Boundary singularities of semilinear elliptic equations with Leray-Hardy potential

Huyuan Chen, Laurent Veron

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Abstract

We study existence and uniqueness of solutions of (E_1) $-\Delta u + \frac{\mu}{|x|^2}u + g(u) = \nu$ in Ω, $u = \lambda$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N_+$ is a bounded smooth domain such that $0 \in \partial\Omega$, $\mu \geq -\frac{N^2}{4}$ is a constant, g a continuous nondecreasing function satisfying some integral growth condition and ν and λ two Radon measures respectively in Ω and on $\partial\Omega$. We show that the situation differs considerably according the measure is concentrated at 0 or not. When g is a power we introduce a capacity framework which provides necessary and sufficient conditions for the solvability of problem (E_1).

Key Words: Hardy Potential, Radon Measure.

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Contents

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1 Introduction

If μ is a real number and $N \geq 2$, the Schrödinger operator $\mathcal{L}_\mu$, defined in a domain $\Omega \subset \mathbb{R}^N$ by

$$\mathcal{L}_\mu u := -\Delta u + \frac{\mu}{|x|^2} u,$$

plays a fundamental role in analysis, because of Hardy's inequality, and in theoretical physics in connexion with uncertainty principle. When the singular point 0 belongs to Ω, there exists a critical value

$$\mu_0 = -\left(\frac{N-2}{2}\right)^2.$$

If $\mu \geq \mu_0$ the operator $\mathcal{L}_\mu$ is bounded from below because of Hardy inequality

$$\int_\Omega |\nabla \phi|^2 + \mu_0 \int_\Omega \frac{\phi^2}{|x|^2} dx \geq 0 \quad \text{for all } \phi \in C_0^\infty(\Omega). \tag{1.1}$$

Sharp properties of this inequality has been studied by Brezis and Vazquez [8]. When $\mu \geq \mu_0$, we studied in [14] the Hardy equation with absorption semi-linearity

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu & \text{in } \Omega \\ \qquad\qquad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

for a Radon measure ν being able to be supported at origin in a bounded smooth domain Ω, where g is a continuous nondecreasing function, by using systematically a notion of weak solutions introduced in [13] associated to a dual formulation with a specific weight function because of the Leray-Hardy potential. In this framework, weak solutions to (1.2) in a class of weighted measures are obtained provided that g satisfies some integrability condition. When this integrability condition is not satisfied by g, not all measures in the above class are suitable for solving (1.2). This is called the *supercritical case*. In the supercritical case and when $g(r) = |r|^{p-1}r$ with $p > 1$, we showed that the set of suitable measures is associated to a property of absolute continuity with respect to some Bessel capacity.

In this article we are interested in the configuration where the singular point of the Leray-Hardy potential lies on the boundary of the domain Ω and we study the following equation

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu & \text{in } \Omega \\ \qquad\qquad u = \lambda & \text{on } \partial\Omega, \end{cases}$$

where ν and λ are bounded Radon measures respectively on Ω and $\partial\Omega$. When $\mu = 0$ the first study is due to Gmira and Véron [17] who proved the existence and uniqueness of a *very weak*

solution. Such a solution u is a function belonging to $L^1(\Omega)$ such that $\rho g(u) \in L^1(\Omega)$, where $\rho(x) = \text{dist}\,(x, \partial\Omega)$, satisfying

$$\int_\Omega (-u\Delta\zeta + g(u)\zeta)\, dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial \mathbf{n}} d\nu$$

for all $\zeta \in C^1_c(\overline{\Omega})$ such that $\Delta\zeta \in L^\infty(\Omega)$. The condition for the existence and uniqueness of a solution is

$$\int_1^\infty (g(s) - g(-s))\, s^{-\frac{2N}{N-1}} ds < \infty. \tag{1.3}$$

When $\mu \neq 0$, a typical domain is $\Omega = \mathbb{R}^N_+ := \{x = (x', x_N) = (x_1, ..., x_N) = x_N > 0\}$. There exists a critical value

$$\mu \geq \mu_1 := -\frac{N^2}{4},$$

which is a fundamental value for the operator $\mathcal{L}_\mu$, being the best constant of the Hardy inequality

$$\int_{\mathbb{R}^N_+} |\nabla\phi|^2 + \mu_1 \int_{\mathbb{R}^N_+} \frac{\phi^2}{|x|^2} dx \geq 0 \qquad \text{for all } \phi \in C^\infty_0(\mathbb{R}^N_+).$$

If $\mathbb{R}^N_+$ is replaced by a bounded domain Ω satisfying the condition

$$(\mathcal{C}_1) \qquad 0 \in \partial\Omega\ ,\ \Omega \subset \mathbb{R}^N_+ \text{ and } \langle x, \mathbf{n}\rangle = O(|x|^2) \text{ for all } x \in \partial\Omega,$$

where $\mathbf{n} = \mathbf{n}_x$ is the outward normal vector at x, this inequality is never achieved and there exists a remainder [9]: if we set $R_\Omega = \max_{z\in\Omega} |z|$, there holds

$$\int_\Omega |\nabla\phi|^2 + \mu_1 \int_\Omega \frac{\phi^2}{|x|^2} dx \geq \frac{1}{4} \int_\Omega \frac{\phi^2}{|x|^2 \ln^2(|x| R_\Omega^{-1})} dx \qquad \text{for all } \phi \in C^\infty_0(\Omega). \tag{1.4}$$

Note that the last condition in $(\mathcal{C}_1)$ holds if Ω is a C^2 domain. Put

$$\alpha_+ := \alpha_+(\mu) = 1 - \frac{N}{2} + \sqrt{\mu + \frac{N^2}{4}} \quad \text{and} \quad \alpha_- := \alpha_-(\mu) = 1 - \frac{N}{2} - \sqrt{\mu + \frac{N^2}{4}}.$$

If Ω satisfies $(\mathcal{C}_1)$ there exists $\ell^\Omega_\mu > 0$ defined by

$$\ell^\Omega_\mu := \min\left\{ \int_\Omega \left(|\nabla v|^2 + \frac{\mu}{|x|^2} v^2 \right) dx : v \in C^1_c(\Omega), \int_\Omega v^2 dx = 1 \right\}.$$

If $\mu \geq \mu_1$ this first eigenvalue is achieved in the space $H_\mu(\Omega)$ which is the closure of $C^1_c(\Omega)$ for the norm

$$v \mapsto \|v\|_{H_\mu(\Omega)} := \sqrt{\int_\Omega \left(|\nabla v|^2 + \frac{\mu}{|x|^2} v^2 \right) dx}.$$

Note that $H_\mu(\Omega)=H^1_0(\Omega)$ if $\mu>\mu_1$, $H^1_0(\Omega)\subsetneqq H_{\mu_1}(\Omega)$ and the imbedding of $H_{\mu_1}(\Omega)$ in $L^2(\Omega)$ is compact. We proved in [15] the positive eigenfunction $\gamma^\Omega_\mu\in H_\mu(\Omega)$ of $\mathcal{L}_\mu$ associated to the first eigenvalue ℓ^Ω_μ satisfies

$$\begin{cases}\mathcal{L}_\mu\gamma^\Omega_\mu=\ell^\Omega_\mu\gamma^\Omega_\mu & \text{in } \Omega\\ \gamma^\Omega_\mu=0 & \text{on } \partial\Omega\setminus\{0\}\end{cases}$$

and there exist $c_1>c_2>0$ and $\tilde c>0$ such that for all $x\in\overline{\Omega}\setminus\{0\}$

$$\begin{aligned}&(i) && c_2|x|^{\alpha_+-1}\rho(x)\le\gamma^\Omega_\mu(x)\le c_1|x|^{\alpha_+-1}\rho(x),\\ &(ii) && |\nabla\gamma^\Omega_\mu(x)|\le\tilde c\frac{\gamma^\Omega_\mu(x)}{\rho(x)}.\end{aligned}\tag{1.5}$$

This function will play the role as a weight function. Inequality (1.4) implies the existence of the Green kernel G^Ω_μ with corresponding Green operator $\mathbb{G}^\Omega_\mu$. The Poisson kernel K^Ω_μ of $\mathcal{L}_\mu$ in $\Omega\times\partial\Omega$ is constructed in [15], by a simple truncation as in [31] if $\mu\ge0$, and by a more elaborate approximation in the general case. When $\mu>0$ the kernel has the property that

$$K^\Omega_\mu(x,0)=0\quad\text{for all } x\in\overline{\Omega}\setminus\{0\},$$

by [31, Theorem A.1]. The singular kernel ϕ^Ω_μ is the analogue in a bounded domain of the explicit singular solution $x\mapsto\phi_\mu(x)=|\,x\,|^{\alpha_--1}\,x_N$ defined in $\mathbb{R}^N_+$, and it satisfies for all $x\in\overline{\Omega}\setminus\{0\}$,

$$c_3|x|^{\alpha_--1}\rho(x)\le\phi^\Omega_\mu(x)\le c_4|x|^{\alpha_--1}\rho(x)\quad\text{if } \mu>\mu_1,\tag{1.6}$$

and

$$c_5|x|^{-\frac{N}{2}}(|\ln|x||+1)\rho(x)\le\phi^\Omega_{\mu_1}(x)\le c_6|x|^{-\frac{N}{2}}(|\ln|x||+1)\rho(x).\tag{1.7}$$

We assume that Ω is a bounded smooth domain such that $0\in\partial\Omega$ and its normal vector $e_N=(0,\cdots,0,1)\in\mathbb{R}^N$ at origin in the sequel. We define the γ^Ω_μ-dual operator $\mathcal{L}^*_\mu$ of $\mathcal{L}_\mu$ by

$$\mathcal{L}^*_\mu\zeta=-\Delta\zeta-\frac{2}{\gamma^\Omega_\mu}\langle\nabla\gamma^\Omega_\mu,\nabla\zeta\rangle+\ell^\Omega_\mu\zeta\quad\text{for all } \zeta\in C^{1,1}(\Omega).$$

It satisfies the following commutating property

$$\mathcal{L}_\mu(\gamma^\Omega_\mu\zeta)=\gamma^\Omega_\mu\mathcal{L}^*_\mu\zeta.$$

Denote by $\mathfrak{M}(\Omega;\gamma^\Omega_\mu)$ the set of Radon measures ν in Ω such that

$$\sup\left\{\int_\Omega\zeta d|\lambda|:\zeta\in C_c(\Omega),\,0\le\zeta\le\gamma^\Omega_\mu\right\}:=\int_\Omega\gamma^\Omega_\mu d|\nu|<\infty.$$

Thus, if $\nu\in\mathfrak{M}_+(\Omega;\gamma^\Omega_\mu)$ the measure $\gamma^\Omega_\mu\nu$ is a bounded measure in Ω. We also set

$$\beta^\Omega_\mu(x)=-\frac{\partial\gamma^\Omega_\mu}{\partial\mathbf{n}}\Big\lfloor_{\partial\Omega}.\tag{1.8}$$

The space of Radon measures λ on $\partial\Omega \setminus \{0\}$ such that

$$\sup\left\{\int_{\partial\Omega\setminus\{0\}} \zeta d|\lambda| : \zeta \in C_c(\partial\Omega \setminus \{0\}),\, 0 \le \zeta \le \beta_\mu^\Omega\right\} := \int_{\partial\Omega\setminus\{0\}} \beta_\mu^\Omega d|\lambda| < \infty,$$

is denoted by $\mathfrak{M}(\partial\Omega; \beta_\mu^\Omega)$. The extension of $\lambda \in \mathfrak{M}_+(\partial\Omega; \beta_\mu^\Omega)$ as a measure $\beta_\mu^\Omega\lambda$ in $\partial\Omega$ is given by

$$\int_{\partial\Omega} \zeta d(\beta_\mu^\Omega\lambda) = \sup\left\{\int_{\partial\Omega} \upsilon\beta_\mu^\Omega d\lambda : \upsilon \in C_c(\partial\Omega \setminus \{0\}),\, 0 \le \upsilon \le \zeta\right\} \quad \text{for all } \zeta \in C(\partial\Omega)\,,\ \zeta \ge 0$$

and $\beta_\mu^\Omega\lambda = \beta_\mu^\Omega\lambda_+ - \beta_\mu^\Omega\lambda_-$ if λ is a signed measure in $\mathfrak{M}(\partial\Omega; \beta_\mu^\Omega)$, and this defines the set $\mathfrak{M}(\partial\Omega; \beta_\mu^\Omega)$ of all such extensions. The Dirac mass at 0 does not belong to $\mathfrak{M}(\partial\Omega; \beta_\mu^\Omega)$, but it is the limit of sequences of measures in this space. We proved in [15] that if $\nu \in \mathfrak{M}_+(\Omega; \gamma_\mu^\Omega)$, $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu^\Omega)$ and $k \in \mathbb{R}$, the function

$$u = \mathbb{G}_\mu^\Omega[\nu] + \mathbb{K}_\mu^\Omega[\lambda] + k\phi_\mu^\Omega := \mathbb{H}_\mu^\Omega[(\nu, \lambda + k\delta_0)]$$

is the unique function belonging to $L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$ satisfying

$$\int_\Omega u\mathcal{L}_\mu^*\zeta d\gamma_\mu^\Omega = \int_\Omega \zeta d(\gamma_\mu^\Omega\nu) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega\lambda) + kc_\mu\zeta(0)$$

for all $\zeta \in \mathbb{X}_\mu(\Omega) = \left\{\zeta \in C(\overline{\Omega}) \text{ s.t. } \gamma_\mu^\Omega\zeta \in H_\mu(\Omega) \text{ and } \rho\mathcal{L}_\mu^*\zeta \in L^\infty(\Omega)\right\}$, where

$$c_\mu = \begin{cases} 2\sqrt{\mu - \mu_1}\displaystyle\int_{\mathbb{S}_+^{N-1}} \phi_1^2 dS & \text{if } \mu > \mu_1, \\ \left(\frac{N}{2} - 1\right)\displaystyle\int_{\mathbb{S}_+^{N-1}} \phi_1^2 dS & \text{if } \mu = \mu_1, \end{cases}$$

and ϕ_1 is the positive eigenfunction of $\Delta_{\mathbb{S}^{N-1}}$ in $\mathbb{S}_+^{N-1} := \{(x', x_N) \in \mathbb{R}^N : |x| = 1,\, x_N > 0\}$ with zero Dirichlet boundary condition with respect to the first eigenvalue.

Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous nondecreasing function satisfying $rg(r) \ge 0$. Thanks to this result we can construct of weak solutions of the problem

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu & \text{in } \Omega \\ \qquad\qquad u = \lambda + k\delta_0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

Definition 1.1 *Let $(\nu, \lambda) \in \mathfrak{M}(\Omega; \gamma_\mu^\Omega) \times \mathfrak{M}(\partial\Omega; \beta_\mu^\Omega)$ and $k \in \mathbb{R}$. A function $u \in L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$ is a weak solution of (1.9) if $g(u) \in L^1(\Omega, d\gamma_\mu^\Omega)$ and*

$$\int_\Omega \left(u\mathcal{L}_\mu^*\zeta + g(u)\zeta\right) d\gamma_\mu = \int_\Omega \zeta d(\gamma_\mu\nu) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega\lambda) + kc_\mu\zeta(0) \quad \textit{for any } \zeta \in \mathbb{X}_\mu(\Omega).$$

We set

$$p^*_\mu = 1 - \frac{2}{\alpha_-} = \frac{N+2+2\sqrt{\mu-\mu_1}}{N-2+2\sqrt{\mu-\mu_1}} \text{ and } \ p^{**}_\mu = 1 - \frac{2}{\alpha_+} = \frac{N+2-2\sqrt{\mu-\mu_1}}{N-2-2\sqrt{\mu-\mu_1}}. \tag{1.10}$$

Note that $p^*_0 = \frac{N+1}{N-1}$, $p^*_{\mu_1} = \frac{N+2}{N-2}$, p^{**}_μ is defined only if $N \geq 3$ and $-\frac{N^2}{4} \leq \mu < 1-N$.

Our first result deals with the existence of a solution with an isolated singularity on boundary:

Theorem A *Assume $N \geq 3$ and $\mu \geq \mu_1$, or $N = 2$ and $\mu > \mu_1$, and let $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous nondecreasing function such that $rg(r) \geq 0$. If there holds*

$$\int_1^\infty (g(s) - g(-s))\, s^{-1-p^*_\mu} ds < \infty \quad \text{if } \mu > \mu_1, \tag{1.11}$$

or

$$\int_1^\infty (g(s\ln s) - g(-s\ln|s|))\, s^{-1-p^*_{\mu_1}} ds < \infty \quad \text{if } \mu = \mu_1, \tag{1.12}$$

then for any $k \in \mathbb{R}$ there exists a unique weak solution $u_{k\delta_0}$ to

$$\begin{cases} \mathcal{L}_\mu u + g(u) = 0 & \text{in } \Omega \\ \qquad\qquad u = k\delta_0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore,

$$\lim_{x\to 0} \frac{u_{k\delta_0}(x)}{\phi^\Omega_\mu(x)} = \frac{k}{c_\mu}.$$

When the measures do not charge the point 0, we have a result which is similar as the one proved in [17].

Theorem B *Assume $N \geq 3$ and $\mu \geq \mu_1$, or $N = 2$ and $\mu > \mu_1$, and let $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous nondecreasing function such that $rg(r) \geq 0$ satisfying*

$$\int_1^\infty (g(s) - g(-s))\, s^{-1-p^*_0} ds < \infty. \tag{1.13}$$

Then for any $(\nu, \lambda) \in \mathfrak{M}(\Omega; \gamma^\Omega_\mu) \times \mathfrak{M}(\partial\Omega; \beta^\Omega_\mu)$ there exists a unique weak solution u to

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu & \text{in } \Omega \\ \qquad\qquad u = \lambda & \text{on } \partial\Omega. \end{cases}$$

Finally we construct a solution to (1.9) without restriction on the measures by gluing solutions corresponding to Theorems A and B *provided g satisfies the weak Δ_2-condition* already introduced in [14]:

There exists a continuous nondecreasing positive function $K : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

$$|g(s+r)| \leq K(|r|)\,(|g(s)| + |g(r)|) \quad \text{for all } \ (s,r) \in \mathbb{R}\times\mathbb{R} \ \ s.t. \ \ sr \geq 0. \tag{1.14}$$

Theorem C *Assume $N \geq 3$ and $\mu \geq \mu_1$, or $N = 2$ and $\mu > \mu_1$, and let $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous nondecreasing function such that $rg(r) \geq 0$ satisfying the weak Δ_2-condition and*

$$\int_1^\infty (g(s) - g(-s))\, s^{-1-\min\{p_\mu^*, p_0^*\}} ds < +\infty. \tag{1.15}$$

Then for any $(\nu, \lambda) \in \mathfrak{M}(\Omega; \gamma_\mu^\Omega) \times \mathfrak{M}(\partial\Omega; \beta_\mu^\Omega)$ and $k \in \mathbb{R}$ there exists a solution u to the problem (1.9).

A nonlinearity g for which problem (1.9) admits a solution is called *subcritical*. A couple of measures (ν, λ) for which problem (1.9) admits a solution is called *g-good*. In the supercritical case all the measures are not g-good. Besides the problem at 0 where (1.11)-(1.12) may or may not be satisfied, the admissibility of a measure depends on its concentration expressed in terms of Bessel capacities. We denote these capacities by $c_{\alpha,q}^{\mathbb{R}^d}$ where $d = N$ or $N-1$. In this framework we consider only the case where $g(r) = g_p(r) := |r|^{p-1}r$ with $p > 1$. The following theorem is proved.

Theorem D *Assume $\mu \geq \mu_1$ and $p > 1$.*
1- A measure $\nu \in \mathfrak{M}(\Omega; \gamma_\mu^\Omega)$ is g_p-good if and only if it is absolutely continuous with respect to the $c_{2,p'}^{\mathbb{R}^N}$-capacity.
2- A measure $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu^\Omega)$ is g_p-good if and only if it is absolutely continuous with respect to the $c_{\frac{2}{p},p'}^{\mathbb{R}^{N-1}}$-capacity.

Similarly we have a characterization of removable singularities.

Theorem E *Assume $\mu \geq \mu_1$, $p > 1$ and $K \subset \overline{\Omega}$ is compact. Then any weak solution of*

$$\begin{cases} \mathcal{L}_\mu u + g_p(u) = 0 & \text{in } \ \Omega \cap K^c \\ u = 0 & \text{on } \ \partial\Omega \cap K^c \end{cases} \tag{1.16}$$

can be extended as a solution of the same equation in Ω vanishing on $\partial\Omega$ if and only if
(i) $c_{2,p'}^{\mathbb{R}^N}(K) = 0$ if $K \subset \Omega$.
(ii) $c_{\frac{2}{p},p'}^{\mathbb{R}^{N-1}}(K) = 0$ if $K \subset \partial\Omega \setminus \{0\}$.
(iii) $c_{2,p'}^{\mathbb{R}^N}(K) = 0$ and $c_{\frac{2}{p},p'}^{\mathbb{R}^{N-1}}(K \cap \partial\Omega)$ if $K \subset \overline{\Omega} \setminus \{0\}$.
(iv) $c_{\frac{2}{p},p'}^{\mathbb{R}^{N-1}}(K) = 0$ and $p \geq p_\mu^*$ if $0 \in K \subset \partial\Omega$ and $K \setminus \{0\} \neq \{\emptyset\}$.
(v) $c_{2,p'}^{\mathbb{R}^N}(K \cap \Omega) = 0$, $c_{\frac{2}{p},p'}^{\mathbb{R}^{N-1}}(K \cap \partial\Omega) = 0$ and $p \geq p_\mu^*$ if $0 \in K \subset \overline{\Omega}$ and $K \cap \Omega \neq \{\emptyset\}$.

At end we characterize the behaviour of solutions of

$$\begin{cases} \mathcal{L}_\mu u + g_p(u) = 0 & \text{in } \ \Omega \\ u = h & \text{on } \ \partial\Omega \setminus \{0\}, \end{cases} \tag{1.17}$$

where $h \in C^3(\partial\Omega)$. When $p \geq p_\mu^*$ we prove that u is indeed the very weak solution of

$$\begin{cases} \mathcal{L}_\mu u + g_p(u) = 0 & \text{in } \ \Omega \\ u = h & \text{on } \ \partial\Omega. \end{cases} \tag{1.18}$$

The techniques we use are extensions of charaterization of singularities developed studies in [17] and [18]. We associate a problem on $\mathbb{S}^{N-1}_+$:

$$\begin{cases} -\Delta'\omega + (\Lambda_{p,N} + \mu)\,\omega + g_p(\omega) = 0 & \text{in } \mathbb{S}^{N-1}_+ \\ \omega = 0 & \text{on } \partial\mathbb{S}^{N-1}_+, \end{cases} \tag{1.19}$$

where

$$\Lambda_{p,N} = \frac{2}{p-1}\left(N - \frac{2p}{p-1}\right).$$

Let $\mathcal{S}_{\mu,p}$ (resp. $\mathcal{S}^+_{\mu,p}$) denote the set of solutions (resp. positive solutions) of (1.19). We set

$$\tilde{p}^*_\mu = 1 + \frac{2}{a_-} = \frac{N+2+2\sqrt{\mu-\mu_2}}{N-2+2\sqrt{\mu-\mu_2}} \text{ and } \tilde{p}^{**}_\mu = 1 + \frac{2}{a_+} = \frac{N+2-2\sqrt{\mu-\mu_2}}{N-2-2\sqrt{\mu-\mu_2}}, \tag{1.20}$$

where $\mu_2 = -\left(\frac{N+2}{2}\right)^2$. Note that $\tilde{p}^{**}_\mu$ is defined only if $N \geq 9$ and $-\frac{N^2}{4} \leq \mu < -2N$. The introduction of the numbers a_+ and a_-, will be explained in the proof of the theorem. Then we have

Theorem F *Assume $\mu \geq \mu_1$ and $p > 1$.*
1- $\mathcal{S}_{\mu,p}$ is not reduced to $\{0\}$ if and only if $\Lambda_{p,N} + \mu + N - 1 < 0$, that is
(i) either $1 < p < p^*_\mu$,
(ii) or $N \geq 3$, $\mu_1 \leq \mu < 1 - N$ and $p > p^{**}_\mu$.
2- If $\mathcal{S}^+_{\mu,p}$ is non-empty, it is reduced to one element ω_μ.
3- All the elements of $\mathcal{S}_{\mu,p}$ have constant sign if $\Lambda_{p,N} + \mu + N - 1 < \Lambda_{p,N} + \mu + 2N \leq 0$, that is:
(i) when $\mu \geq 1 - N$ and $\tilde{p}^*_\mu \leq p < p^*_\mu$,
(ii) when $N \geq 3$, $-2N \leq \mu < 1 - N$ and either $\tilde{p}^*_\mu \leq p < p^*_\mu$ or $p^{**}_\mu < p$,
(iii) when $N \geq 9$ and $\mu_1 \leq \mu < -2N$ and either $\tilde{p}^*_\mu \leq p < p^*_\mu$ or $p^{**}_\mu < p \leq \tilde{p}^{**}_\mu$.

Since any solution of (1.17) satisfies

$$|u(x)| \leq c_7 \rho(x) |x|^{-\frac{p+1}{p-1}} \quad \text{for all } x \in \Omega \cap B_{r_0}, \tag{1.21}$$

for some $r_0 > 0$ and $c_7 > 0$ depending on N, p and Ω, we flatten the boundary as in [17], define the new function $\tilde{u}(y)$ by this change of variable, set $v(t,\sigma) = r^{\frac{2}{p-1}}\tilde{u}(r,\sigma)$ with $t = \ln r$ and study the limit set $\mathcal{E}_v$ of the new equation satisfied by $v(t,.)$ when $t \to -\infty$. This limit set is a connected compact subset of $\mathcal{E}_\mu$. If $u \geq 0$, $\mathcal{E}_v \subset \mathcal{E}^+_\mu$. Thus we prove the following.

Theorem G *Assume $\mu \geq \mu_1$, $h \in C^3(\partial\Omega)$ and $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ is a nonnegative solution of (1.17). If $1 < p < p^*_\mu$ then*
(i) either

$$\lim_{\substack{\Omega \ni x \to 0 \\ \frac{x}{|x|} \to \sigma \in S^{N-1}_+}} |x|^{\frac{2}{p-1}} u(x) = \omega_\mu(\sigma), \tag{1.22}$$

(ii) or there exists $\ell > 0$ such that

$$u(x) = \ell K^\Omega_\mu(x,0)(1 + o(1)) \quad \text{as } x \in \Omega,\ x \to 0, \tag{1.23}$$

and u is the weak solution of

$$\begin{cases} \mathcal{L}_\mu u + g_p(u) = 0 & \text{in } \Omega \\ u = h + c\ell\delta_0 & \text{on } \partial\Omega. \end{cases} \tag{1.24}$$

When u is a signed solution, the situation is more delicate and we obtain only partial results.

Theorem H *Assume $\mu \geq \mu_1$, $h \in C^3(\partial\Omega)$ and $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ is a solution of (1.17). If $\tilde{p}^*_\mu \leq p < p^*_\mu$, then*
(a) either

$$\lim_{\substack{\Omega \ni x \to 0 \\ \frac{x}{|x|} \to \sigma \in \mathbb{S}^{N-1}_+}} |x|^{\frac{2}{p-1}} u(x) = \pm\omega_\mu(\sigma), \tag{1.25}$$

(b) or

$$\lim_{\substack{\Omega \ni x \to 0 \\ \frac{x}{|x|} \to \sigma \in \mathbb{S}^{N-1}_+}} |x|^{\frac{2}{p-1}} u(x) = 0. \tag{1.26}$$

If we assume furthermore that $\tilde{p}^*_\mu < p$ and (1.26) is verified, then there exists $\ell \in \mathbb{R}$ such that (1.23) and (1.24) hold.

In two cases the limit set is reduced to a single element of $\mathcal{E}_\mu$, whatever is the structure of this set.

Theorem I *Assume $\mu \geq \mu_1$, $h \in C^3(\partial\Omega)$ and $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ is a solution of (1.17)).*
1- If $N + 2\sqrt{\mu - \mu_1} < 4$ and $p = 3$, then there exists $\omega \in \mathcal{S}_{\mu,p}$ such that

$$\lim_{\substack{\Omega \ni x \to 0 \\ \frac{x}{|x|} \to \sigma \in \mathbb{S}^{N-1}_+}} |x|^{\frac{2}{p-1}} u(x) = \omega(\sigma).$$

2- If $N = 2$ and $1 < p < 1 + \frac{2}{\sqrt{\mu+1}}$, then

$$\lim_{\substack{\Omega \ni x \to 0 \\ \frac{x}{|x|} \to \sigma \in \mathbb{S}^{1}_+}} |x|^{\frac{2}{p-1}} u(x) = \omega(\sigma),$$

where ω is a solution of

$$\begin{cases} -\omega'' + \left(\mu - \left(\frac{2}{p-1}\right)^2\right)\omega + g_p(\omega) = 0 \quad \text{on } (0,\pi) \\ \omega(0) = \omega(\pi) = 0. \end{cases} \tag{1.27}$$

Furthermore, if $\partial\Omega$ is locally a straigh tline near 0 and the limit in (1.27) is zero, there exists $\ell \in \mathbb{R}$ such that (1.23) holds.

We end this article with a removability result.

Theorem J *Assume $\mu \geq \mu_1$, $p \geq p^*_\mu$, $h \in C^3(\partial\Omega)$ and $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ is a solution of (1.17). Then u is actually the weak solution of (1.18).*

The rest of this paper is organized as follows. In section 2, we recall Kato's inequality and prove the existence and uniqueness of semilinear elliptic equation with measures sources when the nonlinearity is subcritical. Section 3 is devoted to deal with the supcritical case by connecting the measures with Bessel capacities. Finally, we analyze the behaviors of solutions provided regular boundary conditions by considering associated problem on semi-sphere.

2 The subcritical case

2.1 Kato inequality

Proposition 2.1 *Let $N \geq 2$, $\mu \geq \mu_1$ and $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ be a continuous function satisfying*

$$g(s_1, x) \geq g(s_2, x) \quad \text{if } x \in \mathbb{R}^N_+ \text{ and } s_1 \geq s_2.$$

If u and v belong to $C^{1,1}(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ satisfy

$$\begin{cases} \mathcal{L}_\mu u + g(x,u) \geq \mathcal{L}_\mu v + g(x,v) & \text{in } \Omega \\ u \geq v & \text{on } \partial\Omega \setminus \{0\} \end{cases}$$

and

$$\liminf_{r \to 0} \sup_{\substack{x \ni \Omega \\ |x| = r}} \frac{v(x) - u(x)}{\phi^\Omega_\mu(x)} \leq 0,$$

then $v \leq u$ in Ω.

Proof. Set $w = v - u$, then $\mathcal{L}_\mu w + h(x)w = 0$ where

$$h(x) = \begin{cases} \dfrac{g(x,v) - g(x,u)}{w} & \text{if } w \neq 0 \\ 0 & \text{if } w = 0. \end{cases}$$

Hence $h \geq 0$. For $\epsilon > 0$, we set $W_\epsilon = v - u - \epsilon\phi^\Omega_\mu$. Then $W_\epsilon \in C^{0,1}_c(\overline{\Omega} \setminus \{0\})$. There exists a sequence $\{r_n\}$ tending to 0 such that

$$W_\epsilon(x) < 0 \quad \text{for } |x| = r_n,$$

and there holds

$$-\Delta W_\epsilon + \frac{\mu}{|x|^2} W_\epsilon + hW_\epsilon \leq 0.$$

Multiplying by $(W_\epsilon)_+ := \max\{0, W_\epsilon\}$ and integrating yields, since $(W_\epsilon)_+ \in C^{1,1}_c(\overline{\Omega} \setminus \{0\})$

$$\int_{\Omega \setminus B_{r_n}} \left(\mid \nabla (W_\epsilon)_+ \mid^2 + \frac{\mu_1}{|x|^2} (W_\epsilon)^2_+ \right) dx \leq 0.$$

Hence $(W_\epsilon)_+ = 0$ in $\Omega \setminus B_{r_n}$, we get the result by letting $r_n \to 0$ first and then $\epsilon \to 0$. □

The following form of Kato's inequality for Schrödinger operators with Hardy-Leray potential with boundary singularity singularity is important in our approach of the concept of weak solutions to (1.9).

Proposition 2.2 *[15, Lemma 3.1] Assume $N \geq 3$ and $\mu \geq \mu_1$, or $N = 2$ and $\mu > \mu_1$. Then for any $(f, h) \in L^1(\Omega, d\gamma_\mu^\Omega) \times L^1(\partial\Omega, d\beta_\mu^\Omega)$ there exists a unique function $u \in L^1(\Omega, |x|^{-1} d\gamma_\mu^\Omega)$ satisfying*

$$\int_\Omega u\mathcal{L}_\mu^* \zeta \, d\gamma_\mu^\Omega = \int_\Omega \zeta f \, d\gamma_\mu^\Omega + \int_{\partial\Omega} h \, d\beta_\mu^\Omega \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega).$$

Furthermore, for any $\zeta \in \mathbb{X}_\mu^+(\Omega) = \{\zeta \in \mathbb{X}_\mu(\Omega) : \zeta \geq 0\}$, there holds

$$\int_\Omega |u| \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega(x) \leq \int_\Omega \zeta f \mathrm{sgn}(u) d\gamma_\mu^\Omega(x) + \int_{\partial\Omega} |h| \zeta d\beta_\mu^\Omega(x')$$

and

$$\int_\Omega u_+ \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega(x) \leq \int_\Omega \zeta f \mathrm{sgn}_+(u) d\gamma_\mu^\Omega(x) + \int_{\partial\mathbb{R}^N_+} h_+ \zeta d\beta_\mu^\Omega(x').$$

Let $\sigma_\mu^\Omega \in H_\mu(\Omega)$ be the unique variational solution of

$$\mathcal{L}_\mu \sigma_\mu^\Omega = \frac{\gamma_\mu^\Omega}{\min\{l_\mu^\Omega, \rho\}} \quad \text{in } \Omega \quad \text{and} \quad \sigma_\mu^\Omega = 0 \quad \text{on } \partial\Omega,$$

then σ_μ^Ω belongs to $C^2(\overline{\Omega} \setminus \{0\})$ and satisfies (see [15, Appendix]

$$(i) \qquad \gamma_\mu^\Omega \leq \sigma_\mu^\Omega \leq c_7 \gamma_\mu^\Omega \qquad \text{in } \Omega,$$

$$(ii) \qquad \nabla\sigma_\mu^\Omega(x) \sim \nabla\gamma_\mu^\Omega(x) \qquad \text{as } x \to 0.$$

Furthermore $\dfrac{\partial\sigma_\mu^\Omega}{\partial\mathbf{n}} < 0$ on $\partial\Omega \setminus \{0\}$. The function $\eta = \dfrac{\sigma_\mu^\Omega}{\gamma_\mu^\Omega}$ which verifies

$$\mathcal{L}_\mu^* \eta = \frac{1}{\min\{l_\mu^\Omega, \rho\}} \quad \text{in } \Omega, \tag{2.1}$$

plays an important role as a test function because of the following estimates that it satisfies

$$1 \leq \eta \leq c_7 \quad \text{and} \quad |\nabla\eta| \leq c_7 \rho^{-1} \quad \text{in } \Omega. \tag{2.2}$$

2.2 Proof of Theorem A

Assume $\Omega \subset B_1$ and let $k > 0$. If $\mu > \mu_1$, we have by (1.5) and (1.6)

$$\begin{aligned} \int_\Omega g(k\phi_\mu) d\gamma_\mu &\leq c_9 \int_{B_R} g(c_8 |x|^{\alpha_-}) |x|^{\alpha_+} dx \leq c_{10} \int_0^R g(c_8 r^{\alpha_-}) r^{\alpha_+ + N - 1} dr \\ &\leq c_{11} \int_{R^{1/\alpha_-}}^\infty g(s) s^{-1 + \frac{\alpha_+ + N}{\alpha_-}} ds = c_{11} \int_{R^{1/\alpha_-}}^\infty g(s) s^{-1-p_\mu^*} < \infty, \end{aligned} \tag{2.3}$$

where $\phi_\mu(x) = |x|^{\alpha_- - 1}x_N \geq \phi^\Omega_\mu(x)$ in Ω, and p^*_μ is defined in (1.10). If $\mu = \mu_1$ we obtain similarly

$$\int_\Omega g(k\phi_{\mu_1})d\gamma^\Omega_{\mu_1} \leq c_{11}\int_{R^{1/\alpha_-}}^\infty g(s\ln s)s^{-\frac{2N}{N-2}}ds < \infty.$$

For $r > 0$ small enough set $\Omega_r = \Omega \setminus \overline{B}_r$, $\partial\Omega_r = \Gamma_{1,r} \cup \Gamma_{2,r}$ where $\Gamma_{1,r} = B^c_r \cap \partial\Omega$ and $\Gamma_{2,r} = \partial B_r \cap \Omega$. We consider the problem

$$\begin{cases} \mathcal{L}_\mu v + g(v) = 0 & \text{in } \Omega_r \\ \qquad\qquad v = k\phi^\Omega_\mu & \text{on } \partial\Omega_r. \end{cases} \tag{2.4}$$

The associated functional where $G(r) = \displaystyle\int_0^r g(s)ds$ is expressed by

$$J^r_\mu(v) = \int_{\Omega_r}\left(\frac{1}{2}|\nabla v|^2 + \frac{\mu}{2|x|^2}v^2 + G(v)\right)dx$$

and defined over $\mathcal{H}_r = \{v \in H^1(\Omega_\epsilon) : v = k\phi^\Omega_\mu \text{ on } \partial\Omega_r\}$. Any $v \in \mathcal{H}_r$ can be written as $v = k\phi^\Omega_\mu + w$ where $w \in H^1_0(\Omega_r)$, then $J^r_\mu(v) = J^r_\mu(k\phi^\Omega_\mu + w) = \tilde{J}^r_\mu(w)$, where

$$\begin{aligned}
\tilde{J}^r_\mu(w) &= \int_{\Omega_r}\left(\frac{1}{2}|\nabla w|^2 + \frac{\mu}{2|x|^2}w^2 + G(w + k\phi^\Omega_\mu)\right)dx + \frac{k^2}{2}\int_{\Omega_r}\left(|\nabla\phi^\Omega_\mu|^2 + \frac{\mu}{|x|^2}(\phi^\Omega_\mu)^2\right)dx \\
&\quad + \int_{\Omega_r}\left(\langle\nabla\phi^\Omega_\mu, \nabla w\rangle + \frac{\mu}{|x|^2}\phi^\Omega_\mu w\right)dx \\
&= \int_{\Omega_r}\left(\frac{1}{2}|\nabla w|^2 + \frac{\mu}{2|x|^2}w^2 + G(w + k\phi^\Omega_\mu)\right)dx + \frac{k^2}{2}\int_{\Omega_r}\left(|\nabla\phi^\Omega_\mu|^2 + \frac{\mu}{|x|^2}(\phi^\Omega_\mu)^2\right)dx \\
&\quad + \int_{\Omega_r}\eta\mathcal{L}_\mu\phi^\Omega_\mu dx + \int_{\partial\Omega_r}\frac{\partial\phi^\Omega_\mu}{\partial\mathbf{n}}wdS \\
&\geq \frac{1}{4}\int_{\Omega_r}\frac{w^2}{|x|^2\ln^2(|x|)}dx + \frac{k^2}{2}\int_{\Omega_r}\left(|\nabla\phi^\Omega_\mu|^2 + \frac{\mu}{|x|^2}(\phi^\Omega_\mu)^2\right)dx,
\end{aligned}$$

since $w \in H^1_0(\Omega_r)$, (1.4) holds and $G \geq 0$. Hence $\tilde{J}^r_\mu$ and therefore J^r_μ is coercive and since it is convex, it admits a unique minimum u_r, which is the unique classical solution of (2.4) by standard regularity and by Proposition 2.1 such that $0 < u_r \leq \phi^\Omega_\mu$ in Ω_r.

By monotonicity $0 < u_r \leq u_{r'}$ in $\Omega_{r'}$ if $r \in (0, r')$. Let $u_k = \lim_{r\to 0} u_r$. Because of (2.3), $g(u_r) \to g(u_0)$ in $L^1(\Omega; d\gamma^\Omega_\mu)$. Let $\gamma_r := \gamma^{\Omega_r}_\mu$ be the first eigenfuntion of the operator

$$\omega \mapsto -\Delta\omega + \frac{\mu}{|x|^2}\omega \quad \text{in } H^1_0(\Omega_r)$$

with corresponding eigenvalue $\ell_r := \ell^{\Omega_r}_\mu$. We normalize γ_r by $\gamma_r(x_0) = 1$ for some fixed x_0 in $\Omega_{\frac{1}{4}}$. Then $\ell_r > \ell^\Omega_\mu$ and $\ell_r \to \ell^\Omega_\mu$ when $r \to 0$. Furthermore $\gamma_r \to \gamma^\Omega_\mu$ uniformly on $\overline{\Omega}_r$ for any

$\delta > 0$, where $\gamma_\mu^\Omega(x_0) = 1$. If $\zeta \in \mathbb{X}_\mu(\Omega)$, we have

$$\begin{aligned} 0 &= \int_{\Omega_r} \zeta\gamma_r \left(\mathcal{L}_\mu u_r + g(u_r)\right) dx \\ &= \int_{\Omega_r} \Big(\left(-\gamma_r\Delta\zeta - 2\langle\nabla\gamma_r, \nabla\zeta\rangle + \ell_r\zeta\gamma_r\right) u_r + \zeta\gamma_r g(u_r)\Big) dx - k\int_{\Gamma_{2,r}} \zeta \frac{\partial\gamma_r}{\partial\mathbf{n}}\phi_\mu^\Omega dS. \end{aligned} \tag{2.5}$$

Since

$$-\int_{\Gamma_{2,r}} \frac{\partial\gamma_r}{\partial\mathbf{n}}\phi_\mu^\Omega dS = \int_{\Omega_\epsilon} \phi_\mu^\Omega \Delta\gamma_r dS - \int_{\Omega_r} \gamma_r \Delta\phi_\mu^\Omega dS = -\ell_\epsilon \int_{\Omega_r} \gamma_r\phi_\mu^\Omega dx,$$

then, letting $r \to 0$,

$$\lim_{r\to 0} \int_{\Gamma_{2,r}} \frac{\partial\gamma_r}{\partial\mathbf{n}}\phi_\mu^\Omega dS = \ell_\mu^\Omega \int_\Omega \gamma_\mu^\Omega\phi_\mu^\Omega dx.$$

Noting from (2.5) that

$$\lim_{r\to 0} \int_{\Omega_r} \Big(\left(-\gamma_r\Delta\zeta - 2\langle\nabla\gamma_r, \nabla\zeta\rangle + \ell_r\zeta\gamma_r\right) u_r + \zeta\gamma_r g(u_r)\Big) dx = \int_\Omega \left(u_k\mathcal{L}_\mu^*\zeta + \zeta g(u_k)\right) d\gamma_\mu^\Omega,$$

we infer

$$\int_\Omega \left(u_k\mathcal{L}_\mu^*\zeta + \zeta g(u_k)\right) d\gamma_\mu^\Omega = c_{N,\mu,\Omega} k\zeta(0), \tag{2.6}$$

with

$$c_{N,\mu,\Omega} = \ell_\mu^\Omega \int_\Omega \gamma_\mu^\Omega \phi_\mu dx.$$

Since $x \mapsto k\phi_\mu^\Omega(x)$ satisfies (2.4) with $g = 0$, it satisfies also (2.6), always with $g = 0$. Combining this result with the uniqueness and the estimates given in [15, Proposition 2.1], we can compute the explicit value of $c_{N,\mu,\Omega} = c_\mu$. □

2.3 Proof of Theorem B

We first assume that $(\nu, \lambda) \in \mathfrak{M}_+(\Omega; \gamma_\mu^\Omega) \times \mathfrak{M}_+(\partial\Omega; \beta_\mu^\Omega)$. Since g satisfies (1.3) and $\mathcal{L}_\mu$ is uniformly elliptic in Ω_r, it follows from [30, Section 3] that the problem

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu_\epsilon & \text{in } \Omega_r \\ u = \lambda_\epsilon & \text{on } \Gamma_{1,r} := \partial\Omega \cap B_r^c \\ u = 0 & \text{on } \Gamma_{2,r} := \Omega \cap \partial B_r, \end{cases} \tag{2.7}$$

admits a unique weak solution $u_{\epsilon,r}$, where $\nu_\epsilon = \nu_\epsilon\chi_{B_\epsilon^c}$, $\lambda_\epsilon = \lambda_\epsilon\chi_{B_\epsilon^c}$ and $0 < r < \epsilon/2$.

By the comparison principle, for $0 < \epsilon' < \epsilon$ and $0 < r' < r$ there holds

$$(i) \quad 0 < u_{\epsilon,r} < u_{\epsilon',r'} \quad \text{and} \quad (ii) \quad u_{\epsilon,r} \le \mathbb{G}_\mu^{\Omega_r}[\nu_\epsilon] + \mathbb{K}_\mu^{\Omega_r}[\lambda_\epsilon] \le \mathbb{G}_\mu^\Omega[\nu] + \mathbb{K}_\mu^\Omega[\lambda] \quad \text{in } \Omega_r,$$

where $\mathbb{G}_\mu^{\Omega_r}$ and $\mathbb{K}_\mu^{\Omega_r}$ denote respectively the Green and the Poisson potentials of the operator $\mathcal{L}_\mu$ in Ω_r. The mappings $r \mapsto u_{\epsilon,r}$, $r \mapsto \mathbb{G}_\mu^{\Omega_r}$ and $r \mapsto \mathbb{K}_\mu^{\Omega_r}$ are decreasing. We set $u_\epsilon = \lim_{r\to 0} u_{\epsilon,r}$, then

$$0 \le u_\epsilon \le \mathbb{G}_\mu^\Omega[\nu_\epsilon] + \mathbb{K}_\mu^\Omega[\lambda_\epsilon] \le \mathbb{G}_\mu^\Omega[\nu] + \mathbb{K}_\mu^\Omega[\lambda]. \tag{2.8}$$

If $\zeta \in \mathbb{X}_\mu(\Omega)$ vanishes in some neighbourhood of 0, there holds for $r > 0$ small enough,

$$\int_{\Omega_r} \left(u_{\epsilon,r}\mathcal{L}_\mu^*\zeta + g(u_{\epsilon,r})\zeta\right) d\gamma_\mu^\Omega = \int_{\Omega_r} \zeta d(\gamma_\mu^\Omega \nu_\epsilon) + \int_{\Gamma_{1,\delta}} \zeta d(\beta_\mu^\Omega \lambda_\epsilon).$$

Letting $r \to 0$, we obtain the identity

$$\int_{\Omega} \left(u_\epsilon\mathcal{L}_\mu^*\zeta + g(u_\epsilon)\zeta\right) d\gamma_\mu^\Omega = \int_{\Omega} \zeta d(\gamma_\mu^\Omega \nu_\epsilon) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda_\epsilon). \tag{2.9}$$

Because u_ϵ is $\mathcal{L}_\mu$-harmonic in $\Omega \cap B_\epsilon$ and vanishes on $\partial\Omega \cap B_\epsilon$, it satisfies $u_\epsilon(x) \le c_{12}\gamma_\mu^\Omega(x)$ if $x \in \Omega \cap B_{\frac{\epsilon}{2}}$ for some $c_{12} > 0$ depending also on ϵ, and $(\gamma_\mu^\Omega(x))^{-1}u_\epsilon(x) \to c_{13} \ge 0$ when $x \to 0$ by [15, Section 3]. Let $\zeta \in \mathbb{X}_\mu(\Omega)$ and

$$\ell_n(x) = \begin{cases} 0 & \text{if } \ |x| < \frac{1}{n} \\ \frac{1}{2} - \frac{1}{2}\cos\left(n\pi\left(|x| - \frac{1}{n}\right)\right) & \text{if } \ \frac{1}{n} \le |x| \le \frac{2}{n} \\ 1 & \text{if } \ |x| > \frac{2}{n}. \end{cases} \tag{2.10}$$

We set $\zeta_n = \ell_n\zeta$. Then

$$\int_{\Omega} \left(u_\epsilon\mathcal{L}_\mu^*\zeta_n + g(u_\epsilon)\zeta_n\right) d\gamma_\mu^\Omega = \int_{\Omega} \zeta_n d(\gamma_\mu^\Omega \nu_\epsilon) + \int_{\partial\Omega} \zeta_n d(\beta_\mu^\Omega \lambda_\epsilon). \tag{2.11}$$

Firstly we observe that

$$\int_{\Omega} \zeta_n d(\gamma_\mu^\Omega \nu_\epsilon) + \int_{\partial\Omega} \zeta_n d(\beta_\mu^\Omega \lambda_\epsilon) \to \int_{\Omega} \zeta d(\gamma_\mu^\Omega \nu_\epsilon) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda_\epsilon) \quad \text{as } \ n \to \infty.$$

Then, for n large enough,

$$\int_{\Omega} g(u_\epsilon)\zeta_n d\gamma_\mu^\Omega = \int_{\Omega_{\frac{r}{2}}} g(u_\epsilon)\zeta_n d\gamma_\mu^\Omega + \int_{\Omega\cap B_{\frac{r}{2}}} g(u_\epsilon)\zeta_n d\gamma_\mu^\Omega =: A_n + B_n.$$

Because G_μ^Ω and K_μ^Ω are respectively equivalent to G_0^Ω and K_0^Ω in $\Omega_{\frac{r}{2}}$, the condition (1.13), jointly with (2.8), implies that A_n is bounded independently of n and converges to $\displaystyle\int_{\Omega_{\frac{r}{2}}} g(u_\epsilon)\zeta d\gamma_\mu^\Omega$. If $\mu \ge 1-N$, α_+ is nonnegative thus $g(u_\epsilon)\zeta_n\gamma_\mu^\Omega$ is bounded in $B_{\frac{r}{2}}$. If $\mu_1 \le \mu < 1-N$, then $\alpha_+ < 0$ and we have

$$|B_n| \le \int_{\Omega\cap B_{\frac{r}{2}}} g(c_{12})\zeta_n d\gamma_\mu^\Omega \le \int_0^r g(c_{12}r^{\alpha_+})r^{\alpha_+ + N-1}dr \le \frac{1}{\alpha_+}\int_{r^{\frac{1}{\alpha_+}}}^\infty g(c_{12}s)s^{\frac{N}{\alpha_+}}ds < \infty$$

since $\frac{N}{\alpha_+} \le -1 - \frac{N+2}{N-2} < -1 - \frac{N+1}{N-1}$ and (1.13) holds. Therefore

$$\lim_{n\to\infty} \int_{\Omega} g(u_\epsilon)\zeta_n d\gamma_\mu^\Omega = \int_{\Omega} g(u_\epsilon)\zeta d\gamma_\mu^\Omega.$$

Finally, we perform the estimates

$$\int_\Omega u_\epsilon \mathcal{L}_\mu^* \zeta_n d\gamma_\mu^\Omega = C_n + D_n + E_n$$

with

$$C_n = \int_\Omega \ell_n u_\epsilon \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \ , \ D_n = \int_\Omega \zeta u_\epsilon \mathcal{L}_\mu^* \ell_n d\gamma_\mu^\Omega \ , \ E_n = -2\int_\Omega u_\epsilon \langle \nabla\zeta, \nabla\ell_n\rangle d\gamma_\mu^\Omega.$$

Since u_ϵ satisfies (2.8) it follows from [15, Theorem D] that it is bounded in $L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$ independently of ϵ. Hence

$$\lim_{n\to\infty} C_n = \int_\Omega u_\epsilon \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega.$$

Using the fact that $u_\epsilon(x) \sim c_{13}\gamma_\mu^\Omega(x)$ and $\zeta(x) = \zeta(0)(1+o(1))$ when $x \to 0$ we obtain

$$\int_\Omega \zeta u_\epsilon \mathcal{L}_\mu^* \ell_n d\gamma_\mu^\Omega = c_{13}\zeta(0)\int_{\Omega\cap\left(B_{\frac{2}{n}}\setminus B_{\frac{1}{n}}\right)} \left(-(\gamma_\mu^\Omega)^2\Delta\ell_n - 2\gamma_\mu^\Omega\langle\nabla\ell_n, \nabla\gamma_\mu^\Omega\rangle\right) dx + o(1) = o(1),$$

since $\ell_n'(\frac{1}{n}) = \ell_n'(\frac{2}{n}) = 0$ and γ_μ^Ω vanishes on $\partial\Omega$. Similarly

$$\lim_{n\to\infty} E_n = 0.$$

These facts imply that

$$\int_\Omega \left(u_\epsilon \mathcal{L}_\mu^*\zeta + g(u_\epsilon)\zeta\right) d\gamma_\mu^\Omega = \int_\Omega \zeta d(\gamma_\mu^\Omega \nu_\epsilon) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda_\epsilon) \ \text{ for any } \zeta \in \mathbb{X}_\mu(\Omega). \tag{2.12}$$

Notice that from the above derivation, (2.12) holds true for $\zeta = \eta$, where η is defined in (2.12).

Hence u_ϵ is the weak solution of

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu_\epsilon & \text{in } \ \Omega \\ \qquad\qquad u = \lambda_\epsilon & \text{on } \partial\Omega. \end{cases} \tag{2.13}$$

Because of uniqueness, $\epsilon \mapsto u_\epsilon$ is increasing and $u := \lim_{\epsilon\to 0} u_\epsilon$ satisfies

$$0 \le u \le \mathbb{G}_\mu^\Omega[\nu] + \mathbb{K}_\mu^\Omega[\lambda] \quad \text{in } \ \Omega.$$

If we take $\zeta = \eta$ defined by (2.1) we deduce from (2.12)

$$\int_\Omega \left(\frac{u_\epsilon}{\rho} + g(u_\epsilon)\eta\right) d\gamma_\mu^\Omega = \int_\Omega \eta d(\gamma_\mu^\Omega \nu_\epsilon) + \int_{\partial\Omega} \eta d(\beta_\mu^\Omega \lambda_\epsilon).$$

The right-hand side of the above identity converges to $\int_\Omega \eta d(\gamma_\mu^\Omega \nu) + \int_{\partial\Omega} \eta d(\beta_\mu^\Omega \lambda)$. Then by monotone convergence

$$\int_\Omega \left(\frac{u}{\rho} + g(u)\eta\right) d\gamma_\mu^\Omega = \int_\Omega \eta d(\gamma_\mu^\Omega \nu) + \int_{\partial\Omega} \eta d(\beta_\mu^\Omega \lambda).$$

This implies that $u_\epsilon \to u$ in $L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$ and $g(u_\epsilon) \to g(u)$ in $L^1(\Omega, d\gamma_\mu^\Omega)$ as $\epsilon \to 0^+$. Therefore, since any $\zeta \in \mathbb{X}_\mu(\Omega)$, satisfies $|\zeta| \le c\eta$ for some $c > 0$, we infer

$$\int_\Omega \left(u\mathcal{L}_\mu^*\zeta + g(u)\zeta\right) d\gamma_\mu^\Omega = \int_\Omega \zeta d(\gamma_\mu^\Omega \nu) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda),$$

which completes the proof when the two measures are nonnegative.

In the general case we use the Jordan decomposition $\nu = \nu^+ - \nu^-$, $\lambda = \lambda^+ - \lambda^-$ where ν^+, ν^-, λ^+ and λ^- are nonnegative. Let $\nu_\epsilon^\pm$ and $\lambda_\epsilon^\pm$ be $\nu^\pm\chi_{\Omega_\epsilon}$ and $\lambda^\pm\chi_{\partial\Omega\cap B_\epsilon^c}$ respectively. We denote by $u_{\epsilon,r}^+$ the solution of (2.7) corresponding to the couple $(\nu_\epsilon^+, \lambda_\epsilon^+)$ and by $u_{\epsilon,r}^-$ the solution of

$$\begin{cases} \mathcal{L}u - g(u) = \nu_\epsilon^- & \text{in } \Omega_r \\ u = \lambda_\epsilon^- & \text{in } \Gamma_{1,r} \\ u = 0 & \text{in } \Gamma_{2,r}. \end{cases}$$

Then $-u_{\epsilon,r}^- \le \min\{0, u_{\epsilon,r}\} \le \max\{0, u_{\epsilon,r}\} \le u_{\epsilon,r}^+$. The mapping $r \mapsto u_{\epsilon,r}^+$ (resp. $r \mapsto u_{\epsilon,r}^-$) is monotone increasing and we set $u_\epsilon^+ = \lim_{r\to 0} u_{\epsilon,r}^+$ (resp. $u_\epsilon^- = \lim_{r\to 0} u_{\epsilon,r}^-$). The mapping $r \mapsto u_{\epsilon,r}$ has no reason to be monotone, but by standard regularity theory there exists $\{r_j\}$ converging to 0 and $u_\epsilon \in L^q_{loc}$ $(1 < q < \frac{N}{N-1})$ such that $u_{\epsilon,r_j} \to u_\epsilon$ in $L^q_{loc}(\Omega)$ and a.e. in Ω. Hence u_ϵ satisfies (2.9). Since (2.11) holds we derive that u_ϵ satisfies (2.13). We end the proof as in the first case, using dominated convergence theorem. □

2.4 Proof of Theorem C

We first assume that ν, λ and k are nonnegative. For $0 < r < \epsilon/4$ we consider the problem

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu_\epsilon & \text{in } \Omega_r \\ u = \lambda_\epsilon & \text{on } \Gamma_{1,r} \\ u = k\phi_\mu^\Omega & \text{on } \Gamma_{2,r}. \end{cases} \tag{2.14}$$

The solution is denoted by $u_{\epsilon,k,r}$ and we recall that $u_{\epsilon,r}$ is the solution of (2.7). There holds

$$\max\{u_{\epsilon,r}, u_{k\delta_0}\} \le u_{\epsilon,k,r} \le u_\epsilon + k\mathbb{K}_\mu^\Omega[\delta_0] \quad \text{in } \Omega_r. \tag{2.15}$$

Furthermore $u_{\epsilon,k,r} \le u_{\epsilon,k,r'}$ if $0 < r' < r$. Since u_ϵ and $k\mathbb{K}_\mu^\Omega[\delta_0]$ belong to $L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$ it implies that $u_{\epsilon,k,r}$ converges in $L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$ and almost everywhere to $u_{\epsilon,k}$ when $r \to 0$. Since γ_μ^Ω is a supersolution for the equation $\mathcal{L}_\mu u + g(u) = 0$ in Ω_r, for any $0 < \epsilon_0 < \epsilon/4$ there exists $c_{14} > 0$ depending on ϵ_0 such that for $0 < r \le \epsilon_0/4$,

$$u_{\epsilon,r}(x) \le c_{14}\gamma_\mu^\Omega(x) \qquad \text{for all } \ x \in B_{\epsilon_0} \cap \Omega_r.$$

For any $\sigma > 0$ there exists $r_\sigma > 0$ such that for any $r < r_\sigma$, $u_{\epsilon,r} \le \sigma\mathbb{K}_\mu^\Omega[\delta_0]$ in $B_{r_\sigma} \cap \Omega_r$. Therefore $u_\epsilon + k\mathbb{K}_\mu^\Omega[\delta_0] \le (k+\sigma)\mathbb{K}_\mu^\Omega[\delta_0]$ in $B_{r_\sigma} \cap \Omega$. This implies

$$g(u_{\epsilon,k,r}) \le g((k+\sigma)\mathbb{K}_\mu^\Omega[\delta_0]) \qquad \text{in } \ \Omega_r \cap B_{r_\sigma}. \tag{2.16}$$

Then we obtain, with $R = \operatorname{diam}\Omega$ and some $c > 0$,

$$\int_\Omega g((k+\sigma)\mathbb{K}_\mu^\Omega[\delta_0])d\gamma_\mu^\Omega \le \int_0^R g(c|x|^{\alpha_-})|x|^{\alpha_+}dx$$
$$= \frac{1}{|\alpha_-|}\int_{R^{\frac{1}{\alpha_-}}}^\infty g(ct)t^{\frac{N+\alpha_+-\alpha_-}{\alpha_-}}dt \le \frac{1}{|\alpha_-|}\int_{R^{\frac{1}{\alpha_-}}}^\infty g(ct)t^{-1-p_\mu^*} < \infty.$$

This implies in particular that

$$\int_{\Omega_r\cap B_{r_\sigma}} g((k+\sigma)\mathbb{K}_\mu^\Omega[\delta_0])d\gamma_\mu^\Omega \le \frac{1}{|\alpha_-|}\int_{R^{\frac{1}{\alpha_-}}}^\infty g(ct)t^{-1-p_\mu^*}. \tag{2.17}$$

In the set Ω_{r_σ}, we have $k\mathbb{K}_\mu^\Omega[\delta_0] \le cr_\sigma^{\alpha_-}$ for some $c > 0$. By the local Δ_2-condition, we deduce

$$g(u_{\epsilon,r,k}) \le g(u_\epsilon + k\mathbb{K}_\mu^\Omega[\delta_0]) \le K(cr_\sigma^{\alpha_-})\left(g(u_\epsilon) + g(cr_\sigma^{\alpha_-})\right). \tag{2.18}$$

Because $g(u_\epsilon)$ is bounded in $L^1(\Omega_r, d\gamma_\mu^\Omega)$ independently of r by Theorem B, we infer from (2.16), (2.17) and (2.18) that $g(u_{\epsilon,k,r})$ is bounded in $L^1(\Omega_r, d\gamma_\mu^\Omega)$ independently of r. Let $\zeta \in \mathbb{X}_\mu(\Omega)$ vanishing near 0, then for r small enough,

$$\int_\Omega \left(u_{\epsilon,k,r}\mathcal{L}_\mu^*\zeta + g(u_{\epsilon,k,r})\zeta\right)d\gamma_\mu^\Omega = \int_\Omega \zeta d(\gamma_\mu^\Omega\nu_\epsilon) + \int_{\partial\Omega}\zeta d(\beta_\mu^\Omega\lambda_\epsilon).$$

Using the mononoticity of $r \mapsto u_{\epsilon,k,r}$ and the dominated convergence theorem we get

$$\int_\Omega \left(u_{\epsilon,k}\mathcal{L}_\mu^*\zeta + g(u_{\epsilon,k})\zeta\right)d\gamma_\mu^\Omega = \int_\Omega \zeta d(\gamma_\mu^\Omega\nu_\epsilon) + \int_{\partial\Omega}\zeta d(\beta_\mu^\Omega\lambda_\epsilon).$$

As we notice it, the singular measure $k\delta_0$ cannot appear in this formulation. If $\zeta \in \mathbb{X}_\mu(\Omega)$ we set $\zeta_n = \ell_n\zeta$ where ℓ_n is defined in (2.10). Then

$$\int_\Omega \left(u_{\epsilon,k}\mathcal{L}_\mu^*\zeta + g(u_{\epsilon,k})\zeta\right)\ell_n d\gamma_\mu^\Omega - \int_{\Omega\cap\left(B_{\frac{2}{n}}\setminus B_{\frac{1}{n}}\right)} A_n u_{\epsilon,k}d\gamma_\mu^\Omega = \int_\Omega \zeta\ell_n d(\gamma_\mu^\Omega\nu_\epsilon) + \int_{\partial\Omega}\zeta\ell_n d(\beta_\mu^\Omega\lambda_\epsilon),$$

where

$$A_n = \zeta\Delta\ell_n + 2\langle\nabla\ell_n, \nabla\zeta\rangle + 2\alpha_+\zeta\langle\nabla\ell_n, \frac{x}{|x|^2}\rangle.$$

Clearly we have that

$$\lim_{n\to\infty}\int_\Omega \left(u_{\epsilon,k}\mathcal{L}_\mu^*\zeta + g(u_{\epsilon,k})\zeta\right)\ell_n d\gamma_\mu^\Omega = \int_\Omega \left(u_{\epsilon,k}\mathcal{L}_\mu^*\zeta + g(u_{\epsilon,k})\zeta\right)d\gamma_\mu^\Omega$$

and

$$\lim_{n\to\infty}\left(\int_\Omega \zeta\ell_n d(\gamma_\mu^\Omega\nu_\epsilon) + \int_{\partial\Omega}\zeta\ell_n d(\beta_\mu^\Omega\lambda_\epsilon)\right) = \int_\Omega \zeta d(\gamma_\mu^\Omega\nu_\epsilon) + \int_{\partial\Omega}\zeta d(\beta_\mu^\Omega\lambda_\epsilon).$$

Next

$$A_n = \left[\frac{n^2\pi^2}{2}\cos\left(n\pi\left(|x| - \frac{1}{n}\right)\right) + \frac{n\pi(N-1+2\alpha_+)}{2|x|}\sin\left(n\pi\left(|x| - \frac{1}{n}\right)\right)\right](\zeta(0) + o(1)) + O(n).$$

Using (2.15) with $\delta = 0$ and the fact that $u_\epsilon = o(\mathbb{K}^\Omega_\mu[\delta_0])$ near 0, we obtain after a technical but straightforward computation

$$\lim_{n\to\infty} \int_{\Omega\cap\left(B_{\frac{2}{n}}\setminus B_{\frac{1}{n}}\right)} A_n u_{\epsilon,k} d\gamma^\Omega_\mu = kc_\mu\zeta(0). \tag{2.19}$$

By the normalization chosen it follows that $u_{\epsilon,k}$ satisfies

$$\int_\Omega \left(u_{\epsilon,k}\mathcal{L}^*_\mu\zeta + g(u_{\epsilon,k})\zeta\right) d\gamma^\Omega_\mu = \int_\Omega \zeta d(\gamma^\Omega_\mu\nu_\epsilon) + \int_{\partial\Omega} \zeta d(\beta^\Omega_\mu\lambda_\epsilon) + kc_\mu\zeta(0). \tag{2.20}$$

Hence $u_{\epsilon,k}$ is the weak solution of

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu_\epsilon & \text{in } \Omega \\ u = \lambda_\epsilon + k\delta_0 & \text{on } \partial\Omega. \end{cases}$$

The end of the proof in the nonnegative case is standard: we observe that the mapping $\epsilon \mapsto u_{\epsilon,k}$ is nondecreasing. We denote by u_k its limit when $\epsilon \to 0$. If $\zeta \in \mathbb{X}_\mu(\Omega)$, the right-hand side of (2.20) converges to

$$\int_\Omega \zeta d(\gamma^\Omega_\mu\nu) + \int_{\partial\Omega} \zeta d(\beta^\Omega_\mu\lambda) + kc_\mu\zeta(0) \quad \text{as } \epsilon \to 0.$$

If we take $\zeta = \eta$, by property (2.2), (2.19) becomes

$$\limsup_{n\to\infty} \int_{\Omega\cap\left(B_{\frac{2}{n}}\setminus B_{\frac{1}{n}}\right)} A_n u_{\epsilon,k} d\gamma^\Omega_\mu \le kc_\mu \sup_\Omega \eta, \tag{2.21}$$

and when $\epsilon \to 0$,

$$\int_\Omega \left(\frac{u_k}{\rho} + g(u_k)\eta\right) d\gamma^\Omega_\mu \le \int_\Omega \eta d(\gamma^\Omega_\mu\nu_\epsilon) + \int_{\partial\Omega} \eta d(\beta^\Omega_\mu\lambda_\epsilon) + kc_\mu \sup_\Omega \eta. \tag{2.22}$$

Thus, by the monotone convergence theorem we have that $u_{\epsilon,k} \to u_k$ in $L^1(\Omega, \rho^{-1})d\gamma^\Omega_\mu)$ and $g(u_{\epsilon,k}) \to g(u_k)$ in $L^1(\Omega, d\gamma^\Omega_\mu)$ as $\epsilon \to 0$. Therefore, by the dominated convergence theorem we conclude that for any $\zeta \in \mathbb{X}_\mu(\Omega)$ there holds

$$\int_\Omega \left(u_k\mathcal{L}^*_\mu\zeta + g(u_k)\zeta\right) d\gamma^\Omega_\mu = \int_\Omega \zeta d(\gamma^\Omega_\mu\nu) + \int_{\partial\Omega} \zeta d(\beta^\Omega_\mu\lambda) + kc_\mu\zeta(0).$$

Hence u_k is the weak solution of (1.9). When ν and λ are signed measures and k is a real number, we use the Jordan decomposition of $\nu = \nu^+ - \nu^-$ and $\lambda = \lambda^+ - \lambda^-$ and assume for example that k is nonnegative and we construct the solutions $u^+_{\epsilon,k,r}$ of

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu^+_\epsilon & \text{in } \Omega_r \\ u = \lambda^+_\epsilon & \text{on } \Gamma_{1,r} \\ u = u_{k\delta_0} & \text{on } \Gamma_{2,r} \end{cases}$$

and $u^-_{\epsilon,\delta}$ of

$$\begin{cases} \mathcal{L}_\mu u - g(u) = \nu^-_\epsilon & \text{in } \Omega_r \\ u = \lambda^-_\epsilon & \text{on } \Gamma_{1,r} \\ u = 0 & \text{on } \Gamma_{2,r}. \end{cases}$$

Then the function $u_{\epsilon,k,r}$ of (2.14) satisfies $-u^-_{\epsilon,r} \le \min\{0, u_{\epsilon,k,r}\} \le \max\{0, u_{\epsilon,k,r}\} \le u^+_{\epsilon,k,r}$. Since $u^+_{\epsilon,k,r}$ is monotone with respect to r with limit $u^+_{\epsilon,k}$, we obtain, as in the proof of *Theorem B*, the existence of a limit $u_{\epsilon,k}$ of a sequence u_{ϵ,k,r_j}, a.e. and in $L^q_{loc}(\Omega)$, and $u_{\epsilon,k}$ satisfies (2.17) for any $\zeta \in \mathbb{X}_\mu(\Omega)$ which vanishes near 0.

Since $u^-_{\epsilon,r}$ is $\mathcal{L}_\mu$ Harmonic in $\Omega \cap B_\epsilon$, $u^-_{\epsilon,r} = 0$ on $(\partial\Omega \cap B_\epsilon) \setminus \{0\}$ and charges no Dirac mass at origin in the weak sense, then

$$-u^-_{\epsilon,r} \ge -c_{15}\gamma^\Omega_\mu \quad \text{on} \quad \Omega \cap \partial B_{\frac{\epsilon}{2}}$$

for some $c_{15} > 0$ dependent of ϵ. Thus, there exists $c_{16} > 0$ such that

$$u_{\epsilon,k,r} \ge u_{k\delta_0} - c_{16}\gamma^\Omega_\mu := w \qquad \text{for all } \ x \in \Omega \cap B_{\frac{\epsilon}{2}}.$$

Combining these estimates with (2.15) (applied to $u^+_{\epsilon,r,k}$) we obtain

$$u_{k\delta_0} - c_{16}\gamma^\Omega_\mu \le u_{\epsilon,k,r} \le u^+_{\epsilon,r} + k\mathbb{K}^\Omega_\mu[\delta_0] \le u^+_\epsilon + k\mathbb{K}^\Omega_\mu[\delta_0] \quad \text{in } \Omega \cap B_{\frac{\epsilon}{2}}, \tag{2.23}$$

where $u^+_{\epsilon,r}$ and u^+_ϵ are the solutions of (2.13) with $r > 0$ and $r = 0$ respectively with ν_ϵ and λ_ϵ replaced by ν^+_ϵ and λ^+_ϵ. Thanks to estimate (2.23) we infer as in the case where ν_ϵ and λ_ϵ are nonnegative that $u_{\epsilon,k}$ satisfies (2.20). We also have

$$-u^-_\epsilon \le \min\{0, u_{\epsilon,k}\} \le \max\{0, u_{\epsilon,k}\} \le u^+_{\epsilon,k}$$

and

$$g(-u^-_\epsilon) \le \min\{0, g(u_{\epsilon,k})\} \le \max\{0, g(u_{\epsilon,k})\} \le g(u^+_{\epsilon,k}).$$

Then there exist a function $u_k \in L^q_{loc}(\Omega)$ $(1 < q < \frac{N}{N-1})$ and a sequence $\{\epsilon_j\}$ converging to 0 such that $u_{\epsilon_j,k} \to u_k$ $L^q_{loc}(\Omega)$ and a.e. in Ω. Since $g(u^+_{\epsilon,k})$ and $g(-u^-_\epsilon)$ converge in $L^1(\Omega, d\gamma^\Omega_\mu)$ and $u^+_{\epsilon,k}$ and u^-_ϵ in $L^1(\Omega, \rho^{-1}d\gamma^\Omega_\mu)$, it follows that $g(u_{\epsilon,k})$ and $u_{\epsilon,k}$ endow the same properties. This is sufficient to see that (2.20) implies (2.22), which ends the proof. □

3 The supercritical case

3.1 Reduced measures

We present here the notion of reduced measure which has been introduced by Brezis, Marcus and Ponce [7]. This notion turned out to be a very useful tool for analyzing supercritical problems. Since many results are simple adaptations of similar ones used in [14], we will state most of them without detailled proofs. We assume that g is a continuous nondecreasing function vanishing at 0 and for $\ell > 0$, we set

$$g_\ell(r) = \begin{cases} \min\{g(r),\, g(\ell)\} & \text{if } \ r \ge 0 \\ \max\{g(-\ell),\, g(r)\} & \text{if } \ r < 0. \end{cases}$$

If $k \in \mathbb{R}_+$, and $(\nu, \lambda) \in \mathfrak{M}_+(\Omega; \gamma_\mu^\Omega) \times \mathfrak{M}_+(\partial\Omega; \beta_\mu^\Omega)$ we denote by u_ℓ the solution of

$$\begin{cases} \mathcal{L}_\mu u + g_\ell(u) = \nu & \text{in } \Omega \\ u = \lambda + k\delta_0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Existence of u_ℓ comes from *Theorem C.*

Proposition 3.1 *Let $k \in \mathbb{R}_+$, and $(\nu, \lambda) \in \mathfrak{M}_+(\Omega; \gamma_\mu^\Omega) \times \mathfrak{M}_+(\partial\Omega; \beta_\mu^\Omega)$, then $\ell \mapsto u_\ell$ is monotone decreasing and converges to some function u^* when $\ell \to \infty$ and there exists a real number $k^* \in [0, k]$ and two measures $(\nu^*, \lambda^*) \in \mathfrak{M}_+(\Omega; \gamma_\mu^\Omega) \times \mathfrak{M}_+(\partial\Omega; \beta_\mu^\Omega)$ satisfying $0 \le \nu^* \le \nu$ and $0 \le \lambda^* \le \lambda$ such that u^* is a weak solution of*

$$\begin{cases} \mathcal{L}_\mu u + g_\ell(u) = \nu^* & \text{in } \Omega \\ u = \lambda^* + k^*\delta_0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Furthermore the correspondence $(\nu, \lambda, k) \mapsto (\nu^*, \lambda^*, k^*)$ is nondecreasing.

Proof. The monotonicity is clear. By Fatou's lemma $u^* := \lim_{\ell\to\infty} u_\ell$ satisfies

$$\int_\Omega \left(u^* \mathcal{L}_\mu^* \zeta + g(u^*)\zeta\right) d\gamma_\mu^\Omega \le \int_\Omega \zeta d(\gamma_\mu^\Omega \nu) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda) + k c_\mu \zeta(0) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega),\ \zeta \ge 0.$$

The function u^* is the largest subsolution of problem (1.9). Since the mapping

$$\zeta \mapsto \int_\Omega \left(u^* \mathcal{L}_\mu^* \zeta + g(u^*)\zeta\right) d\gamma_\mu^\Omega \quad \text{for all } \zeta \in C_c^\infty(\Omega)$$

is a positive distribution, it is a positive measure denoted by ν^*. It is smaller than ν, hence it belongs to $\mathfrak{M}_+(\Omega; \gamma_\mu^\Omega)$. Similarly the function u^* admits a boundary trace λ^* on $\partial\Omega \setminus \{0\}$ which is a positive Radon measure smaller than λ. Hence $\lambda^* \in \mathfrak{M}_+(\partial\Omega^*; \beta_\mu^\Omega)$. By using (1.1), it is extended as a measure on $\partial\Omega$, still denoted by λ^*. If $\zeta \in \mathbb{X}_\mu(\Omega)$ vanishes near 0, there holds

$$\int_\Omega \left(u^* \mathcal{L}_\mu^* \zeta + g(u^*)\zeta\right) d\gamma_\mu^\Omega = \int_\Omega \zeta d(\gamma_\mu^\Omega \nu^*) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda^*).$$

Let v be the solution of

$$\begin{cases} \mathcal{L}_\mu v + g(v) = \nu^* & \text{in } \Omega \\ v = \lambda^* & \text{on } \partial\Omega. \end{cases}$$

Existence is standard since u^* exists. Furthermore v is a subsolution of problem (1.9) hence it is smaller than u^*. Therefore $w = u^* - v$ is nonnegative and it satisfies

$$\begin{cases} \mathcal{L}_\mu w + g(u^*) - g(v) = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases}$$

Let $\psi \in H_\mu$ be the solution of

$$\begin{cases} \mathcal{L}_\mu \psi = g(u^*) - g(v) & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

then $w+\psi$ is a nonnegative $\mathcal{L}_\mu$-harmonic function vanishing on $\partial\Omega\setminus\{0\}$. By [15, Theorem A] there exists $k^*\geq 0$ such that

$$\lim_{x\to 0}\frac{(w+\psi)(x)}{\gamma_\mu^\Omega(x)}=k^*,$$

and

$$\int_\Omega (w+\psi)\mathcal{L}_\mu^*\zeta d\gamma_\mu^\Omega = k^*c_\mu\zeta(0)\qquad \text{for all } \zeta\in\mathbb{X}_\mu(\Omega).$$

It follows from (3.3) that this implies

$$\lim_{x\to 0}\frac{w(x)}{\phi_\mu^\Omega(x)}=k^*,$$

and

$$\int_\Omega \left(w\mathcal{L}_\mu^*\zeta+\zeta(g(u^*)-g(v))\right)d\gamma_\mu^\Omega = k^*c_\mu\zeta(0)\qquad \text{for all } \zeta\in\mathbb{X}_\mu(\Omega).$$

Since $u^*=w+v$ and

$$\int_\Omega \left(v\mathcal{L}_\mu^*\zeta+\zeta g(v)\right)d\gamma_\mu^\Omega=\int_\Omega\zeta d(\gamma_\mu^\Omega\nu^*)+\int_{\partial\Omega}\zeta d(\beta_\mu^\Omega\lambda^*)\quad \text{for all } \zeta\in\mathbb{X}_\mu(\Omega)$$

we infer

$$\int_\Omega \left(u^*\mathcal{L}_\mu^*\zeta+g(u^*)\zeta\right)d\gamma_\mu^\Omega=\int_\Omega\zeta d(\gamma_\mu^\Omega\nu^*)+\int_{\partial\Omega}\zeta d(\beta_\mu^\Omega\lambda^*)+k^*c_\mu\zeta(0)\quad \text{for all } \zeta\in\mathbb{X}_\mu(\Omega).$$

The last assertion is obvious. □

Definition 3.1 *The triplet of measures $(\nu^*,\lambda^*,k^*\delta_0)$ is called the reduced triplet associated to $(\nu,\lambda,k\delta_0)$. If $(\nu^*,\lambda^*,k^*\delta_0)=(\nu,\lambda,k\delta_0)$ the triplet is called g-good.*

Lemma 3.2 *Let (ν,λ,k) and (ν',λ',k') in $\mathfrak{M}_+(\Omega;\gamma_\mu^\Omega)\times\mathfrak{M}_+(\partial\Omega;\beta_\mu^\Omega)\times\mathbb{R}_+$. If $\nu'\leq\lambda$, $\lambda'\leq\lambda$ and $k'\leq k$ and $(\nu,\lambda,k)=(\nu^*,\lambda^*,k^*)$, then $(\nu',\lambda',k')=(\nu'^*,\lambda'^*,k'^*)$.*

Proof. For $\ell>0$, let $u_\ell=u_{\ell,\nu,\lambda,k}$ be the solution of (3.1). We define similarly $u'_\ell=u'_{\ell,\nu',\lambda',k'}$. Then $u'_\ell\leq u_\ell$ for any $\ell>0$. Then $u_\ell\downarrow u^*$ and $u'_\ell\downarrow u'^*$ as $\ell\to\infty$ where $u^*\,u'^*$ are the solution of (1.9) with sources (ν^*,λ^*,k^*), (ν'^*,λ'^*,k'^*) respectively, and these convergences hold in $L^1(\Omega,\rho^{-1}d\gamma_\mu^\Omega)$ by the previous proposition. Since $(\nu,\lambda,k)=(\nu^*,\lambda^*,k^*)$, then

$$\mathcal{L}_\mu(u_\ell-u^*)+g_\ell(u_\ell)-g_\ell(u^*)=g(u^*)-g_\ell(u^*)$$

and we deduce from Proposition 2.2 that

$$\int_\Omega(u_\ell-u^*)\rho^{-1}d\gamma_\mu^\Omega+\int_\Omega|g_\ell(u_\ell)-g_\ell(u^*)|\eta d\gamma_\mu^\Omega\leq\int_\Omega(g(u)-g_\ell(u))\eta d\gamma_\mu^\Omega.$$

Because $|g_\ell(u_\ell)-g_\ell(u^*)|\leq|g_\ell(u_\ell)-g(u^*)|+g(u^*)-g_\ell(u^*)$ we get

$$\int_\Omega|g_\ell(u_\ell)-g_\ell(u^*)|\eta d\gamma_\mu^\Omega\leq 2\int_\Omega(g(u^*)-g_\ell(u^*))\eta d\gamma_\mu^\Omega\to 0\quad\text{as }\ \ell\to\infty.$$

Since $g_\ell(u'_\ell) \le g_\ell(u_\ell)$, it follows by Vitali's theorem that $g_\ell(u'_\ell)$ converges to $g(u'^*)$ in $L^1(\Omega, d\gamma_\mu^\Omega)$. Letting $\ell \to \infty$ in the weak formulation of the equation satisfied by u'_ℓ we conclude that u'^* verifies

$$\int_\Omega \left(u'^* \mathcal{L}_\mu^* \zeta + g(u'^*)\zeta\right) d\gamma_\mu^\Omega = \int_\Omega \zeta d(\gamma_\mu^\Omega \nu') + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda') + k' c_\mu \zeta(0) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega).$$

This implies the claim. □

As a consequence we have

Proposition 3.3 *The triplet $(\nu^*, \lambda^*, k^*\delta_0)$ is the largest g-good triplet smaller than $(\nu, \lambda, k\delta_0)$.*

Lemma 3.4 *Let (ν, λ, k) in $\mathfrak{M}_+(\Omega; \gamma_\mu^\Omega) \times \mathfrak{M}_+(\partial\Omega; \beta_\mu^\Omega) \times \mathbb{R}_+$. The two next statements are equivalent:*
(i) The triplet (ν, λ, k) is g-good.
(ii) For any $\epsilon > 0$, $0 \le k' \le k$, $(\nu_\epsilon, \lambda_\epsilon, k')$ is g-good.

Proof. We recall that $\nu_\epsilon = \chi_{\Omega_\epsilon}\nu$ and $\lambda_\epsilon = \chi_{\partial\Omega \cap B_\epsilon^c}\lambda$.
(i) implies (ii) by Lemma 3.2.
Conversely, if $(\nu_\epsilon, \lambda_\epsilon, k')$ is g-good for any $\epsilon > 0$ and $k' \in [0, k]$, let $u_{\epsilon,k'}$ be the solution of

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \nu_\epsilon & \text{in } \Omega \\ u = \lambda_\epsilon + k'\delta_0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Then map $(\epsilon, k') \mapsto u_{\epsilon,k'}$ is nonincreasing in ϵ and nondecreasing in k'. There holds

$$\int_\Omega \left(u_{\epsilon,k'} \mathcal{L}_\mu^* \zeta + g(u_{\epsilon,k'})\zeta\right) d\gamma_\mu^\Omega = \int_\Omega \zeta d(\gamma_\mu^\Omega \nu_\epsilon) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda_\epsilon) + k' c_\mu \zeta(0) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega).$$

From (2.22) we have that

$$\int_\Omega \left(u_{\epsilon,k'}\rho^{-1} + g(u_{\epsilon,k'})\eta\right) d\gamma_\mu^\Omega \le \int_\Omega \eta d(\gamma_\mu^\Omega \nu_\epsilon) + \int_{\partial\Omega} \eta d(\beta_\mu^\Omega \lambda_\epsilon) + k' c_\mu \sup_\Omega \eta.$$

Put $u = \lim_{(\epsilon,k')\to(0,k)} u_{\epsilon,k'}$. By the monotone convergence theorem,

$$\int_\Omega \left(u\rho^{-1} + g(u)\eta\right) d\gamma_\mu^\Omega = \int_\Omega \eta d(\gamma_\mu^\Omega \nu) + \int_{\partial\Omega} \eta d(\beta_\mu^\Omega \lambda) + k c_\mu \eta(0).$$

Therefore $u_{\epsilon,k'} \to u$ in $L^1(\Omega, \rho^{-1} d\gamma_\mu^\Omega)$ and $g(u_{\epsilon,k'}) \to g(u)$ in $L^1(\Omega, d\gamma_\mu^\Omega)$ as $\epsilon \to 0^+$. Going to the limit in (3.4) yields the claim. □

Remark. The previous result is a particular case of the following result: If $\{(\nu_n, \lambda_n, k_n)\} \subset \mathfrak{M}_+(\Omega; \gamma_\mu^\Omega) \times \mathfrak{M}_+(\partial\Omega; \beta_\mu^\Omega) \times \mathbb{R}_+$ is an increasing sequence of g-good triplet converging to $(\nu, \lambda, k) \in \mathfrak{M}_+(\Omega; \gamma_\mu^\Omega) \times \mathfrak{M}_+(\partial\Omega; \beta_\mu^\Omega) \times \mathbb{R}_+$, then (ν, λ, k) is g-good.

3.2 Capacitary framework, good measures and removable sets

In the sequel, we set $g(r) = g_p(r) := |r|^{p-1}r$ with $p > 1$. The following a priori estimate of Keller-Osserman type is standard and easy to prove (see e.g. [18], [23]).

Lemma 3.5 *Let $p > 1$, $\mu \in \mathbb{R}$, $G \subset \mathbb{R}^N$ be a domain such that $0 \notin G$. There exist constants $A > 0$, $B \geq 0$ depending on N, p, μ such that any compact subset F of ∂G, possibly empty, and any solution v of*

$$\begin{cases} \mathcal{L}_\mu v + g_p(v) = 0 & \text{in } G \\ v = 0 & \text{on } \partial G \setminus (\{0\} \cup F), \end{cases}$$

there holds

$$|v(x)| \leq A \max\left\{|x|^{-\frac{2}{p-1}}, \left(\operatorname{dist}(x,F)^{-\frac{2}{p-1}}\right)\right\} + B \quad \text{for all } x \in G.$$

Proof of Theorem D. Since g_p satisfies the uniform Δ_2-condition, i.e. $K(|r|)$ is constant in inequality (1.14), if $(\nu, 0, 0)$, $(0, \lambda, 0)$ and $(0, 0, k\delta_0)$ are g_p-good, then $(\nu, \lambda, k\delta_0)$ is also g_p-good, and conversely. Assume now that $(\nu, \lambda, 0)$ is g_p-good, or, equivalently, for any $\epsilon > 0$, $(\nu_\epsilon, \lambda_\epsilon, 0)$, is g_p-good. Let u_ϵ be the solution of (3.4) with $k' = 0$. Let $\tilde{\Omega}_\epsilon$ be a smooth domain such that $\Omega_\epsilon \subset \tilde{\Omega}_\epsilon \subset \Omega_{\frac{\epsilon}{2}}$. Then $\tilde{u}_\epsilon := u_\epsilon\lfloor_{\tilde{\Omega}_\epsilon}$ satisfies

$$\begin{cases} \mathcal{L}_\mu \tilde{u}_\epsilon + g_p(\tilde{u}_\epsilon) = \nu_\epsilon & \text{in } \tilde{\Omega}_\epsilon \\ \tilde{u}_\epsilon = \lambda_\epsilon & \text{on } \partial\Omega \cap \partial\tilde{\Omega}_\epsilon \\ \tilde{u}_\epsilon = u_\epsilon & \text{on } \partial\tilde{\Omega}_\epsilon \cap \Omega. \end{cases}$$

Furthermore $\frac{\mu}{|x|^2}$ is bounded in $\tilde{\Omega}_\epsilon$. Hence the Green operator $\mathbb{G}^{-\Delta+\mu|.|^{-2}}$ relative to $\tilde{\Omega}_\epsilon$ is equivalent of the one relative to $-\Delta$ and $\nu_\epsilon \in \mathfrak{M}_+(\Omega; \rho)$. Let $\tilde{\Omega}_{\epsilon,t} = \{x \in \tilde{\Omega}_\epsilon : \rho(x) > t\}$ and $\nu_{\epsilon,t} = \chi_{\tilde{\Omega}_{\epsilon,t}} \nu_\epsilon$. The bounded measure $\nu_{\epsilon,t}$ is g_p-good in $\tilde{\Omega}_\epsilon$. From [2], this holds if and only if for any Borel set $K \subset \tilde{\Omega}_\epsilon$,

$$c_{2,p'}^{\mathbb{R}^N}(K) = 0 \Longrightarrow \nu_{\epsilon,t}(K) = 0.$$

Assume now $E \subset \Omega$ is a compact set such that $c_{2,p'}^{\mathbb{R}^N}(E) = 0$. Then $c_{2,p'}^{\mathbb{R}^N}(E \cap \tilde{\Omega}_{\epsilon,t}) = 0$ and thus $\nu_{\epsilon,t}(E \cap \tilde{\Omega}_{\epsilon,t}) = 0$. By the monotone convergence theorem, it implies

$$\lim_{\epsilon\to 0}\lim_{t\to 0} \nu_{\epsilon,t}(E \cap \tilde{\Omega}_{\epsilon,t}) = \lim_{\epsilon\to 0} \nu_\epsilon(E \cap \tilde{\Omega}_\epsilon) = \nu(E) = 0.$$

Similarly, using Marcus-Véron results on the boundary trace (see e.g. [23]) λ is g_p-good if and only if λ_ϵ vanishes on compact sets $E \subset \partial\tilde{\Omega}_\epsilon$ such that $c_{\frac{2}{p},p'}^{\mathbb{R}^{N-1}}(E) = 0$. Clearly λ shares this property.

Conversely, if ν (resp. λ) vanishes on compact sets $E \subset \Omega$ (resp. $E \subset \partial\Omega$) such that $c_{2,p'}^{\mathbb{R}^N}(E) = 0$ (resp. $c_{\frac{2}{p},p'}^{\mathbb{R}^N}(E) = 0$), then ν_+ (resp. λ_+) has the same property. Hence we can assume that ν (resp. λ) is nonnegative. Clearly ν_ϵ (resp. λ_ϵ) shares also this property. If $0 < t < \epsilon$ we denote by Ω_t^* a smooth domain such that $\Omega_\epsilon \subset \Omega_t^* \subset \Omega$ and $\Omega_t^* \cap B_{\frac{t}{2}} = 0$ there exists an increasing sequence $\{\nu_{\epsilon,n}\}$ (resp. $\{\lambda_{\epsilon,n}\}$) of positive bounded measures belong

to $W^{-2,p}(\Omega)$ (resp. $W^{-\frac{2}{p},p}(\partial\Omega)$) converging to ν_ϵ (resp. λ_ϵ). The measures $\nu_{\epsilon,n}$ (resp. $\lambda_{\epsilon,n}$) are g_p-good relatively to the open set Ω_t^*. Therefore there exists a sequence of solutions $\{\tilde{u}_{\epsilon,t\,n}\}$ satisfying weakly

$$\begin{cases} \mathcal{L}_\mu \tilde{u}_{\epsilon,t\,n} + g_p(\tilde{u}_{\epsilon,t\,n}) = \nu_{\epsilon,n} & \text{in } \Omega_t^* \\ \tilde{u}_{\epsilon,t\,n} = \lambda_{\epsilon,n} & \text{on } \partial\Omega \cap \partial\Omega_t^* \\ \tilde{u}_{\epsilon,t\,n} = 0 & \text{on } \partial\Omega_t^* \cap \Omega. \end{cases}$$

Letting $n \to \infty$, we infer that $\tilde{u}_{\epsilon,\delta\,n}$ increases and converges to the solution $\tilde{u}_{\epsilon,\delta}$ of

$$\begin{cases} \mathcal{L}_\mu \tilde{u}_{\epsilon,t} + g_p(\tilde{u}_{\epsilon,t}) = \nu_\epsilon & \text{in } \Omega_t^* \\ \tilde{u}_{\epsilon,t} = \lambda_\epsilon & \text{on } \partial\Omega \cap \partial\Omega_t^* \\ \tilde{u}_{\epsilon,t} = 0 & \text{on } \partial\Omega_t^* \cap \Omega. \end{cases}$$

For $0 < t < t'$, $\tilde{u}_{\epsilon,t} \geq \tilde{u}_{\epsilon,t'}$, hence $\tilde{u}_\epsilon := \lim_{t\to 0^+} \tilde{u}_{\epsilon,t}$ satisfies

$$\int_\Omega \left(\tilde{u}_\epsilon \mathcal{L}_\mu^* \zeta + g_p(\tilde{u}_\epsilon)\zeta\right) d\gamma_\mu^\Omega = \int_\Omega \zeta d(\gamma_\mu^\Omega \nu_\epsilon) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda_\epsilon), \tag{3.5}$$

for all $\zeta \in \mathbb{X}_\mu^\Omega$ which vanishes in a neigborhood of 0. We end the proof as in Theorem B. We first obtain that $\tilde{u}_\epsilon$ satisfies (3.5) for all $\zeta \in \mathbb{X}_\mu^\Omega$, and then we let $\epsilon \to 0$ and conclude that $u := \lim_{\epsilon\to 0} \tilde{u}_\epsilon$ satisfies

$$\begin{cases} \mathcal{L}_\mu u + g_p(u) = \nu & \text{in } \Omega \\ u = \lambda & \text{on } \partial\Omega, \end{cases}$$

hence (ν, λ) is g_p good. □

Proof of Theorem E. A particular case of *Theorem E* that we will prove in *Theorem J* is that 0 is a non-removable singularity if and only if $1 < p < p_\mu^*$ for any $\mu \geq \mu_1$ and $N > 2$, or $p > p_\mu^{**}$ with $N \geq 3$ and $\mu < 1 - N$.

(i) Assume $K \subset \Omega$ is compact. It follows from [2, Theorem 3.1] that $c_{2,p'}^{\mathbb{R}^N}(K) = 0$ is a necessary and sufficient condition for K to be removable for the operator $\mathcal{L}_\mu$ (and $p \geq \frac{N}{N-2}$ otherwise K is empty).

(ii) Let $K \subset \partial\Omega \setminus \{0\}$ be compact and, for $\epsilon > 0$, $K_\epsilon = \{x \in \Omega : \text{dist}\,(x, K) < \epsilon\}$. Assume u is a function belonging to $L^1(\Omega \setminus K_\epsilon, \rho^{-1} d\gamma_\mu^\Omega) \cap L^p(\Omega \setminus K_\epsilon, d\gamma_\mu^\Omega)$ for any $\epsilon > 0$ satisfying

$$\int_\Omega \left(u\mathcal{L}_\mu^* \zeta + g_p(u)\zeta\right) d\gamma_\mu(x) = 0, \tag{3.6}$$

for any $\zeta \in \mathbb{X}_\mu(\Omega)$ vanishing in a neighborhood of K. Taking a test function $\zeta \in C^2(\overline{\Omega})$ vanishing on $\partial\Omega$ and in a neighborhood of K we infer by standard regularity theory that $u \in C^2(\overline{\Omega} \setminus (K \cup \{0\})$ is a strong solution of $\mathcal{L}_\mu u + g_p(u) = 0$ in Ω which vanishes on $\partial\Omega \setminus (K \cup \{0\})$. Let $G \subset \Omega$ be a smooth domain such that K is interior to $\partial G \cap \partial\Omega$ relatively to the induced topology on $\partial\Omega$ and such that $0 \notin \overline{G}$. Then $\mu|x|^{-2}$ is bounded in $\overline{G}$. Then there exists $a > 0$ and $b \in \mathbb{R}$ such that

$$g_p(u) + \mu|x|^{-2}u \geq au_+^p - b.$$

Set $m = \max\{u_+(x) : x \in \partial G \cap \Omega\}$. Then implies that $v = \left(u - m - \left(\frac{b_+}{a}\right)^{\frac{1}{p}}\right)$ satisfies $-\Delta v + av^p \leq 0$ in G and vanishes on $\partial G \setminus K$. Since $c^{\mathbb{R}^{N-1}}_{\frac{2}{p},p'}(K) = 0$ (and $p \geq \frac{N+1}{N-1}$ otherwise K is empty), $v = 0$ by [21, Theorem 3.3], which implies $u \leq m + \left(\frac{b_+}{a}\right)^{\frac{1}{p}}$ in G. Similarly u is bounded from below in G and it follows that (3.6) holds for all $\zeta \in \mathbb{X}_\mu(\Omega)$. Hence $u = 0$ by uniqueness.

Conversely, if $c^{\mathbb{R}^{N-1}}_{\frac{2}{p},p'}(K) > 0$, then there exists a capacitary measure λ_K belonging to $W^{-\frac{2}{p},p}(\partial\Omega)$ with support in K. Since λ_K vanishes on Borel set with $c^{\mathbb{R}^{N-1}}_{\frac{2}{p},p'}$-capacity 0, it is g_p-good and there exists a solution u to

$$\begin{cases} \mathcal{L}_\mu u + g_p(u) = 0 & \text{in } \Omega \\ u = \lambda_K & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Hence u satisfies (3.6) for all $\zeta \in \mathbb{X}_\mu(\Omega)$ vanishing in a neighborhood of K. Hence K is not removable.

(iii) If $K \subset \overline{\Omega} \setminus \{0\}$ is such that $c^{\mathbb{R}^{N-1}}_{\frac{2}{p},p'}(K \cap \partial\Omega) > 0$ then $K \cap \partial\Omega$ is not removable by (ii). If $c^{\mathbb{R}^N}_{2,p'}(K \cap \Omega) > 0$, then there exists an increasing sequence of compact sets $K_n \subset K \cap \Omega$ such that $c^{\mathbb{R}^N}_{2,p'}(K_n) > 0$. Hence K_n is not removable, and clearly K inherits the same property as it contains K_n.

(iv) If $0 \in K \subset \partial\Omega$ and $K \setminus \{0\} \neq \emptyset$ and assume that any solution of (1.16) is identically 0, in particular any solution which vanishes on $\partial\Omega \setminus \{0\}$ is zero. By *Theorem J* this is ensured only if $p \geq p^*_\mu$. If $c^{\mathbb{R}^{N-1}}_{\frac{2}{p},p'}(K) > 0$, then either $p < \frac{N+1}{N-1}$, thus $K \setminus \{0\}$ contains at least one point which is not removable, or $p \geq \frac{N+1}{N-1}$, and since $c^{\mathbb{R}^{N-1}}_{\frac{2}{p},p'}(K \setminus \{0\}) = c^{\mathbb{R}^{N-1}}_{\frac{2}{p},p'}(K) > 0 > 0$, there exists a compact subset $K' \subset K \setminus \{0\}$ such that $c^{\mathbb{R}^{N-1}}_{\frac{2}{p},p'}(K') > 0$. Hence K', and therefore K, is not removable. This implies that if K is removable one must have $p \geq p^*_\mu$ and $c^{\mathbb{R}^{N-1}}_{\frac{2}{p},p'}(K) = 0$.

Conversely, if $p \geq p^*_\mu$, we will see at Theorem J that there exists no nonzero solution $u \in C(\overline{\Omega} \setminus \{0\})$ of $\mathcal{L}_\mu u + g_p(u) = 0$ vanishing on $\partial\Omega \setminus \{0\}$. For $0 < t < \epsilon$ we set $K_t = \{x \in \Omega : \text{dist}\,(x, K) < t\}$, $K_{\delta,\epsilon} = K_t \cap B^c_{2\epsilon}$ and $\Omega_{t,\epsilon} = \Omega \setminus \overline{K}_{t,\epsilon}$. We denote by $v_{t,\epsilon}$ the maximal solution of $\mathcal{L}_\mu u + g_p(u) = 0$ in $\Omega_{t,\epsilon}$ which vanishes on $\partial\Omega \setminus K_{t,\epsilon}$; hence it blows-up on $\partial K_{t,\epsilon}$ and it can be easily constructed by Lemma 3.5 by approximation with solutions with finite boundary value on $\partial K_{t,\epsilon}$. We also denote by w_ϵ the maximal solution of the same equation in $\Omega_\epsilon := \Omega \cap \overline{B}^c_\epsilon$ which vanishes on $\partial\Omega \setminus B_\epsilon$. It blows up on $\partial B_\epsilon \cap \Omega$. If u is a solution of (1.16), it is dominated in $\Omega \setminus (\overline{K}_{t,\epsilon} \cup \overline{B}_\epsilon)$ by the supersolution $v_{t,\epsilon} + w_\epsilon$. When $t \to 0$, $v_{t,\epsilon}$ converges to the function $v_{0,\epsilon}$ which satisfies the equation in Ω_ϵ and vanishes on $\partial\Omega_\epsilon \setminus K$. Since $c^{\mathbb{R}^{N-1}}_{\frac{2}{p},p}(K) = 0$, there holds $c^{\mathbb{R}^{N-1}}_{\frac{2}{p},p}(K \cap B^c_{2\epsilon}) = 0$. Therefore $v_{0,\epsilon} = 0$. When $\epsilon \to 0$, w_ϵ decreases and converges to a solution of the equation in Ω which vanishes on $\partial\Omega \setminus \{0\}$, hence this limit is zero and consequently $u = 0$.

(v) If $0 \in K \subset \overline{\Omega}$ and $K \setminus \{0\} \neq \emptyset$ and any solution of (1.16) is identically 0. Then $p \geq p^*_\mu$ as in (iv). Since $K \cap \Omega \neq \emptyset$ then any point in $K \cap \Omega$ is a removable singularity, hence $p \geq \frac{N}{N-2}$

(which implies $p > \frac{N+1}{N-1}$). If $c^{\mathbb{R}^N}_{2,p'}(K \cap \Omega) > 0$, there exists a compact set $K' \subset K \cap \Omega$ such that $c^{\mathbb{R}^N}_{2,p'}(K') > 0$. Then K' is not removable by *Theorem D*, hence K is not removable too. If $c^{\mathbb{R}^{N-1}}_{\frac{2}{p},p'}(K \cap \partial\Omega) > 0$, then K is not removable as in (iv).

Conversely assume that $p \geq p^*_\mu$, $c^{\mathbb{R}^N}_{2,p'}(K \cap \Omega) = 0$, $c^{\mathbb{R}^{N-1}}_{\frac{2}{p},p'}(K \cap \partial\Omega) = 0$ and u satisfies (1.16). For $0 < t < \epsilon$, we define $K_t = \{x \in \Omega : \text{dist}\,(x, K \cap \partial\Omega) < t\}$, $K_{t,\epsilon} = K_\delta \cap B^c_{2\epsilon}$ as in (iv) and $\tilde{K}_{t,\epsilon} = \{x \in \Omega : \text{dist}\,(x, K \cap \Omega) < t\} \cap \{x \in \Omega : \text{dist}\,(x, \partial\Omega) > 2\epsilon\}$. The functions $v_{t,\epsilon}$ and w_ϵ are defined as in (iv). We also denote by $\tilde{v}_{t,\epsilon}$ the maximal solution of $\mathcal{L}_\mu + g_p(u) = 0$ in $\Omega \setminus \tilde{K}_{t,\epsilon}$ which vanishes on $\partial\Omega$. Then $u \leq v_{t,\epsilon} + \tilde{v}_{t,\epsilon} + w_\epsilon$ in $\Omega \setminus (\overline{K}_{t,\epsilon} \cup \tilde{K}_{t,\epsilon} \cup \overline{B}_\epsilon$. When $t \to 0$, $v_{t,\epsilon} \to 0$ since $c^{\mathbb{R}^{N-1}}_{\frac{2}{p},p'}(K \cap B^c_\epsilon \cap \partial\Omega) = 0$ and $\tilde{v}_{t,\epsilon} \to 0$ since $c^{\mathbb{R}^N}_{2,p'}(K \cap \{x \in \Omega : \text{dist}\,(x, \partial\Omega) > 2\epsilon\}) = 0$. Hence $u \leq w_\epsilon$ and we conclude as in (iv) by letting $\epsilon \to 0$. □

4 Isolated boundary singularities

The study of boundary isolated singularities is based upon a technical framework which has been introduced by [17] in the case $\mu = 0$. For the sake of completeness we recall this formalism. Up to a rotation we assume that the inward normal direction to $\partial\Omega$ at 0 is $\mathbf{e}_N = (0', 1) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and that the tangent hyperplane to $\partial\Omega$ at 0 is $\partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}$. For $R > 0$ set $B'_R = \{x' \in \mathbb{R}^{N-1} : |x'| < R\}$ and $D_R = B'_R \times (-R, R)$. Then there exist $R > 0$ and a C^2 function $\theta : B'_R \mapsto \mathbb{R}$ such that $\partial\Omega \cap D_R = \{x = (x', x_N) : x_N = \theta(x') \text{ for } x' \in B'_R\}$ and $\Omega \cap D_R = \{x = (x', x_N) : \theta(x') < x_N < R\}$. Furthermore $\nabla\theta(0) = 0$. Define the function $\Theta = (\Theta_1, ..., \Theta_N)$ on D_R by $y_j = \Theta_j(x) = x_j$ if $1 \leq j \leq N-1$ and $y_N = \Theta_N(x) = x_N - \theta(x')$. Since $D\Theta(0) = Id$ we can assume that Θ is a diffeomorphism from D_R onto $\Theta(D_R)$. Let z be the harmonic extension of h in $B_R \cap \Omega$ vanishing on $\Omega \cap \partial B_R$ and set

$$u(x) - z(x) = \tilde{u}(y),\ z(x) = \tilde{z}(y) \qquad \text{for all } x \in D^+_R = B'_R \times [0, R).$$

Denote by $(r, \sigma) \in (0, \tilde{r}) \times S^{N-1}$ the spherical coordinates in $\mathbb{R}^N$ and set

$$\tilde{u}(y) = \tilde{u}(r, \sigma) = r^{-a} v(t, \sigma)\,,\ \tilde{z}(y) = \tilde{z}(r, \sigma) = r^{-a}\mathcal{Z}(t, \sigma)\,,\ t = \ln r\,,\ a = \frac{2}{p-1}.$$

Then v is bounded and satisfies the following asymptotically autonomous equation in $(-\infty, r_0] \times \mathbb{S}^{N-1}_+$

$$\begin{aligned}(1 + \epsilon_1(t,\cdot))v_{tt} + (N - 2 - 2a + \epsilon_2(t,\cdot))\, v_t + (a(a + 2 - N) - \mu + \epsilon_3(t,\cdot))\, v + \Delta' v \\ + \langle \nabla' v, \epsilon_4(t,\cdot)\rangle + \langle \nabla' v_t, \epsilon_5(t,\cdot)\rangle + \langle \nabla'(\langle \nabla' v, \mathbf{e}_N\rangle), \epsilon_6(t,\cdot)\rangle + \mu\mathcal{Z} - |v + \mathcal{Z}|^{p-1}(v + \mathcal{Z}) = 0,\end{aligned} \tag{4.1}$$

where Δ' is the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$ and the ϵ_j satisfy the estimates

$$|\epsilon_j(t,\cdot)| + |\partial_t \epsilon_j(t,\cdot)| + |\nabla' \epsilon_j(t,\cdot)| \leq c_{17} e^t.$$

As for $\mathcal{Z}$ it verifies

$$|\mathcal{Z}(t,\cdot)| + |\partial_t \mathcal{Z}(t,\cdot)| + |\nabla' \mathcal{Z}(t,\cdot)| \leq c_{17} e^{at}.$$

This is due to the fact that $|\theta(x')| = O(|x'|^2)$ near 0. Furthermore, standard elliptic equations theory implies that there holds, if $k + \ell \leq 3$,

$$\left|\frac{\partial^k \nabla'^\ell v}{\partial t^k}(t,\cdot)\right| \leq c_{18} \quad \text{in } (-\infty, r_0] \times \mathbb{S}^{N-1}_+. \tag{4.2}$$

Proof of Theorem F. We denote by $\mathcal{S}_{\mu,p}$ the set of functions satisfying

$$\begin{cases} -\Delta'\omega + (a(N-2-a)+\mu)\,\omega + g_p(\omega) = 0 & \text{in } \mathbb{S}^{N-1}_+ \\ \omega = 0 & \text{on } \partial\mathbb{S}^{N-1}_+, \end{cases} \tag{4.3}$$

where $a = \frac{2}{p-1}$.
(i) If ω is a solution it satisfies

$$\begin{aligned} 0 &= \int_{\mathbb{S}^{N-1}_+} \left(|\nabla'\omega|^2 + (a(N-2-a)+\mu)\,\omega^2 + |\omega|^{p+1}\right) dS \\ &\geq \int_{\mathbb{S}^{N-1}_+} \left(N-1 + (a(N-2-a)+\mu)\,\omega^2 + |\omega|^{p+1}\right) dS. \end{aligned}$$

If $N - 1 + a(N-2-a) + \mu \geq 0$, then necessarily $\omega = 0$. Next

$$N - 1 + a(N-2-a) + \mu \geq 0 \Longleftrightarrow -\alpha_+ \leq a \leq -\alpha_-$$
$$\Longleftrightarrow$$
$$\begin{cases} \text{(i) either } \ p \geq 1 - \frac{2}{\alpha_-} = p^*_\mu, \\ \text{(ii) or } \ 1 < p \leq 1 - \frac{2}{\alpha_+} = p^{**}_\mu \quad \text{provided } \ N \geq 3 \text{ and } \ \mu_1 \leq \mu < 1 - N. \end{cases}$$

(ii) By minimization $\mathcal{S}_{\mu,p}$ is not empty if the conditions (i) or (ii) of Theorem F are fulfilled, in which case $\mathcal{S}_{\mu,p}$ has a unique positive element (see [17] for a similar situation). This unique positive element is denoted ω_μ.
(iii) The last statement follows an idea introduced in [28]. The hupper hemisphere admits the following representation

$$\mathbb{S}^{N-1}_+ = \left\{x = ((\sin\phi)\sigma', \cos\phi) : \sigma' \in \mathbb{S}^{N-2}, \phi \in \left(0, \frac{\pi}{2}\right)\right\}.$$

The surface measure dS on $\mathbb{S}^{N-1}$ can be decomposed as

$$dS(\sigma) = (\sin\phi)^{N-2} dS'(\sigma') d\phi$$

where dS' is the surface measure on $\mathbb{S}^{N-2}$. If $h(\sigma) = h(\sigma', \phi)$ is defined on $\mathbb{S}^{N-1}$, we put

$$\overline{h}'(\phi) = \frac{1}{|\mathbb{S}^{N-2}|}\int_{\mathbb{S}^{N-2}} h(\sigma', \phi) dS'(\sigma').$$

Let ω be an element of $\mathcal{S}_{\mu,p}$, then, by averaging (4.3),

$$\int_{\mathbb{S}^{N-1}_+} \left(-\Delta'(\omega - \overline{\omega}') + (a(N-2-a)+\mu)\,(\omega - \overline{\omega}') + \left(g_p(\omega) - \overline{g_p(\omega)}'\right)\right)(\omega - \overline{\omega}') dS = 0.$$

By monotonicity

$$\int_{\mathbb{S}^{N-1}_+} \left(g_p(\omega) - \overline{g_p(\omega)}'\right)(\omega - \overline{\omega}')dS = \int_{\mathbb{S}^{N-1}_+} \left(g_p(\omega) - g_p(\overline{\omega}')\right)(\omega - \overline{\omega}')dS$$
$$\geq 2^{1-p}\int_{\mathbb{S}^{N-1}_+} |\omega - \overline{\omega}'|^{p+1}dS.$$

The function $\omega - \overline{\omega}'$ is orthogonal to the first eigenspace of $-\Delta'$ in $H^{1,2}_0(\mathbb{S}^{N-1}_+)$. Since the second eigenvalue of $-\Delta'$ in $H^{1,2}_0(\mathbb{S}^{N-1}_+)$ in $2N$, we have

$$-\int_{\mathbb{S}^{N-1}_+} (\omega - \overline{\omega}')\Delta'(\omega - \overline{\omega}')dS \geq 2N\int_{\mathbb{S}^{N-1}_+} (\omega - \overline{\omega}')^2 dS.$$

Hence

$$\int_{\mathbb{S}^{N-1}_+} \left((a(N-2-a) + \mu + 2N)\,(\omega - \overline{\omega}')^2 + 2^{1-p}\,|\omega - \overline{\omega}'|^{p+1}\right) dS \leq 0.$$

Hence, if $a(N-2-a) + \mu + 2N \geq 0$ it follows that $\omega - \overline{\omega}' = 0$. The polynomial

$$P_2(X) := X^2 + (2-N)X - \mu - 2N$$

admits two real roots provided $\mu \geq -(\frac{N+2}{2})^2 := \mu_2$, which are expressed by

$$a_- = \frac{N}{2} - 1 + \sqrt{\mu - \mu_2}, \qquad a_+ = \frac{N}{2} - 1 - \sqrt{\mu - \mu_2},$$

and

$$P_2\left(\frac{2}{p-1}\right) \leq 0 \Longleftrightarrow a_+ \leq \frac{2}{p-1} \leq a_-.$$

Note that $a_+a_- > 0$ if and only if $-2N > \mu$. Furthermore $P_2(-\alpha_-) < 0$ and $P_2(-\alpha_+) < 0$. Then

(i)	if $\mu \geq 1 - N$	then $a_+ < -\alpha_+ < 0 < -\alpha_- < a_- \Longrightarrow \tilde{p}^*_\mu < p^*_\mu$,
(ii)	if $N \geq 3$ & $-2N \leq \mu < 1 - N$	then $a_+ < 0 < -\alpha_+ \leq -\alpha_- < a_- \Longrightarrow \tilde{p}^*_\mu < p^*_\mu < p^{**}_\mu$,
(iii)	if $N \geq 9$ & $\mu_1 \leq \mu < -2N$	then $0 < a_+ < -\alpha_+ \leq -\alpha_- < a_- \Longrightarrow \tilde{p}^*_\mu < p^*_\mu < p^{**}_\mu < \tilde{p}^{**}_\mu$,

where, we recall it,

$$p^*_\mu = 1 - \frac{2}{\alpha_-}, \quad \tilde{p}^*_\mu = 1 + \frac{2}{a_-}, \quad \tilde{p}^{**}_\mu = 1 + \frac{2}{a_+}, \quad p^{**}_\mu = 1 - \frac{2}{\alpha_+}.$$

Therefore $\omega - \overline{\omega}' = 0$ if the following conditions are satisfied

(i) when $\mu \geq 1 - N$ and $\tilde{p}^*_\mu \leq p < p^*_\mu$,

(ii) when $N \geq 3$, $-2N \leq \mu < 1 - N$ and either $\tilde{p}^*_\mu \leq p < p^*_\mu$ or $p^{**}_\mu < p$,

(iii) when $N \geq 9$ and $\mu_1 \leq \mu < -2N$ and either $\tilde{p}^*_\mu \leq p < p^*_\mu$ or $p^{**}_\mu < p \leq \tilde{p}^{**}_\mu$.

If one of the above conditions is fulfilled, ω depends only on the variable $\phi \in [0, \frac{\pi}{2}]$. It satisfies

$$\begin{cases} -\dfrac{1}{\sin^{N-2}\phi}\left(\omega_\phi \sin^{N-2}\phi\right)_\phi + (a(N-2-a)+\mu)\,\omega + g_p(\omega) = 0 \qquad \text{in } (0,\frac{\pi}{2}) \\ \omega_\phi(0) = 0\,,\ \omega(\frac{\pi}{2}) = 0. \end{cases}$$

Define the operator

$$\mathcal{B}(\psi) = -\frac{1}{\sin^{N-2}\phi}\left(\sin^{N-2}\phi\psi_\phi\right)_\phi$$

among functions ψ in the space $\mathcal{H}_\mathcal{B} \subset C^2([0,\frac{\pi}{2}])$ satisfying $\psi_\phi(0) = 0$ and $\psi(\frac{\pi}{2}) = 0$. The first eigenvalue of $\mathcal{B}$ in $\mathcal{H}_\mathcal{B}$ is $N-1$ and the second in $2N$. Since g_p is nonnecreasing, it is known (see e.g. [4]) that the constant sign solutions ω_p and $-\omega_p$ lie on a branch of bifurcation issued from $N-1$ and there exists no other bifurcation when the parameter $a(N-2-a)+\mu$ belongs to $(N-1, 2N]$. This implies $\mathcal{S}_{\mu,p} = \{\omega_p, -\omega_p, 0\}$ and ends the proof. □

For proving *Theorems G, H, I, J* we recall here the following technical results [17, Theorem 5.1] related to the solutions of (1.17) satisfying

$$\lim_{x\to 0} |x|^{\frac{2}{p-1}} u(x) = 0. \tag{4.4}$$

The statement is easily adapted from the one of the above mentioned theorem. We denote by $\lambda_k = \{k(k+N-2 : k \in \mathbb{N}^*\}$ the set of eigenvalues of $-\Delta'$ in $H^{1,0}(\mathbb{S}^{N-1}_+)$. Any separable $\mathcal{L}_\mu$-harmonic function in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ endows the form

$$x \mapsto u(x) = u(r,\sigma) = r^{\alpha_k}\phi_k(\sigma) \qquad (r,\sigma) \in \mathbb{R}_+ \times \mathbb{S}^{N-1}_+,$$

where $\phi_k \in ker(\Delta' + \lambda_k I)$ and $\alpha_k = \alpha_{k-}$ or α_{k+} the smallest and the largest root of

$$\alpha^2 + (N-2)\alpha - \lambda_k - \mu = 0,$$

which exist for some $k \geq 1$ if and only if $\mu \geq \mu_k := \mu_1 + N - 1 - \lambda_k$. Note that $\alpha_{k-} \leq 0$ for all $k \in \mathbb{N}^*$ and $\alpha_{k+} \leq 0$ if and only if $\mu \geq -\lambda_k$ (which imposes $N \geq 8k(k + \sqrt{2k(k-1)})$).

Theorem 4.1 *Assume $\mu \geq \mu_1$, $1 < p < p^*_\mu$ and $h \in C^3(\partial\Omega)$. If $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega)$ is a solution of (1.17) satisfying (4.4) and*

(A) either $u_-(x) = O(|x|^{-\frac{2}{p-1}+\delta})$ near $x = 0$, for some $\delta > 0$,

(B) or $N = 2$ and Ω is locally a straight line near $x = 0$,

(B) or $-\frac{2}{p-1}$ is not equal to some α_{k-} for some $k \in \mathbb{N}^*$.

Then

(i) either u is the weak solution of (1.18),

(ii) or there exist an integer $k \in \mathbb{N}^*$ such that $-\alpha_{k-} < \frac{2}{p-1}$ and a nonzero spherical harmonic ψ_k of degree k such that

$$\lim_{x\to 0} r^{\alpha_{k-}}\tilde{u}(r,\sigma) = \psi_k(\sigma). \tag{4.5}$$

4.1 Proof of Theorems G, H, I and J

Because of (4.2) the negative trajectory of v in $C^1_0(\overline{\mathbb{S}^{N-1}_+})$ which is defined by

$$\mathcal{T}_-(v) = \bigcup_{t\leq r_0-1} \{v(t,.)\},$$

is relatively compact in the $C^2(\overline{\mathbb{S}^{N-1}_+})$-topology. The limit set $\mathcal{E}_v$ of $\mathcal{T}_-(v)$ at $-\infty$ defined by

$$\mathcal{E}_v = \bigcap_{\tau\leq r_0-1} \overline{\bigcup_{t\leq\tau}\{v(t,.)\}}^{C^1_0(\overline{\mathbb{S}^{N-1}_+})},$$

is non-empty. Since $1 < p < p^*_\mu$ and $\mu \geq \mu_1$, there holds

$$p < \frac{N+2}{N-2}. \tag{4.6}$$

Thus the coefficient of v_t in (4.1) is not zero (asymptotically, when $t \to -\infty$). Then energy damping holds and, in the same way as in [17] up to a shift of μ in the coefficient of v in (4.1), we obtain

$$\int_{-\infty}^{r_0-1}\int_{\mathbb{S}^{N-1}_+} v_t^2 d\sigma dt < \infty.$$

Combining this estimate with (4.2) and some standard manipulations (see [17]) implies that

$$\|v_t(t,.)\|_{C^1(\overline{\mathbb{S}^{N-1}_+})} + \|v_{tt}(t,.)\|_{C(\overline{\mathbb{S}^{N-1}_+})} \to 0 \quad \text{as } t \to -\infty.$$

Hence $\mathcal{E}_v$ is a compact connected component of $\mathcal{S}_{\mu,p}$.

Proof of Theorem G. If ω is nonnegative, either $\mathcal{E}_v = \{\omega_p\}$ and (1.22) holds or

$$\lim_{t\to-\infty} \|v(t,.)\|_{C^2(\overline{\mathbb{S}^{N-1}_+})} \to 0 \quad \text{as } t \to -\infty.$$

If this holds, it follows by Theorem 4.1-A that either $u = 0$ or (4.5) is verified for some $k \geq 1$. Since any spherical harmonics of degree at least two changes sign k must be equal to 1. Then

$$\tilde{u}(x) = \ell\phi_\mu(x)(1 + o(1)) \quad \text{as } x \to 0,$$

which is (1.23). □

Corollary 4.2 *Let $\mu_1 \leq \mu$ and $1 < p < p^*_\mu$. Then for any $h \in C^3(\Omega)$, $h \geq 0$ there exists only one solution of (1.17) with a strong singularity at $x = 0$, that is satisfying (1.22).*

Proof. It is a consequence of *Theorem G* that the limit of $u_{\ell\delta_0,h}$ of the solution of (1.24) when $\ell \to \infty$ is a solution which satisfies (1.22). The method of proof of uniqueness is due to Marcus and Véron [20]. The minimal solution of (1.24) with a strong singularity at $x = 0$ is defined by

$$\underline{u}_{\infty,h} := \lim_{\ell\to\infty} u_{\ell\delta_0,h}.$$

For constructing the maximal solution we define the sequence $\overline{u}_{n,h}$ of solutions of

$$\begin{cases} \mathcal{L}_\mu \overline{u}_{n,h} + g_p(\overline{u}_{n,h}) = 0 & \text{in } \Omega \cap \overline{B}^c_{\frac{1}{n}} \\ \overline{u}_{n,h} = h & \text{in } \partial\Omega \cap B^c_{\frac{1}{n}} \\ \overline{u}_{n,h} = cn^{\frac{2}{p-1}} & \text{in } \Omega \setminus \partial B_{\frac{1}{n}}, \end{cases}$$

where $c > 0$ is some constant large enough. Then $u_{\ell\delta_0,h} \le \overline{u}_{n,h}$. By convexity there holds

$$\overline{u}_{n,h} - u_{\ell\delta_0,h} \le \overline{u}_{n,0} - u_{\ell\delta_0,0}.$$

By monotonicity $\{\overline{u}_{n,h}\}$ decreases and converges to the maximum solution $\overline{u}_{\infty,h}$ of (1.22) and there holds

$$\overline{u}_{\infty,h} - \underline{u}_{\infty,h} \le \overline{u}_{\infty,0} - \underline{u}_{\infty,0}.$$

Furthermore, by (1.22), there exists $K = K(p,\mu,\Omega) > 1$ such that

$$\overline{u}_{\infty,0} \le K\underline{u}_{\infty,0}.$$

If we assume that $\overline{u}_{\infty,0} > \underline{u}_{\infty,0}$, then, again by convexity, the function

$$U = \overline{u}_{\infty,0} - \frac{1}{K}\left(\underline{u}_{\infty,0} - \overline{u}_{\infty,0}\right)$$

is a supersolution for problem (1.17) smaller than $\underline{u}_{\infty,0}$. The function

$$U^* = \left(\frac{1}{2K} + \frac{1}{2}\right)\overline{u}_\infty$$

is a supersolution of the same problem (1.17) smaller than U. By a standard result there exists V solution of the problem such that $U^* \le V \le U$. In particular V has a strong blow-up at $x = 0$ and it is smaller than the minimal solution $\underline{u}_\infty$, contradiction. □

Proof of Theorem H. Since $\mathcal{E}_v$ is a connected subset of the discrete set $\mathcal{S}_{\mu,p}$ which has three connected components $(\{\omega_p\}, \{-\omega_p\}, \{0\})$ by Theorem F-(1) either (1.25) or (4.4) holds. Since $p > \tilde{p}^*_\mu$, $-\frac{2}{p-1}$ which necessarily larger α_{1-} satisfies either $\frac{2}{p-1} < \alpha_{2-}$ or, if $\frac{2}{p-1} > \alpha_{2+}$ in the case $N \ge 9$ and $\mu < -2N$ and $\frac{2}{p-1}$ is not equal to any α_{k-} or α_{k+} for $k > 2$ by the equation. Hence, by Theorem 4.1, (1.23) holds. □

Remark. If $p = \tilde{p}^*_\mu$ or $p = \tilde{p}^{**}_\mu$ the method shows that either (1.25) or (4.4) holds. Since it is the spectral case always difficult to handle we cannot prove that (1.23) also holds, a fact that we conjecture.

Proof of Theorem I. The two statements obey a totaly different approach.

Statement 1- is a consequence of the theory of analytic functionals developped by in [26], [27] and applied to Emden-Fowler equations in [5]. The key point is to consider the equation (4.1) satisfied by $v(t,.) = r^{-\frac{2}{p-1}}\tilde{u}(r,\sigma)$ in $(-\infty, r_0) \times \mathbb{S}^{N-1}_+$ and to verify that, as a function of v, it is *real analytic*. Hence p must be an odd integer. If $1 < p < p^*_\mu$ the only possibility is $p = 3$ which

is in the range if $N+4\sqrt{\mu-\mu_1}$. If $\mu<1-N$ and $p>p_\mu^{**}$ there are infinitely many possibilities for p.

Statement 2- The convergence to one element of $\mathcal{S}_{\mu,p}$ follows from the fact that this set of solutions of (1.27) is discrete. If $\partial\Omega$ is locally a close graph near 0 the paper [12] which use Sturmian arguments and the Jordan closed curve theorem. If $\omega=0$, as quoted in [17, Theorem 5.1-C2] we perform a reflexion through $\partial\Omega$ near 0 and apply the result of [12, Lemma 2.1], the shift of the coefficient by μ playing no role. □

Proof of Theorem J. Since $p>1$, any solution u of (1.17) satisfies the estimate of Lemma 3.5 under the following form

$$|u(x)|\le A|x|^{-\frac{2}{p-1}} \qquad \text{for all } x\in\overline{\Omega}\setminus\{0\}.$$

If $p>p_\mu^*$, then $-\frac{2}{p-1}>\alpha_-$, therefore $u(x)=o(\phi_\mu(x))$ near $x=0$. Let u_+ be the solution of

$$\begin{cases} \mathcal{L}_\mu u+g_p(u)=0 & \text{in } \Omega\\ \qquad\qquad u=h_+ & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

For any $\epsilon>0$, $u_++\epsilon\phi_\mu$ is a supersolution of $\mathcal{L}_\mu u+g_p(u)=0$, larger than u near $x=0$. Then that $u\le u_++\epsilon\phi_\mu$ and, letting $\epsilon\to0$ then $u\le u_+$. Similarly u is larger than $-u_--\epsilon\phi_\mu$, where u_- is the solution of (4.7) with h_+ replaced by h_-. Letting $\epsilon\to0$ yields $-u_-\le u\le u_+$. It follows by the method of Theorem B that u is the weak solution of (1.18).

If $p=p_\mu^*$ and $\mu=\mu_1$, then similarly $u(x)=o(\phi_\mu(x))$ near $x=0$ and the result follows by the same method.

Finally, if $p=p_\mu^*$ and $\mu>\mu_1$, then (4.6) holds. Using the variable $t=\ln r$ and $v(t,.)=r^{\frac{2}{p-1}}\tilde{u}(r,.)$ we obtain from the previous energy method that

$$\mathcal{E}_v\subset\mathcal{S}_{\mu,p}=\{0\}.$$

Hence u satisfies (4.4). Since $p=p_\mu^*$, $\frac{2}{p-1}=-\alpha_-$. Hence $u=o(\phi_\mu)$ near 0 and the conclusion follows as in the previous cases. □

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