. Since τ is a \mathbf{K} -unifier of $\Box x \vee \Box \neg x$, therefore by the modal disjunction property, $\tau(x)$ is in \mathbf{K} , or $\neg \tau(x)$ is in \mathbf{K} . In the former case, since $\tau \preceq \sigma_{\perp}$, therefore $\top \leftrightarrow \perp$ is in \mathbf{K} : a contradiction. In the latter case, since $\tau \preceq \sigma_{\top}$, therefore $\perp \leftrightarrow \top$ is in \mathbf{K} : a contradiction.

The existence of a finitary \mathbf{K} -unifiable formula does not imply that all \mathbf{K} -unifiable formulas are finitary. Witness, the formula $x \rightarrow \Box x$ put forward by Jeřábek [23]. It is \mathbf{K} -unifiable : substitutions like $\sigma_{\top}(x) = \top$ and $\sigma_d(x) = \Box^{<d} x \wedge \Box^d \perp$ for each $d \geq 0$ are unifiers of it. However, it is nullary. Let us see why. Firstly, for all unifiers τ of $x \rightarrow \Box x$, $\tau(x)$ is in \mathbf{K} , or $\tau(x) \rightarrow \Box^{\deg(\tau(x))} \perp$ is in \mathbf{K} . This is a consequence of the fact that \mathbf{K} satisfies the following variant of the rule of margins : for all formulas φ , if $\varphi \rightarrow \Box \varphi$ is in \mathbf{K} then φ is in \mathbf{K} , or $\varphi \rightarrow \Box^{\deg(\varphi)} \perp$ is in \mathbf{K} . Secondly, as the reader can prove, for all substitutions τ , $\tau(x)$ is in \mathbf{K} iff $\sigma_{\top} \preceq \tau$ and for all unifiers τ of $x \rightarrow \Box x$, $\tau(x) \rightarrow \Box^{\deg(\tau(x))} \perp$ is in \mathbf{K} iff $\sigma_{\deg(\tau(x))} \preceq \tau$. Thirdly, there exists no minimal complete set of unifiers of $x \rightarrow \Box x$. Indeed, suppose Σ is a minimal complete set of unifiers of $x \rightarrow \Box x$. Since Σ is complete, therefore let $\tau \in \Sigma$ be such that $\tau \preceq \sigma_0$. Since Σ is a set of unifiers of $x \rightarrow \Box x$, therefore $\tau(x)$ is in \mathbf{K} , or $\tau(x) \rightarrow \Box^{\deg(\tau(x))} \perp$ is in \mathbf{K} . In the former case, $\sigma_{\top} \preceq \tau$. Since $\tau \preceq \sigma_0$, therefore $\sigma_{\top} \preceq \sigma_0$. Hence, $\top \leftrightarrow \perp$ is in \mathbf{K} : a contradiction. In the latter case, since Σ is a set of unifiers of $x \rightarrow \Box x$, therefore $\sigma_{\deg(\tau(x))} \preceq \tau$. Since Σ is complete, therefore let $\mu \in \Sigma$ be such that $\mu \preceq \sigma_{\deg(\tau(x))+1}$. As the reader can

prove, $\sigma_{\deg(\tau(x))+1} \preceq \sigma_{\deg(\tau(x))}$. Since $\sigma_{\deg(\tau(x))} \preceq \tau$ and $\mu \preceq \sigma_{\deg(\tau(x))+1}$, therefore $\mu \preceq \tau$. Since Γ is minimal, therefore $\mu = \tau$. Since $\sigma_{\deg(\tau(x))} \preceq \tau$ and $\mu \preceq \sigma_{\deg(\tau(x))+1}$, therefore $\sigma_{\deg(\tau(x))} \preceq \sigma_{\deg(\tau(x))+1}$. Thus, $\lambda(\sigma_{\deg(\tau(x))}(x)) \leftrightarrow \sigma_{\deg(\tau(x))+1}(x)$ is in \mathbf{K} for some substitution λ . Consequently, $\Box^{\deg(\tau(x))+1} \perp \rightarrow \Box^{\deg(\tau(x))} \perp$ is in \mathbf{K} : a contradiction.

Because of the strong proximity between the modal logics \mathbf{K} , \mathbf{KD} , \mathbf{KT} , \mathbf{KB} , \mathbf{KDB} and \mathbf{KTB} in terms of axiomatization/completeness and decidability/complexity, the reader may wonder whether Jeřábek's line of reasoning can be used as it is for \mathbf{KD} , \mathbf{KT} , \mathbf{KB} , \mathbf{KDB} and \mathbf{KTB} . Obviously, in this line of reasoning, the formulas $\Box^d \perp$ for each $d \geq 0$ play an important role⁷. Unfortunately, when $d \geq 1$, $\Box^d \perp$ is equivalent to \perp in \mathbf{KD} , \mathbf{KT} , \mathbf{KDB} and \mathbf{KTB} and is equivalent to $\Box \perp$ in \mathbf{KB} . It follows that Jeřábek's line of reasoning has to be seriously adapted if one wants to apply it to \mathbf{KD} , \mathbf{KT} , \mathbf{KB} , \mathbf{KDB} and \mathbf{KTB} .

Using a parameter p , Balbiani and Gencer [6] have considered the formula $x \rightarrow (p \wedge [p]x)$ within the context of \mathbf{KD} . This formula is \mathbf{KD} -unifiable : substitutions like $\sigma_p(x) = p$ and $\sigma_d(x) = p \wedge [p]^{<d}x \wedge [p]^d \perp$ for each $d \geq 0$ are unifiers of it. However, it is nullary. To see why, it suffices to firstly prove that for all unifiers τ of $x \rightarrow (p \wedge [p]x)$, $\tau(x) \leftrightarrow p$ is in \mathbf{KD} , or $\tau(x) \rightarrow [p]^{\deg(\tau(x))} \perp$ is in \mathbf{KD} , to secondly prove that for all substitutions τ , $\tau(x) \leftrightarrow p$ is in \mathbf{KD} iff $\sigma_p \preceq \tau$ and for all unifiers τ of $x \rightarrow (p \wedge [p]x)$, $\tau(x) \rightarrow [p]^{\deg(\tau(x))} \perp$ is in \mathbf{KD} iff $\sigma_{\deg(\tau(x))} \preceq \tau$ and to thirdly prove that there exists no minimal complete set of unifiers of $x \rightarrow (p \wedge [p]x)$.

Using distinct parameters p, q , Balbiani [5] and Balbiani and Gencer [7] have respectively considered the formula $(x \rightarrow (p \wedge [q]y)) \wedge (y \rightarrow (q \wedge [p]x))$ within the context of \mathbf{KT} and the formula $(p^0 q^0 \wedge x) \rightarrow [p^0 q^1][p^1 q^0][p^0 q^0]x$ within the context of \mathbf{KB} , \mathbf{KDB} and \mathbf{KTB} . Following a line of reasoning similar to the ones presented above, they have proved that the former formula is unifiable and nullary in \mathbf{KT} and the latter formula is unifiable and nullary in \mathbf{KB} , \mathbf{KDB} and \mathbf{KTB} .

The nullariness character of the modal logics \mathbf{KD} , \mathbf{KT} , \mathbf{KB} , \mathbf{KDB} and \mathbf{KTB} constitutes an answer to a question put forward by Dzik [14, Chapter 5]. Nevertheless, this answer only concerns *UWP*, **the types of \mathbf{KD} , \mathbf{KT} , \mathbf{KB} , \mathbf{KDB} and \mathbf{KTB} for *ELU* remaining unknown**. Moreover, much remains to be done, seeing that, for instance, **the types of simple Church-Rosser modal logics like \mathbf{KG} , \mathbf{KDG} and \mathbf{KTG} are unknown**. As well, **the type of the least modal logic containing $\Box^d \perp$ is unknown for each**

7. As well as the fact that for all $d \geq 0$, $\Box^{d+1} \perp \rightarrow \Box^d \perp$ is not valid.

$d \geq 2$.

In a sense, the existence of a nullary unifiable formula is the worst thing one can imagine about the unification type of a modal logic. Luckily, there exists modal logics where every unifiable formulas can be proved to be unitary. This is the case of the modal logic $\mathbf{S5}$. In order to understand why, it suffices given a unifier σ of a unifiable formula φ , to consider the substitution ϵ_φ^σ such that for all $x \in \mathcal{V}\mathcal{AR}$, if $x \in \mathbf{var}(\varphi)$ then $\epsilon_\varphi^\sigma(x) = (\Box\varphi \wedge x) \vee (\neg\Box\varphi \wedge \sigma(x))$ else $\epsilon_\varphi^\sigma(x) = x$ ⁸. By induction on the formula ψ , the reader may prove that if $\mathbf{var}(\psi) \subseteq \mathbf{var}(\varphi)$ then $\Box\varphi \rightarrow (\epsilon_\varphi^\sigma(\psi) \leftrightarrow \psi)$ and $\neg\Box\varphi \rightarrow (\epsilon_\varphi^\sigma(\psi) \leftrightarrow \sigma(\psi))$ are valid. Since σ is a unifier of φ , we obtain the validity of $\Box\varphi \rightarrow \epsilon_\varphi^\sigma(\varphi)$ and $\neg\Box\varphi \rightarrow \epsilon_\varphi^\sigma(\varphi)$. Moreover, for all unifiers τ of φ and for all variable x , $\tau(\epsilon_\varphi^\sigma(x)) \leftrightarrow \tau(x)$ is valid. Hence, ϵ_φ^σ is a unifier of φ and for all unifiers τ of φ , $\epsilon_\varphi^\sigma \preceq \tau$. Thus, ϵ_φ^σ is a most general unifier of φ : it constitutes by its own a complete set of unifiers of φ . Consequently, φ is unitary.

Additional comments The truth is that the above line of reasoning proving the unitariness of every $\mathbf{S5}$ -unifiable formula can be adapted to each modal logic where a modality \forall ("everywhere, ...") of arity 1 playing the role of a universal modality is definable⁹. The unitariness of every unifiable formula is a remarkable property of a modal logic seeing that it guarantees the existence of a most general unifier for all its unifiable formulas. Nevertheless, it does not constitute a sufficient condition for the decidability of its unification problem and its admissibility problem. After the first results of Rybakov [29, 30] about the decidability of the unification problem and the admissibility problem in modal logics such as $\mathbf{K4}$, $\mathbf{S4}$, etc, it was an open question to determine whether the decidability of a modal logic ensures the decidability of its unification problem and its admissibility problem. This question has been negatively answered by Chagrov [11] who has constructed a — rather artificial — decidable modal logic with an undecidable admissibility problem. Later on, Wolter and Zakharyashev [32] has proved that the unification problem and the admissibility problem are undecidable in \mathbf{K} , $\mathbf{K4}$ and every modal logic between \mathbf{K} and $\mathbf{K4}$ when the above-mentioned modality \forall is definable.

Today, it is possible, without using the universal modality, to construct a very simple decidable modal logic with an undecidable admissibility problem. Let us see

8. The reader will remark how this substitution looks like the Löwenheim substitution considered in Section 3.

9. Enriching modal logics with the universal modality gives rise to an \mathbf{EXP} -complete satisfiability problem in the case of \mathbf{K} and a \mathbf{PSPACE} -complete satisfiability problem in the case of some simple extensions of $\mathbf{K4}$ and $\mathbf{S4}$ [20, 31].

how. In the case of the modal logic $\mathbf{Alt}_1 \times \mathbf{Alt}_1$ — the *product* of \mathbf{Alt}_1 with itself —, on top of the Boolean connectives considered in Section 3, \mathcal{O} contains the modalities $\Box_1 \dots$ and $\Box_2 \dots$ of arity 1. The modalities $\Diamond_1 \dots$ and $\Diamond_2 \dots$ of arity 1 are defined by the abbreviations : $\Diamond_1 \varphi ::= \neg \Box_1 \neg \varphi$ and $\Diamond_2 \varphi ::= \neg \Box_2 \neg \varphi$. Formulas are interpreted in Kripke frames (W, R_1, R_2) where R_1 and R_2 are deterministic binary relations on W such that $R_1 \circ R_2 \subseteq R_2 \circ R_1$, $R_2 \circ R_1 \subseteq R_1 \circ R_2$ and $R_1^{-1} \circ R_2 \subseteq R_2 \circ R_1^{-1}$ — with $\mathcal{P}(W)$ as set of values and $\{W\}$ as set of designated values — by means of truth assignments V such that $V(\Box_1 \varphi) = \{s \in W : \text{for all } t \in W, \text{ if } R_1(s, t) \text{ then } t \in V(\varphi)\}$ and $V(\Box_2 \varphi) = \{s \in W : \text{for all } t \in W, \text{ if } R_2(s, t) \text{ then } t \in V(\varphi)\}$. It is well-known that in $\mathbf{Alt}_1 \times \mathbf{Alt}_1$, the satisfiability problem is *NP*-complete [16, Theorem 8.53]. What is maybe less known is that in $\mathbf{Alt}_1 \times \mathbf{Alt}_1$, the validity problem of rules is undecidable¹⁰ [16, Theorem 8.54]. What is new is that in $\mathbf{Alt}_1 \times \mathbf{Alt}_1$, the admissibility problem is undecidable. This can be proved by a reduction of the *domino-tiling problem* (Π) an instance of which consists of a 7-tuple $(D, S_1, S_2, D_u, D_d, D_l, D_r)$ where D is a finite set of domino-types, S_1 and S_2 are binary relations on D and D_u, D_d, D_l and D_r are subsets of D . A tiling of such 7-tuple is a triple (a_1, a_2, f) where a_1, a_2 are positive integers and f is a function associating an element $f(i_1, i_2) \in D$ to each $(i_1, i_2) \in \{1, \dots, a_1\} \times \{1, \dots, a_2\}$. We shall say that (a_1, a_2, f) is *correct* if the following conditions hold for all $i_1 \in \{1, \dots, a_1\}$ and for all $i_2 \in \{1, \dots, a_2\}$:

- if $i_1 < a_1$ then $f(i_1, i_2) S_1 f(i_1 + 1, i_2)$,
- if $i_2 < a_2$ then $f(i_1, i_2) S_2 f(i_1, i_2 + 1)$,
- $f(a_1, i_2) \in D_u$,
- $f(1, i_2) \in D_d$,
- $f(i_1, 1) \in D_l$,
- $f(i_1, a_2) \in D_r$.

Let $(D, S_1, S_2, D_u, D_d, D_l, D_r)$ be an instance of (Π) with $\delta_1, \dots, \delta_n$ a list of its domino-types. Let y, z and x_1, \dots, x_n be pairwise distinct variables. Let ϕ be the conjunction of the following formulas :

- $\Box_1 \Box_2 \neg(x_k \wedge x_l)$ where $1 \leq k, l \leq n$ and $k \neq l$,
- $\Box_1 \Box_2 (x_k \rightarrow \Box_1 \bigvee \{x_l : 1 \leq l \leq n \text{ and } (\delta_k, \delta_l) \in S_1\})$ where $1 \leq k \leq n$,
- $\Box_1 \Box_2 (x_k \rightarrow \Box_2 \bigvee \{x_l : 1 \leq l \leq n \text{ and } (\delta_k, \delta_l) \in S_2\})$ where $1 \leq k \leq n$,
- $y \rightarrow (\Box_1 y \wedge \Box_2 y)$,
- $z \rightarrow (\Box_1 z \wedge \Box_2 z)$,
- $\neg y \rightarrow \Box_1 \neg y$,
- $\neg z \rightarrow \Box_2 \neg z$,
- $\Box_1 \Box_2 ((y \wedge \Box_1 \perp) \rightarrow \bigvee \{x_k : 1 \leq k \leq n \text{ and } \delta_k \in D_u\})$,
- $\Box_2 ((y \wedge \neg z) \rightarrow \Box_1 (z \rightarrow \bigvee \{x_k : 1 \leq k \leq$

10. The rule $\frac{\varphi_1, \dots, \varphi_n}{\chi}$ is *valid* if the validity of $\varphi_1, \dots, \varphi_n$ implies the validity of χ .

- n and $\delta_k \in D_d\})$,
- $\Box_1 ((\neg y \wedge z) \rightarrow \Box_2 (y \rightarrow \bigvee \{x_k : 1 \leq k \leq n \text{ and } \delta_k \in D_l\}))$,
- $\Box_1 \Box_2 ((z \wedge \Box_2 \perp) \rightarrow \bigvee \{x_k : 1 \leq k \leq n \text{ and } \delta_k \in D_r\})$.

Let ψ be the formula

- $(\Diamond_2 y \wedge \Diamond_1 z \wedge \Box_1 \Box_2 \bigvee \{x_k : 1 \leq k \leq n\}) \rightarrow (y \vee z)$.

What is remarkable is that there exists a correct tiling iff the rule $\frac{\phi}{\psi}$ is not admissible. Seeing that the domino-tiling problem (Π) is undecidable [26], this proves that in $\mathbf{Alt}_1 \times \mathbf{Alt}_1$, the admissibility problem is undecidable. **As for the computability of $\mathbf{Alt}_1 \times \mathbf{Alt}_1$ -*ELU* and $\mathbf{Alt}_1 \times \mathbf{Alt}_1$ -*UWP*, it is open.**

5 Case of multi-agent epistemic logics

Syntax and semantics In the case of multi-agent epistemic logics, on top of the Boolean connectives considered in Section 3, \mathcal{O} consists of the modalities $\Box_a \dots$ (“agent a knows that ...”) of arity 1 where a ranges over a countable set \mathcal{AGT} of agents. We denote the associated language by \mathcal{MAEPL} . The modalities $\Diamond_a \dots$ (“it is compatible with the knowledge of agent a that ...”) of arity 1 are defined by the abbreviations : $\Diamond_a \varphi ::= \neg \Box_a \neg \varphi$. We shall say that a formula φ is *Boolean* if for all $a \in \mathcal{AGT}$, φ contains no occurrence of the modality \Box_a . For all $a \in \mathcal{AGT}$, we shall say that a formula φ is *a-restricted* if for all $b \in \mathcal{AGT}$, if $a \neq b$ then φ contains no occurrence of the modality \Box_b . For all $a \in \mathcal{AGT}$, we shall say that a formula φ is *a-monic* if for all $b \in \mathcal{AGT}$ and for all formulas ψ , if $\Box_b \psi$ is a subformula of φ then $a = b$, or ψ is variable-free. Obviously, for all $a \in \mathcal{AGT}$, every *a-restricted* formula is *a-monic*.

Formulas of \mathcal{MAEPL} are interpreted in multi-agent Kripke frames $(W, \{R_a : a \in \mathcal{AGT}\})$ by means of truth assignments V such that $V(\Box_a \varphi) = \{s \in W : \text{for all } t \in W, \text{ if } R_a(s, t) \text{ then } t \in V(\varphi)\}$. In this section, we will not reopen the debate about the reasonableness of the notion of knowledge corresponding to such-and-such class of multi-agent Kripke frames. We will rather content ourselves with analyzing the effects of the considered classes of multi-agent Kripke frames on the computability of unification and the unification types. In most cases, it is assumed that for all $a \in \mathcal{AGT}$, the accessibility relation R_a is an equivalence relation¹¹. The associated modal logic is denoted $\mathbf{S5}_{\mathcal{AGT}}$. In few cases, it is moreover assumed that for all $a, b \in \mathcal{AGT}$, the accessibility relations R_a and R_b are in local agreement, i.e. for all $s \in W$, $R_a(s) \subseteq R_b(s)$, or

11. In that case, for all $a \in \mathcal{AGT}$ and for all $s \in W$, we will denote by $R_a(s)$ the equivalence class modulo R_a with s as its representative.

$R_b(s) \subseteq R_a(s)$. The associated modal logic is denoted $\mathbf{S5}_{\mathcal{AGT}}^{la}$.

Computability of unification $\mathbf{S5}_{\mathcal{AGT}}\text{-ELU}$ and $\mathbf{S5}_{\mathcal{AGT}}^{la}\text{-ELU}$ are **NP**-complete, seeing that the corresponding classes of multi-agent Kripke frames are such that for all $a \in \mathcal{AGT}$, R_a is serial¹². **Regarding $\mathbf{S5}_{\mathcal{AGT}}\text{-UWP}$ and $\mathbf{S5}_{\mathcal{AGT}}^{la}\text{-UWP}$, their computability is a mystery.**

Unification types In this paragraph, we consider the class of multi-agent Kripke frames where for all $a \in \mathcal{AGT}$, R_a is an equivalence relation. If \mathcal{AGT} is finite then the modality \forall (“every agent knows that ...”) of arity 1 defined by the abbreviation $\forall\varphi ::= \bigwedge\{\Box_a\varphi : a \in \mathcal{AGT}\}$ plays the role of a universal modality when we restrict the discussion to the class of multi-agent Kripke frames where for all $a, b \in \mathcal{AGT}$, R_a and R_b are in local agreement. In this case, following the argument used in Section 4 for modal logics where a modality playing the role of a universal modality is definable, every unifiable formulas can be proved to be unitary.

Otherwise, that is to say if \mathcal{AGT} is not finite, or when we do not restrict the discussion to the class of multi-agent Kripke frames where for all $a, b \in \mathcal{AGT}$, R_a and R_b are in local agreement, the situation is worse. To see why, use pairwise distinct parameters p, q, r and take the formula $(x \rightarrow \Box x) \wedge (\neg x \rightarrow \blacksquare \neg x)$ where for all formulas φ , we write $\Box\varphi$ to mean $[p^0q^0r^0]_1[p^0q^0r^1]_2[p^0q^1r^0]_1[p^0q^1r^1]_2[p^1q^0r^0]_1[p^1q^0r^1]_2\varphi$ and we write $\blacksquare\varphi$ to mean $p^1q^0r^1 \rightarrow [p^1q^0r^0]_2[p^0q^1r^1]_1[p^0q^1r^0]_2[p^0q^0r^1]_1[p^0q^0r^0]_2\Box_1\varphi$. It is $\mathbf{S5}_{\mathcal{AGT}}$ -unifiable : substitutions like $\sigma_d(x) = \Box^d x \wedge \Box^d \perp$ for each $d \geq 0$ and $\tau_d(x) = \neg(\blacksquare^{<d} \neg x \wedge \blacksquare^d \perp)$ for each $d \geq 0$ are unifiers of it. However, following a line of reasoning similar to the ones developed above for **K** and **KD**, the reader may prove that it is nullary.

Additional comments The existence of a nullary $\mathbf{S5}_{\mathcal{AGT}}$ -unifiable formula does not imply that all $\mathbf{S5}_{\mathcal{AGT}}$ -unifiable formulas are nullary. Witness, the monic formulas. Indeed, let $a \in \mathcal{AGT}$ and φ be an a -monic formula. Suppose it is $\mathbf{S5}_{\mathcal{AGT}}$ -unifiable. Since φ is a -monic, therefore for all $b \in \mathcal{AGT}$ and for all formulas ψ , if $\Box_b\psi$ is a subformula of φ then $a = b$, or ψ is variable-free. Given an $\mathbf{S5}_{\mathcal{AGT}}$ -unifier σ of φ , let ϵ_φ^σ be the substitution such that for all $x \in \mathcal{VAR}$, if $x \in \text{var}(\varphi)$ then $\epsilon_\varphi^\sigma(x) = (\Box_a\varphi \wedge x) \vee (\neg\Box_a\varphi \wedge \sigma(x))$ else $\epsilon_\varphi^\sigma(x) = x$. By induction on the a -monic formula ψ , the reader may prove that if $\text{var}(\psi) \subseteq \text{var}(\varphi)$ then

12. The line of reasoning we have developed in Section 4 to prove that ELU is in **NP** for each modal logic containing **KD** can be applied in the multi-agent setting as well.

$\Box_a\varphi \rightarrow (\epsilon_\varphi^\sigma(\psi) \leftrightarrow \psi)$ and $\neg\Box_a\varphi \rightarrow (\epsilon_\varphi^\sigma(\psi) \leftrightarrow \sigma(\psi))$ are valid. Since σ is a unifier of φ , we obtain the validity of $\Box_a\varphi \rightarrow \epsilon_\varphi^\sigma(\varphi)$ and $\neg\Box_a\varphi \rightarrow \epsilon_\varphi^\sigma(\varphi)$. Moreover, for all unifiers τ of φ and for all variable x , $\tau(\epsilon_\varphi^\sigma(x)) \leftrightarrow \tau(x)$ is valid. Hence, ϵ_φ^σ is a unifier of φ and for all unifiers τ of φ , $\epsilon_\varphi^\sigma \preceq \tau$. Thus, ϵ_φ^σ is a most general unifier of φ : it constitutes by its own a complete set of unifiers of φ . Consequently, φ is unitary.

In next section, we will apply this line of reasoning to the monic \mathcal{MAEPCL} -formulas associated to some simple epistemic planning problems.

6 Application to epistemic planning

On top of the Boolean connectives \perp , \neg and \vee considered in Section 3 and the modalities \Box_a considered in Section 5, let us define the modalities $[[\varphi]] \dots$ of arity 1 for each $\varphi \in \mathcal{MAEPCL}$ by the inductive abbreviations $[[\varphi]]p ::= (\varphi \rightarrow p)$, $[[\varphi]]\perp ::= \neg\varphi$, $[[\varphi]]\neg\psi ::= (\varphi \rightarrow \neg[[\varphi]]\psi)$, $[[\varphi]](\psi \vee \chi) ::= ([[\varphi]]\psi \vee [[\varphi]]\chi)$ and $[[\varphi]]\Box_a\psi ::= (\varphi \rightarrow \Box_a[[\varphi]]\psi)$. Let us define the modalities $\langle\langle\varphi\rangle\rangle \dots$ of arity 1 for each $\varphi \in \mathcal{MAEPCL}$ by the abbreviations : $\langle\langle\varphi\rangle\rangle\psi ::= \neg[[\varphi]]\neg\psi$. Within the setting of Public Announcement Logic [12, Chapter 4], for all $\varphi \in \mathcal{MAEPCL}$, the modalities $[[\varphi]] \dots$ (“if φ holds then after it is announced, ... holds”) and $\langle\langle\varphi\rangle\rangle \dots$ (“ φ holds and after it is announced, ... holds”) have been used to formalize the notion of the public announcement of φ which is the atomic action performed by an outside observer, perceived by all agents and consisting of publicly announcing φ . By means of these modalities together with the modality of common knowledge, one can faithfully represent all of the reasoning in examples such as the muddy children puzzle.

The reader will remark that the formulas written by using the modalities $[[\varphi]] \dots$ and $\langle\langle\varphi\rangle\rangle \dots$ where φ ranges over the set of all \mathcal{MAEPCL} -formulas are exponentially more succinct than the \mathcal{MAEPCL} -formulas they are the abbreviations of. Nevertheless, as proved by Lutz [25], the membership in $\mathbf{S5}_{\mathcal{AGT}}$ of formulas written by using these modalities has the same complexity as the membership in $\mathbf{S5}_{\mathcal{AGT}}$ of \mathcal{MAEPCL} -formulas : **PSPACE**.

Suppose the formula $\varphi(\bar{p})$ describes a given initial situation in terms of the knowledge of \mathcal{AGT} -agents about the list of parameter facts $\bar{p} = (p_1, \dots, p_m)$ and the formula $\langle\langle\chi(\bar{p}\bar{x})\rangle\rangle\psi(\bar{p})$ represents the knowledge of these agents about \bar{p} in a desirable final situation (the ψ -part) after an executable public announcement concerning \bar{p} and the list of variable facts $\bar{x} = (x_1, \dots, x_n)$ (the χ -part) has been performed. It may happen that $\varphi(\bar{p}) \rightarrow \langle\langle\chi(\bar{p}\bar{x})\rangle\rangle\psi(\bar{p})$ is not $\mathbf{S5}_{\mathcal{AGT}}$ -valid. Hence, we may ask whether there are tuples $\bar{\phi} = (\phi_1, \dots, \phi_n)$ of formulas such that $\varphi(\bar{p}) \rightarrow$

$\langle\langle\chi(\bar{p}\bar{\phi})\rangle\rangle\psi(\bar{p})$ is valid. Moreover, we may be interested to obtain the most general tuple $\bar{\phi}$ of formulas such that $\varphi(\bar{p}) \rightarrow \langle\langle\chi(\bar{p}\bar{\phi})\rangle\rangle\psi(\bar{p})$ is valid. Introduced in this way, the formula $\varphi(\bar{p}) \rightarrow \langle\langle\chi(\bar{p}\bar{x})\rangle\rangle\psi(\bar{p})$ constitutes a special instance of what is now called an epistemic planning problem. See [10] for a general introduction.

Consider for example, the formula $\psi \rightarrow \langle\langle\Box_a x\rangle\rangle\Box_b \chi$ where ψ is a variable-free $\mathcal{MAEP}\mathcal{L}$ -formula, x is a variable, a and b are distinct agents and χ is a Boolean variable-free $\mathcal{MAEP}\mathcal{L}$ -formula. Considered as an epistemic planning problem, to solve this formula means to ask whether there exists an $\mathcal{MAEP}\mathcal{L}$ -formula ϕ such that $\psi \rightarrow \langle\langle\Box_a \phi\rangle\rangle\Box_b \chi$ — “in every situation where ψ holds, a knows that ϕ holds and after it is announced that a knows that ϕ holds, b knows that χ holds” — is $\mathbf{S5}_{AGT}$ -valid. Since χ is Boolean, therefore taking into account the definition of the modalities $[\langle\langle\varphi\rangle\rangle] \dots$ and $\langle\langle\varphi\rangle\rangle \dots$ for each $\varphi \in \mathcal{MAEP}\mathcal{L}$, this is equivalent to solve the formula $\psi \rightarrow (\Box_a x \wedge \Box_b \chi')$ where χ' is the $\mathcal{MAEP}\mathcal{L}$ -formula $\Box_a x \rightarrow \chi$. Since \Box_b corresponds in Kripke frames to an equivalence relation, therefore this is equivalent to solve the conjunction κ of the formulas $\psi \rightarrow \Box_a x$ and $\Diamond_b \psi \rightarrow \chi'$. Since ψ and χ are variable-free, therefore κ is a -monic. Moreover, as the reader can prove, κ is $\mathbf{S5}_{AGT}$ -unifiable iff $\Diamond_a \psi \wedge \Diamond_b \psi \rightarrow \chi$ is $\mathbf{S5}_{AGT}$ -valid. In that case, any substitution σ such that $\sigma(x) = \Diamond_a \psi$ is a unifier of it and by applying the line of reasoning developed in the paragraph “Additional comments” of Section 5, one can construct a most general unifier of it and of the given formula $\psi \rightarrow \langle\langle\Box_a x\rangle\rangle\Box_b \chi$ too.

More generally, one may be interested by solving formulas like $\psi \rightarrow \langle\langle\Box_a x \wedge \Diamond_a y_1 \wedge \dots \wedge \Diamond_a y_m\rangle\rangle(\Box_{b_1} \chi_1 \wedge \dots \wedge \Box_{b_n} \chi_n)$ where ψ is a variable-free $\mathcal{MAEP}\mathcal{L}$ -formula, $m, n \geq 0$, x and y_1, \dots, y_m are pairwise distinct variables, a and b_1, \dots, b_n are pairwise distinct agents and χ_1, \dots, χ_n are a -restricted variable-free $\mathcal{MAEP}\mathcal{L}$ -formulas. Considered as an epistemic planning problem, to solve this formula means to ask whether there exists $\mathcal{MAEP}\mathcal{L}$ -formulas ϕ and $\theta_1, \dots, \theta_m$ such that $\psi \rightarrow \langle\langle\Box_a \phi \wedge \Diamond_a \theta_1 \wedge \dots \wedge \Diamond_a \theta_m\rangle\rangle(\Box_{b_1} \chi_1 \wedge \dots \wedge \Box_{b_n} \chi_n)$ — “in every situation where ψ holds, a knows that ϕ holds, it is compatible with the knowledge of a that θ_i holds for each $i \in \{1, \dots, m\}$ and after it is announced that a knows that ϕ holds and it is compatible with the knowledge of a that θ_i holds for each $i \in \{1, \dots, m\}$, b_j knows that χ_j holds for each $j \in \{1, \dots, n\}$ ” — is $\mathbf{S5}_{AGT}$ -valid. Since χ_1, \dots, χ_n are a -restricted, therefore taking into account the definition of the modalities $[\langle\langle\varphi\rangle\rangle] \dots$ and $\langle\langle\varphi\rangle\rangle \dots$ for each $\varphi \in \mathcal{MAEP}\mathcal{L}$, this is equivalent to solve the formula $\psi \rightarrow (\Box_a x \wedge \Diamond_a y_1 \wedge \dots \wedge \Diamond_a y_m \wedge \Box_{b_1} \chi'_1 \wedge \dots \wedge \Box_{b_n} \chi'_n)$ where χ'_j is the $\mathcal{MAEP}\mathcal{L}$ -formula $(\Box_a x \wedge \Diamond_a y_1 \wedge \dots \wedge \Diamond_a y_m) \rightarrow \chi_j$ for each $j \in \{1, \dots, n\}$. Since $\Box_{b_1}, \dots, \Box_{b_n}$ corres-

pond in Kripke frames to equivalence relations, therefore this is equivalent to solve the conjunction κ of the formulas $\psi \rightarrow (\Box_a x \wedge \Diamond_a y_1 \wedge \dots \wedge \Diamond_a y_m)$ and $\Diamond_{b_j} \psi \rightarrow \chi'_j$ where j ranges over $\{1, \dots, n\}$. Since ψ and χ_1, \dots, χ_n are variable-free, therefore κ is a -monic. Moreover, as the reader can prove, κ is $\mathbf{S5}_{AGT}$ -unifiable iff $\Diamond_a \psi \wedge \Diamond_{b_j} \psi \rightarrow \chi_j$ is $\mathbf{S5}_{AGT}$ -valid for each $j \in \{1, \dots, n\}$. In that case, any substitution σ such that $\sigma(x) = \Diamond_a \psi$ and $\sigma(y_j) = \psi$ for each $j \in \{1, \dots, n\}$ is a unifier of it and by applying the line of reasoning developed in the paragraph “Additional comments” of Section 5, one can construct a most general unifier of it and of the given formula $\psi \rightarrow \langle\langle\Box_a x \wedge \Diamond_a y_1 \wedge \dots \wedge \Diamond_a y_m\rangle\rangle(\Box_{b_1} \chi_1 \wedge \dots \wedge \Box_{b_n} \chi_n)$ too.

7 Conclusion

The modal logics like the ones considered in this paper (\mathbf{K} , \mathbf{KD} , etc) have now limited mathematical interest for what concerns axiomatization/completeness and decidability/complexity. Nevertheless, with respect to some predetermined propositional logic \mathbf{L} , considering the question of the determination of its unification type and the question of the computability of the unification problem it gives rise to is justified from the following perspectives : methods for deciding the \mathbf{L} -unifiability of formulas can be used to understand what is the overlap between the properties formulas correspond to in \mathbf{L} [2] ; in case \mathbf{L} is a description logic, unification algorithms can be used to detect redundancies in \mathbf{L} -based systems [3, 4] ; methods for deciding the \mathbf{L} -unifiability of formulas can be used to improve the efficiency of theorem provers devoted to solve the membership problem in \mathbf{L} . One readily observes that, while attacking the above-mentioned problems, little, if anything, from the standard tools in modal logics (canonical models, filtrations, etc) is helpful. In order to successfully solve them, new techniques in modal logics must be developed and much remains to be done. The study of unification in modal logics has still many secrets to reveal.

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