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Existence and uniqueness theorem for a 3-dimensional polytope in $\mathbb{R}^3$ with prescribed directions and perimeters of the facets

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Abstract

We give a set of conditions that is necessary and sufficient for the existence and uniqueness up to translations of a 3-dimensional polytope P in $\mathbb{R}^3$ having N facets with given unit outward normal vectors $n_1, \ldots, n_N$ and corresponding facet perimeters $L_1, \ldots, L_N$.

In 1897, Hermann Minkowski studied the problem of prescribing the areas and outer unit normals of the facets of a 3-dimensional polytope. The existence and uniqueness theorem that he obtained is one the most fundamental results in the theory of polytopes. This paper is devoted to the analogous problem of prescribing the perimeters and outer unit normals of the facets of a 3-dimensional polytope. Our main result (Theorem 5) gives a necessary and sufficient condition for the existence and uniqueness up to a translation of a 3-dimensional polytope P in $\mathbb{R}^3$ having N facets with given unit outward normals $n_1, \ldots, n_N$ and corresponding facet perimeters $L_1, \ldots, L_N$.

Introduction to the problem

In this paper, a polytope of $\mathbb{R}^3$ is the convex hull of finitely many points in $\mathbb{R}^3$. The classical Minkowski problem for polytopes in $\mathbb{R}^3$ concerns the following question:

Given a collection $n_1, \ldots, n_N$ of N pairwise distinct unit vectors in $\mathbb{R}^3$ and $F_1, \ldots, F_N$ a collection of N positive real numbers, is there a polytope P in $\mathbb{R}^3$ having the n_i as its facet unit outward normals and the F_i as the corresponding facet areas $(1 \leq i \leq N)$, and, if so, is P unique up to translations?

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H. Minkowski proved the following uniqueness theorem (see [1, Theorem 9, p. 107]):

Theorem 1 *(H. Minkowski, 1897: [4] and [5, pp. 103-121])*
A polytope in $\mathbb{R}^3$ is uniquely determined, up to translations, by the directions and the areas of its facets.

A well-known necessary condition for the existence of a polytope having facet unit outward normals $n_1, \ldots, n_N$ and corresponding facet areas $F_1, \ldots, F_N$ is that:

$$\sum_{i=1}^{N} F_i n_i = 0.$$

An existence theorem of H. Minkowski ensures that this condition is both necessary and sufficient:

Theorem 2 *(H. Minkowski, 1897: [4] and [5, pp. 103-121])*
Let $n_1, \ldots, n_N \in \mathbb{R}^3$ be N pairwise distinct unit vectors linearly spanning $\mathbb{R}^3$ and let $F_1, \ldots, F_N$ be N positive real numbers. There exists a polytope P in $\mathbb{R}^3$ having N facets with unit outward normals $n_1, \ldots, n_N$ and corresponding facet areas $F_1, \ldots, F_N$ if, and only if, we have

$$\sum_{i=1}^{N} F_i n_i = 0.$$

Here, we have to mention that Theorem 2 is only the 3-dimensional version of the classical Minkowski existence and uniqueness theorem [6, p. 455], which is valid in $\mathbb{R}^d$ for all $d \geq 2$. The proof of our main result (Theorem 5) will make use of the 2-dimensional version, which is almost trivial:

The Minkowski theorem for convex polygons in $\mathbb{R}^2$.

Let $n_1, \ldots, n_N \in \mathbb{R}^2$ be N pairwise distinct unit vectors linearly spanning $\mathbb{R}^2$ and let $l_1, \ldots, l_N$ be N positive real numbers. There exists a convex polygon P in $\mathbb{R}^2$ having N edges with unit outward normals $n_1, \ldots, n_N$ and corresponding edge lengths $l_1, \ldots, l_N$ if, and only if,

$$\sum_{i=1}^{N} l_i n_i = 0.$$

This paper is devoted to the analogue of the classical Minkowski problem obtained by replacing areas by perimeters. For this analogue, the following uniqueness result is known (see [1, p. 108]):

Theorem 3 *A polytope in $\mathbb{R}^3$ is uniquely determined, up to translations, by the directions and the perimeters of its facets.*

Theorems 1 and 3 are similar uniqueness theorems which are both corollaries of a same general result by A.D. Alexandrov (see [1, Theorem 8, p. 107]). Thus, we are led to the natural question of the existence of an analogue to Theorem 2 for the existence of a polytope with prescribed directions and perimeters of the facets.

For convenience, we will restrict ourselves to 3-dimensional polytopes in $\mathbb{R}^3$. Recall that the dimension of a convex body in $\mathbb{R}^d$ is simply the dimension of its affine hull. Recall also that a facet of a 3-dimensional polytope P is a (convex) polygonal face of P, and that its perimeter is defined to be the sum of the lengths of all its sides (edges).

Difficulty of the problem

The above problem of prescribing the perimeters and outer unit normals of the facets of a 3-dimensional polytope has attracted the attention of geometers. Recently, a paper by V. Alexandrov highlighted its difficulty in explaining why a simple equation involving the prescribed perimeters cannot suffice to establish an analogue to Theorem 2 [2]. The main result of that paper reads as follows:

Theorem 4 *(V. Alexandrov, 2018: [2])*

Let $n_1, \ldots, n_5$ in $\mathbb{R}^3$ be defined by the formulas

$$n_1 := (0,0,1)\,,\ n_2 := \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),\ n_3 := \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),$$
$$n_4 := \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ n_5 := \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Let $\mathcal{L}(n_1, \ldots, n_5) \subset \mathbb{R}^5$ be the set of all points $(L_1, \ldots, L_5) \in \mathbb{R}^5$ with the following property: there exists a polytope $P \subset \mathbb{R}^3$ such that $n_1, \ldots, n_5$ (and no other vector) are the unit outward normals to the facets of P, and L_k is the perimeter of the face with the outward normal n_k for every $k \in \{1, \ldots, 5\}$. Then the set $\mathcal{L}(n_1, \ldots, n_5) \subset \mathbb{R}^5$ is not locally-analytic.

This result is of course interpreted by V. Alexandrov as an obstacle for finding an existence theorem for a polytope with prescribed directions and perimeters of the facets. This was the major source of inspiration for the work presented in this paper.

Necessary conditions for the existence of a solution

Let $n_1, \ldots, n_N$ be a collection of N pairwise distinct unit vectors linearly spanning $\mathbb{R}^3$ and let $L_1, \ldots, L_N$ be a collection of N positive

real numbers. The following set $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv}), (\mathbf{v})\}$ of conditions is necessary for the existence of a 3-dimensional polytope P in $\mathbb{R}^3$ having the n_i as its facet outward unit normals and the L_i as the corresponding facet perimeters.

(**i**) For each $i \in \{1, \ldots, N\}$, there exists a decomposition of L_i into a sum of N non-negative real numbers,

$$L_i = \sum_{j=1}^{N} l_{ij},$$

in such a way that:

(**ii**) for all $(i, j) \in (\{1, \ldots, N\})^2$, $l_{ij} = 0$ if n_i and n_j are collinear;

(**iii**) for all $(i, j) \in (\{1, \ldots, N\})^2$, $l_{ji} = l_{ij}$.

In other words, conditions (**i**) through (**iii**) require the existence of a symmetric matrix with nonnegative entries $(l_{ij})_{1 \leq i,j \leq N}$, such that row i sums to L_i, $(1 \leq i \leq N)$, and $l_{ij} = 0$ for collinear n_i, n_j, $(1 \leq i, j \leq N)$.

Indeed, if such a polytope P exists, then denoting by $f_1, \ldots, f_N$ the N facets with respective unit outward normals $n_1, \ldots, n_N$, the required relationships hold if we put:

$$l_{ij} := \begin{cases} 0 \text{ if } i = j \text{ or if } f_i \text{ and } f_j \text{ have no common edge} \\ \\ \text{the length of the common edge otherwise.} \end{cases}$$

Our condition (iv) is a consequence of the fact that the edge vectors of a facet (which are perpendicular to the unit normals of both incident facets), oriented in positive direction with respect to the unit normal of the facet, concatenate into a (simple) closed circuit:

$$(\mathbf{iv}) \text{ For every } i \in \{1, \ldots, N\}, \sum_{j \in \{k | l_{ik} \neq 0\}} l_{ij} \left[\frac{n_i \times n_j}{\sin(n_i, n_j)} \right] = 0,$$

where $\times$ denotes the cross product (here, we of course assume that $\mathbb{R}^3$ is oriented by its canonical basis), and (n_i, n_j) denotes the length of the shortest arc of great circle joining n_i to n_j on the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$ (recall that n_i and n_j are non collinear by condition (**ii**) since $l_{ij} \neq 0$). Indeed, each facet f_i of P is a convex polygon the boundary of which is a closed polygonal line. Here, it is worth noting that, for all $j \in \{k \,|\, l_{ik} \neq 0\}$,

$$\overrightarrow{u_{i,j}} = (n_i \times n_j) / \sin(n_i, n_j)$$

is a unit vector that is such that the vector $\overrightarrow{v_{i,j}} := l_{ij}\overrightarrow{u_{i,j}}$ is of the form $\overrightarrow{MM'}$, where M and M' are two consecutive vertices of the oriented boundary of the face f_i (see the figure 1 where $\mathrm{v}_1, \ldots, \mathrm{v}_m$ are the successive $\overrightarrow{v_{i,j}} := l_{ij}\overrightarrow{u_{i,j}}$, with $j \in \{k \,|l_{ik} \neq 0\}$).

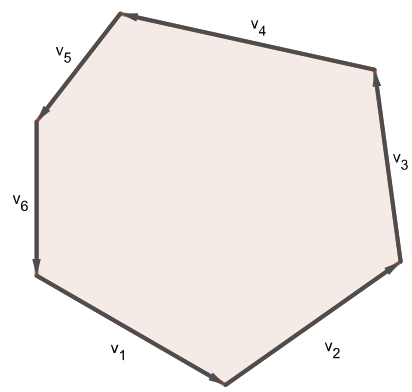

Figure 1. Illustration of the fact that

$$\sum_{j\in\{k|l_{ik}\neq 0\}} l_{ij}\overrightarrow{u_{ij}} = \textstyle\sum_{l=1}^{m} \mathrm{v}_l = 0$$

Our last necessary condition (v) will follow from Steinitz's theorem (e.g. see [7, Chapter 4, p. 103]), which characterizes in purely graph-theoretic terms those graphs that can be represented as the 1-skeleton of some 3-dimensional polytope:

Theorem (Steinitz's theorem). *A graph can be represented as the* 1*-skeleton of some* 3*-dimensional polytope if, and only if, it is simple, planar, and* 3*-connected.*

For the convenience of the reader, we shall summarize some basic definitions and facts on graphs and 1-skeletons just before the proof of Theorem 5. From Steinitz's theorem, the following last condition is also necessary in our case:

(**v**) The datum of the matrix $(l_{ij})_{1\leq i,j\leq N}$ determines as follows a simple 3-connected planar graph G drawn on the unit sphere $\mathbb{S}^2$ (so that no two of the edges intersect at a point other than a vertex): the vertices of G are the unit vectors $n_1, \ldots, n_N$, and any pair of non-collinear vertices $\{n_i, n_j\}$ of G is connected by an edge that is given by the shortest arc of great circle joining the two vertices on $\mathbb{S}^2$ if, and only if, $l_{ij} \neq 0$.

Note that the datum of G, drawn on $\mathbb{S}^2$, simply corresponds to that of the so-called *slope diagram representation* of the desired polytope P, say $SDR(P)$, in which facets, edges and vertices of P are represented by points, spherical arcs and convex spherical polygons on $\mathbb{S}^2$. More precisely, in $SDR(P)$, each facet is represented by the end point of its outward unit normal vector, each edge is represented by the shortest arc of great circle joining the two points corresponding to the adjacent facets of the edge, and each vertex is represented by the convex spherical polygon bounded by the arcs corresponding to the edges of P meeting at the vertex. In this paper, we will also call $SDR(P)$ *the spherical representation of* P.

Our main result

Theorem 5 *Let $n_1, \ldots, n_N \in \mathbb{R}^3$ be N distinct unit vectors linearly spanning $\mathbb{R}^3$ and let $L_1, \ldots, L_N$ be N positive real numbers. There exists a 3-dimensional polytope P in $\mathbb{R}^3$ having facet unit outward normals $n_1, \ldots, n_N$ and corresponding facet perimeters $L_1, \ldots, L_N$ if, and only if, the set of conditions $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv}), (\mathbf{v})\}$ holds.*

Important remark. The set of conditions $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv})\}$ is far from being sufficient to ensure that there exists a 3-dimensional polytope in $\mathbb{R}^3$ having facet unit outward normals $n_1, \ldots, n_N$ and corresponding facet perimeters $L_1, \ldots, L_N$. Indeed, many problems can arise if we drop condition $(\mathbf{v})$ from Theorem 5. If we retain only the first four conditions the data could correspond to a union of several polytopes. It is also possible for the data to be consistent with conditions $(\mathbf{i}) - (\mathbf{iv})$ while corresponding to non-convex polytopes, including such with non-convex facets, if one considers expanding the notion of unit outward normal. Of course, in our context, a reconstruction attempt as a convex polytope will then lead to problems. For instance, if we consider the unit vectors $n_1, \ldots, n_7 \in \mathbb{R}^3$ defined by $n_1 := (0, 0, -1)$, $n_2 := \left(\cos\frac{2\pi}{5}, \sin\frac{2\pi}{5}, 0\right)$, $n_3 := \left(\cos\frac{6\pi}{5}, \sin\frac{6\pi}{5}, 0\right)$, $n_4 := (1, 0, 0)$, $n_5 := \left(\cos\frac{4\pi}{5}, \sin\frac{4\pi}{5}, 0\right)$, $n_6 := \left(\cos\frac{8\pi}{5}, \sin\frac{8\pi}{5}, 0\right)$, $n_7 := (0, 0, 1)$, and the datum of the matrix

$$(l_{ij})_{1 \le i,j \le 7} := \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

for which the set of conditions $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv})\}$ holds with $L_1 = L_7 = 5$ and $L_2 = L_3 = L_4 = L_5 = L_6 = 4$, we obtain a combinatorial prism. As per the given unit outward normals, the prism has parallel top and bottom pentagon facets, and five side facets with dihedral angle $\pi/5$ between adjacent sides. Connecting the side facets necessarily leads to self-intersections, thus there is no 3-dimensional polytope P in $\mathbb{R}^3$ corresponding to these data. There is, however, a non-convex right prism with regular pentagram base (like the one shown in Figure 2) which corresponds to these data with suitable adaptation of the notion of unit outward normal.

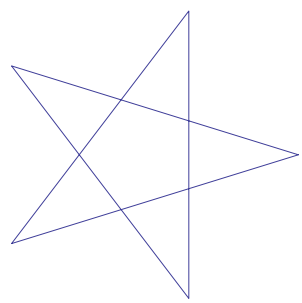

Figure 2. A pentagram

In other words, given N distinct unit vectors $n_1, \ldots, n_N \in \mathbb{R}^3$ linearly spanning $\mathbb{R}^3$ and N positive real numbers $L_1, \ldots, L_N$ such that the set of conditions $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv})\}$ holds, we can of course associate a graph to the matrix $(l_{ij})_{1 \leq i,j \leq N}$ but to be sure that this graph does correspond to the 1-skeleton of some 3-dimensional polytope, it is necessary and sufficient to assume that it satisfies the conditions of Steinitz's theorem. This is essentially what condition $(\mathbf{v})$ requires.

Basic definitions and facts on graphs and 1-skeletons

For the convenience of the reader, we summarize some basic definitions and facts on graphs and 1-skeletons:

∘ The 1-*skeleton* of a polytope P is the graph whose vertices and edges are just the vertices and edges of P with the same incidence relation.
∘ A graph is said to be *polyhedral* if it can be represented as the 1-skeleton of some 3-dimensional polytope.
∘ A graph is said to be *simple* if it contains neither multiple edges nor loops.
∘ A graph is said to be *planar* if it can be drawn in the plane so that no two of the edges intersect at a point other than a vertex.

∘ A graph is called 3-*connected* if it is connected, has at least 4 vertices, and remains connected whenever fewer than 3 vertices are removed.
∘ The dual of a polyhedral graph is also a polyhedral graph. More precisely, every polyhedral graph G has a well-defined dual graph G^* (independent of the plane embedding), corresponding to the 1-skeleton of the dual polytope.

Proof of Theorem 5. We have already seen that this set of conditions $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv}), (\mathbf{v})\}$ is necessary for such a polytope P to exist in $\mathbb{R}^3$.

Conversely assume that the set of conditions $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv}), (\mathbf{v})\}$ holds. Recall that two polytopes P and P' are said to be *combinatorially equivalent* if there is a bijection between their faces that preserves the inclusion relation. It is well-known that the combinatorial structure of a 3-dimensional polytope P is completely determined by its 1-skeleton [7, p. 105].

(a) *There exists a 3-dimensional polytope with the given combinatorial structure*

By Steinitz's theorem, the simple 3-connected planar graph G that is constructed on the sphere $\mathbb{S}^2$ in accordance with condition $(\mathbf{v})$ and its geometric dual graph G^* (which is also simple, planar, and 3-connected) are polyhedral: they can be represented in $\mathbb{R}^3$ as the 1-skeletons of two dual 3-dimensional polytopes, say Q and Q^*, respectively.

(b) *The shape of one facet (resp. of one vertex-figure) can be chosen*

Moreover, by the following refinement by Barnette and Grünbaum, we can preassign the shape of a face of one of these two polytopes [3]:

Theorem (Barnette and Grünbaum). *If one face of a 3-dimensional polytope Q is an n-gon, then there exists a polytope Q' combinatorially equivalent to Q, of which the corresponding face is any prescribed convex n-gon.*

Moreover, as noticed by Barnette and Grünbaum [3, p. 305]: "By an obvious application of duality, it follows from the theorem that the shape of one vertex-figure may be prescribed". In the part (f) of the proof, we will make use of this dual form of the theorem to start our construction. Our desired polytope P will be combinatorially equivalent to Q^*.

(c) *Vocabulary convention*

In the remainder of the proof, the assembly of all the facets of a given 3-dimensional polytope that share a same vertex will be called a *corner*

of the polytope. The *spherical representation* of such a corner, or of a 3-dimensional polytope, is defined as follows:
- A facet f corresponding to a unit normal n is represented on $\mathbb{S}^2$ by n;
- An edge is represented on $\mathbb{S}^2$ by the shortest arc of great circle joining the two points corresponding to the two adjacent facets of the edge;
- A vertex is represented on $\mathbb{S}^2$ by the convex spherical polygon that is bounded by the spherical arcs corresponding to the edges that are adjacent at the vertex.

(d) *The polygons making up the desired facets are uniquely determined up to translations in space*

Our aim is to prove the existence of a 3-dimensional polytope P that satisfies the set of conditions $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv}), (\mathbf{v})\}$, and thus, the spherical representation of which is given by G.

Since condition $(\mathbf{iv})$ is satisfied, the Minkowski existence and uniqueness theorem for convex polygons ensures that, for any $i \in \{1, \ldots, N\}$, there exists in $n_i^{\perp}$ (the linear plane with unit normal n_i, endowed with the induced orientation) a positively oriented convex polygon f_i whose edges e_{ij} are directed by the unit vectors

$$\overrightarrow{u_{i,j}} = (n_i \times n_j) / \sin(n_i, n_j), \quad (j \in \{k \,|\, l_{ik} \neq 0\}),$$

and have corresponding lengths l_{ij}; and moreover, that this polygon f_i, of which the perimeter is L_i by condition $(\mathbf{i})$, is unique up to translations in $n_i^{\perp}$. Note that in the above expression of $\overrightarrow{u_{i,j}}$, the vectors n_i and n_j are non-collinear by condition $(\mathbf{ii})$ since $l_{ij} \neq 0$.

Thus, *for any $i \in \{1, \ldots, N\}$, the desired oriented convex polygon f_i, with unit normal n_i, is well-defined and unique up to translations in $\mathbb{R}^3$.*

(e) *All the corners of the desired polytope are well-defined and unique up to translations in space*

Now, let $\mathcal{P}_n$ be any positively oriented n-gon on $\mathbb{S}^2$ that is the oriented boundary of the closure of a connected component of the complementary of the graph G, which is drawn on $\mathbb{S}^2$ according to condition $(\mathbf{v})$, $(n \geq 3)$. Girard's theorem relates spherical angle excess and area of the spherical n-gon, which allows us to deduce that

$$\sum_{k=1}^{n} \alpha_k = (n-2)\pi + Area(\mathcal{P}_n),$$

where $\alpha_1, \ldots, \alpha_n$ denote the interior angles of $\mathcal{P}_n$. Because of condition $(\mathbf{iv})$, $\mathcal{P}_n$ has no reflex angle so that:

$$\sum_{k=1}^{n} \beta_k = 2\pi - Area(\mathcal{P}_n) < 2\pi,$$

where $\beta_1, \ldots, \beta_n$ denote the exterior angles of $\mathcal{P}_n$, (that is, $\beta_k := \pi - \alpha_k$ for all $k \in \{1, \ldots, n\}$). Now, for every for all $k \in \{1, \ldots, n\}$, the exterior angle β_k at a vertex (say, u_k) of $\mathcal{P}_n$ can also be regarded as the interior angle of a convex polygon f_k (which is the facet with unit outward normal u_k of the desired polytope P) at the vertex of f_k (say, s_n) that corresponds to $\mathcal{P}_n$. [This can be seen as follows. The two oriented sides of $\mathcal{P}_n$ that are adjacent at u_k are part of two oriented great circles of $\mathbb{S}^2$, which are the oriented unit circles of two oriented linear planes of $\mathbb{R}^3$. For each of these two planes, consider the linear line that is orthogonal to the plane, and then the intersection point of this line with $\mathbb{S}^2$ that is on the same side of the plane as $\mathcal{P}_n$. By doing this, we obtain the end points of two unit vectors that are directing the two sides of f_k adjacent at s_n and that are pointing outward from these sides at s_n. Finally, we note that the geometric angle between these two unit vectors is nothing but the exterior angle β_k.].

The above inequality, which says that the sum $\sum_{k=1}^{n} \beta_k$ of the exterior angles of $\mathcal{P}_n$, is less than 2π can be regarded as a *nonnegative curvature condition* that is satisfied from our conditions. Therefore, taking into account condition (**iii**), the convex polygons f_k that correspond to the unit vectors u_k of which the end points are the vertices of $\mathcal{P}_n$ can be assembled (by gluing together their sides that correspond to a same edge of $\mathcal{P}_n$) to form a corner (of a 3-dimensional polytope), the spherical representation of which corresponds to $\mathcal{P}_n$ in $\mathbb{S}^2$. Here, the "convexity" at the corner is of course due to the nonnegative curvature condition.

Thus *all the corners of the desired polytope P are well-defined and unique up to translations in $\mathbb{R}^3$.*

(f) *They can be put together without contradiction*

Starting from any of these corners, we can construct by induction the desired polytope P, which is combinatorially equivalent to Q^* and satisfies the set of conditions $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv}), (\mathbf{v})\}$, by assembling at each step, an adjacent corner to the part of P, say $Part\,(P)$, that has already been constructed. Here, by "an adjacent corner" to $Part\,(P)$ we mean "a corner of P that is not included in $Part\,(P)$ but that shares two facets with $Part\,(P)$". At each step, the spherical representation of the part of P that is constructed is controlled by condition (**v**), and the construction can continue until completion since we made sure that all the pieces had the required shape and dimensions.

It is worth noting that, as soon as the position of the first corner is fixed (with, for example, its vertex placed at the origin O), the position of any other vertex S of P is deduced from that of O by a succession of translations from a vertex of P to another: consider any succession

of adjacent regions of $\mathbb{S}^2 \backslash G$ from that corresponding to O to that corresponding to S (two regions of $\mathbb{S}^2 \backslash G$ are said to be adjacent if their boundaries share an edge of G) and note that each crossing on $\mathbb{S}^2$ from one region $\mathcal{R}_i$ of $\mathbb{S}^2 \backslash G$ to an adjacent one $\mathcal{R}_j$ corresponds on P to the translation from a vertex to another by a translation by a vector $\overrightarrow{v_{ij}}$ whose direction is determined by the arc of great circle γ_{ij} separating the two regions on $\mathbb{S}^2$ ($\overrightarrow{v_{ij}}$ is orthogonal to it and oriented in the sense of the crossing) and whose norm $\|\overrightarrow{v_{ij}}\|$ is the length l_{ij} corresponding to γ_{ij}. Of course, thanks to condition (**iv**), the final position of the vertex (i.e. that of S) does not depend on the succession of adjacent regions that has been considered.

■

Remark. In higher dimensions, the problem should, of course, be much more difficult since there is no known analogue of Steinitz's theorem.

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