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Some New Congruences for l -Regular Partitions Modulo 13, 17, and 23

S. Abinash, T. Kathiravan, K. Srilakshmi

Abstract. A partition of n is l -regular if none of its parts is divisible by l . Let $b_l(n)$ denote the number of l -regular partitions of n . In this paper, using theta function identities due to Ramanujan, we establish some new infinite families of congruences for $b_l(n)$ modulo l , where $l = 17, 23$, and for $b_{65}(n)$ modulo 13.

Keywords. congruences, l -regular partitions, theta function identities.

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1. Introduction

We first recall a few terminologies and notations. Let n be a positive integer. A partition of n is a non-increasing sequence of positive integers whose sum is n , and the members of the sequence are called parts. For a positive integer $l \geq 2$, a partition of n is said to be l -regular if none of its parts is divisible by l . Let $b_l(n)$ denote the number of l -regular partitions of n . By convention, we assume that $b_l(0) = 1$. The generating function for $b_l(n)$ is given by

$$\sum_{n=0}^{\infty} b_l(n)q^n = \frac{f_l}{f_1}, \quad (1.1)$$

where $|q| < 1$ and for any positive integer k , f_k is defined by

$$f_k := \prod_{i=1}^{\infty} (1 - q^{ki}).$$

It can be easily verified that for any prime number l ,

$$f_l \equiv f_1^l \pmod{l}. \quad (1.2)$$

In recent years, numerous congruences satisfied by l -regular partitions for different values of l has been established. For instance, Lovejoy and Penniston [LP01] established criteria for 3-divisibility of $b_3(n)$. Dandurand and Penniston [DP09] employed the theory of complex multiplication to give a precise description of those n such that $l|b_l(n)$ for $l \in \{5, 7, 11\}$. Cui and Gu [CG13] derived infinite families of congruences modulo 2 for $b_l(n)$ where $l \in \{2, 4, 5, 8, 13, 16\}$. Webb [Web11] established an infinite family of congruences modulo 3 for $b_{13}(n)$. Furcy and Penniston [FP12] obtained families

of congruences modulo 3 for other values of l which are congruent to 1 modulo 3. Xia and Yao [XY14a] found some infinite families of congruences for $b_9(n)$ modulo 2, and Cui and Gu [CG15] derived congruences for $b_9(n)$ modulo 3. Xia [Xia15] established infinite families of congruences for $b_l(n)$ modulo l where $l \in \{13, 17, 19\}$, for example, for any $n \geq 0$, $k \geq 0$,

$$b_{13} \left(5^{12k}n + \frac{1}{2} \left(5^{12k} - 1 \right) \right) \equiv b_{13}(n) \pmod{13}.$$

For a more complete literature review, see [Xia15]. For more related works, see [AL01, AHS10, CW14, CG18, DLY14, Kei14, LW14, Pen02, Pen08, Tan18, Wan17, Wan18a, Wan18b, Xia14, XY14b, Yao14, ZJY18].

In this paper, we establish infinite families of arithmetic properties and Ramanujan-type congruences for $b_l(n)$ modulo l , where $l \in \{17, 23\}$, and for $b_{65}(n)$ modulo 13. In fact, no such congruence has been established for 23-regular partitions till now, to the best of the authors' knowledge. The following are the main results.

Theorem 1.1. *For any $n \geq 0$, $k \geq 0$,*

$$b_{17} \left(5^{4k}n + \frac{2}{3} \left(5^{4k} - 1 \right) \right) \equiv 2^k b_{17}(n) \pmod{17}, \quad (1.3)$$

and for $r \in \{0, 1, 2, 4\}$,

$$b_{17} \left(5^{4k+4}n + \frac{1}{3} \left(5^{4k+3}(3r+1) - 2 \right) \right) \equiv 0 \pmod{17}. \quad (1.4)$$

Theorem 1.2. *For any $n \geq 0$, $k \geq 0$,*

$$b_{23} \left(5^{24k}n + \frac{11}{12} \left(5^{24k} - 1 \right) \right) \equiv 14^k b_{23}(n) \pmod{23}, \quad (1.5)$$

and for $r \in \{0, 1, 2, 3\}$,

$$b_{23} \left(5^{24k+24}n + \frac{1}{12} \left(5^{24k+23}(12r+7) - 11 \right) \right) \equiv 0 \pmod{23}. \quad (1.6)$$

Theorem 1.3. *For any $n \geq 0$, $k \geq 0$,*

$$b_{65} \left(5^{12k}n + \frac{8}{3} \left(5^{12k} - 1 \right) \right) \equiv 12^k b_{65}(n) \pmod{13}, \quad (1.7)$$

and for $r \in \{2, 4\}$,

$$b_{65} \left(5^{12k+12}n + \frac{1}{3} \left(5^{12k+11}(3r+1) - 8 \right) \right) \equiv 0 \pmod{13}. \quad (1.8)$$

2. Preliminaries

In this section we shall state a preliminary lemma.

Lemma 2.1. For

$$R(q) := \prod_{n=1}^{\infty} \frac{(1 - q^{5n-4})(1 - q^{5n-1})}{(1 - q^{5n-3})(1 - q^{5n-2})},$$

we have

A. ([Ram62] and [Wat38])

$$f_1 = f_{25} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right). \quad (2.9)$$

B. ([Wat29a, Wat29b])

$$\frac{1}{R(q)^5} - 11q - q^2 R(q)^5 = \frac{f_1^6}{f_5^6}. \quad (2.10)$$

C.

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6} \left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5 R(q^5) + 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4 \right). \quad (2.11)$$

Note that the identity in (2.11) can be easily established by first substituting q by q^5 in (2.10) and then using (2.9).

3. Congruences for 17-Regular Partitions

Proof of Theorem 1.1. Taking $l = 17$ in (1.1), we get

$$\sum_{n=0}^{\infty} b_{17}(n)q^n = \frac{f_{17}}{f_1}.$$

Now by (1.2),

$$\sum_{n=0}^{\infty} b_{17}(n)q^n \equiv f_1^{16} \pmod{17}. \quad (3.12)$$

Replacing f_1 using (2.9) and then extracting the terms involving q^{5n+1} , we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5n+1)q^{5n+1} &\equiv f_{25}^{16} \left(\frac{q}{R(q^5)^{15}} + \frac{13q^6}{R(q^5)^{10}} + \frac{8q^{11}}{R(q^5)^5} - q^{16} - 8q^{21} R(q^5)^5 \right. \\ &\quad \left. + 13q^{26} R(q^5)^{10} - q^{31} R(q^5)^{15} \right) \pmod{17}. \end{aligned}$$

If we divide by q and substitute q by $q^{1/5}$, then we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5n+1)q^n &\equiv f_5^{16} \left(\left(\frac{1}{R(q)^5} - 11q - q^2 R(q)^5 \right)^3 + 12q \left(\frac{1}{R(q)^5} - 11q - q^2 R(q)^5 \right)^2 \right. \\ &\quad \left. + 14q^2 \left(\frac{1}{R(q)^5} - 11q - q^2 R(q)^5 \right) + 7q^3 \right) \pmod{17}. \end{aligned}$$

By (2.10), we get

$$\sum_{n=0}^{\infty} b_{17}(5n+1)q^n \equiv f_5^{16} \left(\frac{f_1^{18}}{f_5^{18}} + 12q \frac{f_1^{12}}{f_5^{12}} + 14q^2 \frac{f_1^6}{f_5^6} + 7q^3 \right) \pmod{17}.$$

In view of (3.12), we get

$$\sum_{n=0}^{\infty} b_{17}(5n+1)q^n \equiv \frac{f_1^{18}}{f_5^2} + 12qf_1^{12}f_5^4 + 14q^2f_1^6f_5^{10} + 7\sum_{n=0}^{\infty} b_{17}(n)q^{5n+3} \pmod{17}. \quad (3.13)$$

Replacing f_1 using (2.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5n+1)q^n &\equiv \frac{f_1^{18}}{f_5^2} \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{18} + 12qf_5^4f_1^{12} \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{12} \\ &\quad + 14q^2f_5^{10}f_1^6 \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^6 + 7\sum_{n=0}^{\infty} b_{17}(n)q^{5n+3} \pmod{17}. \end{aligned}$$

If we extract the terms involving q^{5n+3} , divide by q^3 , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5^2n+16)q^n &\equiv \frac{f_5^{18}}{f_1^2} \left(q^3 \right) + 12f_1^4f_5^{12} \left(3\frac{f_1^{12}}{f_5^{12}} + 3q^2 \right) + 14f_1^{10}f_5^6 \left(11\frac{f_1^6}{f_5^6} + 9q \right) \\ &\quad + 7\sum_{n=0}^{\infty} b_{17}(n)q^n \pmod{17}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{17}(5^2n+16)q^n \equiv 7qf_1^{10}f_5^6 + 2q^2f_1^4f_5^{12} + q^3\frac{f_5^{18}}{f_1^2} + 10\sum_{n=0}^{\infty} b_{17}(n)q^n \pmod{17}.$$

Replacing f_1 and $1/f_1$ using (2.9) and (2.11), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5^2n+16)q^n &\equiv 7qf_5^6f_1^{10} \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{10} + 2q^2f_5^{12}f_1^4 \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^4 \\ &\quad + q^3f_5^6f_1^{10} \left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5R(q^5) \right. \\ &\quad \left. + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4 \right)^2 + 10\sum_{n=0}^{\infty} b_{17}(n)q^n \pmod{17}. \end{aligned}$$

If we extract the terms involving q^{5n+1} , divide by q , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(5^3n+41)q^n &\equiv 7f_1^6f_5^{10} \left(\frac{f_1^{12}}{f_5^{12}} + 12q\frac{f_1^6}{f_5^6} + q^2 \right) + 2f_1^{12}f_5^4 \left(12q \right) + f_1^6f_5^{10} \left(10q\frac{f_1^6}{f_5^6} + 6q^2 \right) \\ &\quad + 10\sum_{n=0}^{\infty} b_{17}(5n+1)q^n \pmod{17}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{17}(5^3n+41)q^n \equiv 7\frac{f_1^{18}}{f_5^2} + 16qf_1^{12}f_5^4 + 13q^2f_1^6f_5^{10} + 10\sum_{n=0}^{\infty} b_{17}(5n+1)q^n \pmod{17}.$$

Now by (3.13), we get

$$\sum_{n=0}^{\infty} b_{17}(5^3n+41)q^n \equiv 2\sum_{n=0}^{\infty} b_{17}(n)q^{5n+3} \pmod{17}. \quad (3.14)$$

Comparing the coefficients of the terms of the form q^{5n+3} in (3.14), we can conclude that for any $n \geq 0$,

$$b_{17}(5^4n + 416) \equiv 2b_{17}(n) \pmod{17},$$

or,

$$b_{17}\left(5^4n + \frac{2}{3}(5^4 - 1)\right) \equiv 2b_{17}(n) \pmod{17}. \quad (3.15)$$

Now (1.3) follows from (3.15), by mathematical induction.

On the other hand, comparing the coefficients of the terms of the form q^{5n+r} where $r \in \{0, 1, 2, 4\}$ in (3.14), we can conclude that for any $n \geq 0$,

$$b_{17}\left(5^3(5n + r) + \frac{5^3 - 2}{3}\right) \equiv 0 \pmod{17},$$

or,

$$b_{17}\left(5^4n + \frac{1}{3}(5^3(3r + 1) - 2)\right) \equiv 0 \pmod{17}. \quad (3.16)$$

Now (1.4) follows from (1.3) and (3.16).

4. Congruences for 23-Regular Partitions

Proof of Theorem 1.2. Taking $l = 23$ in (1.1), we get

$$\sum_{n=0}^{\infty} b_{23}(n)q^n = \frac{f_{23}}{f_1}.$$

Now by (1.2),

$$\sum_{n=0}^{\infty} b_{23}(n)q^n \equiv f_1^{22} \pmod{23}. \quad (4.17)$$

Replacing f_1 using (2.9) and then extracting the terms involving q^{5n+2} , we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5n + 2)q^{5n+2} &\equiv f_5^{22} \left(\frac{2q^2}{R(q^5)^{20}} + \frac{21q^7}{R(q^5)^{15}} + \frac{3q^{12}}{R(q^5)^{10}} + \frac{8q^{17}}{R(q^5)^5} - q^{22} - 8q^{27}R(q^5)^5 \right. \\ &\quad \left. + 3q^{32}R(q^5)^{10} - 21q^{37}R(q^5)^{15} + 2q^{42}R(q^5)^{20} \right) \pmod{23}. \end{aligned}$$

If we divide by q^2 and substitute q by $q^{1/5}$, then we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5n + 2)q^n &\equiv f_5^{22} \left(2 \left(\frac{1}{R(q)^5} - 11q - q^2R(q)^5 \right)^4 + 17q \left(\frac{1}{R(q)^5} - 11q - q^2R(q)^5 \right)^3 \right. \\ &\quad \left. + 17q^2 \left(\frac{1}{R(q)^5} - 11q - q^2R(q)^5 \right)^2 + 14q^4 \right) \pmod{23}. \end{aligned}$$

By (2.10), we get

$$\sum_{n=0}^{\infty} b_{23}(5n + 2)q^n \equiv f_5^{22} \left(2 \frac{f_1^{24}}{f_5^{24}} + 17q \frac{f_1^{18}}{f_5^{18}} + 17q^2 \frac{f_1^{12}}{f_5^{12}} + 14q^4 \right) \pmod{23}.$$

In view of (4.17), we get

$$\sum_{n=0}^{\infty} b_{23}(5n+2)q^n \equiv 2\frac{f_1^{24}}{f_5^2} + 17qf_1^{18}f_5^4 + 17q^2f_1^{12}f_5^{10} + 14\sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}. \quad (4.18)$$

Replacing f_1 using (2.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5n+2)q^n &\equiv 2\frac{f_1^{24}}{f_5^2} \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{24} + 17qf_5^4f_1^{18} \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{18} \\ &\quad + 17q^2f_5^{10}f_1^{12} \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{12} + 14\sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}. \end{aligned}$$

If we extract the terms involving q^{5n+4} , divide by q^4 , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5^2n+22)q^n &\equiv 2\frac{f_5^{24}}{f_1^2} \left(q^4 \right) + 17f_1^4f_5^{18} \left(19\frac{f_1^{18}}{f_5^{18}} + 7q^3 \right) + 17f_1^{10}f_5^{12} \left(8\frac{f_1^{12}}{f_5^{12}} + 3q^2 \right) \\ &\quad + 14\sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23}(5^2n+22)q^n \equiv 5q^2f_1^{10}f_5^{12} + 4q^3f_1^4f_5^{18} + 2q^4\frac{f_5^{24}}{f_1^2} + 13\sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}. \quad (4.19)$$

Replacing f_1 and $1/f_1$ using (2.9) and (2.11), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5^2n+22)q^n &\equiv 5q^2f_5^{12}f_1^{10} \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^{10} + 4q^3f_5^{18}f_1^4 \left(\frac{1}{R(q^5)} - q - q^2R(q^5) \right)^4 \\ &\quad + 2q^4f_5^{12}f_1^{10} \left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5R(q^5) \right. \\ &\quad \left. + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4 \right)^2 + 13\sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}. \end{aligned}$$

If we extract the terms involving q^{5n+2} , divide by q^2 , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5^3n+72)q^n &\equiv 5f_1^{12}f_5^{10} \left(\frac{f_1^{12}}{f_5^{12}} + 20q\frac{f_1^6}{f_5^6} + 16q^2 \right) + 4f_1^{18}f_5^4 \left(18q \right) + 2f_1^{12}f_5^{10} \left(10q\frac{f_1^6}{f_5^6} + 10q^2 \right) \\ &\quad + 13\sum_{n=0}^{\infty} b_{23}(5n+2)q^n \pmod{23}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23}(5^3n+72)q^n \equiv 5\frac{f_1^{24}}{f_5^2} + 8qf_1^{18}f_5^4 + 8q^2f_1^{12}f_5^{10} + 13\sum_{n=0}^{\infty} b_{23}(5n+2)q^n \pmod{23}.$$

Now by (4.18), we get

$$\sum_{n=0}^{\infty} b_{23}(5^3n+72)q^n \equiv 8\frac{f_1^{24}}{f_5^2} + 22qf_1^{18}f_5^4 + 22q^2f_1^{12}f_5^{10} + 21\sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}.$$

Replacing f_1 using (2.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5^3 n + 72)q^n &\equiv 8 \frac{f_5^{24}}{f_5^2} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{24} + 22q f_5^4 f_5^{18} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{18} \\ &\quad + 22q^2 f_5^{10} f_5^{12} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{12} + 21 \sum_{n=0}^{\infty} b_{23}(n)q^{5n+4} \pmod{23}. \end{aligned}$$

If we extract the terms involving q^{5n+4} , divide by q^4 , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23}(5^4 n + 572)q^n &\equiv 8 \frac{f_5^{24}}{f_1^2} \left(q^4 \right) + 22 f_1^4 f_5^{18} \left(19 \frac{f_1^{18}}{f_5^{18}} + 7q^3 \right) + 22 f_1^{10} f_5^{12} \left(8 \frac{f_1^{12}}{f_5^{12}} + 3q^2 \right) \\ &\quad + 21 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23}(5^4 n + 572)q^n \equiv 20q^2 f_1^{10} f_5^{12} + 16q^3 f_1^4 f_5^{18} + 8q^4 \frac{f_5^{24}}{f_1^2} + 17 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Now by (4.19), we get

$$\sum_{n=0}^{\infty} b_{23}(5^4 n + 572)q^n \equiv 4 \sum_{n=0}^{\infty} b_{23}(5^2 n + 22)q^n + 11 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23},$$

or,

$$\sum_{n=0}^{\infty} b_{23} \left(5^4 n + \frac{11}{12} (5^4 - 1) \right) q^n \equiv 4 \sum_{n=0}^{\infty} b_{23}(5^2 n + 22)q^n + 11 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Substituting n by $5^2 n + 22$, we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23} \left(5^6 n + \frac{11}{12} (5^6 - 1) \right) q^n &\equiv 4 \sum_{n=0}^{\infty} b_{23} \left(5^4 n + \frac{11}{12} (5^4 - 1) \right) q^n + 11 \sum_{n=0}^{\infty} b_{23}(5^2 n + 22)q^n \pmod{23}, \\ &\equiv 4 \left(4 \sum_{n=0}^{\infty} b_{23}(5^2 n + 22)q^n + 11 \sum_{n=0}^{\infty} b_{23}(n)q^n \right) + 11 \sum_{n=0}^{\infty} b_{23}(5^2 n + 22)q^n \pmod{23}, \\ &\equiv 4 \sum_{n=0}^{\infty} b_{23}(5^2 n + 22)q^n + 21 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}. \end{aligned}$$

Repeating the same process as above, it follows that

$$\sum_{n=0}^{\infty} b_{23} \left(5^{22} n + \frac{11}{12} (5^{22} - 1) \right) q^n \equiv 18 \sum_{n=0}^{\infty} b_{23}(5^2 n + 22)q^n + 20 \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Now by (4.19), we get

$$\sum_{n=0}^{\infty} b_{23} \left(5^{22} n + \frac{11}{12} (5^{22} - 1) \right) q^n \equiv 21q^2 f_1^{10} f_5^{12} + 3q^3 f_1^4 f_5^{18} + 13q^4 \frac{f_5^{24}}{f_1^2} + \sum_{n=0}^{\infty} b_{23}(n)q^n \pmod{23}.$$

Replacing f_1 and $1/f_1$ using (2.9) and (2.11), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23} \left(5^{22}n + \frac{11}{12} (5^{22} - 1) \right) q^n \\ \equiv 21q^2 f_5^{12} f_{25}^{10} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{10} + 3q^3 f_5^{18} f_{25}^4 \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^4 \\ + 13q^4 f_5^{12} f_{25}^{10} \left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5 R(q^5) + 2q^6 R(q^5)^2 \right. \\ \left. - q^7 R(q^5)^3 + q^8 R(q^5)^4 \right)^2 + \sum_{n=0}^{\infty} b_{23}(n) q^n \pmod{23}. \end{aligned}$$

If we extract the terms involving q^{5n+2} , divide by q^2 , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{23} \left(5^{23}n + \frac{7 \cdot 5^{23} - 11}{12} \right) q^n \equiv 21f_1^{12} f_5^{10} \left(\frac{f_1^{12}}{f_5^{12}} + 20q \frac{f_1^6}{f_5^6} + 16q^2 \right) + 3f_1^{18} f_5^4 (18q) \\ + 13f_1^{12} f_5^{10} \left(10q \frac{f_1^6}{f_5^6} + 10q^2 \right) + \sum_{n=0}^{\infty} b_{23}(5n+2) q^n \pmod{23}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{23} \left(5^{23}n + \frac{7 \cdot 5^{23} - 11}{12} \right) q^n \equiv 21 \frac{f_1^{24}}{f_5^2} + 6q f_1^{18} f_5^4 + 6q^2 f_1^{12} f_5^{10} + \sum_{n=0}^{\infty} b_{23}(5n+2) q^n \pmod{23}.$$

Now by (4.18), we get

$$\sum_{n=0}^{\infty} b_{23} \left(5^{23}n + \frac{7 \cdot 5^{23} - 11}{12} \right) q^n \equiv 14 \sum_{n=0}^{\infty} b_{23}(n) q^{5n+4} \pmod{23}. \quad (4.20)$$

Comparing the coefficients of the terms of the form q^{5n+4} in (4.20), we can conclude that for any $n \geq 0$,

$$b_{23} \left(5^{24}n + \frac{11}{12} (5^{24} - 1) \right) \equiv 14b_{23}(n) \pmod{23}. \quad (4.21)$$

Now (1.5) follows from (4.21), by mathematical induction.

On the other hand, comparing the coefficients of the terms of the form q^{5n+r} where $r \in \{0, 1, 2, 3\}$ in (4.20), we can conclude that for any $n \geq 0$,

$$b_{23} \left(5^{23}(5n+r) + \frac{7 \cdot 5^{23} - 11}{12} \right) \equiv 0 \pmod{23},$$

or,

$$b_{23} \left(5^{24}n + \frac{1}{12} (5^{23}(12r+7) - 11) \right) \equiv 0 \pmod{23}. \quad (4.22)$$

Now (1.6) follows from (1.5) and (4.22).

5. Congruences for 65-Regular Partition

Proof of Theorem 1.3. Taking $l = 65$ in (1.1), we get

$$\sum_{n=0}^{\infty} b_{65}(n)q^n = \frac{f_{65}}{f_1}. \quad (5.23)$$

Replacing $1/f_1$ using (2.11), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{65}(n)q^n &= f_{65} \cdot \frac{f_{25}^5}{f_5^6} \left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 \right. \\ &\quad \left. - 3q^5 R(q^5) + 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4 \right). \end{aligned}$$

If we extract the terms involving q^{5n+4} , divide by q^4 , and substitute q by $q^{1/5}$, then by (2.10), we get

$$\sum_{n=0}^{\infty} b_{65}(5n+4)q^n = 5f_{13} \cdot \frac{f_5^5}{f_1^6}.$$

Now by (1.2),

$$\sum_{n=0}^{\infty} b_{65}(5n+4)q^n \equiv 5f_1^7 f_5^5 \pmod{13}.$$

Replacing f_1 using (2.9), we get

$$\sum_{n=0}^{\infty} b_{65}(5n+4)q^n \equiv 5f_5^5 f_{25}^7 \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^7 \pmod{13}.$$

If we extract the terms involving q^{5n+2} , divide by q^2 , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{65}(5^2 n + 14)q^n \equiv 5f_1^5 f_5^7 \left(\frac{f_1^6}{f_5^6} + 8q \right) \equiv 5f_1^{11} f_5 + qf_1^5 f_5^7 \pmod{13}.$$

Replacing f_1 using (2.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{65}(5^2 n + 14)q^n &\equiv 5f_5 f_{25}^{11} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{11} \\ &\quad + qf_5^7 f_{25}^5 \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^5 \pmod{13}. \end{aligned}$$

If we extract the terms involving q^{5n+1} , divide by q , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{65}(5^3 n + 39)q^n \equiv 5f_1 f_5^{11} \left(2\frac{f_1^{12}}{f_5^{12}} + 11q\frac{f_1^6}{f_5^6} + 8q^2 \right) + f_1^7 f_5^5 \left(\frac{f_1^6}{f_5^6} \right) \pmod{13},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{65}(5^3 n + 39)q^n \equiv 11\frac{f_1^{13}}{f_5} + 3qf_1^7 f_5^5 + q^2 f_1 f_5^{11} \pmod{13},$$

or,

$$\sum_{n=0}^{\infty} b_{65} \left(5^3 n + \frac{5^3 - 8}{3} \right) q^n \equiv 11 \frac{f_1^{13}}{f_5} + 3q f_1^7 f_5^5 + q^2 f_1 f_5^{11} \pmod{13}. \quad (5.24)$$

Replacing f_1 using (2.9), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{65} \left(5^3 n + \frac{5^3 - 8}{3} \right) q^n &\equiv 11 \frac{f_{25}^{13}}{f_5} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{13} + 3q f_5^5 f_{25}^7 \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^7 \\ &\quad + q^2 f_5^{11} f_{25} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right) \pmod{13}. \end{aligned}$$

If we extract the terms involving q^{5n+3} , divide by q^3 , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\sum_{n=0}^{\infty} b_{65} \left(5^4 n + \frac{2 \cdot 5^4 - 8}{3} \right) q^n \equiv 11 \frac{f_5^{13}}{f_1} \left(12q^2 \right) + 3f_1^5 f_5^7 \left(\frac{f_1^6}{f_5^6} + 8q \right) + f_1^{11} f_5 \left(12 \right) \pmod{13},$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{65} \left(5^4 n + \frac{2 \cdot 5^4 - 8}{3} \right) q^n \equiv 2f_1^{11} f_5 + 11q f_1^5 f_5^7 + 2q^2 \frac{f_5^{13}}{f_1} \pmod{13}.$$

Replacing f_1 and $1/f_1$ using (2.9) and (2.11), we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{65} \left(5^4 n + \frac{2 \cdot 5^4 - 8}{3} \right) q^n &\equiv 2f_5 f_{25}^{11} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^{11} + 11q f_5^7 f_{25}^5 \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right)^5 \\ &\quad + 2q^2 f_5^7 f_{25}^5 \left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5 R(q^5) + 2q^6 R(q^5)^2 \right. \\ &\quad \left. - q^7 R(q^5)^3 + q^8 R(q^5)^4 \right) \pmod{13}. \end{aligned}$$

If we extract the terms involving q^{5n+1} , divide by q , and substitute q by $q^{1/5}$, then by (2.10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} b_{65} \left(5^5 n + \frac{5^5 - 8}{3} \right) q^n &\equiv 2f_1 f_5^{11} \left(2 \frac{f_1^{12}}{f_5^{12}} + 11q \frac{f_1^6}{f_5^6} + 8q^2 \right) + 11f_1^7 f_5^5 \left(\frac{f_1^6}{f_5^6} \right) \\ &\quad + 2f_1^7 f_5^5 \left(5q \right) \pmod{13}, \end{aligned}$$

which, after rearranging the similar terms, yields

$$\sum_{n=0}^{\infty} b_{65} \left(5^5 n + \frac{5^5 - 8}{3} \right) q^n \equiv 2 \frac{f_1^{13}}{f_5} + 6q f_1^7 f_5^5 + 2q^2 f_1 f_5^{11} \pmod{13}. \quad (5.25)$$

Repeating the same process as above to derive (5.25) from (5.24), it follows that

$$\sum_{n=0}^{\infty} b_{65} \left(5^{11} n + \frac{5^{11} - 8}{3} \right) q^n \equiv \frac{f_1^{13}}{f_5} \equiv \frac{f_{13}}{f_5} \pmod{13}. \quad (5.26)$$

In (2.9), if we substitute q by q^{13} , then we get

$$f_{13} = f_{325} \left(\frac{1}{R(q^{65})} - q^{13} - q^{26} R(q^{65}) \right).$$

Replacing f_{13} in (5.26), we get

$$\sum_{n=0}^{\infty} b_{65} \left(5^{11}n + \frac{5^{11} - 8}{3} \right) q^n \equiv \frac{f_{325}}{f_5} \left(\frac{1}{R(q^{65})} - q^{13} - q^{26} R(q^{65}) \right) \pmod{13}. \quad (5.27)$$

If we extract the terms involving q^{5n+3} , divide by q^3 , and substitute q by $q^{1/5}$, then we get

$$\sum_{n=0}^{\infty} b_{65} \left(5^{12}n + \frac{2 \cdot 5^{12} - 8}{3} \right) q^n \equiv 12q^2 \frac{f_{65}}{f_1} \pmod{13},$$

which, by (5.23), yields

$$\sum_{n=0}^{\infty} b_{65} \left(5^{12}n + \frac{2 \cdot 5^{12} - 8}{3} \right) q^n \equiv 12 \sum_{n=0}^{\infty} b_{65}(n) q^{n+2} \pmod{13}. \quad (5.28)$$

Comparing the coefficients of the terms of the form q^{n+2} in (5.28), we can conclude that for any $n \geq 0$,

$$b_{65} \left(5^{12}n + \frac{8}{3} (5^{12} - 1) \right) \equiv 12b_{65}(n) \pmod{13}. \quad (5.29)$$

Now (1.7) follows from (5.29), by mathematical induction.

On the other hand, comparing the coefficients of the terms of the form q^{5n+r} where $r \in \{2, 4\}$ in (5.27), we can conclude that for any $n \geq 0$,

$$b_{65} \left(5^{11}(5n+r) + \frac{5^{11} - 8}{3} \right) \equiv 0 \pmod{13},$$

or,

$$b_{65} \left(5^{12}n + \frac{1}{3} (5^{11}(3r+1) - 8) \right) \equiv 0 \pmod{13}. \quad (5.30)$$

Now (1.8) follows from (1.7) and (5.30).

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