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REANALYSIS OF NONLINEAR STRUCTURES BY A REDUCTION METHOD OF COMBINED APPROXIMATIONS

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Abstract

The aim of reanalysis methods is to approximate the responses of a structure whose parameters have been perturbed or even modified without solving the new equilibrium equation system associated to the updated structure: only the initial solutions and the perturbed data are used. In the particular case of non-linear problems, the re-actualization of the tangent stiffness matrix at each time step of the Newton-Raphson integration algorithm implies many reanalysis leading to a high computational time. To mitigate these difficulties, one proposes a reduction method adapted to non-linear and large size dynamic models. This study especially focuses on geometrical non-linearities, i.e. large displacements. The presented reduction method is based on the combined approximations method (CA method) introduced by Kirsch.

Keywords: geometrical non-linearities, large displacements, structural reanalysis, combined approximations, robustness.

1 Introduction

In nonlinear dynamic analysis the solution of the differential equations governing the motion of the system are carried out by time iteration algorithms [1]. Generally, the main computational cost concerns the calculation of the updated tangent stiffness matrix. The effort required for solving the associated linear equation sets at each iteration cycle can then quickly become prohibitive, especially when dealing with large scale structures.

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Thus, approximated reanalysis methods can be considered as an interesting alternative [2, 3]. These are intended to efficiently analyze the modified system without solving the complete set of modified equations: the initial state and the modification data are the only information needed. It has been shown in the literature that, in the case of localized nonlinearities, a Ritz basis reduction method can achieve accurate results when enriched by static residual vectors induced by modification forces [2, 3].

However, when dealing with generalized nonlinearities, these methods remain unable to predict the nonlinear behaviour of the structure. To overcome this difficulty a variant of the combined approximations method [4], initially introduced by Kirsch [5], has been extended to nonlinear analysis.

In this paper, the large displacements behaviour of the structure is studied. Its responses are computed in the time domain by means of a classical iterative method. The accuracy of the proposed method are illustrated by three academic examples and compared to the results found in the literature.

2 Large displacements nonlinear formulation

This section reviews the theory of finite element for geometrically non-linear elastic structures (GNS) based on the total lagrangian formulation [6], which constructs the tangent stiffness matrix with respect to the initial configuration. One should notice that the updated lagrangian is another existing formulation, which is derived from the current configuration and does not include the initial displacement matrix K_U [7].

According to [1] and [8-10], the total lagrangian formulation can be described as follows. At first, the displacements u of structures are given by the product

$$u = N \cdot U \quad (1)$$

where N is the vector of shape functions and U the vector of nodal displacements. Because of large displacements and rotations, Green's strain is adopted for the nonlinear relationships between strains and displacements. The Green's strain ε_G includes both linear and nonlinear terms, respectively B_L and $B_{NL}(U)$, relating strain and nonlinear strain to the nodal displacements:

$$\varepsilon_G = (B_L + B_{NL}(U)) U \quad (2)$$

Using the principle of virtual displacement, the virtual work δW is given by,

$$\delta W = \int_{V_0} (\sigma_S \delta \varepsilon_G) dV_0 - q_{ext} \delta U \quad (3)$$

where σ_S is the second Piola-Kirchhoff stress, $\delta\varepsilon_G$ the incremental form of the strain-displacement relationship, V_0 the volume of initial configuration and q_{ext} the vector of external loads. Since Green's strain is based on a small strain, stress can be given by Hook's law:

$$\sigma_S = E \varepsilon_G = E (B_L + B_{NL}(U)) U \quad (4)$$

where E is the modulus of elasticity. Substituting $\delta\varepsilon_G$ from Eq. (2) into Eq. (3) results in,

$$\delta W = \left[\int_{V_0} (\sigma_S (B_L + B_{NL}(U))) dV_0 - q_{ext} \right] \delta U \quad (5)$$

Since δU is arbitrary, the vector of internal forces q_{int} is,

$$q_{int} = \int_{V_0} (\sigma_S (B_L + B_{NL}(U))) dV_0 \quad (6)$$

Taking the derivative of q_{int} with respect to the nodal displacements U gives the tangent stiffness matrix K_T ,

$$K_T = \frac{\partial q_{int}}{\partial U} = \int_{V_0} \left(\frac{\partial \sigma_S}{\partial U} (B_L + B_{NL}(U)) + \sigma_S \frac{\partial B_{NL}(U)}{\partial U} \right) dV_0 \quad (7)$$

By substituting Eq. (4) into Eq. (7) K_T can be rewritten,

$$\begin{aligned} K_T &= \int_{V_0} E B_L B_L dV_0 \\ &+ \int_{V_0} \sigma_S \frac{\partial B_{NL}(U)}{\partial U} dV_0 \\ &+ \int_{V_0} E (B_L B_{NL}(U) + B_{NL}(U) B_L + B_{NL}(U) B_{NL}(U)) dV_0 \end{aligned} \quad (8)$$

It can be noticed that the three terms in equation (8) respectively stand for: the elastic stiffness matrix K_E , the geometric stiffness matrix K_G , and the initial displacement stiffness matrix K_U [1, 8].

3 Nonlinear time integration algorithm

A mechanical system can be represented in the time domain by the differential equation

$$M \ddot{u}(t) + C \dot{u}(t) + f_{NL}(t) = p(t) \quad (9)$$

with initial conditions,

$$u(t_0) = u_0; \quad \dot{u}(t_0) = \dot{u}_0 \quad (10)$$

where M and C stand for the mass and damping matrices of the system, $f_{NL} = f_{NL}(u, \dot{u})$ its restoring force given as a function of the displacement u and the velocity \dot{u} , p the exciting force.

The solution of equation (9) can be approximated in a set of points t_0, t_1, \dots, t_n using the assumption that the velocity and displacement at t_{i+1} can be expressed as functions of $u_i = u(t_i)$, $\dot{u}_i = \dot{u}(t_i)$, $\ddot{u}_i = \ddot{u}(t_i)$ and \ddot{u}_{i+1} , leading to the following relations:

$$\dot{u}_{i+1} = \dot{u}_i + (1 - \gamma) \Delta t \ddot{u}_i + \gamma \Delta t \ddot{u}_{i+1} \quad (11)$$

$$u_{i+1} = u_i + \Delta t \dot{u}_i + (0.5 - \beta) \Delta t^2 \ddot{u}_i + \beta \Delta t^2 \ddot{u}_{i+1} \quad (12)$$

where $\Delta t = t_i - t_{i-1}$

The parameters β and γ define the variation of the acceleration over a time interval $[t_i, t_{i+1}]$ and determine the stability and accuracy characteristics of the method. The average acceleration method used in this study implies: $\beta = 1/4$ and $\gamma = 1/2$.

Equations (11) and (12) can also be rewritten considering incremental quantities:

$$\Delta u_i = u_{i+1} - u_i; \quad \Delta \dot{u}_i = \dot{u}_{i+1} - \dot{u}_i; \quad \Delta \ddot{u}_i = \ddot{u}_{i+1} - \ddot{u}_i; \quad \Delta p_i = p_{i+1} - p_i \quad (13)$$

Then, the incremental equation of motion is given by,

$$M \Delta \ddot{u}_i + C \Delta \dot{u}_i + (f_{NL}(t+1) - f_{NL}(t)) = \Delta p_i \quad (14)$$

and using the approximation

$$f_{NL}(t+1) - f_{NL}(t) \approx K_{Ti} \Delta u_i \quad (15)$$

we finally obtain the equation to solve to compute the incremental displacement Δu_i :

$$\hat{K} \Delta u_i \approx \Delta \hat{p}_i \quad (16)$$

where $K_{Ti} = \partial f_{NL} / \partial u_i^T$ is the instantaneous (tangent) stiffness matrix, calculated as the Jacobian of the restoring force,

$$\hat{K} = K_{Ti} + \frac{\gamma}{\beta \Delta t} C + \frac{1}{\beta \Delta t^2} M \quad (17)$$

$$\Delta \hat{p}_i = \Delta p_i + \left(\frac{1}{\beta \Delta t} M + \frac{\gamma}{\beta} C \right) \dot{u}_i + \left[\frac{1}{2\beta} M + \Delta t \left(\frac{\gamma}{2\beta} - 1 \right) C \right] \ddot{u}_i \quad (18)$$

One should observed that the incremental restoring force, given equation (15), has to be rigorously calculated using the following relation

$$f_{NL}(t_{i+1}) - f_{NL}(t_i) = K_{sec} \Delta u_i \quad (19)$$

The error induced by substituting the tangent stiffness matrix to the secant stiffness one can corrected by employing an iterative Newton-Raphson scheme [11].

Comparison criteria of the time responses

In order to quantify the obtained predictions, the temporal moments are used as a comparison criterion. These temporal moments $M_i(t_s)$ have been proposed for the transitory analysis [12]. They are similar to the static moments and are calculated as balanced summations of the quadratic temporal signal:

$$M_i(t_s) = \int_{-\infty}^{+\infty} (t - t_s)^i (u(t))^2 dt \quad (20)$$

where t_s corresponds to a temporal shift and the index i represents the order of the moment. For more simplicity, the temporal moments M_i are defined for $t_s = 0$. The central moments are thus defined as follows:

$$\begin{aligned} E &= M_0 && \text{energy(m}^2\text{s)} \\ T &= \frac{M_1}{M_0} && \text{central time "Centroid" (s)} \\ D^2 &= \frac{M_2}{M_0} - \left(\frac{M_1}{M_0} \right)^2 && \text{rms duration (s)} \end{aligned} \quad (21)$$

4 Approximated reanalysis method

Approximated reanalysis methods can broadly be classified in two categories [13]. Local approximations, such as first-order Taylor series expansion or the binomial

series expansion about a given design point, are based on information calculated for a single design. These methods are efficient but they are effective only in cases of small changes in the structure. Global approximations, such as polynomial fitting, response surfaces or reduced basis [2, 3], are obtained by analyzing the structure at a number of design points and are valid on the whole design space. However they require more computational effort, especially in the case of large scale finite element models.

To be able to compare the results carried out in the last section of the article, the method exposed by Masson in [2] is briefly described in what follows. It consists in enriching an initial Ritz basis by a set of vectors of minimal rank, obtained by approximating the static response of the truncated modes. This is achieved by taking into account a priori information concerning the modifications applied to the initial structure by means of static residuals generated by the forces associated to structural modifications.

The equilibrium equation of the modified substructure is given by:

$$\left[Z_0(\omega) + \Delta Z(\omega) \right] u(\omega) = 0 \quad (22)$$

Where the initial dynamic stiffness matrix defined by $Z_0(\omega) = K_0 - \omega^2 M_0$ and the modified one by $\Delta Z(\omega) = \Delta K - \omega^2 \Delta M$

One introduces the notion of force associated to structural modifications by:

$$f_\Delta(\omega) = -\Delta Z(\omega) u(\omega) \quad (23)$$

As the response $u(\omega)$ of the modified structure is unknown, it can be approximated by a standard truncated component modes synthesis transformation matrix T_0 , determined from the initial structure, enriched by static residuals $R[f_\Delta(\omega)]$, such as:

$$u(\omega) \simeq T_0 c(\omega) + R[f_\Delta(\omega)] \quad (24)$$

These residual vectors are derived from a force basis spanning the subspace associated to the whole set of modifications. The static residual matrix is then defined by:

$$R_\Delta^* = K^{-1} F_\Delta \quad (25)$$

where F_Δ is the modification force basis. It must be noticed that, for localized nonlinearities, this basis can be determined by applying unitary forces to the nonlinear degrees of freedom.

Singular value decomposition of the global transformation matrix is finally performed to ensure a minimal rank and an optimal condition number of the reduced basis

$$T = [T_0 \quad R_\Delta] \quad (26)$$

In other works, Balmès [3] proposed a similar formulation in which the residuals are updated by an iterative procedure.

5 Model reduction by a variant of the combined approximations method

The Combined Approximations (CA) method developed by Kirsch [5] uses the terms of the local binomial series expansion to compute the vectors of a global reduced basis. For each parametric modification of the initial structure, a new eigenproblem must be solved.

The normal mode ν of the modified structure verifies the equilibrium relation given by:

$$(K_0 + \Delta K)r^{(\nu)} = \lambda^{(\nu)}(M_0 + \Delta M)r^{(\nu)} \quad (27)$$

An approximated solution can then be expressed by the following binomial expansion:

$$r^{(\nu)} \simeq (1 + B)^{-1} r_0^{(\nu)} \quad (28)$$

depending on the modifications performed on the structure, $B = K_0^{-1} \Delta K$, and its initial behaviour $r_0^{(\nu)}$.

For nonlinear reanalysis, the changes in the stiffness matrix ΔK result from the updating of the tangent stiffness matrix, given by:

$$\Delta K = K_T - K_0 \quad (29)$$

where K_T is the tangent stiffness matrix defined, in the case of large displacement, by equation (8).

The reduction basis is then constructed using the following recurrence relation:

$$\begin{cases} r_1^{(\nu)} = K_0^{-1} (M_0 + \Delta M) r_0^{(\nu)} \\ r_i^{(\nu)} = -B r_{i-1}^{(\nu)} \quad (i = 2, 3, \dots, s) \end{cases} \quad (30)$$

and can be finally be written:

$$r_B^{(\nu)} = \begin{bmatrix} r_1^{(\nu)} & r_2^{(\nu)} & \cdots & r_s^{(\nu)} \end{bmatrix} \quad (31)$$

Applying the CA method to complex structures has enlightened sever limitations and convergence problems of the binomial series expansion. This results in less predictive approximated solutions, especially in the case of large modification and nonlinear structures. The variant of the method proposed in this article consists in retaining only the most relevant part of the information contained in the reduction basis of each considered mode. It amounts to extracting the subspace that best spans the solution space [4, 14].

At first, a global transformation matrix is obtained by concatenation of the reduction basis $r_B^{(\nu)}$ associated to m studied normal modes,

$$r_B^* = \begin{bmatrix} r_B^1 & r_B^2 & \cdots & r_B^{(\nu)} & \cdots & r_B^{(m)} \end{bmatrix} \quad (32)$$

Singular value decomposition is then performed:

$$\begin{aligned} r_B^* &= U \Sigma V^T \\ &= U_1 \Sigma_1 V_1^T + U_2 \Sigma_2 V_2^T \end{aligned} \quad (33)$$

The robust Ritz basis, with regard to the parametric modifications, is finally constructed with the column vectors of matrix U_1 ,

$$r_B = U_1 \quad (34)$$

It must be noticed that the proposed CA variant has been successfully extended to linear dynamic substructuring in order to perform structural reanalysis on large size finite element models [14].

6 Numerical simulations

Example 1

To illustrate the proposed method, one considers a damped nonlinear array of spring-mass systems. It consists of a ten degree-of-freedom (dof) damped spring-mass system with localized nonlinear Duffing type oscillator. Its physical parameters are: $m_i = 100$ kg, $k_i = 40 \cdot 10^6$ N/m, $c_i = 2 \xi_i \sqrt{m_i k_i}$ and $\xi_i = 0.04$, for $i = 1, 2, \dots, 10$. The governing differential equation of motion is given by:

$$M \ddot{u}(t) + C \dot{u}(t) + \bar{K}(u(t)) u(t) = p(t) \quad (35)$$

with initial conditions: $u_0(t) = 0$, $\dot{u}_0(t) = 0$

The assembled global stiffness matrix is expressed as:

$$\bar{K}(u(t)) = \begin{pmatrix} \bar{k}_1 + \bar{k}_2 & -\bar{k}_2 & 0 & \cdots & 0 \\ -\bar{k}_2 & \bar{k}_2 + \bar{k}_3 & -\bar{k}_3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & 0 & -\bar{k}_{10} & \bar{k}_{10} \end{pmatrix} \quad (36)$$

with: $\bar{k}_i = k_i \left(1 + \alpha (u_i(t) - u_{i-1}(t))^2\right)$, for $i = 1, 2, \dots, 10$, where α and $\delta u_i(t) = u_i(t) - u_{i-1}(t)$ respectively represent the nonlinear stiffness coefficient and the relative displacements between adjacent degrees of freedom. In this example, one considers 2 localized stiffness nonlinearities at dof 5 and 10, with associated coefficient $\alpha = 1e7$. The structure is excited by a harmonic force $p(t) = 1e4 \times \cos(300t)$.

The response of the system is computed over a time interval [0- T=2s] with a sampling step fixed at $\Delta t = 5e-4$ s.

Firstly, the comparison method described in section 4 (further referred to as LMA) is applied. The initial model of the system, containing 10 dofs, is reduced to a condensed one with 4 dofs by retaining 2 eigenmodes and 2 static residual vectors and the transformation matrix (noted LMA (2, 2)). Secondly, the variant of the CA method is implemented by reanalysing the first 2 normal modes with 3 vectors in the reduced basis (noted CA (2, 3)) associated to each mode (see equation (31)). After concatenation and singular value decomposition of the global basis, the robust reduced model becomes of size 4.

Table 1 recalls the size of the problems to be solved and gives the values of the associated energy criterion, defined equation (21). Figure 1 illustrates the results obtained by an exact reanalysis, compared to the implemented reduction methods. Figure 1 compares the nonlinear dynamic responses of the full model (reference) and the 4 dof condensed model by the LMA method or the alternative of the method CA.

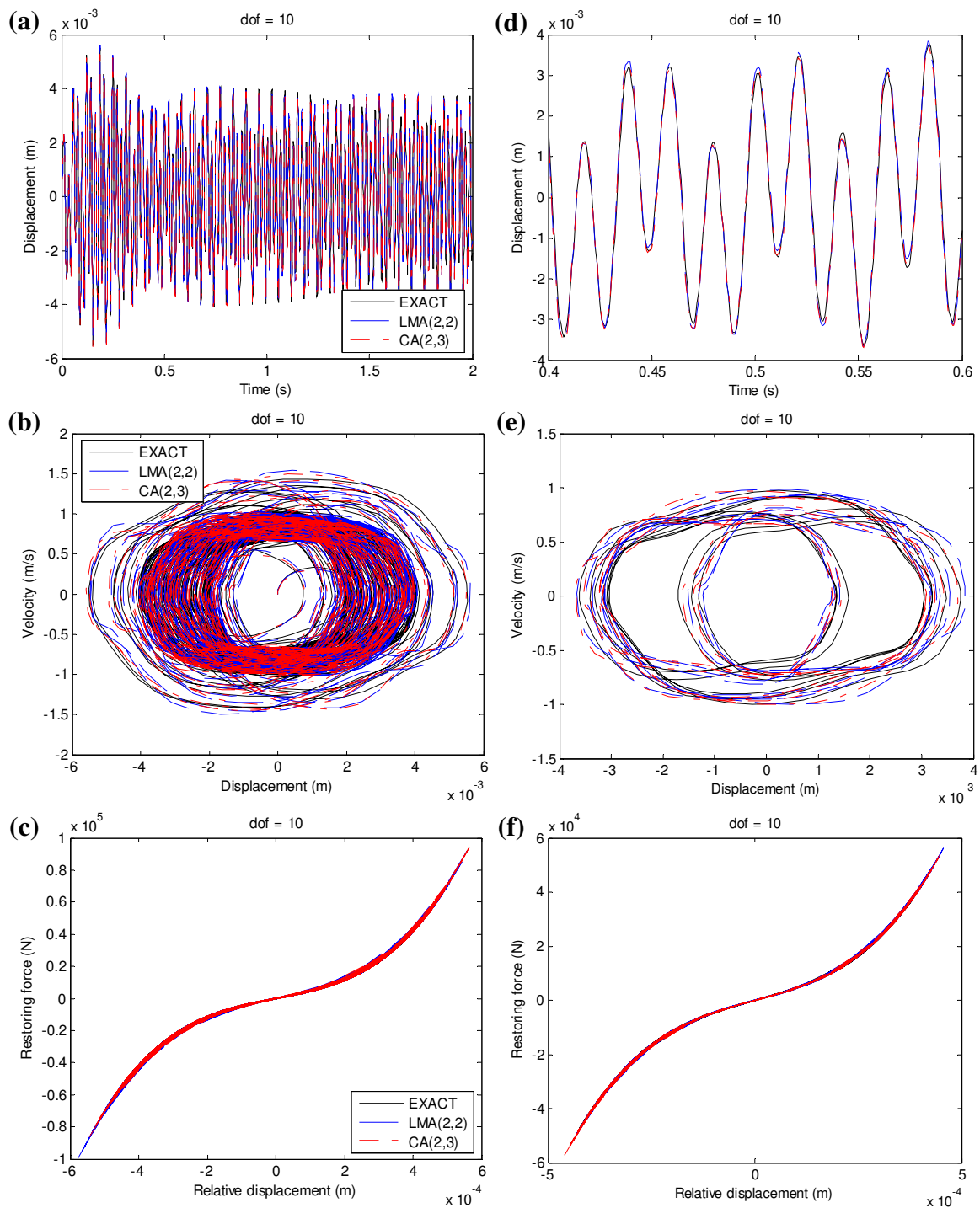


Figure 1: Nonlinear responses of the reference and condensed models at dof 10 (a, b, c); (d, e, f) zoom between [0.4, 0.6 s] for the same responses.

It can be observed, on the time responses and the phase portraits, that the response of the system is correctly predicted by both reduction methods. The nonlinear behaviour is well returned by the condensed stiffness matrix, has shown by the

restoring force diagrams. One can conclude that, in this regime, the proposed method correctly approximates the reference response on an a priori selected time interval.

	Size (dofs)	$E \text{ (m}^2\text{s)} \times 10^{-2}$	$T \text{ (s)}$	$D^2 \text{ (s)}$
Exact	10	2.0911	0.91682	0.35259
CA (2, 3)	4	2.0982	0.91886	0.35357
LMA (2, 2)	4	2.1145	0.91996	0.35253

Table 1: Reduced basis size and associated energy criteria

Example 2

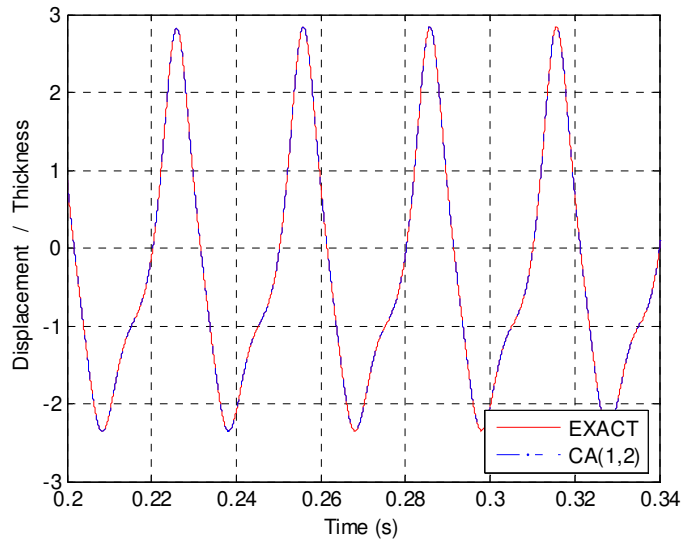
The method is then applied to a clamped–clamped beam having the following geometrical and material properties (Aluminum DTDSO 70) [15]:

$$h = 2 \times 10^{-3} \text{ m}; b = 20 \times 10^{-3} \text{ m}; L = 0.580 \text{ m}; E = 7 \times 10^{10} \text{ N/m}^2; \rho = 2778 \text{ kg/m}^3.$$

The value of the damping parameter is $\xi_i = 0.001$. Here, the aim is to study the large displacement behavior of the beam. The assumption is made to neglect the longitudinal inertia of the beam but not its longitudinal displacements.

A point harmonic excitation of frequency 33 Hz and amplitude 10 N is transversally applied at the middle of the beam.

The results, exposed figure 2, show that the CA variant method applied to the first mode of the structure, with 2 vectors computed in the reduced basis, accurately approximates the exact solution, in accordance with the literature [15].



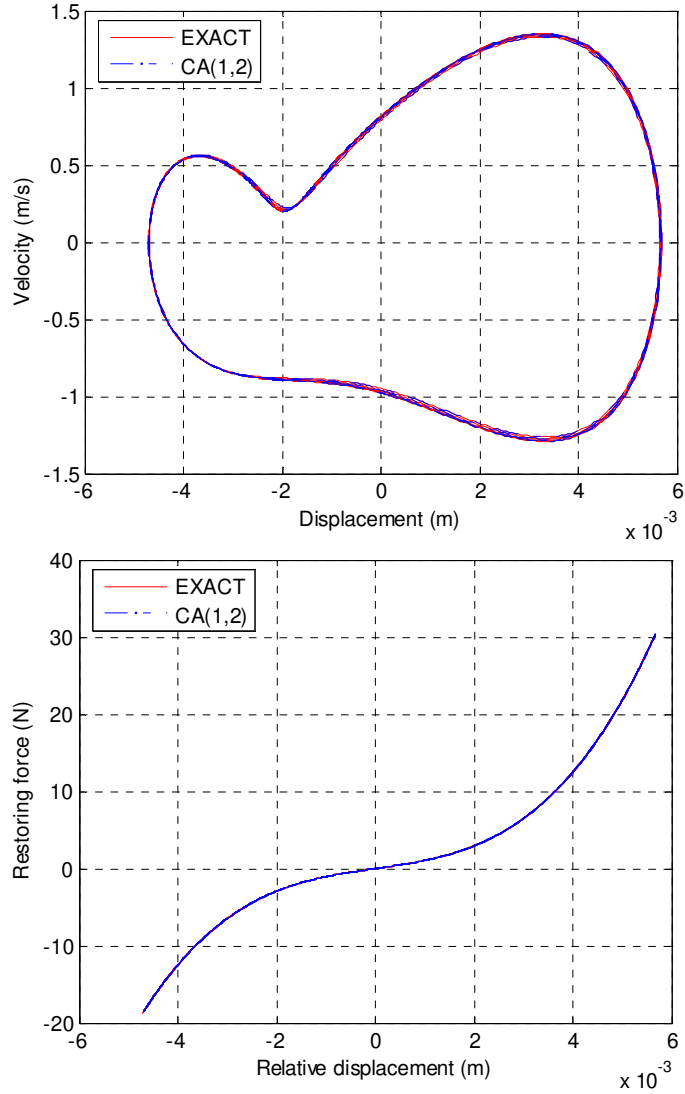


Figure 2: Nonlinear responses of the reference and condensed models

Example 3

This example concerns a clamped frame structure represented in figure 3, which is discretized using a two-dimensional beam element (3 dof per node: U_x, U_y, θ_z). The finite element model contains 510 dof. The mechanical and geometrical characteristics are given by:

$$b = 36 \times 10^{-3} \text{ m}; h = 25 \times 10^{-3} \text{ m}; \text{Area} = b \times h; E = 2.1 \times 10^{11} \text{ N/m}^2; \rho = 7800 \text{ kg/m}^3.$$

The structure is excited at node N_f by a localized choc excitation force, in the U_x direction. The observation point is considered at node N_o , also in the U_x direction.

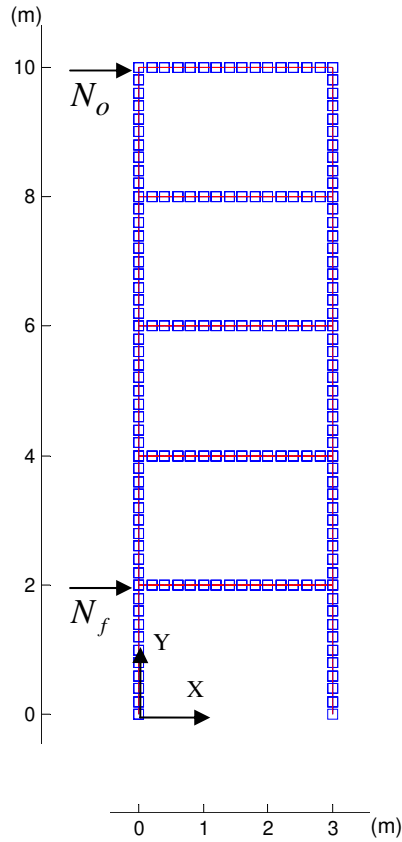


Figure 3: Finite element model of the frame

Table 2 shows that even if the size of the nonlinear system has been reduced by a factor of 50, the energy criteria of the reference and the CA variant methods are identical. The responses plotted in figure 4, on the time interval $[0-50\text{ s}]$ clearly state the accuracy of the proposed method, computed on the first 5 modes, with 2 basis vectors, with regard to the exact solution.

	Size (dofs)	$E (\text{m}^2\text{s}) \times 10^{-6}$	T (s)	D^2 (s)
Exact	510	2.4197	2.1484	1.1748
CA (5, 2)	10	2.4195	2.1484	1.1748

Table 2: Reduced basis size and associated energy criteria

7 Conclusions

In this paper, a variant of the combined approximation reanalysis method is presented. The main contribution concerns the determination of an optimal reduction basis using a singular value decomposition procedure. This method has been successfully applied to large displacement nonlinear analysis. It allows a significant

reduction of the tangent matrix, which needs to be updated at each step of the iterative procedure, especially when dealing with large scale models.

In the case of localized nonlinearities, compared to a classical linear enriched Ritz basis method, the proposed variant leads to better predictions of the response. Furthermore, for generalized nonlinearities, i.e. large displacements, the CA variant still gives accurate results, while the enriched Ritz basis method cannot be applied anymore.

However, some limitations need to be investigated as, for example, the convergence criteria of the binomial series expansion, and the condition number of the reduction transformation. Other simulations also need to be performed to examine the robustness of the method with regard to different levels of nonlinearities and frequency band of interest.

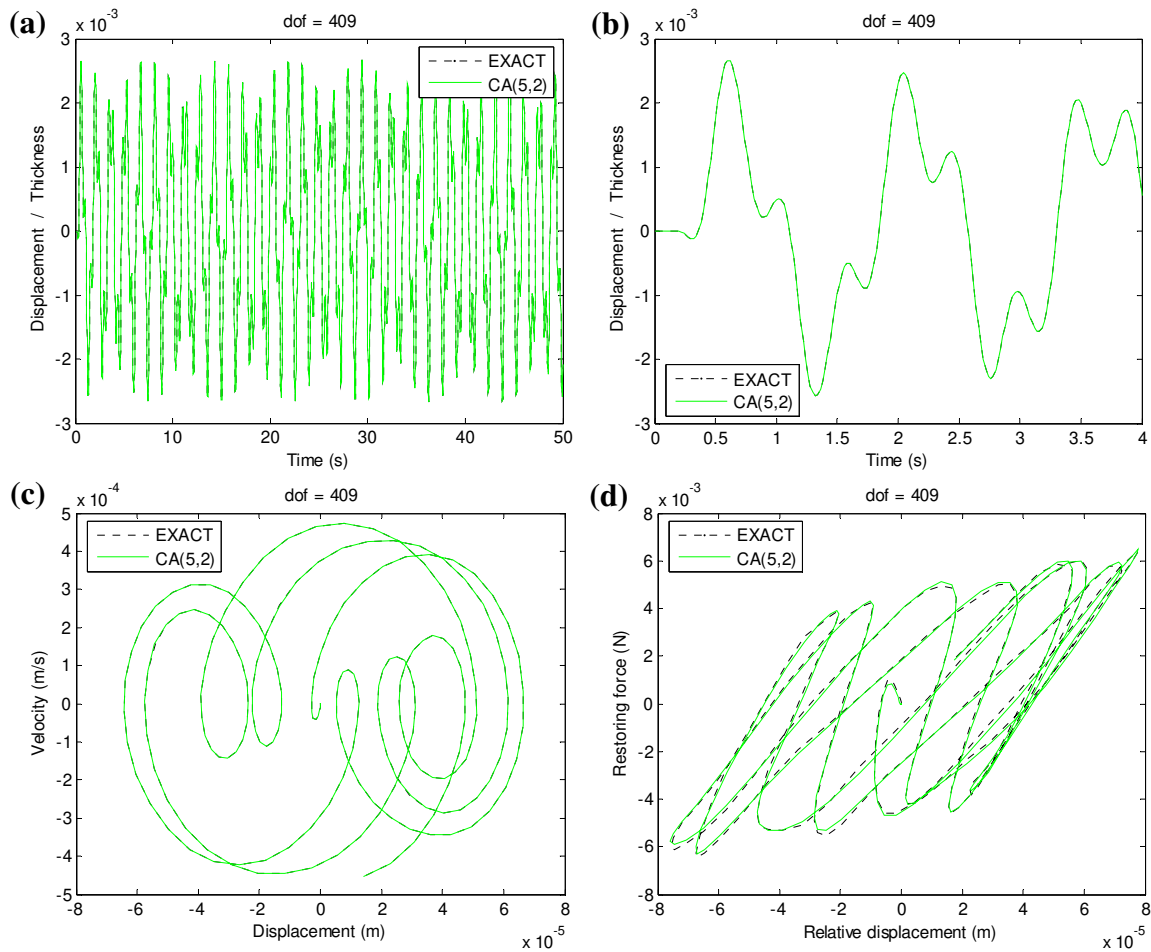


Figure 4: Nonlinear responses of the reference and condensed models at dof 409 (a); (b, c, d) zoom between [0, 4 s] for the same responses.

References

- [1] O.C. Zienkiewicz, R.L. Taylor, "The Finite Element Method, Solid mechanics", 4th edition, vol. 2, McGraw-Hill, New York, 2000.
- [2] G. Masson, B. Ait Brik, S. Cogan, N. Bouhaddi, "Component mode synthesis (CMS) based on an enriched Ritz approach for efficient structural optimization", *Journal of Sound and Vibration*, 2006, 296: 845-860.
- [3] E. Balmès, "Optimal Ritz vectors for component mode synthesis using the singular value decomposition", *AIAA Journal*, 1996, 34: 1256-1260.
- [4] M. Guedri, T. Weisser, N. Bouhaddi, "Réanalyse de structures non-linéaires par une méthode de réduction par approximations combinées", Premier Colloque International IMPACT 2010, 22-24 Mars 2010, Djerba, Tunisie.
- [5] U. Kirsch, "A unified reanalysis approach for structural analysis, design, and optimization", *Structural and Multidisciplinary Optimization*, 2003, 25: 67-85.
- [6] K. Imai, D.M. Frangopol, "Geometrically nonlinear finite element reliability analysis of structural systems. I: theory", *Computers and Structures*, 2000, 77: 677-691
- [7] K.J. Bathe, S. Bolourchi, "Large displacement analysis of three-dimensional beam structures", *International Journal for Numerical Methods in Engineering*, 1979, 14:961-86.
- [8] M.A. Crisfield, "Non-linear finite element analysis of solid and structures", Chichester (UK): Wiley, 1991.
- [9] M.B. Kanchi, "Matrix methods of structural analysis", New Delhi (India): Wiley-Eastern, 1993.
- [10] C.A. Felippa, "Lecture notes in nonlinear finite element methods", Center for Aerospace Structures, University of Colorado, Boulder, CO, 1996.
- [11] P. Paultre, "Dynamiques des structures : application aux ouvrages de génie civil", Ed. Hermès, 2004.
- [12] D.O. Smallwood, "Characterization and simulation of transient vibrations using band limited moments", *Shock and Vibration*, 1994, 1(6): 507-527.
- [13] A.M. Abu Kassim, B.H.V. Topping, "Static reanalysis: a review", *Journal of Structural Engineering*, 1987, 113: 1029-1045.
- [14] T. Weisser, N. Bouhaddi, "Réanalyse dynamique de structures par une variante de la méthode des approximations combinées", 9ème Colloque National en Calcul des Structures, GIENS, 25-29 mai 2009, France.
- [15] P. Ribeiro, "Hierarchical finite element analyses of geometrically non-linear vibration of beams and plane frames", *Journal of Sound and Vibration*, 2001, 246(2), 225-244.