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## Generalized Mertens sums\*

Gérald Tenenbaum

*To Krishna Alladi, half-way,  
as a token of a life-long friendship.*

Let

$$S_k(x) := \sum_{p_1 \cdots p_k \leq x} \frac{1}{p_1 \cdots p_k} \quad (x \geq 2),$$

where  $p_j$  denotes a prime number. It is a well known result of Mertens that

$$S_1(x) = \log_2 x + c_1 + O\left(\frac{1}{\log x}\right) \quad (x \geq 3),$$

with (see, e.g., [3], p. 18)

$$(1) \quad c_1 := \gamma - \sum_p \left\{ \log \left( \frac{1}{1 - 1/p} \right) - \frac{1}{p} \right\} \approx 0.261497.$$

Here and in the sequel,  $\gamma$  is Euler's constant,  $p$  stands for a prime number and  $\log_2$  denotes the two-fold iterated logarithm. The number  $c_1$  is called Mertens' constant, also known as the Meissel-Mertens, or the Kronecker, or the Hadamard-La Vallée-Poussin constant.

In [1], [2], Popa used elementary techniques to derive similar asymptotic formulae in the cases  $k = 2$  and  $3$ , with a main term equal to a polynomial of degree  $k$  in  $\log_2 x$  and a remainder term  $\ll (\log_2 x)^k / \log x$ . In this note we investigate the general case. We define classically  $\Gamma$  as the Euler gamma function.

**Theorem 1.** *Let  $k \geq 1$ . We have*

$$S_k(x) = P_k(\log_2 x) + O\left(\frac{(\log_2 x)^{k-1}}{\log x}\right) \quad (x \geq 3),$$

where  $P_k(X) := \sum_{0 \leq j \leq k} \lambda_{j,k} X^j$ , and

$$\lambda_{j,k} := \sum_{0 \leq m \leq k-j} \binom{k}{m, j, k-m-j} (c_1 - \gamma)^{k-m-j} \left(\frac{1}{\Gamma}\right)^{(m)}(1) \quad (0 \leq j \leq k).$$

*Proof.* Write  $P(s) := \sum_p 1/p^s$ , so that we have

$$P(s) = \log \zeta(s) - g(s), \quad g(s) := \sum_{m \geq 2} \frac{1}{m} \sum_p \frac{1}{p^{ms}}$$

in any simply connected zero and pole-free region of the zeta function where the series  $g(s)$  converges. (Here  $\log \zeta(s)$  is the branch that is real for real  $s > 1$ .) Moreover, for  $s+1$  in the same region, we have

$$P(s+1) = \log(1/s) + h(s),$$

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\* We include here some corrections with respect to the published version.

with  $h(s) = \log\{s\zeta(s+1)\} - g(s+1)$  and where  $\log(1/s)$  is understood as the principal branch. The function  $h(s)$  is clearly holomorphic in a disk around  $s = 0$ .

Now, for any  $c > 0$ , we have

$$(2) \quad S_k(x) = \frac{1}{2\pi i} \int_{c+i\mathbb{R}} P(s+1)^k x^s \frac{ds}{s} \quad (x \in \mathbb{R}^+ \setminus \mathbb{N}).$$

By following, *mutatis mutandis*, the argument of the Selberg-Delange method (see [3], ch. II.5 & II.6) we readily obtain

$$S_k(x) = \frac{1}{2\pi i} \int_{\mathcal{H}} \left\{ \log\left(\frac{1}{s}\right) + h(0) \right\}^k x^s \frac{ds}{s} + O\left(\frac{(\log_2 x)^{k-1}}{\log x}\right) \quad (x \geq 2),^{(1)}$$

where  $\mathcal{H}$  is a Hankel contour around  $\mathbb{R}^-$ , positively oriented.

We also observe that, by (1), we have

$$h(0) = - \sum_p \left\{ \log\left(\frac{1}{1-1/p}\right) - \frac{1}{p} \right\} = c_1 - \gamma.$$

It remains to compute

$$I_m(x) := \frac{1}{2\pi i} \int_{\mathcal{H}} \left\{ \log\frac{1}{s} \right\}^m x^s \frac{ds}{s} \quad (m \geq 0).$$

To this end, we consider Hankel's formula (see, e.g., [3], th. II.0.17)

$$\frac{1}{2\pi i} \int_{\mathcal{H}} \frac{x^s}{s^{1+z}} ds = \frac{(\log x)^z}{\Gamma(z+1)} \quad (z \in \mathbb{C})$$

and derive

$$I_m(x) = \sum_{0 \leq j \leq m} \binom{m}{j} (\log_2 x)^j \left(\frac{1}{\Gamma}\right)^{(m-j)}(1).$$

Rearranging the terms, we arrive at the announced formula for  $P_k(X)$ .  $\square$

*Specialization.* Noting that  $(1/\Gamma)'(1) = \gamma$ ,  $(1/\Gamma)''(1) = \gamma^2 - \frac{1}{6}\pi^2$ ,  $(1/\Gamma)'''(1) = 2\zeta(3) - \frac{1}{2}\pi^2\gamma + \gamma^3$ ,  $(1/\Gamma)^{(4)}(1) = \frac{1}{60}\pi^4 + 8\gamma\zeta(3) + \pi^2\gamma^2 + \gamma^4$ , as may be deduced from classical formulae for the logarithmic derivative of the Euler function (see, e.g., [3], chap. II.0), we find

$$\begin{aligned} P_1(X) &= X + c_1, & P_2(X) &= (X + c_1)^2 - \frac{1}{6}\pi^2, & P_3(X) &= (X + c_1)^3 - \frac{1}{2}\pi^2(X + c_1) + 2\zeta(3), \\ P_4(X) &= (X + c_1)^4 - \pi^2(X + c_1)^2 + 8\zeta(3)(X + c_1) + \frac{1}{60}\pi^4. \end{aligned}$$

*Remark.* By retaining, in the integrand of (2), the first  $N+1$  terms of the Taylor expansion of  $h(s)$  at the origin, the above method readily yields, for arbitrary integer  $N \geq 0$ , an asymptotic formula of the type

$$S_k(x) = \sum_{0 \leq j \leq N} \frac{P_{j,k}(\log_2 x)}{(\log x)^j} + O\left(\frac{(\log_2 x)^{k-1}}{(\log x)^{N+1}}\right)$$

where  $P_{j,k}$  is an explicit polynomial of degree  $\leq k-1$ .

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## References

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1. Due to an oversight, the exponent of  $\log_2 x$  has been set to  $k$  instead of  $k-1$  in the published version of this work.