On Zeta Functions and Tetrahedral Numbers

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September 25, 2019

Abstract

In this short paper we analyze a possible relation between ζ(n) (for n odd and greater than one) and the Tetrahedral Numbers (number of balls in a triangular pyramid in which each edge contains n balls).[1], also the irrationality of all these (ζ) functions.

Introduction.

The Zeta Functions, are functions of the form:

$$\zeta(2n) = \sum_{r=1}^{\infty} \frac{1}{r^{2n}}$$

$$\zeta(2n+1) = \sum_{r=1}^{\infty} \frac{1}{r^{2n+1}}$$

in this paper, we will study the second one. In 2000, T. Rivoal proved that there are infinitely many integers n such that $\zeta(2n+1)$ is irrational, and subsequently in 2001 that at least one of $\zeta(5), \zeta(7), \zeta(9), ...\zeta(21)$ is irrational. Again in 2001 this result has been tightened by Zudilin to replace 21 by 11.[2]
In the case of $\zeta(3)$, it was proven to be irrational in 1978 by Roger Apery and is called Apery's constant.

Zeta Functions for (n) odd.

The first ten values of the Zeta Functions of the form: $\zeta(2n+1) = \sum_{r=1}^{\infty} \frac{1}{r^{2n+1}}$ are next.

$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + ... + \frac{1}{r^3} = 1.202056903$

$\zeta(5) = \frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + ... + \frac{1}{r^5} = 1.036927755$

$\zeta(7) = \frac{1}{1^7} + \frac{1}{2^7} + \frac{1}{3^7} + ... + \frac{1}{r^7} = 1.008349277$

$\zeta(9) = \frac{1}{1^9} + \frac{1}{2^9} + \frac{1}{3^9} + ... + \frac{1}{r^9} = 1.002008392$

$\zeta(11) = \frac{1}{1^{11}} + \frac{1}{2^{11}} + \frac{1}{3^{11}} + ... + \frac{1}{r^{11}} = 1.000494188$

$\zeta(13) = \frac{1}{1^{13}} + \frac{1}{2^{13}} + \frac{1}{3^{13}} + ... + \frac{1}{r^{13}} = 1.000122713$

$\zeta(15) = \frac{1}{1^{15}} + \frac{1}{2^{15}} + \frac{1}{3^{15}} + ... + \frac{1}{r^{15}} = 1.000030588$

$\zeta(17) = \frac{1}{1^{17}} + \frac{1}{2^{17}} + \frac{1}{3^{17}} + ... + \frac{1}{r^{17}} = 1.000007637$

$\zeta(19) = \frac{1}{1^{19}} + \frac{1}{2^{19}} + \frac{1}{3^{19}} + ... + \frac{1}{r^{19}} = 1.000001908$

$\zeta(21) = \frac{1}{1^{21}} + \frac{1}{2^{21}} + \frac{1}{3^{21}} + ... + \frac{1}{r^{21}} = 1.000000476$

if we order them, we have:

$$\begin{bmatrix} \frac{1}{1^3} & \frac{1}{2^3} & \frac{1}{3^3} & \frac{1}{4^3} & \frac{1}{5^3} & \frac{1}{6^3} & \frac{1}{7^3} & \frac{1}{8^3} & \cdots & \frac{1}{r^3} \\ \frac{1}{1^5} & \frac{1}{2^5} & \frac{1}{3^5} & \frac{1}{4^5} & \frac{1}{5^5} & \frac{1}{6^5} & \frac{1}{7^5} & \frac{1}{8^5} & \cdots & \frac{1}{r^5} \\ \frac{1}{1^7} & \frac{1}{2^7} & \frac{1}{3^7} & \frac{1}{4^7} & \frac{1}{5^7} & \frac{1}{6^7} & \frac{1}{7^7} & \frac{1}{8^7} & \cdots & \frac{1}{r^7} \\ \frac{1}{1^9} & \frac{1}{2^9} & \frac{1}{3^9} & \frac{1}{4^9} & \frac{1}{5^9} & \frac{1}{6^9} & \frac{1}{7^9} & \frac{1}{8^9} & \cdots & \frac{1}{r^9} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ \frac{1}{1^{2n+1}} & \frac{1}{2^{2n+1}} & \frac{1}{3^{2n+1}} & \frac{1}{4^{2n+1}} & \frac{1}{5^{2n+1}} & \frac{1}{6^{2n+1}} & \frac{1}{7^{2n+1}} & \frac{1}{8^{2n+1}} & & \end{bmatrix}$$

now, we divide column by another column, and we have:

$\sum_{r=1}^{\infty} \frac{1}{r^{2n+1}} \div \sum_{r=1}^{\infty} \frac{1}{(r+1)^{2n+1}}$

assigning values:

$\sum_{r=1}^{\infty} \frac{1}{2^{2n+1}} \div \sum_{r=1}^{\infty} \frac{1}{3^{2n+1}} = 4$

$\sum_{r=1}^{\infty} \frac{1}{3^{2n+1}} \div \sum_{r=1}^{\infty} \frac{1}{4^{2n+1}} = 2.5$

$\sum_{r=1}^{\infty} \frac{1}{4^{2n+1}} \div \sum_{r=1}^{\infty} \frac{1}{5^{2n+1}} = 2$

$\sum_{r=1}^{\infty} \frac{1}{5^{2n+1}} \div \sum_{r=1}^{\infty} \frac{1}{6^{2n+1}} = 1.75$

$\sum_{r=1}^{\infty} \frac{1}{6^{2n+1}} \div \sum_{r=1}^{\infty} \frac{1}{7^{2n+1}} = 1.6$

$\sum_{r=1}^{\infty} \frac{1}{7^{2n+1}} \div \sum_{r=1}^{\infty} \frac{1}{8^{2n+1}} = 1.5$

$\sum_{r=1}^{\infty} \frac{1}{8^{2n+1}} \div \sum_{r=1}^{\infty} \frac{1}{9^{2n+1}} = 1.4285714 = \frac{10}{7}$

$\sum_{r=1}^{\infty} \frac{1}{9^{2n+1}} \div \sum_{r=1}^{\infty} \frac{1}{10^{2n+1}} = 1.375$

multiplying results in sequence we obtain the Tetrahedral Numbers.

Table 1: Tetrahedral Numbers.

Multiplying Results	**Tetrahedral Number**
(4)(2.5)	10
(4)(2.5)(2)	20
(4)(2.5)(2)(1.75)	35
(4)(2.5)(2)(1.75)(1.6)	56
(4)(2.5)(2)(1.75)(1.6)(1.5)	84
(4)(2.5)(2)(1.75)(1.6)(1.5)(1.4285714)	120
(4)(2.5)(2)(1.75)(1.6)(1.5)(1.4285714)(1.375)	165

The Tetrahedral Numbers, are numbers of the form $\frac{(n)(n+1)(n+2)}{6}$

Now, if we divide row by column, we have:

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^n}}{\frac{1}{r^{2n+1}}}$

For $\zeta(3)$ (knowing that $\zeta(3)$ is irrational.)

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^3}}{\frac{1}{2^{2n+1}}} = 7.21204440396$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^3}}{\frac{1}{3^{2n+1}}} = 28.8481776158$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^3}}{\frac{1}{4^{2n+1}}} = 72.1204440396$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^3}}{\frac{1}{5^{2n+1}}} = 144.240888079$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^3}}{\frac{1}{6^{2n+1}}} = 252.421554139$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^3}}{\frac{1}{7^{2n+1}}} = 403.874486622$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^3}}{\frac{1}{8^{2n+1}}} = 605.811729933$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^3}}{\frac{1}{9^{2n+1}}} = 865.445328475$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^3}}{\frac{1}{10^{2n+1}}} = 1189.98732665$

If we divide: $[\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^3}}{\frac{1}{(r+1)^{2n+1}}}] \div [\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^3}}{\frac{1}{r^{2n+1}}}]$

We have: $\frac{28.8481776158}{7.21204440396}, \frac{72.1204440396}{28.8481776158}, \frac{144.240888079}{72.1204440396} \cdots$

that gives: $4, 2.5, 2, 1.75 \cdots$ etc,

multiplyng them in sequence, we obtain the Tetrahedral Numbers.

For $\zeta(5)$ and $\zeta(7)$, we have:

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^5}}{\frac{1}{2^{2n+1}}} = 6.22156651616$ and $\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^7}}{\frac{1}{2^{2n+1}}} = 6.05009566429$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^5}}{\frac{1}{3^{2n+1}}} = 24.8862660646$ and $\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^7}}{\frac{1}{3^{2n+1}}} = 24.2003826572$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^5}}{\frac{1}{4^{2n+1}}} = 62.2215665161$ and $\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^7}}{\frac{1}{4^{2n+1}}} = 60.5009566429$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^5}}{\frac{1}{5^{2n+1}}} = 124.431330323$ and $\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^7}}{\frac{1}{5^{2n+1}}} = 121.001913286$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^5}}{\frac{1}{6^{2n+1}}} = 217.754828065$ and $\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^7}}{\frac{1}{6^{2n+1}}} = 211.75334825$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^5}}{\frac{1}{7^{2n+1}}} = 348.407724905$ and $\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^7}}{\frac{1}{7^{2n+1}}} = 338.8053572$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^5}}{\frac{1}{8^{2n+1}}} = 522.611587357$ and $\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^7}}{\frac{1}{8^{2n+1}}} = 508.2080358$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^5}}{\frac{1}{9^{2n+1}}} = 746.587981939$ and $\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^7}}{\frac{1}{9^{2n+1}}} = 726.011479715$

$\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^5}}{\frac{1}{10^{2n+1}}} = 1026.55847517$ and $\sum_{r=1}^{\infty}, \sum_{n=1}^{\infty} \frac{\frac{1}{r^7}}{\frac{1}{10^{2n+1}}} = 998.265784608$

As $\zeta(2n+1)$ grows, the whole function approaches to: $n(n-1)(n-2) = \frac{n!}{(n-3)!}$ [3].

$\lim_{\zeta(2n+1)\to\infty}[\frac{\sum_{r=1}^{\infty} \frac{1}{r^{2n+1}}}{\sum_{r=1}^{\infty} \frac{1}{(r+1)^{2n+1}}}] \approx n \cdot (n-1) \cdot (n-2) \approx \frac{n!}{(n-3)!}$

The sequence of this numbers is:
6,24,60,120,210,336,504,720,990,1320... We can also obtain this sequence with $\frac{f(n+1)}{f(n)}$ where $f(n) = (n+2)! \cdot (n+1)! \cdot \frac{n!}{2}$ [4].

References

[1] *N. J. A. Sloane, A Handbook of Integer Sequences, Academic Press, 1973.*

[2] *Julian Havil, Gamma, Exploring Euler's Constant, Princeton Science Library, 2003.*

[3] *R. K. Guy, Unsolved Problems in Theory of Numbers, Section D25.*

[4] *Dr. Wolfdieter Lang.Universitt Karlsruhe (now Karlsruhe Institute of Technology, KIT), Germany. OEIS Foundation Inc. (2019), The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A090443.*