Spatial properties of Wohlhart symmetric mechanism

Laurentiu Racila, Marc Dahan

Spatial properties of Wohlhart symmetric mechanism

Laurentiu Racila · Marc Dahan

Abstract The aim of this paper is to present a new analytical resolution to find closure equations of Wohlhart symmetric mechanism and to give some interesting spatial properties of this mechanism. These properties are used to put in evidence possible industrial applications for this type of mechanism. First, it concerns the analytic resolution of the twelve equations system obtained from the closure equation, system particularized for the case of Wohlhart symmetric mechanism. The matrix form of closure equations is written for a single loop overconstrained mechanism composed by six revolute joints, mechanisms called 6R. A kinematical analysis is made, putting in evidence some interesting kinematical and geometrical properties. These properties allow us to consider a new spatial disposition for this mechanism, by imposing three non-successive joints to remain in a determined plan. Thus, we obtain a translator, a possible future industrial application for this type of overconstrained mechanism.

L. Racila
Department of Applied Mechanics, University of Craiova, Craiova, Romania
e-mail: racila_laurentiu@yahoo.com
url: http://mecanica.ucv ro

M. Dahan (✉)
FEMTO-ST, University of Franche-Comté, Besançon, France
e-mail: marc.dahan@univ-fcomte.fr
url: http://www.femto-st.fr

Keywords Overconstrained mechanisms · Kinematical analysis · Closure equation · Industrial application · Translator

1 Introduction

Single loop overconstrained mechanisms are composed of four, five or six revolute joints. In this paper, we study only six-bars overconstrained mechanisms with revolute joints. These types of overconstrained mechanisms are known from over 150 years, since Sarrus [16] presents in 1853 his 6R mechanism. This mechanism, considered as the first spatial overconstrained mechanism, transforms the circular movement in linear movement and vice versa. Some years later, in 1897 and 1927, Raoul Bricard [5, 6] presents the "paradoxical" mechanisms with revolute joints. Since then, several authors have presented overconstrained mechanisms: Bennett [4], Myard [13], Goldberg [11], Altmann [1], Schatz [17], Waldron [18, 19], Mavroidis and Roth [12], Wohlhart [20, 21], Dietmaier [8]... Today several of these mechanisms are considered as particular cases of the other overconstrained mechanisms.

Among these mechanisms, the mechanisms presented by Bricard are the only ones that are independent, the other 6R overconstrained mechanisms being combinations of the other mechanisms with four or five revolute joints. In his paper, Bricard describes

three types of deformable octahedrons: with axial symmetry, plane symmetry and double collapsible and three six-bar mechanisms: with axial symmetry, plane symmetry and the rectangular mechanism. Later, in 1987, Karl Wohlhart [20] presents a new 6R overconstrained mechanism. This mechanism is considered as a generalization of Bricard rectangular mechanism. We will call this mechanism the "Wohlhart mechanism". It's a mechanism that presents three partial symmetry planes. In the case when the six elements are the same length, the mechanism is called Wohlhart symmetric mechanism.

A first deep analyze of Bricard mechanisms made by J.E. Baker [2], gives the closure equations of these mechanisms. Baker gives also a list of all independent overconstrained mechanisms [3]. To analyze these mechanisms Baker uses trigonometric relationships observed on the mechanisms. Wohlhart also, to make the analysis of Bricard rectangular mechanism, uses spherical trigonometry relationships, finally obtaining implicit relationships for closure equations [21].

In the present paper, we give a complete analytical solution for finding the closure equations of Wohlhart symmetric mechanism. Making a kinematical analysis of this mechanism, some interesting properties are put in evidence and a new spatial disposition is proposed, for a possible industrial applications based on overconstrained mechanisms. In the industry area, the industrial applications based on these mechanisms are only a few. Excepting the Schatz 6R mechanism, called "Turbula" [17] and some deployable structures of Gan [10] and Chen [7], we have no knowledge of existence of other industrial applications of 6R overconstrained mechanisms.

2 Closure equations

We consider the Denavit-Hartenberg formalization for a mechanism with revolute joints [9]. The twist angle between two successive joints (i) and $(i+1)$ is noted α_i, the bar lengths is noted a_i and the offset distance between two elements $(i-1)$ and (i) is noted d_i. The fourth parameter is the angle between two successive elements $(i-1)$ and (i), noted θ_i.

Figure 1 shows the schematic representation of the revolute joints with the Denavit and Hartenberg notations, in which a_i, α_i and d_i are geometrics parameters and θ_i is the kinematical variable.

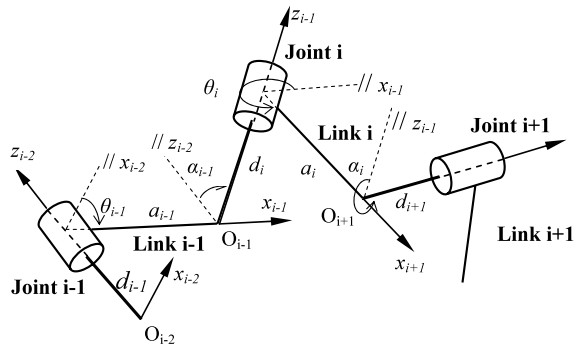


Fig. 1 Denavit–Hartenberg parameters

The homogeneous form for the transfer matrix is:

$$
{}^{i-1}Q_{i_i} = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i \cdot \sin\theta_i & \sin\alpha_i \cdot \sin\theta_i & a_i \cdot \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cdot \cos\theta_i & -\sin\alpha_i \cdot \cos\theta_i & a_i \cdot \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}
$$

The closure condition for a 6R single loop mechanism expresses that the six transfer matrix product is equal with the unity matrix. This condition can also be written:

$$
{}^3Q_4 \cdot {}^4Q_5 \cdot {}^5Q_6 = {}^3Q_2 \cdot {}^2Q_1 \cdot {}^1Q_6
$$

Developing this equation we obtain a twelve equations system. This system is identical with the system obtained by Waldron and cited by Baker [2].

We consider now Wohlhart symmetric mechanism (Fig. 2) with the next geometrical conditions:

$$
\begin{cases} d_1 = d_2 = d_3 = d_4 = d_5 = d_6 = 0 \\ \alpha_1 = \alpha_3 = \alpha_5 = \alpha \\ \alpha_2 = \alpha_4 = \alpha_6 = 2\pi - \alpha \\ a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a \end{cases} \tag{2}
$$

With these conditions (2), the twelve equations system is apparently simplified (Appendix 1).

From (A.11) and (A.12) relations we obtain the two first closure equations:

$$
\theta_2 = \theta_6 \tag{3}
$$

$$
\theta_3 = \theta_5 \tag{4}
$$

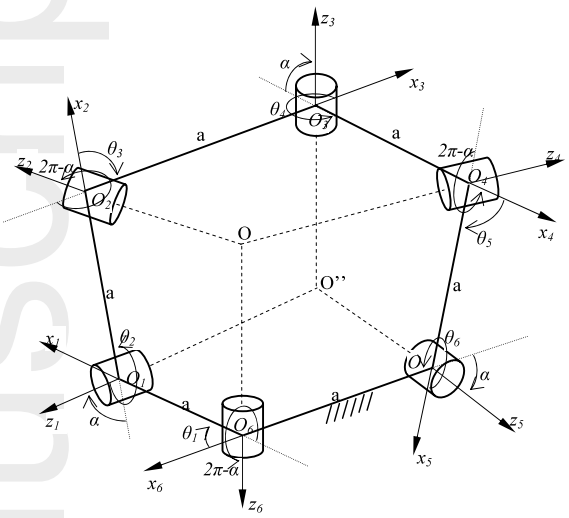


Fig. 2 Wohlhart symmetric mechanism

These two relations, introduced in (A.9) and (A.10), prove that these relations are dependent, so:

$$4 \cdot \sin\frac{\theta_2}{2} \cdot \sin\frac{\theta_3}{2} \cdot \cos\alpha \cdot \left(\cos\frac{\theta_2}{2} \cdot \sin\frac{\theta_1+\theta_3}{2} - \sin\frac{\theta_1}{2} \cdot \sin\frac{\theta_2}{2} \cdot \sin\frac{\theta_3}{2} \cdot \cos\alpha\right) = \cos\frac{\theta_1}{2} \cdot \cos\theta_2 \cdot \sin\theta_3 + \sin\frac{\theta_1}{2} \cdot \cos\theta_3 \tag{5}$$

Introducing the relations (3) and (4) in (A.7) and (A.8) we find:

$$4 \cdot s\frac{\theta_2}{2} \cdot s\frac{\theta_3}{2} \cdot c\alpha \times \left(c\frac{\theta_3}{2} \cdot s\frac{\theta_2+\theta_4}{2} - s\frac{\theta_2}{2} \cdot s\frac{\theta_3}{2} \cdot s\frac{\theta_4}{2} \cdot c\alpha\right) = s\theta_2 \cdot c\theta_3 \cdot c\frac{\theta_4}{2} + c\theta_2 \cdot s\frac{\theta_4}{2} \tag{6}$$

$$4 \cdot c\frac{\theta_2}{2} \cdot s\frac{\theta_3}{2} \cdot c\alpha \times \left(c\frac{\theta_3}{2} \cdot s\frac{\theta_2+\theta_4}{2} - s\frac{\theta_2}{2} \cdot s\frac{\theta_3}{2} \cdot s\frac{\theta_4}{2} \cdot c\alpha\right) = 2 \cdot c^2\frac{\theta_3}{2} \cdot c\frac{\theta_4}{2} + c\theta_2 \cdot c\theta_3 \cdot c\frac{\theta_4}{2} - s\theta_2 \cdot s\frac{\theta_4}{2} \tag{7}$$

These two relations (6) and (7) can be divided, and making some elementary calculus we find the third closure equation:

$$\theta_2 = \theta_4 \tag{8}$$

Introducing the first three closure equations in (A.1), (A.2), (A.5) and (A.6) relations, we obtain that these four relations are dependent to each other.

More precisely, between (A.1), (A.2) and (A.5) relations, the next relation is verified:

$$(\text{A1}) \cdot (\cos\theta_1 + \cos\theta_2) = (\text{A2}) \cdot \sin\theta_1 - (\text{A5}) \cdot \sin\theta_2 \tag{9}$$

and between (A.1), (A.2) and (A.6) we have the relation:

$$(\text{A1}) = (\text{A2}) \cdot (\cos\theta_1 - \cos\theta_2) + (\text{A6}) \cdot \sin\theta_2 \tag{10}$$

Introducing the relations (3), (4) and (8) in (A.3) and (A.9) we can write:

$$\cos^2\alpha \cdot (1-\cos\theta_2) \cdot (1-\cos\theta_3) = 2 \cdot \sin\theta_2 \cdot \sin\theta_3 \cdot \cos\alpha - \cos\theta_2 - \cos\theta_3 - \cos\theta_2 \cdot \cos\theta_3 \tag{11}$$

$$\cos\theta_1 \cdot \cos\theta_2 \cdot \sin\theta_3 + \cos\theta_2 \cdot \sin\theta_3 + \sin\theta_1 \cdot \cos\theta_3 = \sin\theta_2 \cdot \cos\alpha \cdot (\sin\theta_1 \cdot \sin\theta_3 + 1 + \cos\theta_1 - \cos\theta_3 - \cos\theta_1 \cdot \cos\theta_3) - \sin\theta_1 \cdot \cos^2\alpha \cdot (1-\cos\theta_2) \cdot (1-\cos\theta_3) \tag{12}$$

After some simplifications in these two relations we obtain the fourth closure equation:

$$\theta_1 = \theta_5 \tag{13}$$

We have now four closure equations, and for clarity we note θ the odd angles (input angle) and ϕ the even angles (output angle):

$$\begin{cases} \theta_1 = \theta_3 = \theta_5 = \theta \\ \theta_2 = \theta_4 = \theta_6 = \varphi \end{cases} \tag{14}$$

The fifth (and the last) closure equation (input-output equation) is obtaining by introducing the first four closure equations in (A.1) relation:

$$\cos\theta \cdot \cos\varphi \cdot (1+\cos^2\alpha) + (\cos\theta + \cos\varphi) \cdot \sin^2\alpha - 2 \cdot \sin\theta \cdot \sin\varphi \cdot \cos\alpha + \cos^2\alpha = 0 \tag{15}$$

These closure equations are identical with the equations obtained by Cheng [7] using a method based on the mechanism's symmetry.

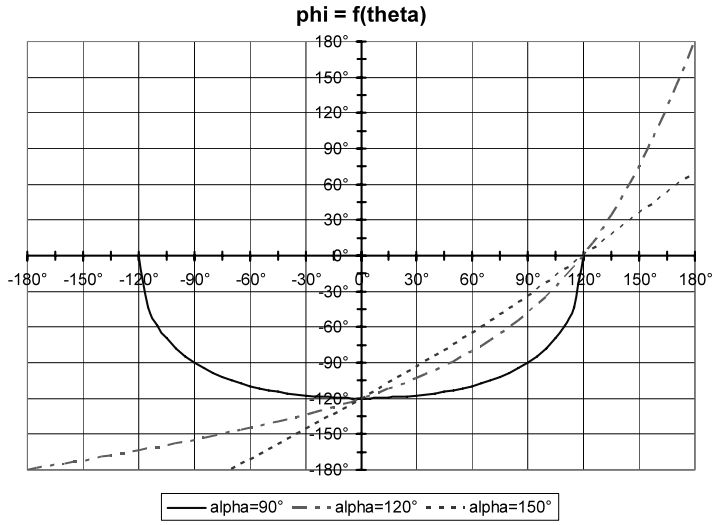


Fig. 3 Input-output curves for $\alpha = 90°$, $\alpha = 120°$ and $\alpha = 150°$

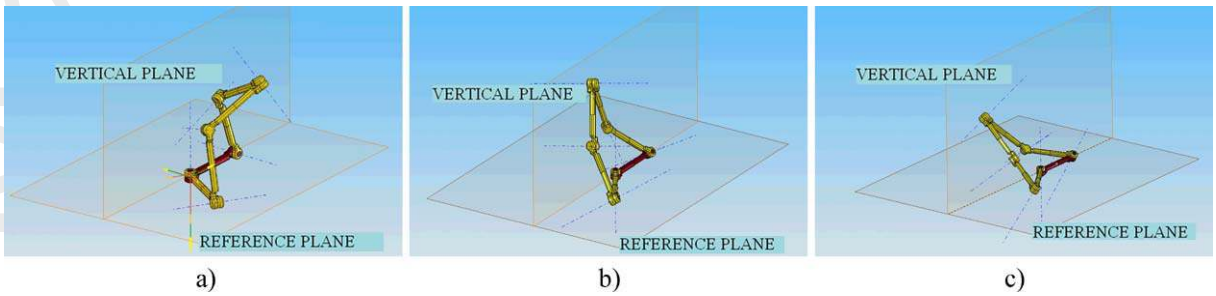


Fig. 4 Wohlhart symmetric mechanisms when: (**a**) $\alpha = 90°$; (**b**) $\alpha = 120°$; (**c**) $\alpha = 150°$

An interesting observation is that the input-output equation is symmetrical in θ and ϕ, so we can consider the output as input and vice-versa, the input-output equation remaining the same. A detailed calculus for finding all these closure equations is available in Racila [14].

Figure 3 shows the curves of input-output equation $\phi = f(\theta)$ for three twist angle: $\alpha = 90°$, $\alpha = 120°$ and $\alpha = 150°$.

The variation curve for a twist angle $\alpha = 90°$ is shown only in the $z < 0$ half space, for the other half space $z > 0$ this variation is the same if we impose a identical back way, or horizontal axis symmetrically if the mechanism follows its movement under the horizontal plane $z = 0$, the movement plane of the first element.

We must precise that this particular twist angle (Bricard symmetric rectangular mechanism) is the only angle that allows a continuous movement of the mechanism, for the other twist angles the mechanism's elements being in collision for some particular input angles.

One observes that for a twist angle of 120° (Fig. 4b) the input angles can vary between $\theta = -180°$ and $\theta = +180°$, but the mechanism cannot follow the movement beyond of $\theta = 180°$ because of the collision of mechanism's elements.

For a twist angle of 150° (Fig. 4c) the input angle cannot take smaller values to about $-70°$, more precisely $\theta = -70.53°$. This limitation is due to the fact that, for an input angle $\theta < -70.53°$, the mechanism's elements are in collision.

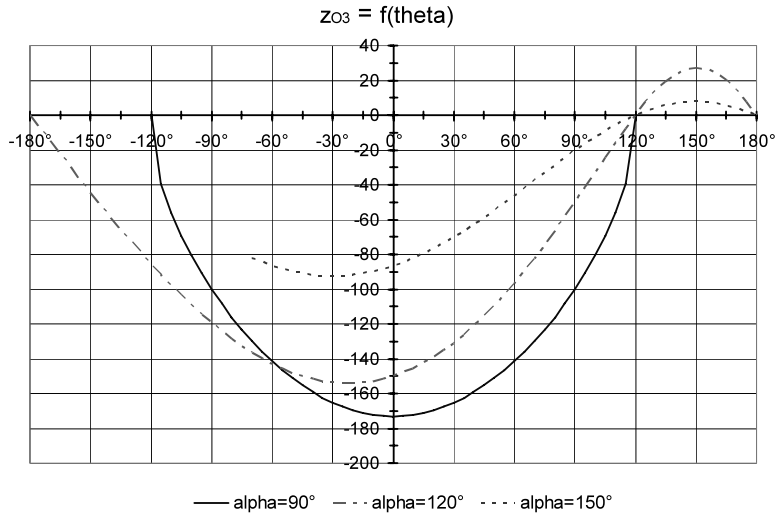


Fig. 5 Variation curves of z_{O_3} for 90°, 120° and 150° twist angles

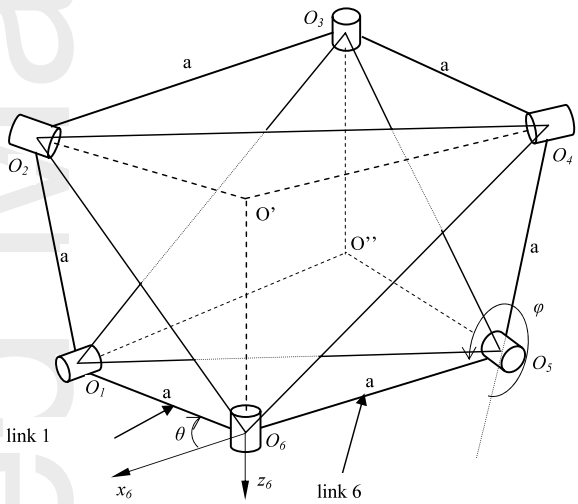


Fig. 6 The two parallels planes

With the help of the input-output equation we can also find the coordinates for some particular points of the mechanism. Figure 5 shows the variation of z coordinate for point O_3.

3 Spatial properties of 6R Wohlhart symmetric mechanism

During this analytical resolution of the Wohlhart symmetric mechanism, we have observed some interesting properties of this mechanism.

The most interesting properties highlight that two planes are always parallel during the deformation of the mechanism.

These two planes are the plane defined by odd joints and the plane defined by even joints (Fig. 6).

To prove this important property, we use an analytical method. We know the input-output equation and the coordinates of joints centers (Appendix 2).

The $(O_2O_4O_6)$ and $(O_1O_3O_5)$ planes equations are obtained with the help of O_2, O_4 and O_6 coordinates, respectively O_1, O_3, and O_5 coordinates.

The $(O_2O_4O_6)$ plane equation is:

$$A_3 \cdot x + B_3 \cdot y + C_3 \cdot z + D_3 = 0 \tag{16}$$

with:

$$\begin{cases} A_3 = a^2 \cdot \sin\varphi \cdot \sin\alpha \cdot (\sin\theta \cdot \cos\varphi \\ \quad + \cos\theta \cdot \sin\varphi \cdot \cos\alpha \\ \quad - \sin\varphi \cdot \cos\alpha + \sin\theta) \\ B_3 = -a^2 \cdot \sin\varphi \cdot \sin\alpha \cdot (\cos\theta + \cos\varphi \\ \quad + \cos\theta \cdot \cos\varphi \\ \quad - \sin\theta \cdot \sin\varphi \cdot \cos\alpha + 1) \\ C_3 = a^2 \cdot \sin\varphi \cdot \cos\alpha \cdot (\cos\theta \cdot \cos\varphi \\ \quad - \sin\theta \cdot \sin\varphi \cdot \cos\alpha + \cos\theta) \\ \quad + a^2 \cdot (\cos\theta + 1) \cdot (\sin\theta \cdot \cos\varphi \\ \quad + \cos\theta \cdot \sin\varphi \cdot \cos\alpha + \sin\theta) \\ D_3 = 0 \end{cases} \tag{17}$$

And the $(O_1O_3O_5)$ plane equation is:

$$A_4 \cdot x + B_4 \cdot y + C_4 \cdot z + D_4 = 0 \tag{18}$$

with:

$$\begin{cases} A_4 = a^2 \cdot \sin\theta \cdot \sin\alpha \cdot (\cos\theta \cdot \sin\varphi \\ \quad + \sin\theta \cdot \cos\varphi \cdot \cos\alpha \\ \quad - \sin\theta \cdot \cos\alpha + \sin\varphi) \\ B_4 = -a^2 \cdot (\cos\theta + 1) \cdot \sin\alpha \cdot (\cos\theta \cdot \sin\varphi \\ \quad + \sin\theta \cdot \cos\varphi \cdot \cos\alpha \\ \quad - \sin\theta \cdot \cos\alpha + \sin\varphi) \\ C_4 = a^2 \cdot \sin\theta \cdot (\cos\theta \cdot \cos\varphi \\ \quad - \sin\theta \cdot \sin\varphi \cdot \cos\alpha + \cos\varphi) \\ \quad + a^2 \cdot (\cos\theta + 1) \cdot (\cos\theta \cdot \sin\varphi \cdot \cos\alpha \\ \quad + \sin\theta \cdot \cos\varphi \cdot \cos^2\alpha \\ \quad + \sin\theta \cdot \sin^2\alpha + \sin\varphi \cdot \cos\alpha) \\ D_4 = a^3 \cdot \sin\theta \cdot \sin\alpha \cdot (\cos\theta \cdot \sin\varphi \\ \quad + \sin\theta \cdot \cos\varphi \cdot \cos\alpha \\ \quad - \sin\theta \cdot \cos\alpha + \sin\varphi) \end{cases} \tag{19}$$

Introducing half-angles $\theta/2$ and $\phi/2$, the $(O_2O_4O_6)$ and $(O_1O_3O_5)$ planes equations are written:

$$\left[\sin\frac{\theta}{2} \cdot \sin\frac{\varphi}{2} \cdot \sin\alpha\right]x + \left[-\cos\frac{\theta}{2} \cdot \sin\frac{\varphi}{2} \cdot \sin\alpha\right]y + \left[\sin\frac{\theta}{2} \cdot \cos\frac{\varphi}{2} + \cos\frac{\theta}{2} \cdot \sin\frac{\varphi}{2} \cdot \cos\alpha\right]z = 0 \tag{20}$$

$$\left[\sin\frac{\theta}{2} \cdot \sin\frac{\varphi}{2} \cdot \sin\alpha\right]x + \left[-\cos\frac{\theta}{2} \cdot \sin\frac{\varphi}{2} \cdot \sin\alpha\right]y + \left[\sin\frac{\theta}{2} \cdot \cos\frac{\varphi}{2} + \cos\frac{\theta}{2} \cdot \sin\frac{\varphi}{2} \cdot \cos\alpha\right]z + a \cdot \sin\frac{\theta}{2} \cdot \sin\frac{\varphi}{2} \cdot \sin\alpha = 0 \tag{21}$$

The two planes $(O_2O_4O_6)$ and $(O_1O_3O_5)$ have equations which are only different in the independent term. They are therefore parallel.

A second important property of this mechanism is that the intersecting point of even joints axes O' and odd joints axes O'' formed a line $O'O''$ which is always perpendicular on the planes (O_2, O_4, O_6) and (O_1, O_3, O_5).

In order to demonstrate this property we must know the coordinates of the points O' and O''. These coordinates will be dependent on the input angle θ and the twist angle α.

For this, we consider four points A, B, C and D necessary to define the rotation axes equations.

In Fig. 7 the A, B, C and D coordinates are written in the mobile reference frame linked to the own joint, $x = 0$, $y = 0$ and $z = -1$.

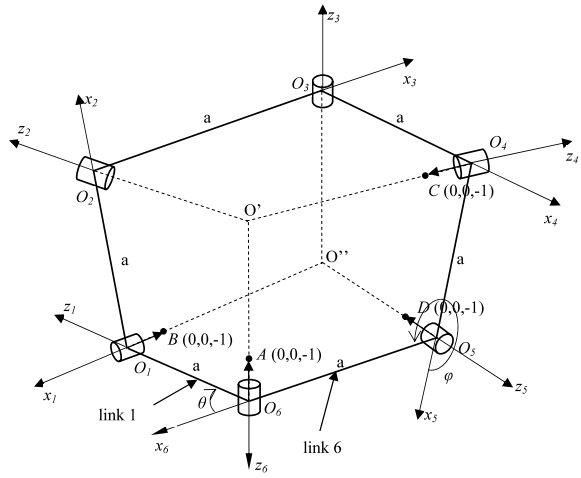


Fig. 7 Unity vectors of O_6O', O_1O'', O_4O' and O_5O'' lines

The coordinates of the points A, B, C and D in the fixed reference frame $R_6(O_6x_6y_6z_6)$ are given in the Appendix 3.

With these coordinates, we write the equations of the lines O_6O' and O_4O', respectively O_1O'' and O_5O''. The coordinates of the points O' and O'' satisfied the two first equations, respectively the two last ones.

We finally obtain the coordinates for these two points in the fixed reference frame R_6:

For the point O':

$$\begin{cases} x_{O'} = 0 \\ y_{O'} = 0 \\ z_{O'} = \frac{a \cdot (\cos\varphi + 1)}{\sin\varphi \cdot \sin\alpha} \end{cases} \tag{22}$$

For the point O'':

$$\begin{cases} x_{O''} = -a \\ y_{O''} = \frac{a \cdot \sin\theta}{1 - \cos\theta} \\ z_{O''} = -\frac{a \cdot (\cos\theta + 1)}{\sin\theta} \cdot \cot\alpha \end{cases} \tag{23}$$

By taking account of (22) and (23) relations, the $O'O''$ equation can be written:

$$\frac{x}{a} = \frac{y}{-\frac{a \cdot \sin\theta}{1 - \cos\theta}} = \frac{z - \frac{a \cdot (\cos\varphi + 1)}{\sin\varphi \cdot \sin\alpha}}{\frac{a \cdot (\cos\theta + 1)}{\sin\theta} \cot\alpha + \frac{a \cdot (\cos\varphi + 1)}{\sin\varphi \cdot \sin\alpha}} \tag{24}$$

The unity vector $\vec{n}$ of this line has the next components:

$$\begin{cases} n_x = 1 \\ n_y = -\frac{\sin\theta}{1-\cos\theta} \\ n_z = \frac{(\cos\theta+1)}{\sin\theta}\cot\alpha + \frac{(\cos\varphi+1)}{\sin\varphi\cdot\sin\alpha} \end{cases} \tag{25}$$

The unity vector $\vec{N}$ orthogonal on the plane $(O_2O_4O_6)$ has the components given by the relation (20).

In order that the $O'O''$ line to be perpendicular to the plane $(O_2O_4O_6)$, it is necessary that vectors $\vec{n}$ and $\vec{N}$ are parallel, so their components must satisfy the relation:

$$\begin{aligned} &\frac{\sin\frac{\theta}{2}\cdot\sin\frac{\varphi}{2}\cdot\sin\alpha}{1} \\ &= \frac{\cos\frac{\theta}{2}\cdot\sin\frac{\varphi}{2}\cdot\sin\alpha}{\frac{\sin\theta}{1-\cos\theta}} \\ &= \frac{\sin\frac{\theta}{2}\cdot\cos\frac{\varphi}{2}+\cos\frac{\theta}{2}\cdot\sin\frac{\varphi}{2}\cdot\cos\alpha}{\frac{(\cos\theta+1)}{\sin\theta}\cot\alpha + \frac{(\cos\varphi+1)}{\sin\varphi\cdot\sin\alpha}} \end{aligned} \tag{26}$$

Some elementary calculations on the previous relation show that the three fractions are equals.

The relation (26) proves that the line $O'O''$ is perpendicular to the plane (O_2, O_4, O_6), so implicit to the plane (O_1, O_3, O_5), in any position during the movement of the mechanism.

4 New spatial disposition

The important properties from above led us to consider a new spatial disposition.

We constrain three revolute joints, by example O_1, O_3 and O_5, to remain in a fixed plane, and we consider this plane as the reference, the plane of the movement.

The three joints O_1, O_3 and O_5 have displacements on three line segments, to the center of their circumscribed circle.

The other three joints O_2, O_4 and O_6 remain always in a parallel plane with the first one, the reference plane.

Figure 8 shows Wohlhart symmetric mechanism in this new spatial disposition, when the twist angle is $\alpha = \pi/2$.

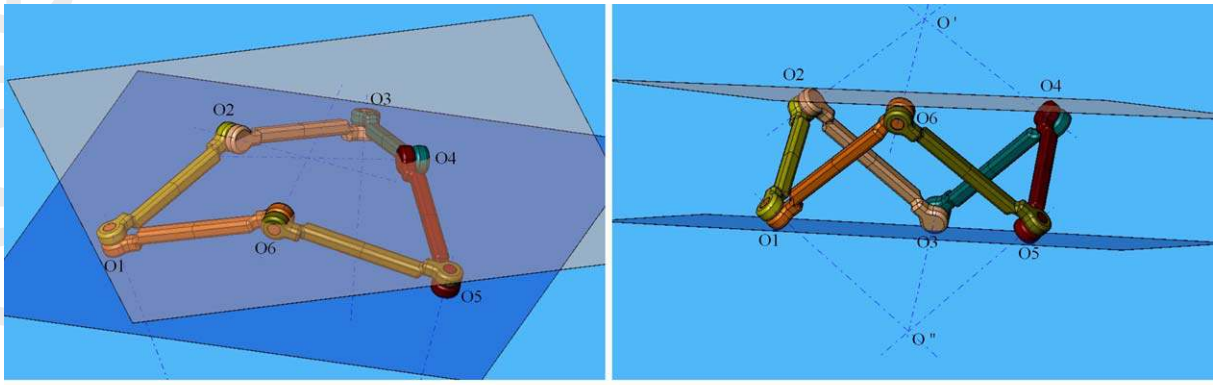
To facilitate the analysis of this mechanism, we consider the next notations in the horizontal plane:

- with subscript the joints number;
- with superscript the position of the joints in its displacements.

We note with b the distances $O_1^1O_1^2$, $O_3^1O_3^2$ and $O_5^1O_5^2$. These distances represent the joints displacements in the horizontal plane of movement (Fig. 9).

The joints O_2, O_4 and O_6 grow up to a maximal distance from the horizontal plane. By continuing the movement, the joints O_2, O_4 and O_6 come back to the horizontal plane.

Due to the mechanism symmetry the triangles $O_5O_6O_1$, $O_1O_2O_3$ and $O_3O_4O_5$ are congruent, and the triangles $O_6O_1O_2$, $O_2O_3O_4$ and $O_4O_5O_6$ are also congruent.



a) b)

Fig. 8 New spatial disposition

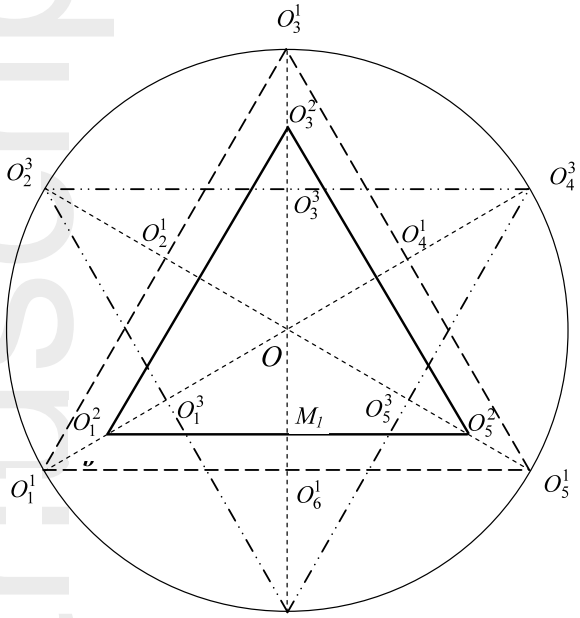


Fig. 9 Horizontal plane of the movement

The consequence is that, when the joints O_1, O_3 and O_5 are constrained to make displacements to the long of the heights of the triangle $O_1^1O_3^1O_5^1$, the other three joints O_2, O_4 and O_6 are always in perpendicular planes to the horizontal plane.

More precisely, when $b = 0$, all the six joints are in the initial position $O_1^1O_2^1O_3^1O_4^1O_5^1O_6^1$ in the horizontal plane, when the input angle θ has zero degrees. When b increases, which means when the three "actuated" joints O_1, O_3 and O_5 follow the heights of the initial triangle $O_1^1O_3^1O_5^1$, the other three joints O_2, O_4 and O_6 are in perpendicular planes to the horizontal plane, and the input angle θ increases.

In Fig. 10 the mechanism is represented in an intermediary position. For the clarity of figure, the joint number 3 is not represented.

The distance between the horizontal plane and the translated plane is noted $H = OT$.

Continuing the movement, the mechanism reaches its final position $O_1^3O_2^3O_3^3O_4^3O_5^3O_6^3$, where all the six joints are again in the fixed horizontal plane but in a different position considering the initial one. In this final position, the input angles has its maximal value $\theta = 2\pi/3$.

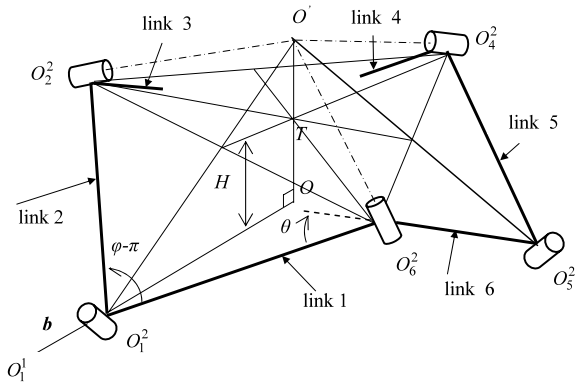


Fig. 10 Intermediary position of the mechanism

In the triangles $O_1^2O_6^2M_1$ and $O_1^2OM_1$ we can write the next expression:

$$O_1^2M_1 = a \cdot \cos\frac{\theta}{2} = \frac{2a - b\sqrt{3}}{2} \tag{27}$$

We obtain the dependence of the variable distance b according to the input angle θ:

$$b = \frac{2a}{\sqrt{3}} \cdot \left(1 - \cos\frac{\theta}{2}\right) \tag{28}$$

We can obtain the variation between the two planes, the fixed one and the translated plane [15].

$$H = \frac{|a \cdot \sin\frac{\theta}{2} \cdot \sin\frac{\varphi}{2} \cdot \sin\alpha|}{\sqrt{\sin^2\frac{\varphi}{2} \cdot \sin^2\alpha + (\sin\frac{\theta}{2} \cdot \cos\frac{\varphi}{2} + \cos\frac{\theta}{2} \cdot \sin\frac{\varphi}{2} \cdot \cos\alpha)^2}} \tag{29}$$

The previous relation permits to draw the variation of this distance according to the input angle θ (Fig. 11a) or to the distance b (Fig. 11b), knowing the variation of this distance (18) according to the input angle. These curves are drawn for three link lengths $a = 40$, $a = 80$ and $a = 120$, and for a twist angle $\alpha = 150°$.

We can also draw the variation of the distance H according to the input distance b for different twist angles. Figure 12 shown this variation for three twist angles $\alpha = 90°$, $\alpha = 120°$ and $\alpha = 150°$.

A prototype model (Fig. 13) was developed at the Applied Mechanical Laboratory in Besançon (France), for a twist angle $\alpha = \pi/2$, and a second prototype with a twist angle $\alpha = 150°$ is under realization.

5 Conclusions

In this paper we have presented a new and complete method for the calculation of closure equations

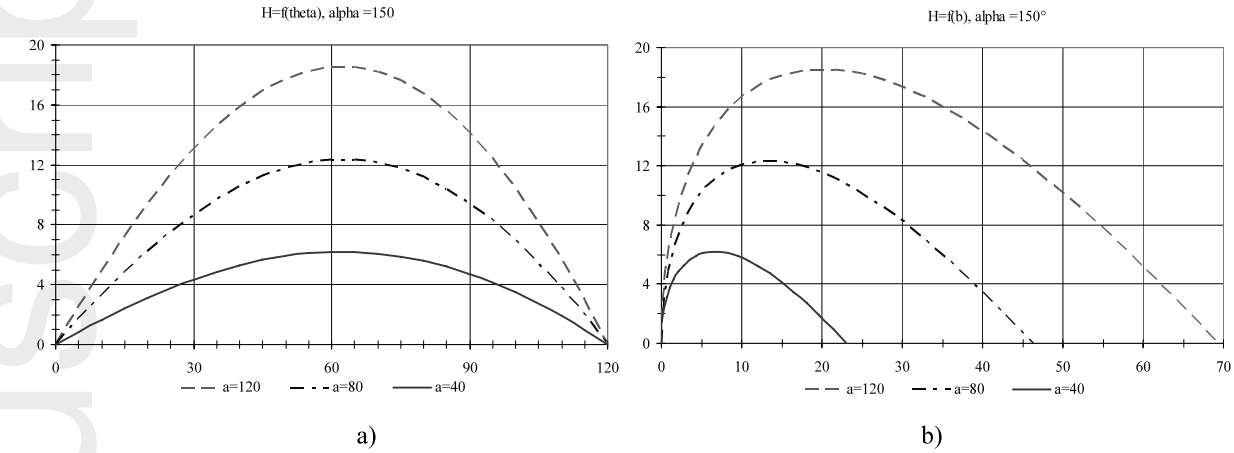


Fig. 11 Variation of $H = f(\theta)$ and $H = f(b)$ for a twist angle $\alpha = 150°$ and for three links length: $a = 40$, $a = 80$ and $a = 120$

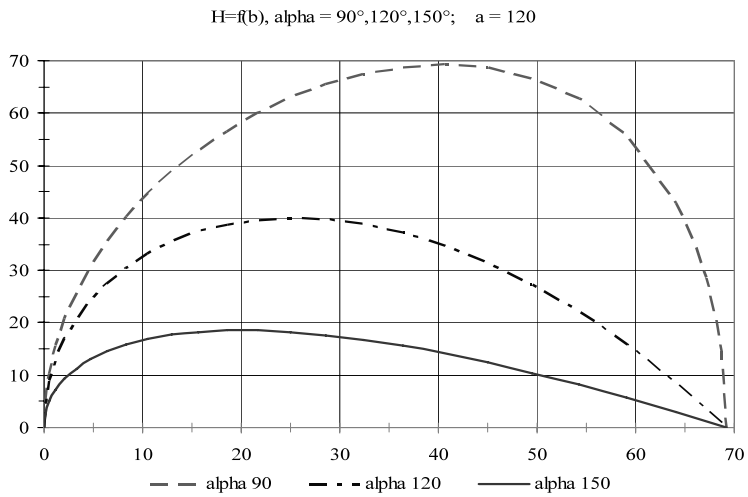


Fig. 12 Variation $H = f(b)$ for a links length $a = 120$

for Wohlhart symmetric mechanism. This analytic method permits an accurate analysis of the mechanism's kinematics and gives a viable alternative to geometrical methods used by the other authors. All these calculus have allowed to observe and to demonstrate new geometrical and kinematical properties of these mechanisms.

An important property specific to these mechanisms: parallelism between even joints plane and odd joints plane, has permitted to use the Wohlhart symmetric mechanism in a new spatial disposition and to obtain a translator.

The biggest advantage of this translator is to present an important surface in translation, a plane surface defined with a good precision by three points.

The second prototype of the translator, built for a twist angle $\alpha = 150°$, will assure a better precision, as we can clearly observe on Fig. 12. Of course, the working domain of this translator is on the second part of the curves, on the lighter (smaller) slope. This second prototype is currently developed in Applied Mechanical Laboratory.

Another important property of this translator concerns the line $O'O''$, the line formed by the intersection points of even and odd axes joints. During the movement this line remains fixed in space and it is orthogonal to the horizontal and translated planes.

This property can be used in the future for other possible applications based on these mechanisms, knowing the lack of industrial applications for overconstrained mechanisms.

Fig. 13 The prototype of the translator for a twist angle $\alpha = 90°$

Appendix 1

$$
\begin{aligned}
&\cos\theta_4 \cdot \cos\theta_5 \cdot \cos\theta_6 - \sin\theta_4 \cdot \sin\theta_5 \cdot \cos\theta_6 \cdot \cos\alpha \\
&\quad - \cos\theta_4 \cdot \sin\theta_5 \cdot \sin\theta_6 \cdot \cos\alpha \\
&\quad - \sin\theta_4 \cdot \cos\theta_5 \cdot \sin\theta_6 \cdot \cos^2\alpha \\
&\quad - \sin\theta_4 \cdot \sin\theta_6 \cdot \sin^2\alpha \\
&= \cos\theta_1 \cdot \cos\theta_2 \cdot \cos\theta_3 \\
&\quad - \cos\theta_1 \cdot \sin\theta_2 \cdot \sin\theta_3 \cdot \cos\alpha \\
&\quad - \sin\theta_1 \cdot \sin\theta_2 \cdot \cos\theta_3 \cdot \cos\alpha \\
&\quad - \sin\theta_1 \cdot \cos\theta_2 \cdot \sin\theta_3 \cdot \cos^2\alpha \\
&\quad - \sin\theta_1 \cdot \sin\theta_3 \cdot \sin^2\alpha \qquad \text{(A.1)} \\
&- \cos\theta_4 \cdot \cos\theta_5 \cdot \sin\theta_6 \cdot \cos\alpha \\
&\quad + \sin\theta_4 \cdot \sin\theta_5 \cdot \sin\theta_6 \cdot \cos^2\alpha \\
&\quad - \cos\theta_4 \cdot \sin\theta_5 \cdot \cos\theta_6 \cdot \cos^2\alpha \\
&\quad - \sin\theta_4 \cdot \cos\theta_5 \cdot \cos\theta_6 \cdot \cos^3\alpha \\
&\quad - \sin\theta_4 \cdot \cos\theta_6 \cdot \cos\alpha \cdot \sin^2\alpha \\
&\quad - \cos\theta_4 \cdot \sin\theta_5 \cdot \sin^2\alpha \\
&\quad - \sin\theta_4 \cdot \cos\theta_5 \cdot \sin^2\alpha \cdot \cos\alpha \\
&\quad + \sin\theta_4 \cdot \sin^2\alpha \cdot \cos\alpha \\
&= \sin\theta_1 \cdot \cos\theta_2 \cdot \cos\theta_3 \\
&\quad - \sin\theta_1 \cdot \sin\theta_2 \cdot \sin\theta_3 \cdot \cos\alpha \\
&\quad + \cos\theta_1 \cdot \sin\theta_2 \cdot \cos\theta_3 \cdot \cos\alpha \\
&\quad + \cos\theta_1 \cdot \cos\theta_2 \cdot \sin\theta_3 \cdot \cos^2\alpha \\
&\quad + \cos\theta_1 \cdot \sin\theta_3 \cdot \sin^2\alpha \qquad \text{(A.2)} \\
&- \cos\theta_4 \cdot \cos\theta_5 \cdot \sin\theta_6 \cdot \sin\alpha \\
&\quad + \sin\theta_4 \cdot \sin\theta_5 \cdot \sin\theta_6 \cdot \sin\alpha \cdot \cos\alpha \\
&\quad - \cos\theta_4 \cdot \sin\theta_5 \cdot \cos\theta_6 \cdot \sin\alpha \cdot \cos\alpha \\
&\quad - \sin\theta_4 \cdot \cos\theta_5 \cdot \cos\theta_6 \cdot \sin\alpha \cdot \cos^2\alpha \\
&\quad - \sin\theta_4 \cdot \cos\theta_6 \cdot \sin^3\alpha \\
&\quad + \cos\theta_4 \cdot \sin\theta_5 \cdot \sin\alpha \cdot \cos\alpha \\
&\quad + \sin\theta_4 \cdot \cos\theta_5 \cdot \sin\alpha \cdot \cos^2\alpha \\
&\quad - \sin\theta_4 \cdot \sin\alpha \cdot \cos^2\alpha \\
&= \sin\theta_2 \cdot \cos\theta_3 \cdot \sin\alpha \\
&\quad + \cos\theta_2 \cdot \sin\theta_3 \cdot \sin\alpha \cdot \cos\alpha \\
&\quad - \sin\theta_3 \cdot \sin\alpha \cdot \cos\alpha \qquad \text{(A.3)} \\
&\cos\theta_4 \cdot \cos\theta_5 \cdot \cos\theta_6 \\
&\quad - \sin\theta_4 \cdot \sin\theta_5 \cdot \cos\theta_6 \cdot \cos\alpha \\
&\quad - \cos\theta_4 \cdot \sin\theta_5 \cdot \sin\theta_6 \cdot \cos\alpha \\
&\quad - \sin\theta_4 \cdot \cos\theta_5 \cdot \sin\theta_6 \cdot \cos^2\alpha \\
&\quad - \sin\theta_4 \cdot \sin\theta_6 \cdot \sin^2\alpha + \cos\theta_4 \cdot \cos\theta_5 \\
&\quad - \sin\theta_4 \cdot \sin\theta_5 \cdot \cos\alpha + \cos\theta_4
\end{aligned}
$$

$$
\begin{aligned}
&= -\cos\theta_2 \cdot \cos\theta_3 + \sin\theta_2 \cdot \sin\theta_3 \cdot \cos\alpha \\
&\quad -\cos\theta_3 - 1 \qquad (A.4)\\
&\sin\theta_4 \cdot \cos\theta_5 \cdot \cos\theta_6 \\
&\quad +\cos\theta_4 \cdot \sin\theta_5 \cdot \cos\theta_6 \cdot \cos\alpha \\
&\quad -\sin\theta_4 \cdot \sin\theta_5 \cdot \sin\theta_6 \cdot \cos\alpha \\
&\quad +\cos\theta_4 \cdot \cos\theta_5 \cdot \sin\theta_6 \cdot \cos^2\alpha \\
&\quad +\cos\theta_4 \cdot \sin\theta_6 \cdot \sin^2\alpha \\
&= -\cos\theta_1 \cdot \cos\theta_2 \cdot \sin\theta_3 \cdot \cos\alpha \\
&\quad -\cos\theta_1 \cdot \sin\theta_2 \cdot \cos\theta_3 \cdot \cos^2\alpha \\
&\quad -\cos\theta_1 \cdot \sin\theta_2 \cdot \sin^2\alpha \\
&\quad +\sin\theta_1 \cdot \sin\theta_2 \cdot \sin\theta_3 \cdot \cos^2\alpha \\
&\quad -\sin\theta_1 \cdot \cos\theta_2 \cdot \cos\theta_3 \cdot \cos^3\alpha \\
&\quad -\sin\theta_1 \cdot \cos\theta_2 \cdot \sin^2\alpha \cdot \cos\alpha \\
&\quad -\sin\theta_1 \cdot \cos\theta_3 \cdot \sin^2\alpha \cdot \cos\alpha \\
&\quad +\sin\theta_1 \cdot \sin^2\alpha \cdot \cos\alpha \qquad (A.5)\\
&-\sin\theta_4 \cdot \cos\theta_5 \cdot \sin\theta_6 \cdot \cos\alpha \\
&\quad -\cos\theta_4 \cdot \sin\theta_5 \cdot \sin\theta_6 \cdot \cos^2\alpha \\
&\quad -\sin\theta_4 \cdot \sin\theta_5 \cdot \cos\theta_6 \cdot \cos^2\alpha \\
&\quad +\cos\theta_4 \cdot \cos\theta_5 \cdot \cos\theta_6 \cdot \cos^3\alpha \\
&\quad +\cos\theta_4 \cdot \cos\theta_6 \cdot \sin^2\alpha \cdot \cos\alpha \\
&\quad -\sin\theta_4 \cdot \sin\theta_5 \cdot \sin^2\alpha \\
&\quad +\cos\theta_4 \cdot \cos\theta_5 \cdot \sin^2\alpha \cdot \cos\alpha \\
&\quad -\cos\theta_4 \cdot \sin^2\alpha \cdot \cos\alpha \\
&= -\sin\theta_1 \cdot \cos\theta_2 \cdot \sin\theta_3 \cdot \cos\alpha \\
&\quad -\sin\theta_1 \cdot \sin\theta_2 \cdot \cos\theta_3 \cdot \cos^2\alpha \\
&\quad -\sin\theta_1 \cdot \sin\theta_2 \cdot \sin^2\alpha \\
&\quad -\cos\theta_1 \cdot \sin\theta_2 \cdot \sin\theta_3 \cdot \cos^2\alpha \\
&\quad +\cos\theta_1 \cdot \cos\theta_2 \cdot \cos\theta_3 \cdot \cos^3\alpha \\
&\quad +\cos\theta_1 \cdot \cos\theta_2 \cdot \sin^2\alpha \cdot \cos\alpha \\
&\quad +\cos\theta_1 \cdot \cos\theta_3 \cdot \sin^2\alpha \cdot \cos\alpha \\
&\quad -\cos\theta_1 \cdot \sin^2\alpha \cdot \cos\alpha \qquad (A.6)\\
&-\sin\theta_4 \cdot \cos\theta_5 \cdot \sin\theta_6 \cdot \sin\alpha \\
&\quad -\cos\theta_4 \cdot \sin\theta_5 \cdot \sin\theta_6 \cdot \sin\alpha \cdot \cos\alpha \\
&\quad -\sin\theta_4 \cdot \sin\theta_5 \cdot \cos\theta_6 \cdot \sin\alpha \cdot \cos\alpha \\
&\quad +\cos\theta_4 \cdot \cos\theta_5 \cdot \cos\theta_6 \cdot \sin\alpha \cdot \cos^2\alpha \\
&\quad +\cos\theta_4 \cdot \cos\theta_6 \cdot \sin^3\alpha \\
&\quad +\sin\theta_4 \cdot \sin\theta_5 \cdot \sin\alpha \cdot \cos\alpha \\
&\quad -\cos\theta_4 \cdot \cos\theta_5 \cdot \sin\alpha \cdot \cos^2\alpha \\
&\quad +\cos\theta_4 \cdot \sin\alpha \cdot \cos^2\alpha \\
&= -\sin\theta_2 \cdot \sin\theta_3 \cdot \sin\alpha \cdot \cos\alpha \\
&\quad +\cos\theta_2 \cdot \cos\theta_3 \cdot \sin\alpha \cdot \cos^2\alpha \\
&\quad +\cos\theta_2 \cdot \sin^3\alpha - \cos\theta_3 \cdot \sin\alpha \cdot \cos^2\alpha \\
&\quad +\sin\alpha \cdot \cos^2\alpha \qquad (A.7)\\
&\sin\theta_4 \cdot \cos\theta_5 \cdot \cos\theta_6 \\
&\quad +\cos\theta_4 \cdot \sin\theta_5 \cdot \cos\theta_6 \cdot \cos\alpha \\
&\quad -\sin\theta_4 \cdot \sin\theta_5 \cdot \sin\theta_6 \cdot \cos\alpha \\
&\quad +\cos\theta_4 \cdot \cos\theta_5 \cdot \sin\theta_6 \cdot \cos^2\alpha \\
&\quad +\cos\theta_4 \cdot \sin\theta_6 \cdot \sin^2\alpha + \sin\theta_4 \cdot \cos\theta_5 \\
&\quad +\cos\theta_4 \cdot \sin\theta_5 \cdot \cos\alpha + \sin\theta_4 \\
&= \cos\theta_2 \cdot \sin\theta_3 \cdot \cos\alpha \\
&\quad +\sin\theta_2 \cdot \cos\theta_3 \cdot \cos^2\alpha \\
&\quad +\sin\theta_2 \cdot \sin^2\alpha + \sin\theta_3 \cdot \cos\alpha \qquad (A.8)\\
&-\sin\theta_5 \cdot \cos\theta_6 \cdot \sin\alpha \\
&\quad -\cos\theta_5 \cdot \sin\theta_6 \cdot \sin\alpha \cdot \cos\alpha \\
&\quad +\sin\theta_6 \cdot \sin\alpha \cdot \cos\alpha \\
&= \cos\theta_1 \cdot \cos\theta_2 \cdot \sin\theta_3 \cdot \sin\alpha \\
&\quad +\cos\theta_1 \cdot \sin\theta_2 \cdot \cos\theta_3 \cdot \sin\alpha \cdot \cos\alpha \\
&-\cos\theta_1 \cdot \sin\theta_2 \cdot \sin\alpha \cdot \cos\alpha \\
&\quad -\sin\theta_1 \cdot \sin\theta_2 \cdot \sin\theta_3 \cdot \sin\alpha \cdot \cos\alpha \\
&\quad +\sin\theta_1 \cdot \cos\theta_2 \cdot \cos\theta_3 \cdot \sin\alpha \cdot \cos^2\alpha \\
&\quad -\sin\theta_1 \cdot \cos\theta_2 \cdot \sin\alpha \cdot \cos^2\alpha \\
&\quad +\sin\theta_1 \cdot \cos\theta_3 \cdot \sin^3\alpha \\
&\quad +\sin\theta_1 \cdot \sin\alpha \cdot \cos^2\alpha \qquad (A.9)\\
&\sin\theta_5 \cdot \sin\theta_6 \cdot \sin\alpha \cdot \cos\alpha \\
&\quad -\cos\theta_5 \cdot \cos\theta_6 \cdot \sin\alpha \cdot \cos^2\alpha \\
&\quad +\cos\theta_6 \cdot \sin\alpha \cdot \cos^2\alpha - \cos\theta_5 \cdot \sin^3\alpha \\
&\quad -\sin\alpha \cdot \cos^2\alpha \\
&= \sin\theta_1 \cdot \cos\theta_2 \cdot \sin\theta_3 \cdot \sin\alpha
\end{aligned}
$$

$$
\begin{aligned}
&+\sin\theta_1\cdot\sin\theta_2\cdot\cos\theta_3\cdot\sin\alpha\cdot\cos\alpha\\
&-\sin\theta_1\cdot\sin\theta_2\cdot\sin\alpha\cdot\cos\alpha\\
&+\cos\theta_1\cdot\sin\theta_2\cdot\sin\theta_3\cdot\sin\alpha\cdot\cos\alpha\\
&-\cos\theta_1\cdot\cos\theta_2\cdot\cos\theta_3\cdot\sin\alpha\cdot\cos^2\alpha\\
&+\cos\theta_1\cdot\cos\theta_2\cdot\sin\alpha\cdot\cos^2\alpha\\
&-\cos\theta_1\cdot\cos\theta_3\cdot\sin^3\alpha\\
&-\cos\theta_1\cdot\sin\alpha\cdot\cos^2\alpha \qquad (A.10)\\
&\sin\theta_5\cdot\sin\theta_6\cdot\sin^2\alpha\\
&-\cos\theta_5\cdot\cos\theta_6\cdot\cos\alpha\cdot\sin^2\alpha\\
&+\cos\theta_6\cdot\cos\alpha\cdot\sin^2\alpha\\
&+\cos\theta_5\cdot\sin^2\alpha\cdot\cos\alpha+\cos^3\alpha\\
&=\sin\theta_2\cdot\sin\theta_3\cdot\sin^2\alpha\\
&-\cos\theta_2\cdot\cos\theta_3\cdot\sin^2\alpha\cdot\cos\alpha\\
&+\cos\theta_2\cdot\sin^2\alpha\cdot\cos\alpha\\
&+\cos\theta_3\cdot\sin^2\alpha\cdot\cos\alpha+\cos^3\alpha \qquad (A.11)\\
&-\sin\theta_5\cdot\cos\theta_6\cdot\sin\alpha\\
&-\cos\theta_5\cdot\sin\theta_6\cdot\sin\alpha\cdot\cos\alpha\\
&+\sin\theta_6\cdot\sin\alpha\cdot\cos\alpha-\sin\theta_5\cdot\sin\alpha\\
&=-\cos\theta_2\cdot\sin\theta_3\cdot\sin\alpha\\
&-\sin\theta_2\cdot\cos\theta_3\cdot\sin\alpha\cdot\cos\alpha\\
&+\sin\theta_2\cdot\sin\alpha\cdot\cos\alpha-\sin\theta_3\cdot\sin\alpha \qquad (A.12)
\end{aligned}
$$

Appendix 2

– The coordinates for the point O_1:

$$
\begin{bmatrix} x_{O_1}\\ y_{O_1}\\ z_{O_1}\\ 1\end{bmatrix}_6=\begin{bmatrix} a\cdot\cos\theta\\ a\cdot\sin\theta\\ 0\\ 1\end{bmatrix} \qquad (B.1)
$$

– The coordinates for the point O_2:

$$
\begin{bmatrix} x_{O_2}\\ y_{O_2}\\ z_{O_2}\\ 1\end{bmatrix}_6=\begin{bmatrix} a\cdot\cos\theta\cdot(\cos\varphi+1)-a\cdot\sin\theta\cdot\sin\varphi\cdot\cos\alpha\\ a\cdot\sin\theta\cdot(\cos\varphi+1)+a\cdot\cos\theta\cdot\sin\varphi\cdot\cos\alpha\\ a\cdot\sin\varphi\cdot\sin\alpha\\ 1\end{bmatrix} \qquad (B.2)
$$

– The coordinates for the point O_3:

$$
\begin{bmatrix} x_{O_3}\\ y_{O_3}\\ z_{O_3}\\ 1\end{bmatrix}_6=\begin{bmatrix} -a\cdot(-\sin\theta\cdot\sin\varphi\cdot\cos\alpha+\cos\theta\cdot\cos\varphi+\cos\varphi+1)\\ a\cdot(\cos\theta\cdot\sin\varphi\cdot\cos\alpha+\sin\theta\cdot\cos\varphi\cdot\cos^2\alpha+\sin\theta\cdot\sin^2\alpha+\sin\varphi\cdot\cos\alpha)\\ a\cdot\sin\alpha\cdot(\cos\theta\cdot\sin\varphi+\sin\theta\cdot\cos\varphi\cdot\cos\alpha-\sin\theta\cdot\cos\alpha+\sin\varphi)\\ 1\end{bmatrix} \qquad (B.3)
$$

– The coordinates for the point O_4:

$$
\begin{bmatrix} x_{O_4}\\ y_{O_4}\\ z_{O_4}\\ 1\end{bmatrix}_6=\begin{bmatrix} -a\cdot(\cos\varphi+1)\\ a\cdot\sin\varphi\cdot\cos\alpha\\ a\cdot\sin\varphi\cdot\sin\alpha\\ 1\end{bmatrix} \qquad (B.4)
$$

– The coordinates for the point O_5:

$$
\begin{bmatrix} x_{O_5}\\ y_{O_5}\\ z_{O_5}\\ 1\end{bmatrix}_6=\begin{bmatrix} -a\\ 0\\ 0\\ 1\end{bmatrix} \qquad (B.5)
$$

– The coordinates for the point O_6:

$$
\begin{bmatrix} x_{O_6}\\ y_{O_6}\\ z_{O_6}\\ 1\end{bmatrix}_6=\begin{bmatrix} 0\\ 0\\ 0\\ 1\end{bmatrix} \qquad (B.6)
$$

Appendix 3

The coordinates for the point A:

$$\begin{cases} x_A = 0 \\ y_A = 0 \\ z_A = -1 \end{cases} \tag{C.1}$$

The coordinates for the point B:

$$\begin{cases} x_B = -\sin\theta \cdot \sin\alpha + a \cdot \cos\theta \\ y_B = \cos\theta \cdot \sin\alpha + a \cdot \sin\theta \\ z_B = -\cos\alpha \end{cases} \tag{C.2}$$

The coordinates for the point C:

$$\begin{cases} x_C = -\sin\varphi \cdot \sin\alpha - a \cdot \cos\varphi - a \\ y_C = \cos\varphi \cdot \sin\alpha \cdot \cos\alpha + \sin\alpha \cdot \cos\alpha \\ \qquad + a \cdot \sin\varphi \cdot \cos\alpha \\ z_C = \cos\varphi \cdot \sin^2\alpha - \cos^2\alpha + a \cdot \sin\varphi \cdot \sin\alpha \end{cases} \tag{C.3}$$

The coordinates for the point D:

$$\begin{cases} x_D = -a \\ y_D = \sin\alpha \\ z_D = -\cos\alpha \end{cases} \tag{C.4}$$

References

1. Altmann PG (1954) Link mechanisms in modern kinematics. Communications to Paul Grodzinski and Ewen M'Ewen. Proc Inst Mech Eng 168(37):889–896
2. Baker JE (1980) An analysis of the Bricard linkages. Mech Mach Theory 15:267–286
3. Baker JE (2002) Displacement-closure equations of the unspecialised double-Hooke's-joint linkage. Mech Mach Theory 37:1127–1144
4. Bennett GT (1903) A new mechanism. Engineering 76:777–778
5. Bricard R (1897) Mémoire sur la théorie de l'octaèdre articulé. J Math Pures Appl Liouville 3:113–148
6. Bricard R (1927) Leçons de cinématique, tome II: cinématique appliquée. Gauthier–Villars, Paris
7. Chen Y, You Z, Tarnai T (2005) Threefold-symmetric Bricard linkages for deployable structures. Int J Solids Struct 42:2287–2301
8. Dietmaier P (1995) A new 6R space mechanism. In: Proceedings of the 9th world congress on the theory of machines and mechanisms, Milano, Italy, vol 1, pp 52–56
9. Denavit J, Hartenberg RS (1955) A kinematic notation for lower-pair mechanism based on matrices. J Appl Mech 22:215–221
10. Gan WW, Pellegrino S (2003) Closed-loop deployable structures. In: Proceedings of 44th AIAA/ASME/ASCE/AHS/ASC structures, structural dynamics, and materials conference, Norfolk, 7–10 April 2003, AIAA 2003-1450
11. Goldberg M (1943) New five-bar and six-bar linkages in three dimensions. Trans ASME 65:649–663
12. Mavroidis C, Roth B (1994) Analysis and synthesis of overconstrained mechanisms. In: Proceedings of the 1994 ASME, design technical conferences, Minneapolis, September 1994, vol DE-70, pp 115–133
13. Myard FE (1931) Chaîne fermée à cinq couples rotoïdes, déformable au premier degré de liberté. C R Hebd Séances Acad Sci 192:1352–1354
14. Racila L (2006), Contributions sur les mécanismes surcontraints à six liaisons rotoïdes. PhD dissertation, University of Franche Comte, Besancon
15. Racila L, Dahan M (2007) Bricard mechanism used as translator. In: Proceedings 12th world congress IFToMM, Besançon, vol 3, pp 337–341
16. Sarrus PF (1853) Note sur la transformation des mouvements rectilignes alternatifs, en mouvements circulaires; et réciproquement. C R Séances Acad Sci 36:1036–1038
17. Schatz P (1975) Rhytmusforschung und Technik. Verlag freies Geistesleben, Stuttgart
18. Waldron KJ (1969) Symmetric overconstrained linkages. Trans ASME, J Eng Ind V91:158–162
19. Waldron KJ (1979) Overconstrained linkages. Env Plann B 6:393–402
20. Wohlhart K (1987) A new 6R space mechanism. In: Proceeding 7th world congress IFToMM, Sevilla, vol 1, pp 193–198
21. Wohlhart K (1993) The two types of the orthogonal Bricard linkage. Mech Mach Theory 28:809–817