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A Hamiltonian structure of the Isobe–Kakinuma model for water waves

Dedicated to the late Professor Walter L. Craig

Vincent Duchêne and Tatsuo Iguchi

Abstract

We consider the Isobe–Kakinuma model for water waves, which is obtained as the system of Euler–Lagrange equations for a Lagrangian approximating Luke's Lagrangian for water waves. We show that the Isobe–Kakinuma model also enjoys a Hamiltonian structure analogous to the one exhibited by V. E. Zakharov on the full water wave problem and, moreover, that the Hamiltonian of the Isobe–Kakinuma model is a higher order shallow water approximation to the one of the full water wave problem.

1 Introduction

We consider a model for the motion of water in a moving domain of the $(n+1)$-dimensional Euclidean space. The water wave problem is mathematically formulated as a free boundary problem for an irrotational flow of an inviscid, incompressible, and homogeneous fluid under a vertical gravitational field. Let t be the time, $\boldsymbol{x} = (x_1, \dots, x_n)$ the horizontal spatial coordinates, and z the vertical spatial coordinate. We assume that the water surface and the bottom are represented as $z = \eta(\boldsymbol{x}, t)$ and $z = -h + b(\boldsymbol{x})$, respectively, where $\eta(\boldsymbol{x}, t)$ is the surface elevation, h is the mean depth, and $b(\boldsymbol{x})$ represents the bottom topography. We denote by $\Omega(t)$, $\Gamma(t)$, and Σ the water region, the water surface, and the bottom of the water at time t, respectively. Then, the motion of the water is described by the velocity potential $\Phi(\boldsymbol{x}, z, t)$ satisfying Laplace's equation

$$\Delta\Phi + \partial_z^2\Phi = 0 \quad \text{in} \quad \Omega(t),\ t > 0, \tag{1.1}$$

where $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$. The boundary conditions on the water surface are given by

$$\begin{cases} \partial_t\eta + \nabla\Phi \cdot \nabla\eta - \partial_z\Phi = 0 & \text{on} \quad \Gamma(t),\ t > 0, \\ \partial_t\Phi + \dfrac{1}{2}\big(|\nabla\Phi|^2 + (\partial_z\Phi)^2\big) + g\eta = 0 & \text{on} \quad \Gamma(t),\ t > 0, \end{cases} \tag{1.2}$$

where $\nabla = (\partial_{x_1}, \dots, \partial_{x_n})^{\mathrm{T}}$, and g is the gravitational constant. The first equation is the kinematic condition on the water surface and the second one is Bernoulli's equation. Finally, the boundary condition on the bottom of the water is given by

$$\nabla\Phi \cdot \nabla b - \partial_z\Phi = 0 \quad \text{on} \quad \Sigma,\ t > 0, \tag{1.3}$$

which is the kinematic condition on the fixed and impermable bottom. These are the basic equations for the water wave problem.

We put

$$\phi(\boldsymbol{x}, t) = \Phi(\boldsymbol{x}, \eta(\boldsymbol{x}, t), t), \tag{1.4}$$

which is the trace of the velocity potential on the water surface. Then, the basic equations for water waves (1.1)–(1.3) are transformed equivalently into

$$\begin{cases} \partial_t \eta - \Lambda(\eta, b)\phi = 0 & \text{on} \quad \mathbf{R}^n,\ t > 0, \\ \partial_t \phi + g\eta + \dfrac{1}{2}|\nabla\phi|^2 - \dfrac{1}{2}\dfrac{\big(\Lambda(\eta,b)\phi + \nabla\eta\cdot\nabla\phi\big)^2}{1+|\nabla\eta|^2} = 0 & \text{on} \quad \mathbf{R}^n,\ t > 0, \end{cases} \tag{1.5}$$

where $\Lambda(\eta, b)$ is the Dirichlet-to-Neumann map for Laplace's equation. Namely, it is defined by

$$\Lambda(\eta, b)\phi = (\partial_z \Phi)|_{z=\eta} - \nabla\eta \cdot (\nabla\Phi)|_{z=\eta},$$

where Φ is the unique solution to the boundary value problem of Laplace's equation (1.1) under the boundary conditions (1.3)–(1.4).

As is well-known, the water wave problem has a conserved energy $E = E_{\text{kin}} + E_{\text{pot}}$, where E_{kin} is the kinetic energy

$$\begin{aligned} E_{\text{kin}} &= \frac{1}{2}\rho \iint_{\Omega(t)} \big(|\nabla\Phi(\boldsymbol{x}, z, t)|^2 + (\partial_z\Phi(\boldsymbol{x}, z, t))^2\big)\,\mathrm{d}\boldsymbol{x}\mathrm{d}z \\ &= \frac{1}{2}\rho \int_{\mathbf{R}^n} \phi(\boldsymbol{x}, t)(\Lambda(\eta, b)\phi)(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}, \end{aligned}$$

and E_{pot} is the potential energy

$$E_{\text{pot}} = \frac{1}{2}\rho g \int_{\mathbf{R}^n} \eta(\boldsymbol{x}, t)^2\,\mathrm{d}\boldsymbol{x}$$

due to the gravity. Here, ρ is a constant density of the water.

V. E. Zakharov [24] found that the water wave system has a Hamiltonian structure and η and ϕ are the canonical variables. The Hamiltonian $\mathscr{H}$ is essentially the total energy, that is, $\mathscr{H} = \frac{1}{\rho}E$. He showed that the basic equations for water waves (1.1)–(1.3) are transformed equivalently into Hamilton's canonical equations

$$\partial_t \eta = \frac{\delta\mathscr{H}}{\delta\phi}, \quad \partial_t\phi = -\frac{\delta\mathscr{H}}{\delta\eta}.$$

Although V. E. Zakharov did not use explicitly the Dirichlet-to-Neumann map $\Lambda(\eta, b)$, the above canonical equations are exactly the same as (1.5). W. Craig and C. Sulem [9] introduced the Dirichlet-to-Neumann map explicitly and derived (1.5). Therefore, nowadays (1.5) is often called the Zakharov–Craig–Sulem formulation of the water wave problem. Since then, W. Craig and his collaborators [3, 4, 5, 6, 7, 8] have used the Hamiltonian structure of the water wave problem in order to analyze long-wave and modulation approximations. Let us also mention the recent work of W. Craig [2], which generalizes the Hamiltonian formulation of water waves described above to a general coordinatization of the free surface allowing overturning wave profiles.

On the other hand, as was shown by J. C. Luke [19], the water wave problem has also a variational structure. His Lagrangian density is of the form

$$\mathscr{L}(\Phi, \eta) = \int_{-h+b(\boldsymbol{x})}^{\eta(\boldsymbol{x},t)} \left(\partial_t\Phi(\boldsymbol{x}, z, t) + \frac{1}{2}\big(|\nabla\Phi(\boldsymbol{x}, z, t)|^2 + (\partial_z\Phi(\boldsymbol{x}, z, t))^2\big)\right)\mathrm{d}z + \frac{1}{2}g\big(\eta(\boldsymbol{x}, t)\big)^2 \tag{1.6}$$

and the action function is given by

$$\mathscr{J}(\Phi, \eta) = \int_{t_0}^{t_1}\int_{\mathbf{R}^n} \mathscr{L}(\Phi, \eta)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

In fact, the corresponding Euler–Lagrange equations are exactly the basic equations for water waves (1.1)–(1.3). We refer to J. W. Miles [21] for the relation between Zakharov's Hamiltonian and Luke's Lagrangian.

M. Isobe [13, 14] and T. Kakinuma [15, 16, 17] obtained a family of systems of equations after replacing the velocity potential Φ in Luke's Lagrangian by

$$\Phi^{\mathrm{app}}(\boldsymbol{x}, z, t) = \sum_{i=0}^{N} \Psi_i(z; b)\phi_i(\boldsymbol{x}, t),$$

where $\{\Psi_i\}$ is a given appropriate function system in the vertical coordinate z and may depend on the bottom topography b and $(\phi_0, \phi_1, \ldots, \phi_N)$ are unknown variables. The Isobe–Kakinuma model is a system of Euler–Lagrange equations corresponding to the action function

$$\mathscr{J}^{\mathrm{app}}(\phi_0, \phi_1, \ldots, \phi_N, \eta) = \int_{t_0}^{t_1}\int_{\mathbf{R}^n} \mathscr{L}(\Phi^{\mathrm{app}}, \eta)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t. \tag{1.7}$$

We have to choose the function system $\{\Psi_i\}$ carefully for the Isobe–Kakinuma model to produce good approximations to the water wave problem. One possible choice is the bases of the Taylor series of the velocity potential $\Phi(\boldsymbol{x}, z, t)$ with respect to the vertical spatial coordinate z around the bottom. Such an expansion has been already used by J. Boussinesq [1] in the case of the flat bottom and, for instance, by C. C. Mei and B. Le Méhauté [20] for general bottom topographies. The corresponding choice of the function system is given by

$$\Psi_i(z; b) = \begin{cases} (z+h)^{2i} & \text{in the case of the flat bottom,} \\ (z+h-b(\boldsymbol{x}))^i & \text{in the case of the variable bottom.} \end{cases}$$

Here we note that the latter choice is valid also for the case of the flat bottom. However, it turns out that the terms of odd degree do not play any important role in such a case so that the former choice is more adequate. In order to treat both cases at the same time, we adopt the approximation

$$\Phi^{\mathrm{app}}(\boldsymbol{x}, z, t) = \sum_{i=0}^{N} (z+h-b(\boldsymbol{x}))^{p_i}\phi_i(\boldsymbol{x}, t), \tag{1.8}$$

where $p_0, p_1, \ldots, p_N$ are nonnegative integers satisfying $0 = p_0 < p_1 < \cdots < p_N$. Plugging (1.8) into the action function (1.7), the corresponding Euler–Lagrange equation yields the Isobe–Kakinuma model of the form

$$\begin{cases} H^{p_i}\partial_t\eta + \displaystyle\sum_{j=0}^{N}\Bigg\{\nabla\cdot\left(\frac{1}{p_i+p_j+1}H^{p_i+p_j+1}\nabla\phi_j - \frac{p_j}{p_i+p_j}H^{p_i+p_j}\phi_j\nabla b\right) \\ \qquad\qquad + \dfrac{p_i}{p_i+p_j}H^{p_i+p_j}\nabla b\cdot\nabla\phi_j - \dfrac{p_ip_j}{p_i+p_j-1}H^{p_i+p_j-1}(1+|\nabla b|^2)\phi_j\Bigg\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 0, 1, \ldots, N, \\ \displaystyle\sum_{j=0}^{N} H^{p_j}\partial_t\phi_j + g\eta + \frac{1}{2}\left\{\left|\sum_{j=0}^{N}(H^{p_j}\nabla\phi_j - p_jH^{p_j-1}\phi_j\nabla b)\right|^2 + \left(\sum_{j=0}^{N} p_jH^{p_j-1}\phi_j\right)^2\right\} = 0, \end{cases} \tag{1.9}$$

where $H(\boldsymbol{x}, t) = h + \eta(\boldsymbol{x}, t) - b(\boldsymbol{x})$ is the depth of the water. Here and in what follows we use the notational convention $0/0 = 0$. This system consists of $(N+1)$ evolution equations

for η and only one evolution equation for $(N+1)$ unknowns $(\phi_0, \phi_1, \dots, \phi_N)$, so that this is an overdetermined and underdetermined composite system. However, the total number of the unknowns is equal to the total number of the equations.

The main purpose of this paper is to show that the Isobe–Kakinuma model (1.9) also enjoys a canonical Hamiltonian structure which is analogous to the one of the water waves problem. In particular, the Hamiltonian is a higher order shallow water approximation of the original Hamiltonian of the water waves problem.

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2 Preliminaries

Since the hypersurface $t = 0$ in the space-time $\mathbf{R}^n \times \mathbf{R}$ is characteristic for the Isobe–Kakinuma model (1.9), the initial value problem to the model is not solvable in general. In fact, if the problem has a solution $(\eta, \phi_0, \dots, \phi_N)$, then by eliminating the time derivative $\partial_t \eta$ from the equations we see that the solution has to satisfy the relations

$$\begin{aligned} &H^{p_i} \sum_{j=0}^{N} \nabla \cdot \left(\frac{1}{p_j+1} H^{p_j+1} \nabla \phi_j - H^{p_j} \phi_j \nabla b \right) \\ &= \sum_{j=0}^{N} \left\{ \nabla \cdot \left(\frac{1}{p_i+p_j+1} H^{p_i+p_j+1} \nabla \phi_j - \frac{p_j}{p_i+p_j} H^{p_i+p_j} \phi_j \nabla b \right) \right. \\ &\qquad \left. + \frac{p_i}{p_i+p_j} H^{p_i+p_j} \nabla b \cdot \nabla \phi_j - \frac{p_i p_j}{p_i+p_j-1} H^{p_i+p_j-1} (1+|\nabla b|^2) \phi_j \right\} \end{aligned} \tag{2.1}$$

for $i = 1, \dots, N$. Therefore, the initial data have to satisfy these relations in order to allow the existence of a solution. Y. Murakami and T. Iguchi [22] and R. Nemoto and T. Iguchi [23] showed that the initial value problem to the Isobe–Kakinuma model (1.9) is well-posed locally in time in a class of initial data for which the relations (2.1) and a generalized Rayleigh–Taylor sign condition are satisfied. Moreover, T. Iguchi [11, 12] showed that the Isobe–Kakinuma model (1.9) is a higher order shallow water approximation for the water wave problem in the strongly nonlinear regime. The Isobe–Kakinuma model (1.9) has also a conserved energy, which is the total energy given by

$$\begin{aligned} E^{\mathrm{IK}}(\eta, \boldsymbol{\phi}) &= \frac{1}{2} \rho \iint_{\Omega(t)} \left(|\nabla \Phi^{\mathrm{app}}(\boldsymbol{x}, z, t)|^2 + (\partial_z \Phi^{\mathrm{app}}(\boldsymbol{x}, z, t))^2 \right) \mathrm{d}\boldsymbol{x} \mathrm{d}z + \frac{1}{2} \rho g \int_{\mathbf{R}^n} \eta(\boldsymbol{x}, t)^2 \, \mathrm{d}\boldsymbol{x} \\ &= \frac{\rho}{2} \int_{\mathbf{R}^n} \left\{ \sum_{i,j=0}^{N} \left(\frac{1}{p_i+p_j+1} H^{p_i+p_j+1} \nabla \phi_i \cdot \nabla \phi_j - \frac{2p_i}{p_i+p_j} H^{p_i+p_j} \phi_i \nabla b \cdot \nabla \phi_j \right. \right. \\ &\qquad \left. \left. + \frac{p_i p_j}{p_i+p_j-1} H^{p_i+p_j-1} (1+|\nabla b|^2) \phi_i \phi_j \right) + g \eta^2 \right\} \mathrm{d}\boldsymbol{x}, \end{aligned} \tag{2.2}$$

where $\boldsymbol{\phi} = (\phi_0, \phi_1, \dots, \phi_N)^{\mathrm{T}}$.

We introduce second order differential operators $L_{ij} = L_{ij}(H, b)$ $(i, j = 0, 1, \dots, N)$ depending on the water depth H and the bottom topography b by

$$\begin{aligned} L_{ij}\psi_j = &-\nabla \cdot \left(\frac{1}{p_i + p_j + 1} H^{p_i+p_j+1} \nabla \psi_j - \frac{p_j}{p_i + p_j} H^{p_i+p_j} \psi_j \nabla b \right) \\ &- \frac{p_i}{p_i + p_j} H^{p_i+p_j} \nabla b \cdot \nabla \psi_j + \frac{p_i p_j}{p_i + p_j - 1} H^{p_i+p_j-1} (1 + |\nabla b|^2) \psi_j. \end{aligned} \tag{2.3}$$

Then, we have $L_{ij}^* = L_{ji}$, where L_{ij}^* is the adjoint operator of L_{ij} in $L^2(\mathbf{R}^n)$. Moreover, the Isobe–Kakinuma model (1.9) and the relations (2.1) can be written simply as

$$\begin{cases} H^{p_i} \partial_t \eta - \displaystyle\sum_{j=0}^{N} L_{ij}(H, b) \phi_j = 0 \qquad \text{for} \quad i = 0, 1, \dots, N, \\ \displaystyle\sum_{j=0}^{N} H^{p_j} \partial_t \phi_j + g\eta + \frac{1}{2} \big(|(\nabla \Phi^{\mathrm{app}})|_{z=\eta}|^2 + ((\partial_z \Phi^{\mathrm{app}})|_{z=\eta})^2 \big) = 0 \end{cases} \tag{2.4}$$

and

$$\sum_{j=0}^{N} (L_{ij}(H, b) - H^{p_i} L_{0j}(H, b)) \phi_j = 0 \qquad \text{for} \quad i = 1, \dots, N, \tag{2.5}$$

respectively. It is easy to calculate the variational derivative of the energy function $E^{\mathrm{IK}}(\eta, \boldsymbol{\phi})$ and to obtain

$$\begin{cases} \dfrac{1}{\rho} \delta_{\phi_i} E^{\mathrm{IK}} = \displaystyle\sum_{j=0}^{N} L_{ij}(H, b) \phi_j \quad j = 0, 1, \dots, N, \\ \dfrac{1}{\rho} \delta_{\eta} E^{\mathrm{IK}} = \dfrac{1}{2} \big(|(\nabla \Phi^{\mathrm{app}})|_{z=\eta}|^2 + ((\partial_z \Phi^{\mathrm{app}})|_{z=\eta})^2 \big) + g\eta. \end{cases} \tag{2.6}$$

Therefore, introducing $\boldsymbol{l}(H) = (H^{p_0}, H^{p_1}, \dots, H^{p_N})^{\mathrm{T}}$, the Isobe–Kakinuma model (1.9) can also be written simply as

$$\begin{pmatrix} 0 & -\boldsymbol{l}(H)^{\mathrm{T}} \\ \boldsymbol{l}(H) & O \end{pmatrix} \partial_t \begin{pmatrix} \eta \\ \boldsymbol{\phi} \end{pmatrix} = \frac{1}{\rho} \begin{pmatrix} \delta_\eta E^{\mathrm{IK}} \\ \delta_{\boldsymbol{\phi}} E^{\mathrm{IK}} \end{pmatrix}. \tag{2.7}$$

In view of (2.5) we introduce also linear operators $\mathcal{L}_i = \mathcal{L}_i(H, b)$ $(i = 1, \dots, N)$ depending on the water depth H and the bottom topography b, and acting on $\boldsymbol{\varphi} = (\varphi_0, \varphi_1, \dots, \varphi_N)^{\mathrm{T}}$ by

$$\mathcal{L}_i \boldsymbol{\varphi} = \sum_{j=0}^{N} \big(L_{ij}(H, b) - H^{p_i} L_{0j}(H, b) \big) \varphi_j \quad \text{for} \quad i = 1, \dots, N, \tag{2.8}$$

and put $\mathcal{L}\boldsymbol{\varphi} = (\mathcal{L}_1 \boldsymbol{\varphi}, \dots, \mathcal{L}_N \boldsymbol{\varphi})^{\mathrm{T}}$. Then, the conditions (2.1) can be written simply as

$$\mathcal{L}(H, b) \boldsymbol{\phi} = \mathbf{0}. \tag{2.9}$$

For later use, we also put $L = L(H, b) = (L_{ij}(H, b))_{0 \le i, j \le N}$ and define $L_0 = L_0(H, b)$ by

$$L_0(H, b) \boldsymbol{\varphi} = \sum_{j=0}^{N} L_{0j}(H, b) \varphi_j. \tag{2.10}$$

Then, the conditions (2.1) are also equivalent to

$$L(H, b) \boldsymbol{\phi} = \big(L_0(H, b) \boldsymbol{\phi} \big) \boldsymbol{l}(H). \tag{2.11}$$

Now, for given functions F_0 and $\boldsymbol{F} = (F_1, \dots, F_N)^{\mathrm{T}}$ we consider the equations

$$\begin{cases} \boldsymbol{l}(H) \cdot \boldsymbol{\varphi} = F_0, \\ \mathcal{L}(H, b)\boldsymbol{\varphi} = \boldsymbol{F}. \end{cases} \tag{2.12}$$

Let $W^{m,p} = W^{m,p}(\mathbf{R}^n)$ be the L^p-based Sobolev space of order m on $\mathbf{R}^n$ and $H^m = W^{m,2}$. The norms of the Sobolev space H^m and of a Banach space $\mathscr{X}$ are denoted by $\|\cdot\|_m$ and by $\|\cdot\|_{\mathscr{X}}$, respectively. Set $\mathring{H}^m = \{\phi\,;\, \nabla\phi \in H^{m-1}\}$. The following lemma was proved in [23].

Lemma 2.1. *Let h, c_0, M be positive constants and m an integer such that $m > \frac{n}{2} + 1$. There exists a positive constant C such that if η and b satisfy*

$$\begin{cases} \|\eta\|_m + \|b\|_{W^{m,\infty}} \le M, \\ c_0 \le H(\boldsymbol{x}) = h + \eta(\boldsymbol{x}) - b(\boldsymbol{x}) \quad \textit{for} \quad \boldsymbol{x} \in \mathbf{R}^n, \end{cases} \tag{2.13}$$

then for any $F_0 \in \mathring{H}^k$ and $\boldsymbol{F} = (F_1, \dots, F_N)^{\mathrm{T}} \in (H^{k-2})^N$ with $1 \le k \le m$ there exists a unique solution $\boldsymbol{\varphi} = (\varphi_0, \varphi_1, \dots, \varphi_N)^{\mathrm{T}} \in \mathring{H}^k \times (H^k)^N$ to (2.12). Moreover, the solution satisfies

$$\|\nabla\varphi_0\|_{k-1} + \|(\varphi_1, \dots, \varphi_N)\|_k \le C\,(\|\nabla F_0\|_{k-1} + \|(F_1, \dots, F_N)\|_{k-2}).$$

3 Hamiltonian structure

In the following, we will fix $b \in W^{m,\infty}$ with $m > \frac{n}{2} + 1$. Let $(\eta, \phi_0, \dots, \phi_N)$ be a solution to the Isobe–Kakinuma model (1.9). As we will see later, the canonical variables of the Isobe–Kakinuma model are the surface elevation η and the trace of the approximated velocity potential on the water surface

$$\phi = \Phi^{\mathrm{app}}|_{z=\eta} = \sum_{j=0}^{N} H^{p_j}\phi_j = \boldsymbol{l}(H) \cdot \boldsymbol{\phi}. \tag{3.1}$$

Then, the relations (2.1) and the above equation are written in the simple form

$$\begin{cases} \boldsymbol{l}(H) \cdot \boldsymbol{\phi} = \phi, \\ \mathcal{L}(H, b)\boldsymbol{\phi} = \mathbf{0}. \end{cases} \tag{3.2}$$

Therefore, it follows from Lemma 2.1 that once the canonical variables (η, ϕ) are given in an appropriate class of functions, $\boldsymbol{\phi} = (\phi_0, \phi_1, \dots, \phi_N)^{\mathrm{T}}$ can be determine uniquely. In other words, these variables $(\phi_0, \phi_1, \dots, \phi_N)$ depend on the canonical variables (η, ϕ) and furthermore they depend on ϕ linearly so that we can write

$$\boldsymbol{\phi} = \mathbf{S}(\eta, b)\phi$$

with a linear operator $\mathbf{S}(\eta, b)$ depending on η and b. Since we fixed b, we simply write $\mathbf{S}(\eta)$ in place of $\mathbf{S}(\eta, b)$ for simplicity.

We proceed to analyze this operator $\mathbf{S}(\eta)$ more precisely. We put

$$U_b^m = \{\eta \in H^m\,;\, \inf_{\boldsymbol{x} \in \mathbf{R}^n}(h + \eta(\boldsymbol{x}) - b(\boldsymbol{x})) > 0\},$$

which is an open set in H^m. For Banach spaces $\mathscr{X}$ and $\mathscr{Y}$, we denote by $B(\mathscr{X}; \mathscr{Y})$ the set of all linear and bounded operators from $\mathscr{X}$ into $\mathscr{Y}$. In view of (2.11), (3.1), and Lemma 2.1, we see easily the following lemma.

Lemma 3.1. *Let m be an integer such that $m > \frac{n}{2} + 1$ and $b \in W^{m,\infty}$. For each $\eta \in U_b^m$ and for $k = 1, 2, \dots, m$, the linear operator*

$$\mathbf{S}(\eta) : \mathring{H}^k \ni \phi \mapsto \boldsymbol{\phi} \in \mathring{H}^k \times (H^k)^N$$

is defined, where $\boldsymbol{\phi} = (\phi_0, \phi_1, \dots, \phi_N)^{\mathrm{T}}$ is the unique solution to (3.2). Moreover, we have $\mathbf{S}(\eta) \in B(\mathring{H}^k; \mathring{H}^k \times (H^k)^N)$ and

$$L(H, b)\boldsymbol{\phi} = \big(L_0(H, b)\boldsymbol{\phi}\big)\boldsymbol{l}(H).$$

Formally, $D_\eta \mathbf{S}(\eta)[\dot{\eta}]$ the Fréchet derivative of $\mathbf{S}(\eta)$ with respect to η is given by

$$\begin{cases} \boldsymbol{l}(H) \cdot \dot{\boldsymbol{\psi}} = -\big(\boldsymbol{l}'(H) \cdot \boldsymbol{\phi}\big)\dot{\eta}, \\ \mathcal{L}(H, b)\dot{\boldsymbol{\psi}} = -D_H \mathcal{L}(H, b)[\dot{\eta}]\boldsymbol{\phi}, \end{cases} \tag{3.3}$$

with $\boldsymbol{\phi} = \mathbf{S}(\eta)\phi$ and $\dot{\boldsymbol{\psi}} = D_\eta \mathbf{S}(\eta)[\dot{\eta}]\phi$, where $\boldsymbol{l}'(H) \cdot \boldsymbol{\phi} = \sum_{j=1}^N p_j H^{p_j - 1}\phi_j$,

$$D_H \mathcal{L}_i(H)[\dot{\eta}]\boldsymbol{\phi} = \sum_{j=0}^N \big(D_H L_{ij}(H, b)[\dot{\eta}] - H^{p_i} D_H L_{0j}(H, b)[\dot{\eta}] - p_i H^{p_i - 1}\dot{\eta} L_{0j}(H, b)\big)\phi_j,$$

and

$$\begin{aligned} D_H L_{ij}(H, b)[\dot{\eta}]\phi_j = &-\nabla \cdot \big\{\dot{\eta}(H^{p_i + p_j}\nabla\phi_j - p_j H^{p_i + p_j - 1}\phi_j \nabla b)\big\} \\ &+ \dot{\eta}\big\{-p_i H^{p_i + p_j - 1}\nabla b \cdot \nabla\phi_j + p_i p_j H^{p_i + p_j - 2}(1 + |\nabla b|^2)\phi_j\big\}. \end{aligned}$$

By using these equations together with Lemma 2.1 and standard arguments, we can justify the Fréchet differentiability of $\mathbf{S}(\eta)$ with respect to η. More precisely, we have the following lemma.

Lemma 3.2. *Let m be an integer such that $m > \frac{n}{2} + 1$ and $b \in W^{m,\infty}$. Then, the map $U_b^m \ni \eta \mapsto \mathbf{S}(\eta) \in B(\mathring{H}^k; \mathring{H}^k \times (H^k)^N)$ is Fréchet differentiable for $k = 1, 2, \dots, m$, and (3.3) holds.*

As mentioned before, the Isobe–Kakinuma model (1.9) has a conserved quantity $E^{\mathrm{IK}}(\eta, \boldsymbol{\phi})$ given by (2.2), which is the total energy. Now, we define a Hamiltonian $\mathscr{H}^{\mathrm{IK}}(\eta, \phi)$ to the Isobe–Kakinuma model by

$$\mathscr{H}^{\mathrm{IK}}(\eta, \phi) = \frac{1}{\rho} E^{\mathrm{IK}}(\eta, \mathbf{S}(\eta)\phi), \tag{3.4}$$

which is essentially the total energy in terms of the canonical variables (η, ϕ).

Lemma 3.3. *Let m be an integer such that $m > \frac{n}{2} + 1$ and $b \in W^{m,\infty}$. Then, the map $U_b^m \times \mathring{H}^1 \ni (\eta, \phi) \mapsto \mathscr{H}^{\mathrm{IK}}(\eta, \phi) \in \mathbf{R}$ is Fréchet differentiable and the variational derivatives of the Hamiltonian are*

$$\begin{cases} \delta_\phi \mathscr{H}^{\mathrm{IK}}(\eta, \phi) = L_0(H, b)\boldsymbol{\phi}, \\ \delta_\eta \mathscr{H}^{\mathrm{IK}}(\eta, \phi) = \frac{1}{\rho}(\delta_\eta E^{\mathrm{IK}})(\eta, \boldsymbol{\phi}) - (\boldsymbol{l}'(H) \cdot \boldsymbol{\phi}) L_0(H, b)\boldsymbol{\phi}, \end{cases}$$

where $\boldsymbol{\phi} = \mathbf{S}(\eta)\phi$.

Proof. Let us calculate Fréchet derivatives of the Hamiltonian $\mathscr{H}^{\mathrm{IK}}(\eta,\phi)$. Let us consider first $U_b^m \times H^2 \ni (\eta,\phi) \mapsto \mathscr{H}^{\mathrm{IK}}(\eta,\phi)$. For any $\dot\phi \in H^2$, we see that

$$\begin{aligned}
D_\phi \mathscr{H}^{\mathrm{IK}}(\eta,\phi)[\dot\phi] &= \frac{1}{\rho}(D_{\boldsymbol{\phi}} E^{\mathrm{IK}})(\eta,\mathbf{S}(\eta)\phi)[\mathbf{S}(\eta)\dot\phi] \\
&= \frac{1}{\rho}((\delta_{\boldsymbol{\phi}} E^{\mathrm{IK}})(\eta,\boldsymbol{\phi}),\mathbf{S}(\eta)\dot\phi)_{L^2} \\
&= (L(H,b)\boldsymbol{\phi},\mathbf{S}(\eta)\dot\phi)_{L^2} \\
&= (\big(L_0(H,b)\boldsymbol{\phi}\big)\boldsymbol{l}(H),\mathbf{S}(\eta)\dot\phi)_{L^2} \\
&= (L_0(H,b)\boldsymbol{\phi},\boldsymbol{l}(H)\cdot\mathbf{S}(\eta)\dot\phi)_{L^2} \\
&= (L_0(H,b)\boldsymbol{\phi},\dot\phi)_{L^2},
\end{aligned}$$

where we used (2.6) and Lemma 3.1. The above calculations are also valid when $(\phi,\dot\phi) \in \mathring{H}^1 \times \mathring{H}^1$, provided we replace the L^2 inner products with the $\mathscr{X}'$–$\mathscr{X}$ duality product where $\mathscr{X} = \mathring{H}^1 \times (H^1)^N$ for the first lines, and $\mathscr{X} = \mathring{H}^1$ for the last line. This gives the first equation of the lemma.

Similarly, for any $(\eta,\phi) \in U_b^m \times \mathring{H}^2$ and $\dot\eta \in H^m$ we see that

$$D_\eta \mathscr{H}^{\mathrm{IK}}(\eta,\phi)[\dot\eta] = \frac{1}{\rho}(D_\eta E^{\mathrm{IK}})(\eta,\mathbf{S}(\eta)\phi)[\dot\eta] + \frac{1}{\rho}(D_{\boldsymbol{\phi}} E^{\mathrm{IK}})(\eta,\mathbf{S}(\eta)\phi)[D_\eta\mathbf{S}(\eta)[\dot\eta]\phi].$$

Here, we have

$$\begin{aligned}
\frac{1}{\rho}(D_{\boldsymbol{\phi}} E^{\mathrm{IK}})(\eta,\mathbf{S}(\eta)\phi)[D_\eta\mathbf{S}(\eta)[\dot\eta]\phi] &= \frac{1}{\rho}((\delta_{\boldsymbol{\phi}} E^{\mathrm{IK}})(\eta,\boldsymbol{\phi}),D_\eta\mathbf{S}(\eta)[\dot\eta]\phi)_{L^2} \\
&= (L(H,b)\boldsymbol{\phi},D_\eta\mathbf{S}(\eta)[\dot\eta]\phi)_{L^2} \\
&= (L_0(H,b)\boldsymbol{\phi},\boldsymbol{l}(H)\cdot D_\eta\mathbf{S}(\eta)[\dot\eta]\phi)_{L^2} \\
&= -(L_0(H,b)\boldsymbol{\phi},(\boldsymbol{l}'(H)\cdot\mathbf{S}(\eta)\phi)\dot\eta)_{L^2} \\
&= -((\boldsymbol{l}'(H)\cdot\boldsymbol{\phi})L_0(H,b)\boldsymbol{\phi},\dot\eta)_{L^2},
\end{aligned}$$

where we used the identity

$$\boldsymbol{l}(H)\cdot D_\eta\mathbf{S}(\eta)[\dot\eta]\phi + (\boldsymbol{l}'(H)\cdot\mathbf{S}(\eta)\phi)\dot\eta = 0,$$

stemming from (3.3). Again, the above identities are still valid for $(\eta,\phi) \in U_b^m \times \mathring{H}^1$ provided we replace the L^2 inner products with suitable duality products. This concludes the proof of the Fréchet differentiability, and the second equation of the lemma. □

Now, we are ready to show our main result in this section.

Theorem 3.4. *Let m be an integer such that $m > \frac{n}{2}+1$ and $b \in W^{m,\infty}$. Then, the Isobe–Kakinuma model* (1.9) *is equivalent to Hamilton's canonical equations*

$$\partial_t\eta = \frac{\delta\mathscr{H}^{\mathrm{IK}}}{\delta\phi}, \quad \partial_t\phi = -\frac{\delta\mathscr{H}^{\mathrm{IK}}}{\delta\eta}, \tag{3.5}$$

with $\mathscr{H}^{\mathrm{IK}}$ defined in (3.4) *as long as $\eta(\cdot,t) \in U_b^m$ and $\phi(\cdot,t) \in \mathring{H}^1$. More precisely, for any regular solution $(\eta,\boldsymbol{\phi})$ to the Isobe–Kakinuma model* (1.9)*, if we define ϕ by* (3.1)*, then (η,ϕ) satisfies Hamilton's canonical equations* (3.5)*. Conversely, for any regular solution (η,ϕ) to Hamilton's canonical equations* (3.5)*, if we define $\boldsymbol{\phi}$ by $\boldsymbol{\phi} = \mathbf{S}(\eta)\phi$, then $(\eta,\boldsymbol{\phi})$ satisfies the Isobe–Kakinuma model* (1.9)*.*

Proof. Suppose that $(\eta, \boldsymbol{\phi})$ is a solution to the Isobe–Kakinuma model (1.9). Then, it satisfies (2.7), particularly, we have

$$\partial_t \eta = L_0(H, b)\boldsymbol{\phi}. \tag{3.6}$$

It follows from (3.1) and (2.7) that

$$\begin{aligned}\partial_t \phi &= \boldsymbol{l}(H) \cdot \partial_t \boldsymbol{\phi} + (\boldsymbol{l}'(H) \cdot \boldsymbol{\phi})\partial_t \eta \\ &= -\frac{1}{\rho}(\delta_\eta E^{\mathrm{IK}})(\eta, \boldsymbol{\phi}) + (\boldsymbol{l}'(H) \cdot \boldsymbol{\phi})L_0(H, b)\boldsymbol{\phi}.\end{aligned}$$

These equations together with Lemma 3.3 show that (η, ϕ) satisfies (3.5).

Conversely, suppose that (η, ϕ) satisfies Hamilton's canonical equations (3.5) and put $\boldsymbol{\phi} = \mathbf{S}(\eta)\phi$. Then, it follows from (3.5) and Lemma 3.3 that we have (3.6). This fact and Lemma 3.1 imply the equation

$$\boldsymbol{l}(H)\partial_t \eta = L(H, b)\boldsymbol{\phi} = \frac{1}{\rho}\delta_{\boldsymbol{\phi}} E^{\mathrm{IK}}(\eta, \boldsymbol{\phi}).$$

We see also that

$$\boldsymbol{l}(H) \cdot \partial_t \boldsymbol{\phi} = \partial_t \phi - (\boldsymbol{l}'(H) \cdot \boldsymbol{\phi})\partial_t \eta = -\frac{1}{\rho}\delta_\eta E^{\mathrm{IK}}(\eta, \boldsymbol{\phi}),$$

where we used (3.5) and Lemma 3.3. Therefore, $(\eta, \boldsymbol{\phi})$ satisfies (2.7), that is, the Isobe–Kakinuma model (1.9). □

4 Consistency

As aforementioned, it was shown in [11, 12] that the Isobe–Kakinuma model (1.9) is a higher order shallow water approximation for the water wave problem in the strongly nonlinear regime. In this section, we will show that the canonical Hamiltonian structure exhibited in the previous section is consistent with this approximation, in the sense that the Hamiltonian of the Isobe–Kakinuma model, $\mathscr{H}^{\mathrm{IK}}(\eta, \phi)$, approximates the Hamiltonian of the water wave problem, $\mathscr{H}(\eta, \phi)$, in the shallow water regime.

In order to provide quantitative results, we first rewrite the equations in a nondimensional form. Let λ be the typical wavelength of the wave. Recalling that h is the mean depth, we introduce the nondimensional aspect ratio

$$\delta = \frac{h}{\lambda},$$

measuring the shallowness of the water. We then rescale the physical variables by

$$\boldsymbol{x} = \lambda\tilde{\boldsymbol{x}}, \quad z = h\tilde{z}, \quad t = \frac{\lambda}{\sqrt{gh}}\tilde{t}, \quad \eta = h\tilde{\eta}, \quad b = h\tilde{b}, \quad \Phi = \lambda\sqrt{gh}\tilde{\Phi}.$$

Under these rescaling, after dropping the tildes for the sake of readability, the basic equations for water waves (1.1)–(1.3) are rewritten in a non-dimensional form

$$\Delta\Phi + \delta^{-2}\partial_z^2\Phi = 0 \quad \text{in} \quad \Omega(t),\ t > 0, \tag{4.1}$$

$$\begin{cases} \partial_t \eta + \nabla\Phi \cdot \nabla\eta - \delta^{-2}\partial_z\Phi = 0 & \text{on} \quad \Gamma(t),\ t > 0, \\ \partial_t \Phi + \dfrac{1}{2}\big(|\nabla\Phi|^2 + \delta^{-2}(\partial_z\Phi)^2\big) + \eta = 0 & \text{on} \quad \Gamma(t),\ t > 0, \end{cases} \tag{4.2}$$

$$\nabla\Phi\cdot\nabla b-\delta^{-2}\partial_z\Phi=0\quad\text{on}\quad\Sigma,\ t>0, \tag{4.3}$$

denoting $\Omega(t)$, $\Gamma(t)$, and Σ the rescaled water region, water surface, and bottom of the water at time t, respectively. Specifically, the rescaled water surface and bottom of the water are represented as $z=\eta(\boldsymbol{x},t)$ and $z=-1+b(\boldsymbol{x})$, respectively. The corresponding dimensionless Zakharov–Craig–Sulem formulation is

$$\begin{cases}\partial_t\eta-\Lambda^\delta(\eta,b)\phi=0 & \text{on}\quad\mathbf{R}^n,\ t>0,\\ \partial_t\phi+\eta+\dfrac{1}{2}|\nabla\phi|^2-\dfrac{\delta^2}{2}\dfrac{\left(\Lambda^\delta(\eta,b)\phi+\nabla\eta\cdot\nabla\phi\right)^2}{1+\delta^2|\nabla\eta|^2}=0 & \text{on}\quad\mathbf{R}^n,\ t>0,\end{cases} \tag{4.4}$$

where

$$\phi(\boldsymbol{x},t)=\Phi(\boldsymbol{x},\eta(\boldsymbol{x},t),t) \tag{4.5}$$

is the trace of the velocity potential on the water surface, and $\Lambda^\delta(\eta,b)$ is the dimensionless Dirichlet-to-Neumann map for Laplace's equation, namely, it is defined by

$$\Lambda^\delta(\eta,b)\phi=\delta^{-2}(\partial_z\Phi)|_{z=\eta}-\nabla\eta\cdot(\nabla\Phi)|_{z=\eta},$$

where Φ is the unique solution to the boundary value problem of the scaled Laplace's equation (4.1) under the boundary conditions (4.3) and (4.5). With this rescaling and definitions, the Hamiltonian of the water wave system is given by

$$\mathscr{H}^\delta(\eta,\phi)=\frac{1}{2}\iint_{\Omega(t)}\left(|\nabla\Phi|^2+\delta^{-2}(\partial_z\Phi)^2\right)\mathrm{d}\boldsymbol{x}\mathrm{d}z+\frac{1}{2}\int_{\mathbf{R}^n}\eta^2\,\mathrm{d}\boldsymbol{x}.$$

In order to rewrite the Isobe–Kakinuma model (1.9) in dimensionless form, we need to rescale the unknown variables $(\phi_0,\phi_1,\ldots,\phi_N)$, depending on the choice of function system $\{\Psi_i\}$. In view of (1.8), we rescale them by

$$\phi_i=\frac{\lambda\sqrt{gh}}{\lambda^{p_i}}\tilde{\phi}_i\qquad\text{for}\quad i=0,1,\ldots,N,$$

so that

$$\Phi^{\mathrm{app}}(\boldsymbol{x},z,t)=\lambda\sqrt{gh}\,\tilde{\Phi}^{\mathrm{app}}(\tilde{\boldsymbol{x}},\tilde{z},\tilde{t})=\lambda\sqrt{gh}\left(\sum_{i=0}^N\delta^{p_i}(\tilde{z}+1-\tilde{b}(\tilde{\boldsymbol{x}}))^{p_i}\phi_i(\tilde{\boldsymbol{x}},\tilde{t})\right). \tag{4.6}$$

As before, we will henceforth drop the tildes for the sake of readability. It is also convenient to introduce the notation

$$\phi_i^\delta=\delta^{p_i}\tilde{\phi}_i\qquad\text{for}\quad i=0,1,\ldots,N,$$

so that the Isobe–Kakinuma model (1.9) in rescaled variables is

$$\begin{cases}H^{p_i}\partial_t\eta+\displaystyle\sum_{j=0}^N\Bigg\{\nabla\cdot\left(\frac{1}{p_i+p_j+1}H^{p_i+p_j+1}\nabla\phi_j^\delta-\frac{p_j}{p_i+p_j}H^{p_i+p_j}\phi_j^\delta\nabla b\right)\\ \qquad\qquad+\dfrac{p_i}{p_i+p_j}H^{p_i+p_j}\nabla b\cdot\nabla\phi_j^\delta-\dfrac{p_ip_j}{p_i+p_j-1}H^{p_i+p_j-1}(\delta^{-2}+|\nabla b|^2)\phi_j^\delta\Bigg\}=0\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for}\quad i=0,1,\ldots,N,\\ \displaystyle\sum_{j=0}^N H^{p_j}\partial_t\phi_j^\delta+\eta+\frac{1}{2}\left\{\left|\sum_{j=0}^N(H^{p_j}\nabla\phi_j^\delta-p_jH^{p_j-1}\phi_j^\delta\nabla b)\right|^2+\delta^{-2}\left(\sum_{j=0}^N p_jH^{p_j-1}\phi_j^\delta\right)^2\right\}=0,\end{cases} \tag{4.7}$$

where $H(\boldsymbol{x},t) = 1+\eta(\boldsymbol{x},t)-b(\boldsymbol{x})$. We also use the notations $\boldsymbol{\phi}^\delta = (\phi_0^\delta, \phi_1^\delta, \dots, \phi_N^\delta)^{\mathrm{T}}$ and $L^\delta = L^\delta(H,b) = (L_{ij}^\delta(H,b))_{0\le i,j\le N}$, where

$$\begin{aligned} L_{ij}^\delta \psi_j = &-\nabla\cdot\left(\frac{1}{p_i+p_j+1}H^{p_i+p_j+1}\nabla\psi_j - \frac{p_j}{p_i+p_j}H^{p_i+p_j}\psi_j\nabla b\right) \\ &- \frac{p_i}{p_i+p_j}H^{p_i+p_j}\nabla b\cdot\nabla\psi_j + \frac{p_ip_j}{p_i+p_j-1}H^{p_i+p_j-1}(\delta^{-2}+|\nabla b|^2)\psi_j. \end{aligned} \tag{4.8}$$

Then, (4.7) can be written in a compact form

$$\begin{pmatrix} 0 & -\boldsymbol{l}(H)^{\mathrm{T}} \\ \boldsymbol{l}(H) & O \end{pmatrix} \partial_t \begin{pmatrix} \eta \\ \boldsymbol{\phi}^\delta \end{pmatrix} = \begin{pmatrix} \delta_\eta E^{\mathrm{IK},\delta} \\ \delta_{\boldsymbol{\phi}^\delta} E^{\mathrm{IK},\delta} \end{pmatrix}, \tag{4.9}$$

where

$$\begin{aligned} E^{\mathrm{IK},\delta}(\eta,\boldsymbol{\phi}^\delta) = \frac{1}{2}\int_{\mathbf{R}^n} \Bigg\{ \sum_{i,j=0}^N \Bigg(&\frac{1}{p_i+p_j+1}H^{p_i+p_j+1}\nabla\phi_i^\delta\cdot\nabla\phi_j^\delta - \frac{2p_i}{p_i+p_j}H^{p_i+p_j}\phi_i^\delta\nabla b\cdot\nabla\phi_j^\delta \\ &+ \frac{p_ip_j}{p_i+p_j-1}H^{p_i+p_j-1}(\delta^{-2}+|\nabla b|^2)\phi_i^\delta\phi_j^\delta\Bigg) + \eta^2\Bigg\}\,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{4.10}$$

Then, we define the Hamiltonian

$$\mathscr{H}^{\mathrm{IK},\delta}(\eta,\phi) = E^{\mathrm{IK},\delta}(\eta,\boldsymbol{\phi}^\delta),$$

where $\boldsymbol{\phi}^\delta$ is the solution to

$$\begin{cases} \boldsymbol{l}(H)\cdot\boldsymbol{\phi}^\delta = \phi, \\ L^\delta(H,b)\boldsymbol{\phi}^\delta = \big(L_0^\delta(H,b)\boldsymbol{\phi}^\delta\big)\boldsymbol{l}(H). \end{cases} \tag{4.11}$$

Here, we used the notation $L_0^\delta = (L_{00}^\delta, \dots, L_{0N}^\delta)$. We recall that $\boldsymbol{\phi}^\delta$ is uniquely determined by (4.11) thanks to Lemma 3.1.

To analyze the consistency of the Hamiltonian in the shallow water regime, we will further restrict ourselves to the following two cases:

(H1) In the case of the flat bottom $b(\boldsymbol{x}) \equiv 0$, $p_i = 2i$ for $i = 0,1,\dots,N$.

(H2) In the case with general bottom topographies, $p_i = i$ for $i = 0,1,\dots,N$.

We are now in position to state the consistency of the Hamiltonian of the Isobe–Kakinuma model with respect to Zakharov's Hamiltonian of the water wave problem in the shallow water regime.

Theorem 4.1. *Let c_0, M be positive constants and $m > \frac{n}{2}+1$ an integer such that $m \ge 4(N+1)$ in the case* (H1) *and $m \ge 4([\frac{N}{2}]+1)$ in the case* (H2)*. There exists a positive constant C such that if $\eta \in H^m$ and $b \in W^{m+1,\infty}$ satisfy*

$$\begin{cases} \|\eta\|_m + \|b\|_{W^{m+1,\infty}} \le M, \\ c_0 \le H(\boldsymbol{x}) = 1+\eta(\boldsymbol{x})-b(\boldsymbol{x}) \quad \textit{for} \quad \boldsymbol{x}\in\mathbf{R}^n, \end{cases}$$

then for any $\delta \in (0,1]$ and any $\phi \in \mathring{H}^m$, we have

$$|\mathscr{H}^\delta(\eta,\phi) - \mathscr{H}^{\mathrm{IK},\delta}(\eta,\phi)| \le \begin{cases} C\|\nabla\phi\|_{4N+3}\|\nabla\phi\|_0\,\delta^{4N+2} & \textit{in the case } (\mathrm{H1}), \\ C\|\nabla\phi\|_{4[\frac{N}{2}]+3}\|\nabla\phi\|_0\,\delta^{4[\frac{N}{2}]+2} & \textit{in the case } (\mathrm{H2}). \end{cases}$$

Remark 4.2. *Theorem 2.4 in [12] in fact states the stronger result that the difference between exact solutions of the water wave problem obtained in [10, 18] and the corresponding solutions of the Isobe–Kakinuma model is bounded with the same order of precision as above on the relevant timescale.*

Remark 4.3. *It is important to notice that the order of the approximation given in Theorem 4.1 is greater than what we could expect based on* (4.6)*, and in particular greater than the one obtained when using the Boussinesq expansion in the flat bottom case (H1):*

$$\phi_{\mathrm{B}}(\tilde{t},\tilde{\boldsymbol{x}}) = \tilde{\Phi}_{\mathrm{B}}^{\mathrm{app}}(\tilde{\boldsymbol{x}},\eta(\tilde{\boldsymbol{x}},\tilde{t}),\tilde{t}) \quad \textit{with} \quad \tilde{\Phi}_{\mathrm{B}}^{\mathrm{app}}(\tilde{\boldsymbol{x}},\tilde{z},\tilde{t}) = \sum_{i=0}^{N} \delta^{2i}(\tilde{z}+1)^{2i}\frac{(-\Delta)^i\phi_0(\tilde{\boldsymbol{x}},\tilde{t})}{(2i)!}$$

where ϕ_0 is the trace of the velocity potential at the bottom. Here we can only expect that the approximation is valid up to an error of order $O(\delta^{2N+2})$, which coincides with the precision of Theorem 4.1 only when $N=0$. When $N=0$, we recover that the Saint-Venant or shallow-water equations provide approximate solutions with precision $O(\delta^2)$; see [10, 18].

Proof of Theorem 4.1. We will modify slightly the strategy in [12]. We first notice that

$$\mathscr{H}^{\delta}(\eta,\phi) = \frac{1}{2}\iint_{\Omega}\big(|\nabla\Phi|^2 + \delta^{-2}(\partial_z\Phi)^2\big)\mathrm{d}\boldsymbol{x}\mathrm{d}z + \frac{1}{2}\int_{\mathbf{R}^n}\eta^2\mathrm{d}\boldsymbol{x},$$
$$\mathscr{H}^{\mathrm{IK},\delta}(\eta,\phi) = \frac{1}{2}\iint_{\Omega}\big(|\nabla\Phi^{\mathrm{app}}|^2 + \delta^{-2}(\partial_z\Phi^{\mathrm{app}})^2\big)\mathrm{d}\boldsymbol{x}\mathrm{d}z + \frac{1}{2}\int_{\mathbf{R}^n}\eta^2\mathrm{d}\boldsymbol{x},$$

where Φ is the unique solution to the boundary value problem of the scaled Laplace's equation (4.1) under the boundary conditions (4.3) and (4.5), and the approximate velocity potential Φ^{app} is defined by

$$\Phi^{\mathrm{app}}(\boldsymbol{x},z) = \sum_{i=0}^{N}(z+1-b(\boldsymbol{x}))^{p_i}\phi_i^{\delta}(\boldsymbol{x}),$$

where $\boldsymbol{\phi}^{\delta} = (\phi_0^{\delta},\phi_1^{\delta},\dots,\phi_N^{\delta})^{\mathrm{T}}$ is the solution to

$$\begin{cases} \displaystyle\sum_{i=0}^{N} H^{p_i}\phi_i^{\delta} = \phi, \\ \displaystyle\sum_{j=0}^{N} L_{ij}^{\delta}(H,b)\phi_j^{\delta} = H^{p_i}\sum_{j=0}^{N} L_{0j}^{\delta}(H,b)\phi_j^{\delta} \quad \text{for} \quad i=0,1,\dots,N. \end{cases}$$

We will denote with tildes, as in [12], the functions obtained when replacing N with $2N+2$. Hence, $\widetilde{\boldsymbol{\phi}}^{\delta} = (\widetilde{\phi}_0^{\delta},\widetilde{\phi}_1^{\delta},\dots,\widetilde{\phi}_{2N+2}^{\delta})^{\mathrm{T}}$ is the solution to

$$\begin{cases} \displaystyle\sum_{i=0}^{2N+2} H^{p_i}\widetilde{\phi}_i^{\delta} = \phi, \\ \displaystyle\sum_{j=0}^{2N+2} L_{ij}^{\delta}(H,b)\widetilde{\phi}_j^{\delta} = H^{p_i}\sum_{j=0}^{2N+2} L_{0j}^{\delta}(H,b)\widetilde{\phi}_j^{\delta} \quad \text{for} \quad i=0,1,\dots,2N+2. \end{cases}$$

We also introduce, as in [12], a modified approximate velocity potential $\widetilde{\Phi}^{\mathrm{app}}$ by

$$\widetilde{\Phi}^{\mathrm{app}}(\boldsymbol{x},z) = \sum_{i=0}^{2N+2}(z+1-b(\boldsymbol{x}))^{p_i}\widetilde{\phi}_i^{\delta}(\boldsymbol{x}), \tag{4.12}$$

and set $\Phi^{\rm res} = \Phi - \widetilde{\Phi}^{\rm app}$ and $\boldsymbol{\varphi}^\delta = (\varphi_0^\delta, \varphi_1^\delta, \dots, \varphi_N^\delta)^{\rm T}$ with $\varphi_j^\delta = \phi_j^\delta - \widetilde{\phi}_j^\delta$ for $j = 0, 1, \dots, N$. Then, we decompose the difference $\mathscr{H}^\delta - \mathscr{H}^{{\rm IK},\delta}$ as

$$
\begin{aligned}
&\mathscr{H}^\delta(\eta,\phi) - \mathscr{H}^{{\rm IK},\delta}(\eta,\phi) \\
&= \frac{1}{2}\iint_\Omega \big\{ \big(|\nabla\Phi|^2 + \delta^{-2}(\partial_z\Phi)^2\big) - \big(|\nabla\widetilde{\Phi}^{\rm app}|^2 + \delta^{-2}(\partial_z\widetilde{\Phi}^{\rm app})^2\big)\big\}{\rm d}\boldsymbol{x}{\rm d}z \\
&\quad + \frac{1}{2}\iint_\Omega \big\{ \big(|\nabla\widetilde{\Phi}^{\rm app}|^2 + \delta^{-2}(\partial_z\widetilde{\Phi}^{\rm app})^2\big) - \big(|\nabla\Phi^{\rm app}|^2 + \delta^{-2}(\partial_z\Phi^{\rm app})^2\big)\big\}{\rm d}\boldsymbol{x}{\rm d}z \\
&= I_1 + I_2.
\end{aligned}
$$

We first evaluate I_1. It is easy to see that

$$
\begin{aligned}
|I_1| \le \frac{1}{2}\big\{ &\|\nabla\Phi^{\rm res}\|_{L^2(\Omega)}\big(\|\nabla\Phi\|_{L^2(\Omega)} + \|\nabla\widetilde{\Phi}^{\rm app}\|_{L^2(\Omega)}\big) \\
&+ \delta^{-2}\|\partial_z\Phi^{\rm res}\|_{L^2(\Omega)}\big(\|\partial_z\Phi\|_{L^2(\Omega)} + \|\partial_z\widetilde{\Phi}^{\rm app}\|_{L^2(\Omega)}\big)\big\}.
\end{aligned} \tag{4.13}
$$

By using [12, Lemma 8.1] with $k = 0$ as well as [12, Lemma 6.4] with $(k, j) = (0, 2N+1)$ in the case (H1) and [12, Lemma 6.9] with $(k, j) = (0, 2[\frac{N}{2}] + 1)$ in the case (H2), we find

$$
\|\nabla\Phi^{\rm res}\|_{L^2(\Omega)} + \delta^{-1}\|\partial_z\Phi^{\rm res}\|_{L^2(\Omega)} \le \begin{cases} C\|\nabla\phi\|_{4N+3}\,\delta^{4N+3} & \text{in the case (H1)}, \\ C\|\nabla\phi\|_{4[\frac{N}{2}]+3}\,\delta^{4[\frac{N}{2}]+3} & \text{in the case (H2)}, \end{cases}
$$

provided that $m \ge 4(N+1)$ in the case (H1), and $m \ge 4([\frac{N}{2}] + 1)$ in the case (H2). Here and in what follows, C denotes a positive constant depending on N, m, c_0, and M, which changes from line to line. On the other hand, it follows from elliptic estimates given in [10, 18] that

$$
\|\nabla\Phi\|_{L^2(\Omega)} + \delta^{-1}\|\partial_z\Phi\|_{L^2(\Omega)} \le C\|\nabla\phi\|_0.
$$

Moreover, by the definition (4.12) and using [12, Lemma 3.4] with $k = 0$, we see that

$$
\begin{aligned}
\|\nabla\widetilde{\Phi}^{\rm app}\|_{L^2(\Omega)} + \delta^{-1}\|\partial_z\widetilde{\Phi}^{\rm app}\|_{L^2(\Omega)} &\le C\big(\|\nabla\widetilde{\phi}_0^\delta\|_0 + \|(\widetilde{\phi}_1^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_1 + \delta^{-1}\|(\widetilde{\phi}_1^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_0\big) \\
&\le C\big(\|\nabla\widetilde{\phi}_0^\delta\|_0 + \delta^{-1}\|(1-\delta^2\Delta)^{\frac{1}{2}}(\widetilde{\phi}_1^\delta, \dots, \widetilde{\phi}_{2N+2}^\delta)\|_0\big) \\
&\le C\|\nabla\phi\|_0.
\end{aligned}
$$

Plugging the above estimates into (4.13), we obtain

$$
|I_1| \le \begin{cases} C\|\nabla\phi\|_{4N+3}\|\nabla\phi\|_0\,\delta^{4N+3} & \text{in the case (H1)}, \\ C\|\nabla\phi\|_{4[\frac{N}{2}]+3}\|\nabla\phi\|_0\,\delta^{4[\frac{N}{2}]+3} & \text{in the case (H2)}, \end{cases} \tag{4.14}
$$

provided that $m \ge 4(N+1)$ in the case (H1), and $m \ge 4([\frac{N}{2}] + 1)$ in the case (H2).

We proceed to evaluate I_2 by noticing that, after the calculations in [12, p. 2009],

$$
\begin{aligned}
I_2 &= \frac{1}{2}\sum_{i=0}^{2N+2}\sum_{j=0}^{2N+2}\big(L_{ij}^\delta(H,b)\tilde{\phi}_j^\delta, \tilde{\phi}_i^\delta\big)_{L^2} - \frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{N}\big(L_{ij}^\delta(H,b)\phi_j^\delta, \phi_i^\delta\big)_{L^2} \\
&= \frac{1}{2}\sum_{j=0}^{N}\sum_{i=N+1}^{2N+2}\big((L_{ij}^\delta(H,b) - H^{p_i}L_{0j}^\delta(H,b))\varphi_j^\delta, \widetilde{\phi}_i^\delta\big)_{L^2} \\
&\quad - \frac{1}{2}\sum_{j=N+1}^{2N+2}\sum_{i=N+1}^{2N+2}\big((L_{ij}^\delta(H,b) - H^{p_i}L_{0j}^\delta(H,b))\widetilde{\phi}_j^\delta, \widetilde{\phi}_i^\delta\big)_{L^2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|I_2| \le C\big\{&\|\boldsymbol{\varphi}^\delta\|_{2N^*+3} + \|(\widetilde{\phi}^\delta_{N+1},\dots,\widetilde{\phi}^\delta_{2N+2})\|_{2N^*+3} \\
&+ \delta^{-2}\big(\|\boldsymbol{\varphi}^\delta\|_{2N^*+1} + \|(\widetilde{\phi}^\delta_{N+1},\dots,\widetilde{\phi}^\delta_{2N+2})\|_{2N^*+1}\big)\big\}\|(\widetilde{\phi}^\delta_{N+1},\dots,\widetilde{\phi}^\delta_{2N+2})\|_{-(2N^*+1)}
\end{aligned}
$$

with $N^\star = N$ in the case (H1) and $N^\star = [\frac{N}{2}]$ in the case (H2). Using [12, Lemma 6.2] with $(k,j) = (2N+3, N), (2N+1, N+1), (-2N-1, N+1)$ in the case (H1) and [12, Lemma 6.7] with $(k,j) = (2[\frac{N}{2}]+3, [\frac{N}{2}]), (2[\frac{N}{2}]+1, [\frac{N}{2}]+1), (-2[\frac{N}{2}]-1, [\frac{N}{2}]+1)$ in the case (H2), we obtain

$$
|I_2| \le \begin{cases} C\|\nabla\phi\|_{4N+2}\|\nabla\phi\|_0\ \delta^{4N+2} & \text{in the case (H1)}, \\ C\|\nabla\phi\|_{4[\frac{N}{2}]+2}\|\nabla\phi\|_0\ \delta^{4[\frac{N}{2}]+2} & \text{in the case (H2)}, \end{cases} \tag{4.15}
$$

provided that $m \ge 4N+3$ in the case (H1), and $m \ge 4[\frac{N}{2}]+3$ in the case (H2). Now, (4.14) and (4.15) give the desired estimate. □

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