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CHEBYSHEV’S BIAS AND GENERALIZED Riemann Hypothesis

ADEL ALAMADHI∗, MICHEL PLANAT†, AND PATRICK SOLÉ‡,*

Abstract. It is well known that $\text{li}(x) > \pi(x)$ (i) up to the (very large) Skewes’ number $x_1 \sim 1.40 \times 10^{316}$ [1]. But, according to a Littlewood’s theorem, there exist infinitely many $x$ that violate the inequality, due to the specific distribution of non-trivial zeros $\gamma$ of the Riemann zeta function $\zeta(s)$, encoded by the equation $\text{li}(x) - \pi(x) \approx \sqrt{x} \log x \left[ 1 + 2 \sum_{\gamma \neq 0} \frac{\sin(\gamma \log x)}{\gamma} \right] (1)$. If Riemann hypothesis (RH) holds, (i) may be replaced by the equivalent statement $\text{li}[\psi(x)] > \pi(x)$ (ii) due to Robin [2]. A statement similar to (i) was found by Chebyshev that $\pi(x; 4, 3) - \pi(x; 4, 1) > 0$ (iii) holds for any $x < 26861$ [3] (the notation $\pi(x; k, l)$ means the number of primes up to $x$ and congruent to $l$ mod $k$). The Chebyshev’s bias (iii) is related to the generalized Riemann hypothesis (GRH) and occurs with a logarithmic density $\approx 0.9959$ [3]. In this paper, we reformulate the Chebyshev’s bias for a general modulus $q$ as the inequality $B(x; q, R) > B(x; q, N)$, where $B(x; k, l) = [\phi(k) * \psi(x; k, l)] - [\phi(k) * \pi(x; k, l)]$ is a counting function introduced in Robin’s paper [2] and $R$ (resp. $N$) is a quadratic residue modulo $q$ (resp. a non-quadratic residue). We investigate numerically the case $q = 4$ and a few prime moduli $p$. Then, we prove that (iv) is equivalent to GRH for the modulus $q$.

1. Introduction

In the following, we denote $\pi(x)$ the prime counting function and $\pi(x; q, a)$ the number of primes not exceeding $x$ and congruent to $a$ mod $q$. The asymptotic law for the distribution of primes is the prime number theorem $\pi(x) \sim \frac{x}{\log x}$. Correspondingly, one gets [4] eq. (14), p. 125

\begin{equation}
\pi(x; q, a) \sim \frac{\pi(x)}{\phi(q)}
\end{equation}

that is, one expects the same number of primes in each residue class $a$ mod $q$, if $(a, q) = 1$. Chebyshev’s bias is the observation that, contrarily to expectations, $\pi(x; q, N) > \pi(x; q, R)$ most of the times, when $N$ is a non-square modulo $q$, but $R$ is.

Let us start with the bias

\begin{equation}
\delta(x, 4) := \pi(x; 4, 3) - \pi(x; 4, 1)
\end{equation}

found between the number of primes in the non-quadratic residue class $N = 3$ mod 4 and the number of primes in the quadratic one $R = 3$ mod 4. The values

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δ(10^n, 4), n ≤ 1, form the increasing sequence

\[ A091295 = \{1, 2, 7, 10, 25, 147, 218, 446, 551, 5960, \ldots\}. \]

The bias is found to be negative in thin zones of size

\[ \{2, 410, 15\,358, 41346, 42\,233\,786, 416\,889\,978, \ldots\} \]

spread over the location of primes of maximum negative bias \[5\]

\[ \{26861, 623\,681, 12\,366\,589, 951\,867\,937, 6\,345\,026\,833, 18\,699\,356\,321\ldots\}. \]

It has been proved that there are are infinitely many sign changes in the Chebyshev’s bias (1.2). This follows from the Littlewood’s oscillation theorem \[6, 7\]

\[ \delta(x, 4) := \Omega_+ \left( \frac{x^{1/2}}{\log x} \log \frac{3}{x} \right). \]

A useful measure of the Chebyshev’s bias is the logarithmic density \[3, 6, 8\]

\[ d(A) = \lim_{x \to \infty} \frac{1}{\log x} \sum_{a \in A, a \leq x} \frac{1}{a} \]

for the positive \( \Delta^+ \) and negative \( \Delta^- \) regions calculated as \( d(\Delta^+) = 0.9959 \) and \( d(\Delta^-) = 0.0041 \).

In essence, Chebyshev’s bias \( \delta(x, 4) \) is similar to the bias

\[ \delta(x) := \text{Li}(x) - \pi(x). \]

It is known that \( \delta(x) > 0 \) up to the (very large) Skewes’ number \( x_1 \sim 1.40 \times 10^{316} \) but, according to Littlewood’s theorem, there also are infinitely many sign changes of \( \delta(x) \) \[7\].

The reason why the asymmetry in \[1.5\] is so much pronounced is encoded in the following approximation of the bias \[3, 9\] \[1\]

\[ \delta(x) \sim \frac{\sqrt{x}}{\log x} \left( 1 + 2 \sum_\gamma \frac{\sin(\gamma \log x + \alpha_\gamma)}{\sqrt{1/4 + \gamma^2}} \right), \]

where \( \alpha_\gamma = \cot^{-1}(2\gamma) \) and \( \gamma \) is the imaginary part of the non-trivial zeros of the Riemann zeta function \( \zeta(s) \). The smallest value of \( \gamma \) is quite large, \( \gamma_1 \sim 14.134 \), and leads to a large asymmetry in \[1.5\].

Under the assumption that the generalized Riemann hypothesis (GRH) holds that is, if the Dirichlet L-function with non-trivial real character \( \kappa_4 \)

\[ L(s, \kappa_4) = \sum_{n \geq 0} \frac{(-1)^n}{(2n + 1)^s}, \]

has all its non-trivial zeros located on the vertical axis \( \Re(s) = \frac{1}{2} \), then the formula \[1.6\] also holds for the Chebyshev’s bias \( \delta(x, 4) \). The smallest non-trivial zero of \( L(s, \kappa_4) \) is at \( \gamma_1 \sim 6.02 \), a much smaller value than than the one corresponding to \( \zeta(s) \), so that the bias is also much smaller.

\[1\] The bias may also be approached in a different way by relating it to the second order Landau-Ramanujan constant \[10\].
A second factor controls the aforementioned asymmetry of a $L$-function of real non-trivial character $\kappa$, it is the variance \[ V(\kappa) = \sum_{\gamma > 0} \frac{2}{1 + \gamma^2}. \]

For the function $\zeta(s)$ and $L(s, \kappa_4)$ one gets $V = 0.045$ and $V = 0.155$, respectively.

Our main goal. In a groundbreaking paper, Robin reformulated the unconditional bias (1.5) as a conditional one involving the second Chebyshev function $\psi(x) = \sum_{p \leq x} \log p$

The equality $\delta'(x) := \text{li}[\psi(x)] - \pi(x) > 0$ is equivalent to RH.

This statement is given as Corollary 1.2 in [12] and led the second and third author of the present work to derive a good prime counting function

$$\pi(x) = \sum_{n=1}^{3} \mu(n) \text{li}[\psi(x)^{1/n}].$$

Here, we are interested in a similar method to regularize the Chebyshev’s bias in a conditional way similar to (1.9). In [2], Robin introduced the function

$$B(x; q, a) = \text{li}[\phi(q)\psi(x; q, a)] - \phi(q)\pi(x; q, a),$$

that generalizes (1.9) and applies it to the residue class $a \mod q$, with $\psi(x, q, a)$ the generalized second Chebyshev’s function. Under GRH he proved that [2, Lemma 2, p. 265]

$$B(x; q, a) = \Omega_{\pm} \left( \sqrt{x} \log^2 x \right),$$

that is

$$B(x; q, a) > 0$$

is equivalent to GRH.

For the Chebyshev’s bias, we now need a proposition taking into account two residue classes such that $a = N$ (a non-quadratic residue) and $a = R$ (a quadratic one).

**Proposition 1.1.** Let $B(x; q, a)$ be the Robin $B$-function defined in (1.11), and $R$ (resp. $N$) be a quadratic residue modulo $q$ (resp. a non-quadratic residue), then the statement $\delta'(x, q) := B(x; q, R) - B(x; q, N) > 0$, $\forall x$ (i), is equivalent to GRH for the modulus $q$.

The present paper deals about the numerical justification of proposition 1.1 in Sec. 2 and its proof in Sec. 3. The calculations are performed with the software Magma [13] available on a 96 MB segment of the cluster at the University of Franche-Comté.

2. The regularized Chebyshev’s bias

All over this section, we are interested in the prime champions of the Chebyshev’s bias $\delta(x, q)$ (as defined in (1.2) or (2.3), depending on the context). We separate the prime champions leading to a positive/negative bias. Thus, the $n$-th prime champion satisfies

$$\delta^{(e)}(x_n, q) = \epsilon n, \ \epsilon = \pm 1.$$
Figure 1. The normalized regularized bias $\delta'(x,4)/\sqrt{x}$ versus the Chebyshev’s bias $\delta(x,4)$ at the prime champions of $\delta(x,4)$ (when $\delta(x,4) > 0$) and at the prime champions of $-\delta(x,4)$ (when $\delta(x,4) < 0$). The extremal prime champions in the plot are $x = 359327$ (with $\delta = 105$) and $x = 951867937$ (with $\delta = -48$). The curve is asymmetric around the vertical axis, a fact that reflects the asymmetry of the Chebyshev’s bias. As explained in the text, a violation of GRH would imply a negative value of the regularized bias $\delta'(x,4)$. The small dot curve corresponds to the fit of $\delta'(x,4)/\sqrt{x}$ by $2/\log x$ calculated in Sec. 3.

We also introduce a new measure of the overall bias $b(q)$, dedicated to our plots, as follows

$$(2.2) \quad b(q) = \sum_{n,\epsilon} \frac{\delta'(\epsilon)(x_n,q)}{x_n}.$$ 

Indeed, smaller is the bias lower is the value of $b(q)$. Anticipating over the results presented below, Table 1 summarizes the calculations.

<table>
<thead>
<tr>
<th>modulus $q$</th>
<th>bias $b(q)$</th>
<th>log density $d(\Delta^+)$</th>
<th>first zero $\gamma_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.7926</td>
<td>0.9959 [3]</td>
<td>14.134</td>
</tr>
<tr>
<td>11</td>
<td>0.1841</td>
<td>0.9167 [3]</td>
<td>0.2029</td>
</tr>
<tr>
<td>13</td>
<td>0.2803</td>
<td>0.9443 [3]</td>
<td>3.119</td>
</tr>
<tr>
<td>163</td>
<td>0.0809</td>
<td>0.55 [9]</td>
<td>2.477</td>
</tr>
</tbody>
</table>
Chebyshev’s bias for the modulus $q = 4$. As explained in the introduction, our goal in this paper is to reexpress a standard Chebyshev’s bias $\delta(x, q)$ into a regularized one $\delta'(x, q)$, that is always positive under the condition that GRH holds. Indeed we do not discover any numerical violation of GRH and we always obtains a positive $\delta'(x, q)$. The asymmetry of Chebyshev’s bias arises in the plot $\delta$ vs $\delta'$, where the fall of the normalized bias $\frac{\delta}{\sqrt{x}}$ is faster for negative values of $\delta$ than for positive ones. Fig. 1 clarifies this effect for the historic modulus $q = 4$.

We restricted our plot to the champions of the bias $\delta$ and separated positive and negative champions.

Chebyshev’s bias for a prime modulus $p$. For a prime modulus $p$, we define the bias so as to obtain an averaging over all differences $\pi(x; p, N) - \pi(x; p, R)$, where as above $N$ and $R$ denote a non-quadratic and a quadratic residue, respectively

\begin{equation}
\delta(x, p) = -\sum a \left( \frac{a}{p} \right) \pi(x; p, a),
\end{equation}

where $\left( \frac{a}{p} \right)$ is the Legendre symbol. Correspondingly, we define the regularized bias as

\begin{equation}
\delta'(x, p) = \frac{1}{[p/2]} \sum a \left( \frac{a}{p} \right) B(x; p, a).
\end{equation}

**Proposition 2.1.** Let $p$ be a selected prime modulus and $\delta'(x, p)$ as in (2.4) then the statement $\delta'(x, p) > 0$, $\forall x$, is equivalent to GRH for the modulus $p$.

As mentioned in the introduction, the Chebyshev’s bias is much influenced by the location of the first non-trivial zero of the function $L(s, \kappa_q)$, $\kappa_q$ being the real non-principal character modulo $q$. This is especially true for $L(s, \kappa_{163})$ with its smaller non-trivial zero at $\gamma \sim 0.2029$ [9]. The first negative values occur at $\{15073, 15077, 15083, \ldots\}$.

Fig. 2 represents the Chebyshev’s bias $\delta'$ for the modulus $q = 163$ versus the standard one $\delta$ (thick dots). The asymmetry of the Chebyshev’s bias is revealed at small values of $|\delta|$ where the the fit of the regularized bias by the curve $2/\log x$ is not good (thin dots).

For the modulus $q = 13$, the imaginary part of the first zero is not especially small, $\gamma_1 \sim 3.119$, but the variance [7.8] is quite high, $V(\kappa_{-13}) \sim 0.396$. The first negative values of $\delta(x, 13)$ at primes occur when $\{2083, 2089, 10531, \ldots\}$. Fig. 3 represents the Chebyshev’s bias $\delta'$ for the modulus $q = 13$ versus the standard one $\delta$ (thick dots) as compared to the fit by $2/\log x$ (thin dots).

Finally, for the modulus $q = 11$, the imaginary part of the first zero is quite small, $\gamma_1 \sim 0.209$, and the variance is high, $V(\kappa_{-11}) \sim 0.507$. In such a case, as shown in Fig. 4 the approximation of the regularized bias by $2/\log x$ is good in the whole range of values of $x$.

### 3. Proof of Proposition [1.1]

For approaching the proposition [1.1] we reformulate it in a simpler way as

**Proposition 3.1.** One introduces the regularized counting function $\pi'(x; q, l) := \pi(x; q, l) - \psi(x; q, l)/\log x$. The statement $\pi'(x; q, N) > \pi'(x; q, R)$, $\forall x$ (ii), is equivalent to GRH for the modulus $q$. 


Figure 2. The normalized regularized bias $\delta'(x,163)/\sqrt{x}$ versus the Chebyshev’s bias $\delta(x,163)$ at all the prime champions of $|\delta(x,163)|$. From $|\delta(x,163)| > 74$ the bias is $\delta(x,163) < 0$ negative, superimposed to the curve at the prime champions of $-\delta(x,163)$ (when $\delta(x,163) < 0$). The extremal prime champions in the plot are $x = 68491$ (with $\delta = 74$) and $x = 174637$ (with $\delta = -86$). The asymmetry is still clearly visible in the range of small values of $|\delta|$ but tends to disappear in the range of high values of $|\delta|$. The small dot curve corresponds to the fit of $\delta'(x,163)/\sqrt{x}$ by $2/\log x$ calculated in Sec. 3.

Proof. First observe that proposition 1.1 follows from proposition 3.1. This is straightforward because according to [2, p. 260], the prime number theorem for arithmetic progressions leads to the approximation

$$\text{Li}\left[\phi(q)\psi(x; q, l)\right] \sim \text{Li}(x) + \frac{\phi(q)\psi(x; q, l) - x}{\log x}.\tag{3.1}$$

As a result

$$\delta'(x, q) = B(x; q, R) - B(x; q, N)$$

$$= \text{Li}[\phi(q)\psi(x; q, R)] - \text{Li}[\phi(q)\psi(x; q, N)] + \phi(q)\delta(x, q)$$

$$\sim \phi(q)[\pi'(x; q, N) - \pi'(x; q, R)].$$

The asymptotic equivalence in (3.1) holds up to the error term [2, p. 260] $O(R(x)/x \log x)$, with

$$\theta_q = \max_{\kappa \bmod q} (\sup \Re(\rho), \rho \text{ a zero of } L(s, \kappa)).$$

Let us now look at the statement GRH $\Rightarrow (i)$. Following [3, p 178-179], one has

$$\psi(x; q, a) = \frac{1}{\phi(q)} \sum_{\kappa \bmod q} \tilde{\kappa}(a)\psi(x, \kappa)$$
Figure 3. The normalized regularized bias $\delta'(x, 13)/\sqrt{x}$ versus the Chebyshev’s bias $\delta(x, 13)$ at the prime champions of $\delta(x, 13)$ (when $\delta(x, 13) > 0$), and the curve at the prime champions of $-\delta(x, 13)$ (when $\delta(x, 13) < 0$). The extremal prime champions in the plot are $x = 263881$ (with $\delta = 123$) and $x = 905761$ (with $\delta = -40$). The small dot curve corresponds to the fit of $\delta'(x, 13)/\sqrt{x}$ by $2/\log x$ calculated in Sec. 3.

Figure 4. The normalized regularized bias $\delta'(x, 11)/\sqrt{x}$ versus the Chebyshev’s bias $\delta(x, 11)$ at the prime champions of $\delta(x, 11)$ (when $\delta(x, 11) > 0$), and the curve at the prime champions of $-\delta(x, 11)$ (when $\delta(x, 11) < 0$). The extremal prime champions in the plot are $x = 638567$ (with $\delta = 158$) and $x = 1867321$ (with $\delta = -32$). The small dot curve corresponds to the (very good) fit of $\delta'(x, 11)/\sqrt{x}$ by $2/\log x$ calculated in Sec. 3.
and under GRH
\[ \pi(x; q, a) = \frac{\pi(x)}{\phi(q)} - \frac{c(q, a)}{\phi(q)} \sqrt{x} \log x + \frac{1}{\phi(q) \log x} \sum_{\kappa \neq \kappa_0} \kappa(a) \psi(x, \kappa) + O\left(\frac{\sqrt{x}}{\log^2 x}\right), \]
where \( \kappa_0 \) is the principal character modulo \( q \) and
\[ c(q, a) = -1 + \#\{1 \leq b \leq q : b^2 = a \mod q\} \]
for coprimes integers \( a \) and \( q \). Note that for an odd prime \( q = p \), one has \( c(p, a) = \left(\frac{a}{p}\right) \).

Thus, under GRH
\[ \pi(x; q, N) - \pi(x; q, R) = \frac{1}{\phi(q) \log x} \left[ \sqrt{x}(c(q, R) - c(q, N)) + \sum_{\kappa \mod q} [\kappa(N) - \kappa(R)] \psi(x, \kappa) \right] + O\left(\frac{\sqrt{x}}{\log^2 x}\right). \]
(3.2)

The sum could be taken over all characters because \( \kappa_0(N) = \kappa_0(R) \). In addition, we have
\[ \psi(x; q, N) - \psi(x; q, R) = \frac{1}{\phi(q)} \sum_{\kappa \mod q} [\kappa(N) - \kappa(R)] \psi(x, \kappa). \]
(3.3)

Using (3.2) and (3.3) the regularized bias reads
\[ \delta'(x, q) \sim \pi'(x; q, N) - \pi'(x; q, R) \]
(3.4)
\[ = \frac{\sqrt{x}}{\log x} [c(q, R) - c(q, N)] + O\left(\frac{\sqrt{x}}{\log^2 x}\right). \]

For the modulus \( q = 4 \), we have \( c(q, 1) = -1 + 2 = 1 \) and \( c(q, 3) = -1 \) so that
\[ \delta'(x, 4) = \frac{2\sqrt{x}}{\log x}. \]
The same result is obtained for a prime modulus \( q = p \) since
\[ c(p, N) = -1 \text{ and } c(p, R) = c(p, 1) = \left(\frac{1}{p}\right) = 1. \]

This finalizes the proof that under GRH, one has the inequality \( \pi'(x; q, N) > \pi'(x; q, R) \).

If GRH does not hold, then using [2] lemma 2, one has
\[ B(x; q, a) = \Omega_\pm(x^\xi) \text{ for any } \xi < \theta_q. \]

Applying this asymptotic result to the residue classes \( a = R \) and \( a = N \), there exist infinitely many values \( x = x_1 \) and \( x = x_2 \) satisfying
\[ B(x_1; q, R) < -x_1^\xi \text{ and } B(x_2; q, N) > x_2^\xi \text{ for any } \xi < \theta_q, \]
so that one obtains
\[ B(x_1; q, R) - B(x_2; q, N) < -x_1^\xi - x_2^\xi < 0. \]
(3.5)

Selecting a pair \((x_1, x_2)\) either
\[ B(x_1; q, R) > B(x_2; q, R) \]
so that \( B(x_2; q, R) - B(x_2; q, N) < 0 \) and (i) is violated at \( x_2 \), or
\[ B(x_1; q, R) < B(x_2; q, R). \]
(3.6)

In the last case, either \( B(x_1; q, N) > B(x_2; q, N) \), so that \( B(x_1; q, R) - B(x_1; q, N) < 0 \) and the inequality (i) is violated at \( x_1 \), or simultaneously
\[ B(x_1; q, N) < B(x_2; q, N) \text{ and } B(x_1; q, R) < B(x_2; q, R), \]
which implies (3.6) and the violation of (i) at \( x = x_1 = x_2 \).
To finalize the proof of 3.1 and simultaneously that of 1.1, one makes use of the asymptotic equivalence of (i) and (ii), that is if GRH is true $\Rightarrow$ (ii) $\Rightarrow$ (i), and if GRH is wrong, (i) may be violated and (ii) as well.

Then, proposition 2.1 also follows as a straightforward consequence of proposition 1.1.

4. Summary

We have found that the asymmetry in the prime counting function $\pi(x; q, a)$ between the quadratic residues $a = R$ and the non-quadratic residues $a = N$ for the modulus $q$ can be encoded in the function $B(x; q, a)$ [defined in 1.11] introduced by Robin the context of GRH [2], or into the regularized prime counting function $\pi'(x; q, a)$, as in Proposition 3.1. The bias in $\pi'$ reflects the bias in $\pi$ conditionally under GRH for the modulus $q$. Our conjecture has been initiated by detailed computer calculations presented in Sec. 2 and proved in Sec. 3. Further work could follow the work about the connection of $\pi$, and thus of $\pi'$, to the sum of squares function $r_2(n)$ [10].

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