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# Some new congruences for $l$ -regular partitions modulo 13, 17, and 23

S. Abinash, T. Kathiravan and K. Srilakshmi

*Dedicated to the memory of Alan Baker*

**Abstract.** A partition of  $n$  is  $l$ -regular if none of its parts is divisible by  $l$ . Let  $b_l(n)$  denote the number of  $l$ -regular partitions of  $n$ . In this paper, using theta function identities due to Ramanujan, we establish some new infinite families of congruences for  $b_l(n)$  modulo  $l$ , where  $l = 17, 23$ , and for  $b_{65}(n)$  modulo 13.

**Keywords.** congruences,  $l$ -regular partitions, theta function identities.

**2010 Mathematics Subject Classification.** 11P83, 05A17

## 1. Introduction

We first recall a few terminologies and notations. Let  $n$  be a positive integer. A partition of  $n$  is a non-increasing sequence of positive integers whose sum is  $n$ , and the members of the sequence are called parts. For a positive integer  $l \geq 2$ , a partition of  $n$  is said to be  $l$ -regular if none of its parts is divisible by  $l$ . Let  $b_l(n)$  denote the number of  $l$ -regular partitions of  $n$ . By convention, we assume that  $b_l(0) = 1$ . The generating function for  $b_l(n)$  is given by

$$\sum_{n=0}^{\infty} b_l(n)q^n = \frac{f_l}{f_1}, \quad (1.1)$$

where  $|q| < 1$  and for any positive integer  $k$ ,  $f_k$  is defined by

$$f_k := \prod_{i=1}^{\infty} (1 - q^{ki}).$$

It can be easily verified that for any prime number  $l$ ,

$$f_l \equiv f_1^l \pmod{l}. \quad (1.2)$$

In recent years, various congruences satisfied by  $l$ -regular partitions for different values of  $l$  has been established. For instance, Lovejoy and Penniston [LoPe01] established criteria for 3-divisibility of  $b_3(n)$ . Dandurand and Penniston [DaPe09] employed the theory of complex multiplication to give a precise description of those  $n$  such that  $l|b_l(n)$  for  $l \in \{5, 7, 11\}$ . Cui and Gu [CuiGu13] derived infinite families of congruences modulo 2 for  $b_l(n)$  where  $l \in \{2, 4, 5, 8, 13, 16\}$ . Webb [Web11] established an infinite family of congruences modulo 3 for  $b_{13}(n)$ . Furcy and Penniston [FuPe12] obtained families of congruences modulo 3 for other values of  $l$  which are congruent to 1 modulo 3. Xia and Yao [XiYa14a] found some infinite families of congruences for  $b_9(n)$  modulo 2, and Cui and Gu [CuiGu15] derived congruences for  $b_9(n)$  modulo 3. Xia [Xia15] established infinite families of congruences for  $b_l(n)$  modulo  $l$  where  $l \in \{13, 17, 19\}$ , for example, for any  $n \geq 0$ ,  $k \geq 0$ ,

$$b_{13} \left( 5^{12k}n + \frac{1}{2} (5^{12k} - 1) \right) \equiv b_{13}(n) \pmod{13}.$$

For a complete literature review, see [Xia15]. For more related works, see [AdDa18, AhLo01, AHS10, CaWe14, CuiGu18, DLY14, Kei14, LiWa14, Pen02, Pen08, Ran17, Tan18, Wan17, Wan18a, Wan18b, Xia14, XiYa14b, Yao14, ZJY18].

In this paper, we establish infinite families of Ramanujan-type congruences for  $b_l(n)$  modulo  $l$ , where  $l \in \{17, 23\}$ , and for  $b_{65}(n)$  modulo 13. In fact, no such congruence has been established for 23-regular partitions till now, to the best of the authors' knowledge. The following are the main results.

**Theorem 1.1.** *For any  $n \geq 0$ ,  $k \geq 0$ ,*

$$b_{17} \left( 5^{4k}n + \frac{2}{3}(5^{4k} - 1) \right) \equiv 2^k b_{17}(n) \pmod{17}, \quad (1.3)$$

and for  $r \in \{0, 1, 2, 4\}$ ,

$$b_{17} \left( 5^{4k+4}n + \frac{1}{3}(5^{4k+3}(3r+1) - 2) \right) \equiv 0 \pmod{17}. \quad (1.4)$$

**Theorem 1.2.** *For any  $n \geq 0$ ,  $k \geq 0$ ,*

$$b_{23} \left( 5^{24k}n + \frac{11}{12}(5^{24k} - 1) \right) \equiv 14^k b_{23}(n) \pmod{23}, \quad (1.5)$$

and for  $r \in \{0, 1, 2, 3\}$ ,

$$b_{23} \left( 5^{24k+24}n + \frac{1}{12}(5^{24k+23}(12r+7) - 11) \right) \equiv 0 \pmod{23}. \quad (1.6)$$

**Theorem 1.3.** *For any  $n \geq 0$ ,  $k \geq 0$ ,*

$$b_{65} \left( 5^{12k}n + \frac{8}{3}(5^{12k} - 1) \right) \equiv 12^k b_{65}(n) \pmod{13}, \quad (1.7)$$

and for  $r \in \{2, 4\}$ ,

$$b_{65} \left( 5^{12k+12}n + \frac{1}{3}(5^{12k+11}(3r+1) - 8) \right) \equiv 0 \pmod{13}. \quad (1.8)$$

## 2. Preliminaries

In this section we shall state a preliminary lemma.

**Lemma 2.1.** *For*

$$R(q) := \prod_{n=1}^{\infty} \frac{(1 - q^{5n-4})(1 - q^{5n-1})}{(1 - q^{5n-3})(1 - q^{5n-2})},$$

we have

A. ([Ram62], [Wat38])

$$f_1 = f_{25} \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right). \quad (2.9)$$

B. ([Wat29a],[Wat29b])

$$\frac{1}{R(q)^5} - 11q - q^2 R(q)^5 = \frac{f_1^6}{f_5^6}. \tag{2.10}$$

C.

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6} \left( \frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5 R(q^5) + 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4 \right). \tag{2.11}$$

Note that the identity in (2.11) can be easily established by first substituting  $q$  by  $q^5$  in (2.10) and then using (2.9).

### 3. Congruences for 17-Regular Partitions

*Proof of Theorem 1.1.* Taking  $l = 17$  in (1.1), we get

$$\sum_{n=0}^{\infty} b_{17}(n)q^n = \frac{f_{17}}{f_1}.$$

Now by (1.2),

$$\sum_{n=0}^{\infty} b_{17}(n)q^n \equiv f_1^{16} \pmod{17}. \tag{3.12}$$

Replacing  $f_1$  using (2.9) and then extracting the terms involving  $q^{5n+1}$ , we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5n+1)q^{5n+1} &\equiv f_{25}^{16} \left( \frac{q}{R(q^5)^{15}} + \frac{13q^6}{R(q^5)^{10}} + \frac{8q^{11}}{R(q^5)^5} - q^{16} - 8q^{21} R(q^5)^5 \right. \\ &\quad \left. + 13q^{26} R(q^5)^{10} - q^{31} R(q^5)^{15} \right) \pmod{17}. \end{aligned}$$

If we divide by  $q$  and substitute  $q$  by  $q^{1/5}$ , then we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5n+1)q^n &\equiv f_5^{16} \left( \left( \frac{1}{R(q)^5} - 11q - q^2 R(q)^5 \right)^3 + 12q \left( \frac{1}{R(q)^5} - 11q - q^2 R(q)^5 \right)^2 \right. \\ &\quad \left. + 14q^2 \left( \frac{1}{R(q)^5} - 11q - q^2 R(q)^5 \right) + 7q^3 \right) \pmod{17}. \end{aligned}$$

By (2.10), we get

$$\sum_{n=0}^{\infty} b_{17}(5n+1)q^n \equiv f_5^{16} \left( \frac{f_1^{18}}{f_5^{18}} + 12q \frac{f_1^{12}}{f_5^{12}} + 14q^2 \frac{f_1^6}{f_5^6} + 7q^3 \right) \pmod{17}.$$

In view of (3.12), we get

$$\sum_{n=0}^{\infty} b_{17}(5n+1)q^n \equiv \frac{f_1^{18}}{f_5^{18}} + 12q f_1^{12} f_5^4 + 14q^2 f_1^6 f_5^{10} + 7 \sum_{n=0}^{\infty} b_{17}(n)q^{5n+3} \pmod{17}. \tag{3.13}$$

Replacing  $f_1$  using (2.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5n+1)q^n &\equiv \frac{f_{25}^{18}}{f_5^2} \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{18} + 12q f_5^4 f_{25}^{12} \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{12} \\ &\quad + 14q^2 f_5^{10} f_{25}^6 \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^6 + 7 \sum_{n=0}^{\infty} b_{17}(n)q^{5n+3} \pmod{17}. \end{aligned}$$

If we extract the terms involving  $q^{5n+3}$ , divide by  $q^3$ , and substitute  $q$  by  $q^{1/5}$ , then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5^2n+16)q^n &\equiv \frac{f_5^{18}}{f_1^2} \left( q^3 \right) + 12f_1^4 f_5^{12} \left( 3\frac{f_1^{12}}{f_5^{12}} + 3q^2 \right) + 14f_1^{10} f_5^6 \left( 11\frac{f_1^6}{f_5^6} + 9q \right) \\ &\quad + 7 \sum_{n=0}^{\infty} b_{17}(n)q^n \pmod{17}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{17}(5^2n+16)q^n \equiv 7q f_1^{10} f_5^6 + 2q^2 f_1^4 f_5^{12} + q^3 \frac{f_5^{18}}{f_1^2} + 10 \sum_{n=0}^{\infty} b_{17}(n)q^n \pmod{17}.$$

Replacing  $f_1$  and  $1/f_1$  using (2.9) and (2.11), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5^2n+16)q^n &\equiv 7q f_5^6 f_{25}^{10} \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{10} + 2q^2 f_5^{12} f_{25}^4 \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^4 \\ &\quad + q^3 f_5^6 f_{25}^{10} \left( \frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5 R(q^5) \right. \\ &\quad \left. + 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4 \right)^2 + 10 \sum_{n=0}^{\infty} b_{17}(n)q^n \pmod{17}. \end{aligned}$$

If we extract the terms involving  $q^{5n+1}$ , divide by  $q$ , and substitute  $q$  by  $q^{1/5}$ , then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5^3n+41)q^n &\equiv 7f_1^6 f_5^{10} \left( \frac{f_1^{12}}{f_5^{12}} + 12q \frac{f_1^6}{f_5^6} + q^2 \right) + 2f_1^{12} f_5^4 \left( 12q \right) + f_1^6 f_5^{10} \left( 10q \frac{f_1^6}{f_5^6} + 6q^2 \right) \\ &\quad + 10 \sum_{n=0}^{\infty} b_{17}(5n+1)q^n \pmod{17}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{17}(5^3n+41)q^n \equiv 7\frac{f_1^{18}}{f_5^2} + 16q f_1^{12} f_5^4 + 13q^2 f_1^6 f_5^{10} + 10 \sum_{n=0}^{\infty} b_{17}(5n+1)q^n \pmod{17}.$$

Now by (3.13), we get

$$\sum_{n=0}^{\infty} b_{17}(5^3n+41)q^n \equiv 2 \sum_{n=0}^{\infty} b_{17}(n)q^{5n+3} \pmod{17}. \quad (3.14)$$

Comparing the coefficients of the terms of the form  $q^{5n+3}$  in (3.14), we can conclude that for any  $n \geq 0$ ,

$$b_{17}(5^4n+416) \equiv 2b_{17}(n) \pmod{17},$$

or,

$$b_{17} \left( 5^4 n + \frac{2}{3} (5^4 - 1) \right) \equiv 2b_{17}(n) \pmod{17}. \tag{3.15}$$

Now (1.3) follows from (3.15), by mathematical induction.

On the other hand, comparing the coefficients of the terms of the form  $q^{5n+r}$  where  $r \in \{0, 1, 2, 4\}$  in (3.14), we can conclude that for any  $n \geq 0$ ,

$$b_{17} \left( 5^3(5n+r) + \frac{5^3-2}{3} \right) \equiv 0 \pmod{17},$$

or,

$$b_{17} \left( 5^4 n + \frac{1}{3} (5^3(3r+1) - 2) \right) \equiv 0 \pmod{17}. \tag{3.16}$$

Now (1.4) follows from (1.3) and (3.16).

### 4. Congruences for 23-Regular Partitions

*Proof of Theorem 1.2.* Taking  $l = 23$  in (1.1), we get

$$\sum_{n=0}^{\infty} b_{23}(n)q^n = \frac{f_{23}}{f_1}.$$

Now by (1.2),

$$\sum_{n=0}^{\infty} b_{23}(n)q^n \equiv f_1^{22} \pmod{23}. \tag{4.17}$$

Replacing  $f_1$  using (2.9) and then extracting the terms involving  $q^{5n+2}$ , we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5n+2)q^{5n+2} &\equiv f_5^{22} \left( \frac{2q^2}{R(q^5)^{20}} + \frac{21q^7}{R(q^5)^{15}} + \frac{3q^{12}}{R(q^5)^{10}} + \frac{8q^{17}}{R(q^5)^5} - q^{22} - 8q^{27}R(q^5)^5 \right. \\ &\quad \left. + 3q^{32}R(q^5)^{10} - 21q^{37}R(q^5)^{15} + 2q^{42}R(q^5)^{20} \right) \pmod{23}. \end{aligned}$$

If we divide by  $q^2$  and substitute  $q$  by  $q^{1/5}$ , then we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5n+2)q^n &\equiv f_5^{22} \left( 2 \left( \frac{1}{R(q)^5} - 11q - q^2 R(q)^5 \right)^4 + 17q \left( \frac{1}{R(q)^5} - 11q - q^2 R(q)^5 \right)^3 \right. \\ &\quad \left. + 17q^2 \left( \frac{1}{R(q)^5} - 11q - q^2 R(q)^5 \right)^2 + 14q^4 \right) \pmod{23}. \end{aligned}$$

By (2.10), we get

$$\sum_{n=0}^{\infty} b_{23}(5n+2)q^n \equiv f_5^{22} \left( 2 \frac{f_1^{24}}{f_5^{24}} + 17q \frac{f_1^{18}}{f_5^{18}} + 17q^2 \frac{f_1^{12}}{f_5^{12}} + 14q^4 \right) \pmod{23}.$$

In view of (4.17), we get

$$\sum_{n=0}^{\infty} b_{23}(5n+2)q^n \equiv 2 \frac{f_1^{24}}{f_5^{24}} + 17q f_1^{18} f_5^4 + 17q^2 f_1^{12} f_5^{10} + 14 \sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}. \tag{4.18}$$

Replacing  $f_1$  using (2.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5n+2)q^n &\equiv 2\frac{f_{25}^{24}}{f_5^2} \left( \frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{24} + 17qf_5^4f_{25}^{18} \left( \frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{18} \\ &\quad + 17q^2f_5^{10}f_{25}^{12} \left( \frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{12} + 14 \sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}. \end{aligned}$$

If we extract the terms involving  $q^{5n+4}$ , divide by  $q^4$ , and substitute  $q$  by  $q^{1/5}$ , then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5^2n+22)q^n &\equiv 2\frac{f_5^{24}}{f_1^2} \left( q^4 \right) + 17f_1^4f_5^{18} \left( 19\frac{f_1^{18}}{f_5^{18}} + 7q^3 \right) + 17f_1^{10}f_5^{12} \left( 8\frac{f_1^{12}}{f_5^{12}} + 3q^2 \right) \\ &\quad + 14 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23}(5^2n+22)q^n \equiv 5q^2f_1^{10}f_5^{12} + 4q^3f_1^4f_5^{18} + 2q^4\frac{f_5^{24}}{f_1^2} + 13 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}. \quad (4.19)$$

Replacing  $f_1$  and  $1/f_1$  using (2.9) and (2.11), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5^2n+22)q^n &\equiv 5q^2f_5^{12}f_{25}^{10} \left( \frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{10} + 4q^3f_5^{18}f_{25}^4 \left( \frac{1}{R(q^5)} - q - q^2R(q^5) \right)^4 \\ &\quad + 2q^4f_5^{12}f_{25}^{10} \left( \frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5R(q^5) \right. \\ &\quad \left. + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4 \right)^2 + 13 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}. \end{aligned}$$

If we extract the terms involving  $q^{5n+2}$ , divide by  $q^2$ , and substitute  $q$  by  $q^{1/5}$ , then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5^3n+72)q^n &\equiv 5f_1^{12}f_5^{10} \left( \frac{f_1^{12}}{f_5^{12}} + 20q\frac{f_1^6}{f_5^6} + 16q^2 \right) + 4f_1^{18}f_5^4 \left( 18q \right) + 2f_1^{12}f_5^{10} \left( 10q\frac{f_1^6}{f_5^6} + 10q^2 \right) \\ &\quad + 13 \sum_{n=0}^{\infty} b_{23}(5n+2)q^n \pmod{23}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23}(5^3n+72)q^n \equiv 5\frac{f_1^{24}}{f_5^2} + 8qf_1^{18}f_5^4 + 8q^2f_1^{12}f_5^{10} + 13 \sum_{n=0}^{\infty} b_{23}(5n+2)q^n \pmod{23}.$$

Now by (4.18), we get

$$\sum_{n=0}^{\infty} b_{23}(5^3n+72)q^n \equiv 8\frac{f_1^{24}}{f_5^2} + 22qf_1^{18}f_5^4 + 22q^2f_1^{12}f_5^{10} + 21 \sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}.$$

Replacing  $f_1$  using (2.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5^3n+72)q^n &\equiv 8\frac{f_{25}^{24}}{f_5^2} \left( \frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{24} + 22qf_5^4f_{25}^{18} \left( \frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{18} \\ &\quad + 22q^2f_5^{10}f_{25}^{12} \left( \frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{12} + 21 \sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}. \end{aligned}$$

If we extract the terms involving  $q^{5n+4}$ , divide by  $q^4$ , and substitute  $q$  by  $q^{1/5}$ , then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{23}(5^4n + 572)q^n \equiv 8 \frac{f_5^{24}}{f_1^2} \left( q^4 \right) + 22 f_1^4 f_5^{18} \left( 19 \frac{f_1^{18}}{f_5^{18}} + 7q^3 \right) + 22 f_1^{10} f_5^{12} \left( 8 \frac{f_1^{12}}{f_5^{12}} + 3q^2 \right) + 21 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23}(5^4n + 572)q^n \equiv 20q^2 f_1^{10} f_5^{12} + 16q^3 f_1^4 f_5^{18} + 8q^4 \frac{f_5^{24}}{f_1^2} + 17 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Now by (4.19), we get

$$\sum_{n=0}^{\infty} b_{23}(5^4n + 572)q^n \equiv 4 \sum_{n=0}^{\infty} b_{23}(5^2n + 22)q^n + 11 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23},$$

or,

$$\sum_{n=0}^{\infty} b_{23} \left( 5^4n + \frac{11}{12} (5^4 - 1) \right) q^n \equiv 4 \sum_{n=0}^{\infty} b_{23}(5^2n + 22)q^n + 11 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Substituting  $n$  by  $5^2n + 22$ , we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23} \left( 5^6n + \frac{11}{12} (5^6 - 1) \right) q^n &\equiv 4 \sum_{n=0}^{\infty} b_{23} \left( 5^4n + \frac{11}{12} (5^4 - 1) \right) q^n + 11 \sum_{n=0}^{\infty} b_{23}(5^2n + 22)q^n \pmod{23}, \\ &\equiv 4 \left( 4 \sum_{n=0}^{\infty} b_{23}(5^2n + 22)q^n + 11 \sum_{n=0}^{\infty} b_{23}(n)q^n \right) + 11 \sum_{n=0}^{\infty} b_{23}(5^2n + 22)q^n \pmod{23}, \\ &\equiv 4 \sum_{n=0}^{\infty} b_{23}(5^2n + 22)q^n + 21 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}. \end{aligned}$$

Repeating the same process as above, it follows that

$$\sum_{n=0}^{\infty} b_{23} \left( 5^{22}n + \frac{11}{12} (5^{22} - 1) \right) q^n \equiv 18 \sum_{n=0}^{\infty} b_{23}(5^2n + 22)q^n + 20 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Now by (4.19), we get

$$\sum_{n=0}^{\infty} b_{23} \left( 5^{22}n + \frac{11}{12} (5^{22} - 1) \right) q^n \equiv 21q^2 f_1^{10} f_5^{12} + 3q^3 f_1^4 f_5^{18} + 13q^4 \frac{f_5^{24}}{f_1^2} + \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Replacing  $f_1$  and  $1/f_1$  using (2.9) and (2.11), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23} \left( 5^{22}n + \frac{11}{12} (5^{22} - 1) \right) q^n &\equiv 21q^2 f_5^{12} f_{25}^{10} \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{10} + 3q^3 f_5^{18} f_{25}^4 \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^4 \\ &\quad + 13q^4 f_5^{12} f_{25}^{10} \left( \frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5 R(q^5) + 2q^6 R(q^5)^2 \right. \\ &\quad \left. - q^7 R(q^5)^3 + q^8 R(q^5)^4 \right)^2 + \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}. \end{aligned}$$



If we extract the terms involving  $q^{5n+2}$ , divide by  $q^2$ , and substitute  $q$  by  $q^{1/5}$ , then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23} \left( 5^{23}n + \frac{7 \cdot 5^{23} - 11}{12} \right) q^n &\equiv 21f_1^{12}f_5^{10} \left( \frac{f_1^{12}}{f_5^{12}} + 20q \frac{f_1^6}{f_5^6} + 16q^2 \right) + 3f_1^{18}f_5^4 \left( 18q \right) \\ &\quad + 13f_1^{12}f_5^{10} \left( 10q \frac{f_1^6}{f_5^6} + 10q^2 \right) + \sum_{n=0}^{\infty} b_{23}(5n+2)q^n \pmod{23}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23} \left( 5^{23}n + \frac{7 \cdot 5^{23} - 11}{12} \right) q^n \equiv 21 \frac{f_1^{24}}{f_5^2} + 6q f_1^{18} f_5^4 + 6q^2 f_1^{12} f_5^{10} + \sum_{n=0}^{\infty} b_{23}(5n+2)q^n \pmod{23}.$$

Now by (4.18), we get

$$\sum_{n=0}^{\infty} b_{23} \left( 5^{23}n + \frac{7 \cdot 5^{23} - 11}{12} \right) q^n \equiv 14 \sum_{n=0}^{\infty} b_{23}(n) q^{5n+4} \pmod{23}. \quad (4.20)$$

Comparing the coefficients of the terms of the form  $q^{5n+4}$  in (4.20), we can conclude that for any  $n \geq 0$ ,

$$b_{23} \left( 5^{24}n + \frac{11}{12} (5^{24} - 1) \right) \equiv 14b_{23}(n) \pmod{23}. \quad (4.21)$$

Now (1.5) follows from (4.21), by mathematical induction.

On the other hand, comparing the coefficients of the terms of the form  $q^{5n+r}$  where  $r \in \{0, 1, 2, 3\}$  in (4.20), we can conclude that for any  $n \geq 0$ ,

$$b_{23} \left( 5^{23}(5n+r) + \frac{7 \cdot 5^{23} - 11}{12} \right) \equiv 0 \pmod{23},$$

or,

$$b_{23} \left( 5^{24}n + \frac{1}{12} (5^{23}(12r+7) - 11) \right) \equiv 0 \pmod{23}. \quad (4.22)$$

Now (1.6) follows from (1.5) and (4.22).

## 5. Congruences for 65-Regular Partition

*Proof of Theorem 1.3.* Taking  $l = 65$  in (1.1), we get

$$\sum_{n=0}^{\infty} b_{65}(n) q^n = \frac{f_{65}}{f_1}. \quad (5.23)$$

Replacing  $1/f_1$  using (2.11), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{65}(n) q^n &= f_{65} \cdot \frac{f_{25}^5}{f_5^6} \left( \frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 \right. \\ &\quad \left. - 3q^5 R(q^5) + 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4 \right). \end{aligned}$$

If we extract the terms involving  $q^{5n+4}$ , divide by  $q^4$ , and substitute  $q$  by  $q^{1/5}$ , then by (2.10), we get

$$\sum_{n=0}^{\infty} b_{65}(5n+4)q^n = 5f_{13} \cdot \frac{f_5^5}{f_1^6}.$$

Now by (1.2),

$$\sum_{n=0}^{\infty} b_{65}(5n+4)q^n \equiv 5f_1^7 f_5^5 \pmod{13}.$$

Replacing  $f_1$  using (2.9), we get

$$\sum_{n=0}^{\infty} b_{65}(5n+4)q^n \equiv 5f_5^5 f_{25}^7 \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^7 \pmod{13}.$$

If we extract the terms involving  $q^{5n+2}$ , divide by  $q^2$ , and substitute  $q$  by  $q^{1/5}$ , then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{65}(5^2n+14)q^n \equiv 5f_1^5 f_5^7 \left( \frac{f_1^6}{f_5^6} + 8q \right) \equiv 5f_1^{11} f_5 + qf_1^5 f_5^7 \pmod{13}.$$

Replacing  $f_1$  using (2.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{65}(5^2n+14)q^n &\equiv 5f_5 f_{25}^{11} \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{11} \\ &\quad + qf_5^7 f_{25}^5 \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^5 \pmod{13}. \end{aligned}$$

If we extract the terms involving  $q^{5n+1}$ , divide by  $q$ , and substitute  $q$  by  $q^{1/5}$ , then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{65}(5^3n+39)q^n \equiv 5f_1 f_5^{11} \left( 2\frac{f_1^{12}}{f_5^{12}} + 11q\frac{f_1^6}{f_5^6} + 8q^2 \right) + f_1^7 f_5^5 \left( \frac{f_1^6}{f_5^6} \right) \pmod{13},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{65}(5^3n+39)q^n \equiv 11\frac{f_1^{13}}{f_5} + 3qf_1^7 f_5^5 + q^2 f_1 f_5^{11} \pmod{13},$$

or,

$$\sum_{n=0}^{\infty} b_{65} \left( 5^3n + \frac{5^3 - 8}{3} \right) q^n \equiv 11\frac{f_1^{13}}{f_5} + 3qf_1^7 f_5^5 + q^2 f_1 f_5^{11} \pmod{13}. \quad (5.24)$$

Replacing  $f_1$  using (2.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{65} \left( 5^3n + \frac{5^3 - 8}{3} \right) q^n &\equiv 11\frac{f_{25}^{13}}{f_5} \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{13} + 3qf_5^5 f_{25}^7 \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^7 \\ &\quad + q^2 f_5^{11} f_{25} \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right) \pmod{13}. \end{aligned}$$

If we extract the terms involving  $q^{5n+3}$ , divide by  $q^3$ , and substitute  $q$  by  $q^{1/5}$ , then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{65} \left( 5^4 n + \frac{2 \cdot 5^4 - 8}{3} \right) q^n \equiv 11 \frac{f_5^{13}}{f_1} \left( 12q^2 \right) + 3f_1^5 f_5^7 \left( \frac{f_1^6}{f_5^6} + 8q \right) + f_1^{11} f_5 \left( 12 \right) \pmod{13},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{65} \left( 5^4 n + \frac{2 \cdot 5^4 - 8}{3} \right) q^n \equiv 2f_1^{11} f_5 + 11q f_1^5 f_5^7 + 2q^2 \frac{f_5^{13}}{f_1} \pmod{13}.$$

Replacing  $f_1$  and  $1/f_1$  using (2.9) and (2.11), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{65} \left( 5^4 n + \frac{2 \cdot 5^4 - 8}{3} \right) q^n & \equiv 2f_5 f_{25}^{11} \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{11} + 11q f_5^7 f_{25}^5 \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^5 \\ & + 2q^2 f_5^7 f_{25}^5 \left( \frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5 R(q^5) + 2q^6 R(q^5)^2 \right. \\ & \left. - q^7 R(q^5)^3 + q^8 R(q^5)^4 \right) \pmod{13}. \end{aligned}$$

If we extract the terms involving  $q^{5n+1}$ , divide by  $q$ , and substitute  $q$  by  $q^{1/5}$ , then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{65} \left( 5^5 n + \frac{5^5 - 8}{3} \right) q^n & \equiv 2f_1 f_5^{11} \left( 2 \frac{f_1^{12}}{f_5^{12}} + 11q \frac{f_1^6}{f_5^6} + 8q^2 \right) + 11f_1^7 f_5^5 \left( \frac{f_1^6}{f_5^6} \right) \\ & + 2f_1^7 f_5^5 \left( 5q \right) \pmod{13}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{65} \left( 5^5 n + \frac{5^5 - 8}{3} \right) q^n \equiv 2 \frac{f_1^{13}}{f_5} + 6q f_1^7 f_5^5 + 2q^2 f_1 f_5^{11} \pmod{13}. \quad (5.25)$$

Repeating the same process as above to derive (5.25) from (5.24), it follows that

$$\sum_{n=0}^{\infty} b_{65} \left( 5^{11} n + \frac{5^{11} - 8}{3} \right) q^n \equiv \frac{f_1^{13}}{f_5} \equiv \frac{f_{13}}{f_5} \pmod{13}. \quad (5.26)$$

In (2.9), if we substitute  $q$  by  $q^{13}$ , then we get

$$f_{13} = f_{325} \left( \frac{1}{R(q^{65})} - q^{13} - q^{26} R(q^{65}) \right).$$

Replacing  $f_{13}$  in (5.26), we get

$$\sum_{n=0}^{\infty} b_{65} \left( 5^{11} n + \frac{5^{11} - 8}{3} \right) q^n \equiv \frac{f_{325}}{f_5} \left( \frac{1}{R(q^{65})} - q^{13} - q^{26} R(q^{65}) \right) \pmod{13}. \quad (5.27)$$

If we extract the terms involving  $q^{5n+3}$ , divide by  $q^3$ , and substitute  $q$  by  $q^{1/5}$ , then we get

$$\sum_{n=0}^{\infty} b_{65} \left( 5^{12}n + \frac{2 \cdot 5^{12} - 8}{3} \right) q^n \equiv 12q^2 \frac{f_{65}}{f_1} \pmod{13},$$

which, by (5.23), yields

$$\sum_{n=0}^{\infty} b_{65} \left( 5^{12}n + \frac{2 \cdot 5^{12} - 8}{3} \right) q^n \equiv 12 \sum_{n=0}^{\infty} b_{65}(n) q^{n+2} \pmod{13}. \quad (5.28)$$

Comparing the coefficients of the terms of the form  $q^{n+2}$  in (5.28), we can conclude that for any  $n \geq 0$ ,

$$b_{65} \left( 5^{12}n + \frac{8}{3}(5^{12} - 1) \right) \equiv 12b_{65}(n) \pmod{13}. \quad (5.29)$$

Now (1.7) follows from (5.29), by mathematical induction.

On the other hand, comparing the coefficients of the terms of the form  $q^{5n+r}$  where  $r \in \{2, 4\}$  in (5.27), we can conclude that for any  $n \geq 0$ ,

$$b_{65} \left( 5^{11}(5n+r) + \frac{5^{11} - 8}{3} \right) \equiv 0 \pmod{13},$$

or,

$$b_{65} \left( 5^{12}n + \frac{1}{3}(5^{11}(3r+1) - 8) \right) \equiv 0 \pmod{13}. \quad (5.30)$$

Now (1.8) follows from (1.7) and (5.30).

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