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Finite series expressions to evaluate the beam shape coefficients of a Laguerre-Gauss beam freely propagating.

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Abstract

When using some analytical and semi-analytical scattering theories such as Generalized Lorenz-Mie Theories (GLMTs) and Extended Boundary Condition Method (EBCM) for structured beams, electromagnetic fields, expanded over a set of vector wave functions, may be encoded in some coefficients known as Beam Shape Coefficients (BSCs). In the case of Laguerre-Gauss beams freely propagating, the evaluation of BSCs by using numerical quadratures is time-consuming while localized approximations have been shown to be of limited validity. The present paper therefore establishes finite series expressions to evaluate the BSCs of Laguerre-Gauss beams freely propagating in a rigorous framework, which has the known advantage of speeding up computations with respect to the use of a numerical quadrature technique.

Keywords: Generalized Lorenz-Mie theories; structured beams; T-matrix; beam shape coefficients; finite series; Laguerre-Gauss beams.

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1 Introduction.

When dealing with T-matrix methods for the scattering of light and other electromagnetic radiation by particles, such as analytical Generalized Lorenz-Mie Theories (GLMTs) for particles having a sufficient degree of symmetry to allow for the use of a method of separation of variables [1], [2], [3], or semi-analytical Extended Boundary Condition Method (EBCM) for arbitrary shaped particles [4], [5] particularly in the case of structured beams [6], [7], electromagnetic fields may be expanded over a set of vector wave functions. In spherical coordinates, expansions are carried out over a set of vector spherical wave functions (VSWFs) and expansion coefficients are expressed in terms of coefficients known as Beam Shape Coefficients (BSCs) usually denoted as $g_{n,TM}^{m}$ and $g_{n,TE}^{m}$ (TM: Transverse Magnetic; TE: Transverse Electric).

The original method to evaluate the BSCs of electromagnetic beams is by using quadratures, e.g. [8]. In some cases, quadratures may be analytically solved to produce closed form expressions of the BSCs, such as in the cases of zeroth-order Bessel beams [9], higher-order Bessel beams [10] or zeroth-order Mathieu beams [11]. In the case of Gaussian beams, it has never been possible to analytically solve the quadratures and, therefore, since these Gaussian beams may be viewed as a special case of Laguerre-Gauss beams, there is little chance to achieve analytical quadratures to the evaluation of BSCs of Laguerre-Gauss beams as well. Hence, quadratures to the evaluation of BSCs of Laguerre-Gauss beams have to be carried out numerically, which is a time-consuming process. To circumvent this difficulty, we may think of using a localized approximation (with several variants) which may speed up the computations by several orders of magnitude as reviewed in [12], see also [13] for an up-dated terminology, and [14], [15] for complements. Unfortunately, it has recently been demonstrated that any existing localized approximation has a limited domain of validity when dealing with helical beams, i.e. having a topological charge [16], as is the case for Laguerre-Gauss beams whether they are freely propagating [17] or focused by a lens [18]. The arsenal of methods to evaluate BSCs however contains another technique to speed up the evaluation of BSCs, namely the use of finite series which is a rigorous analytical method [19], [20]. A disadvantage of the finite series method is nevertheless its lack of flexibility insofar as it requires a fairly heavy analytical work before numerical implementation when the description of the beam is modified. Up to now, finite series expressions are known only in the case of Gaussian beams. To circumvent the aforementioned difficulties in the evaluation of BSCs of Laguerre-Gauss beams, the present paper establishes the finite series expressions required for the use of the finite series technique in the case of Laguerre-Gauss beams freely propagating.

Up to now, we have expressed our motivation to the study of the finite series method to evaluate the BSCs of Laguerre-Gauss beams. This motivation is reinforced by the many applications of such beams, in particular due to the fact that they possess a helical wavefront allowing for the transfer of angular momentum to an illuminated object, e.g. Padgett and Allen [21] who discussed orbital

angular momentum associated with helical wavefronts and applications, Garbin *et al.* [22] who experimentally and numerically studied the scattering of highly focused Laguerre-Gauss beams by dielectric and metal spheres, O'Holleran *et al.* [23] who provided a 3D view of optical vortices associated with Laguerre-Gauss modes, Friese *et al.* [24] who studied the transfer of optical angular momentum to trapped absorbing particles, a study relevant to the field of optical information. Laguerre-Gauss beams were discussed as well in the framework of a review devoted to optical tweezers by Molloy and Padgett [25]. Other applications concerned quantum information and the entanglement of the polarization of a single photon with its orbital angular momentum by Nagali *et al.* [26], entanglement again but involving many orthogonal quantum states rather than only two states by Mair *et al.* [27], object identification by measuring the joint orbital angular momentum spectrum of two-photon states by Uribe-Patarroyo *et al.* [28], optical traps to study vortices in Bose-Einstein condensates by Tempere *et al.* [29], optical communication systems in atmospheric turbulence by Malik *et al.* [30], use of vortex masks to observe dim exoplanets by Foo *et al.* [31], to provide a few examples without pretending to exhaustiveness.

The paper is organized as follows. Section 2 recalls a background concerning the method to be used to establish finite series expressions and concerning the electromagnetic fields of a Laguerre-Gauss beam freely propagating. Section 3 deals with the finite series expressions for TM- and TE-BSCs of a Laguerre-Gauss beam freely propagating. Section 4 is a conclusion.

2 Background.

2.1 Neumann Expansion Theorem (NET).

Although the way to use the NET, relying on Neumann or Bessel function expansions, has already been published in the literature, e.g. [3], pp. 121-124, it is concisely reproduced here for the convenience of the reader. It starts with a result published by Watson [32], pp. 524-525. Let us consider an equation of the form:

$$x^{1/2}g(x) = \sum_{n=0}^{\infty} c_n J_{n+1/2}(x) \tag{1}$$

in which $J_{n+1/2}(.)$ are classical half-order Bessel functions. The Maclaurin expansion of the function $g(x)$ reads as:

$$g(x) = \sum_{n=0}^{\infty} b_n x^n \tag{2}$$

Then the NET states that the coefficients c_n are given by:

$$c_n = (n + \frac{1}{2}) \sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m} \frac{\Gamma(\frac{1}{2}+n-m)}{m!} b_{n-2m} \tag{3}$$

in which Γ is the celebrated Gamma function satisfying $\Gamma(n) = (n-1)!$ and $\Gamma(z+1) = z\Gamma(z)$. We then have two ways of expressing the coefficients c_n, either from the Bessel function expansion of Eq.1 or from the Maclaurin expansion of Eq.2, using Eq.3. Equating the two resulting expressions will allow one to establish finite series expressions for the BSCs. This is done using what may be called the NET-procedure.

2.2 The NET-procedure.

The procedure starts with the expressions of the radial electric E_r and magnetic H_r field components in spherical coordinates (r, θ, φ) which may be expressed as, e.g. Eqs.(3.10) and (3.19) of [3]:

$$\begin{pmatrix} E_r \\ H_r \end{pmatrix} = \begin{pmatrix} E_0 \\ H_0 \end{pmatrix} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} c_n^{pw} \begin{pmatrix} g_{n,TM}^m \\ g_{n,TE}^m \end{pmatrix} \frac{n(n+1)}{r} \Psi_n^{(1)}(kr) P_n^{|m|}(\cos\theta) \exp(im\varphi) \tag{4}$$

in which E_0, H_0 are field strengths, $g_{n,TM}^m$ and $g_{n,TE}^m$ are the BSCs to be evaluated within the framework of the NET-procedure, k is the wavenumber, $\Psi_n^{(1)}(.)$ denote the first-order spherical Bessel functions also denoted as $j_n(.)$, and $P_n^{|m|}(.)$ are associated Legendre functions defined by using Hobson's notation [33]. Furthermore c_n^{pw} (with pw standing for "plane wave") denotes expansion coefficients which appear in the Bromwich formulation of the classical Lorenz-Mie theory [34] according to:

$$c_n^{pw} = \frac{(-i)^n}{ik} \frac{2n+1}{n(n+1)} \tag{5}$$

Afterward, we discard the φ-dependency by using the orthogonality relation:

$$\int_0^{2\pi} \exp[i(m-m')\varphi] d\varphi = 2\pi\delta_{mm'} \tag{6}$$

to obtain:

$$\begin{aligned} & \int_0^{2\pi} \begin{pmatrix} E_r \\ H_r \end{pmatrix} \exp(-im\varphi) d\varphi \qquad (7) \\ = \quad & 2\pi \begin{pmatrix} E_0 \\ H_0 \end{pmatrix} \sum_{n=|m|}^{\infty} c_n^{pw} \begin{pmatrix} g_{n,TM}^m \\ g_{n,TE}^m \end{pmatrix} \frac{n(n+1)}{r} \Psi_n^{(1)}(kr) P_n^{|m|}(\cos\theta) \end{aligned}$$

in which we took account of the fact that $P_n^{|m|} = 0$ if $n < |m|$. The next step is to discard the θ-dependency as well. This may be done in two ways. The first way is to specify $\theta = \pi/2$ in Eq.7 and to invoke the following relations for associated Legendre functions [33]:

$$P_n^m(0) = (-1)^{\frac{n+m}{2}} \frac{(n+m-1)!!}{2^{\frac{n-m}{2}} (\frac{n-m}{2})!}, \; (n-m) \text{ even} \qquad (8)$$

$$P_n^m(0) = 0, \; (n-m) \text{ odd} \qquad (9)$$

in which:

$$\left.\begin{array}{c} n!! = 1.3.5...n \\ (-1)!! = 1 \end{array}\right\} \qquad (10)$$

to be completed with, for a better-looking expression:

$$n!! = \frac{2^{\frac{n+1}{2}}}{\sqrt{\pi}} \Gamma(\frac{n}{2} + 1) \qquad (11)$$

Furthermore, we have [32], [35]:

$$\Psi_n^{(1)}(kr) = \sqrt{\frac{\pi}{2kr}} J_{n+\frac{1}{2}}(kr) \qquad (12)$$

Eq.7 may then be rewritten as:

$$(kr)^{1/2}\int_0^{2\pi} r\begin{pmatrix} E_r(\theta=\pi/2) \\ H_r(\theta=\pi/2)\end{pmatrix}\exp(-im\varphi)d\varphi \tag{13}$$
$$= \pi\sqrt{2\pi}\begin{pmatrix} E_0 \\ H_0\end{pmatrix}\sum_{n=|m|,(n-m)\text{ even}}^{\infty} c_n^{pw} n(n+1)\begin{pmatrix} g_{n,TM}^m \\ g_{n,TE}^m\end{pmatrix} P_n^{|m|}(0)J_{n+1/2}(kr)$$

Later on, this equation, specified for the Laguerre-Gauss beams under study, will be given the form of Eq.1 and will allow one to express the BSCs using finite series expressions, with however the restriction that we must have $(n-m)$ even. For $(n-m)$ odd, Eq.7 is differentiated with respect to $\cos\theta$, in order to take advantage of the relations:

$$[\frac{dP_n^m(\cos\theta)}{d\cos\theta}]_{\cos\theta=0} = 0,\ (n-m)\text{ even} \tag{14}$$

$$[\frac{dP_n^m(\cos\theta)}{d\cos\theta}]_{\cos\theta=0} = (-1)^{\frac{n+m-1}{2}}\frac{(n+m)!!}{2^{\frac{n-m-1}{2}}(\frac{n-m-1}{2})!},\ (n-m)\text{ odd} \tag{15}$$

Then, instead of Eq.13, we obtain:

$$(kr)^{1/2}\int_0^{2\pi} r[\frac{\partial}{\partial\cos\theta}\begin{pmatrix} E_r \\ H_r\end{pmatrix}]_{\theta=\pi/2}\exp(-im\varphi)d\varphi \tag{16}$$
$$= \pi\sqrt{2\pi}\begin{pmatrix} E_0 \\ H_0\end{pmatrix}\sum_{n=|m|,(n-m)\text{ odd}}^{\infty} c_n^{pw} n(n+1)\begin{pmatrix} g_{n,TM}^m \\ g_{n,TE}^m\end{pmatrix}[\frac{dP_n^m(\cos\theta)}{d\cos\theta}]_{\cos\theta=0}J_{n+1/2}(kr)$$

which will later be given the form of Eq.1 and will allow one to obtain finite series expressions for the BSCs of Laguerre-Gauss beams freely propagating, for $(n-m)$ odd.

2.3 Laguerre-Gauss beams freely propagating.

The radial component of the electric field of a Laguerre-Gauss beam freely propagating, relying on books by Siegmann [36], [37], for a harmonic beam having a time-dependency of the form $\exp(+i\omega t)$, is given by [17]:

$$E_r(r,\theta,\varphi) = \frac{E_0}{kw(r\cos\theta)}\left(\frac{r\sin\theta\sqrt{2}}{w(r\cos\theta)}\right)^{\nu} L_{\mu}^{\nu}\left(\frac{2r^2\sin^2\theta}{w^2(r\cos\theta)}\right) \exp[-\frac{ikr^2\sin^2\theta}{2\widetilde{q}(r\cos\theta)}]\exp[i(2\mu+\nu+1)\psi(r\cos\theta)]\exp(i\nu\varphi)\exp(-ikr\cos\theta)\sin\theta\cos\varphi \tag{17}$$

in which:

$$w(r\cos\theta) = w_0\sqrt{1+\left(\frac{2r\cos\theta}{kw_0^2}\right)^2} \tag{18}$$

$$\widetilde{q}(r\cos\theta) = r\cos\theta + \frac{ikw_0^2}{2} \tag{19}$$

$$\psi(r\cos\theta) = \tan^{-1}\frac{2r\cos\theta}{kw_0^2} \tag{20}$$

in which w_0 is the beam waist radius of the beam, while μ and ν define the type of Laguerre-Gauss beam under study. Also, L_{μ}^{ν} denotes associated Laguerre polynomials defined as follows [38], [39]:

$$L_{\mu}^{\nu}(x) = \frac{1}{\mu!}\sum_{i=0}^{\mu}\frac{\mu!}{i!}\begin{pmatrix}\nu+\mu\\ \mu-i\end{pmatrix}(-x)^i \tag{21}$$

in which () denotes a binomial coefficient. The "usual" Laguerre polynomials $L_{\mu}(x)$ are $L_{\mu}^{0}(x)$. The associated Laguerre polynomials can be evaluated recursively from the first two polynomials as follows:

$$L_0^{\alpha}(x) = 1 \tag{22}$$

$$L_1^{\alpha}(x) = 1+\alpha-x \tag{23}$$

$$L_{k+1}^{\alpha}(x) = \frac{(2k+1+\alpha-x)L_k^{\alpha}(x)-(k+\alpha)L_{k-1}^{\alpha}(x)}{k+1} \tag{24}$$

The radial component of the electric field will allow one to provide finite series expressions for the TM-BSCs. For the TE-BSCs, we need the corresponding expression for the radial component of the magnetic field. For this, we have to take into account the fact that Eq.17 represents a paraxial approximation which therefore does not satisfy Maxwell's equations. Hence, using the Faraday law to express the magnetic field from the electric field will generate artefacts which will have to be removed to obtain a proper paraxial approximation of the radial component of the magnetic field. Having this in mind, the radial component E_r is actually obtained from an electric field $\mathbf{E}$ which may be written as [17]:

$$\mathbf{E} = (E_x, E_y, E_z) = (E_0 \mathcal{E} e^{-ikz}, 0, 0) \tag{25}$$

leading to:

$$E_r = E_0 \mathcal{E} e^{-ikz} \sin\theta \cos\varphi \tag{26}$$

which, identifying with Eq.17, defines $\mathcal{E}$. According to Faraday's law, we then have:

$$\mathbf{H} = \frac{1}{-i\omega\mu} \operatorname{curl} \mathbf{E} = \frac{1}{-i\omega\mu}(0, \frac{\partial E_x}{\partial z}, 0) \tag{27}$$

in which μ is the permeability, e.g. Eq.(1.41) in [3]. From Eqs.17, 25, 27 and removing artefacts which include $\partial\mathcal{E}/\partial z$, we obtain:

$$\mathbf{H} = (0, H_0 \mathcal{E} e^{-ikz}, 0) \tag{28}$$

in which we have used $E_0/H_0 = \omega\mu/k$. Hence:

$$H_r = H_0 \mathcal{E} e^{-ikz} \sin\theta \sin\varphi \tag{29}$$

which will be used to determine the TE-BSCs. The removal of the artefacts aimed to symmetrize expressions for the radial electric and radial magnetic components, as shown in Eqs.26 and 29. They are easily identified because their presence would break the symmetry exhibited by these equations.

3 Expressions to evaluate BSCs.

3.1 TM-BSCS, $(n-m)$ even.

For $\theta = \pi/2$, we readily have $w(r\cos\theta) = w_0$, $\widetilde{q}(r\cos\theta) = ikw_0^2/2$, and $\psi(r\cos\theta) = \tan^{-1} 0 = 0$, see Eqs.18-20. Then Eq.17 becomes:

$$E_r(\theta = \pi/2) = \mathcal{F}[e^{i(\nu+1)\varphi} + e^{i(\nu-1)\varphi}] \tag{30}$$

in which:

$$\mathcal{F} = \frac{E_0}{2kw_0}\left(\frac{r\sqrt{2}}{w_0}\right)^{\nu} L_\mu^\nu(\frac{2r^2}{w_0^2})\exp(\frac{-r^2}{w_0^2}) \tag{31}$$

We then insert Eq.30 into Eq.13, so that, using Eq.6, the left-hand-side of the electric part of Eq.13 becomes:

$$LHS = 2\pi(kr)^{1/2} r\mathcal{F}[\delta_{m,\nu+1} + \delta_{m,\nu-1}] \tag{32}$$

We now introduce $x = kr$ (not to be confused with a Cartesian coordinate) and the beam confinement factor $s = 1/(kw_0)$, and invoke Eq.5. Rearranging, Eq.13 may then be written as:

$$\begin{aligned} & x^{1/2} g(x)[\delta_{m,\nu+1} + \delta_{m,\nu-1}] \\ = & \sum_{n=|m|}^{\infty} (-i)^n (2n+1) g_{n,TM}^m P_n^{|m|}(0) J_{n+1/2}(x),\ (n-m) \text{ even} \end{aligned} \tag{33}$$

which is of the form of Eq.1 to be used in the NET-procedure, with the function $g(x)$ given by:

$$g(x) = \frac{i}{\sqrt{2\pi}} xs(x\sqrt{2}s)^\nu L_\mu^\nu(2x^2s^2)\exp(-x^2s^2) \tag{34}$$

Note that Eq.33 shows that we shall only deal with the modes $m = \nu+1$ and $m = \nu - 1$ in agreement with what we already learnt in [17]. In this Ref., p.46, correcting an obvious typo, we noted: "we provided results only for $m = \nu \pm 1$, in

agreement with the fact that BSCs for m$\neq \nu \pm 1$ are zero for localized beams". This property could not be positively stated by using numerical quadratures because there is necessarily the possibility of round-off errors , but it is indeed here obtained formally within the framework of the NET-procedure.

To account for the modulus of m in Eq.33, restricting ourselves to $\nu \geq 0$, we have to distinguish three cases: (i) $m = \nu + 1$, $\forall \nu \geq 0$ (ii) $m = \nu - 1$, $\nu > 0$ (iii) $m = \nu - 1$, $\nu = 0$ i.e. $m = -1$. Furthermore, we shall complement various quantities already introduced with an extra-subscript μ, for convenience. We then have:

(i) $m = \nu + 1$, $\forall \nu \geq 0$

$$x^{1/2}[g(x)]_\mu = \sum_{n=\nu+1}^{\infty} (-i)^n (2n+1)[g_{n,TM}^{\nu+1}]_\mu P_n^{\nu+1}(0) J_{n+1/2}(x),\ (n-\nu) \text{ odd} \quad (35)$$

which, using Eq.1, leads to:

$$c_n = (-i)^n (2n+1)[g_{n,TM}^{\nu+1}]_\mu P_n^{\nu+1}(0),\ n \geq \nu + 1,\ n - \nu \text{ odd} \quad (36)$$

$$c_n = 0, \text{ otherwise} \quad (37)$$

(ii) $m = \nu - 1$, $\nu > 0$

$$x^{1/2}[g(x)]_\mu = \sum_{n=\nu-1}^{\infty} (-i)^n (2n+1)[g_{n,TM}^{\nu-1}]_\mu P_n^{\nu-1}(0) J_{n+1/2}(x),\ (n-\nu) \text{ odd} \quad (38)$$

leading to:

$$c_n = (-i)^n (2n+1)[g_{n,TM}^{\nu-1}]_\mu P_n^{\nu-1}(0),\ n \geq \nu - 1,\ n - \nu \text{ odd} \quad (39)$$

$$c_n = 0, \text{ otherwise} \quad (40)$$

(iii) $m = \nu - 1$, $\nu = 0$

$$x^{1/2}[g(x)]_\mu = \sum_{n=1}^{\infty}(-i)^n(2n+1)[g^{-1}_{n,TM}]_\mu P^1_n(0)J_{n+1/2}(x),\ n \text{ odd} \tag{41}$$

leading to:

$$c_n = (-i)^n(2n+1)[g^{-1}_{n,TM}]_\mu P^1_n(0),\ n \geq 1,\ n \text{ odd} \tag{42}$$

$$c_n = 0, \text{ otherwise} \tag{43}$$

We then have a first set of expressions for the coefficients c_n. We are now going to establish a second set of expressions for the same coefficients relying on Eqs.2 and 3. For this, we rewrite $[g(x)]_\mu$ under the form:

$$[g(x)]_\mu = G_\nu L^\nu_\mu(2x^2s^2)\exp(-x^2s^2)x^{\nu+1} \tag{44}$$

in which G_ν, which does not depend on μ, reads as:

$$G_\nu = \frac{is}{\sqrt{2\pi}}(s\sqrt{2})^\nu \tag{45}$$

We shall use the recurrence relations for L^ν_μ, Eqs.22–24, under the form:

$$L^\nu_0(2x^2s^2) = 1 \tag{46}$$

$$L^\nu_1(2x^2s^2) = 1 + \nu - 2x^2s^2 \tag{47}$$

$$L^\nu_{k+1}(2x^2s^2) = \frac{(2k+1+\nu-2x^2s^2)L^\nu_k(2x^2s^2) - (k+\nu)L^\nu_{k-1}(2x^2s^2)}{k+1} \tag{48}$$

the last one being better rewritten as:

$$L_\mu^v(2x^2s^2) = \frac{2\mu-1+\nu}{\mu}L_{\mu-1}^\nu(2x^2s^2) - \frac{\mu-1+\nu}{\mu}L_{\mu-2}^\nu(2x^2s^2) - \frac{2x^2s^2}{\mu}L_{\mu-1}^\nu(2x^2s^2) \tag{49}$$

Let us first consider the cases $\mu = 0$ and 1 which have to be used to initialize a recurrence process. From Eqs.46 and 44:

$$[g(x)]_0 = G_\nu \exp(-x^2s^2)x^{\nu+1} = G_\nu \sum_{t=0}^{\infty} \frac{(-x^2s^2)^t}{t!}x^{\nu+1} = G_\nu \sum_{t=0}^{\infty} \frac{(-s^2)^t}{t!}x^{2t+\nu+1} \tag{50}$$

To display this in the form of Eq.2, we set $n = 2t + \nu + 1$ to obtain:

$$[g(x)]_0 = G_\nu \sum_{n=\nu+1}^{\infty} E(1)x^n \tag{51}$$

in which we introduced the notation:

$$E(u) = \frac{(-s^2)^{\frac{n-\nu-u}{2}}}{(\frac{n-\nu-u}{2})!} \tag{52}$$

Eq.51 is of the form of Eq.2:

$$[g(x)]_0 = \sum_{n=0}^{\infty} [b_n]_0 x^n \tag{53}$$

with:

$$[b_n]_0 = \varepsilon(n; 0, 1, ..., \nu)G_\nu E(1) \tag{54}$$

in which $\varepsilon(n; \alpha_j)$=0 if n is equal to one of the α_j's, and is equal to 1 otherwise.

For $\mu = 1$, we use Eq.47 instead of 46 and, after a few manipulations similar to the ones used previously, we obtain:

$$[b_n]_1 = G_\nu[\varepsilon(n; 0, 1, ..., \nu)(1+\nu)E(1) - 2s^2\varepsilon(n; 0, 1, ..., \nu+2)E(3) \tag{55}$$

We now deal with a general recurrence relation. Using Eqs.44 and 49, we have:

$$[g(x)]_\mu = \frac{2\mu-1+\nu}{\mu}[g(x)]_{\mu-1} - \frac{\mu-1+\nu}{\mu}[g(x)]_{\mu-2} - \frac{2s^2}{\mu}\{x^2[g(x)]_{\mu-1}\} \quad (56)$$

Let us focus on the term $\{.\}$ denoted $\mathcal{A}$. Recalling Eq.2, it can be evaluated as follows:

$$\mathcal{A} = x^2\sum_{n=0}^{\infty}[b_n]_{\mu-1}x^n = \sum_{n=0}^{\infty}[b_n]_{\mu-1}x^{n+2} = \sum_{t=0}^{\infty}[b_t]_{\mu-1}x^{t+2} \quad (57)$$

which, using $(n = t+2)$, becomes:

$$\mathcal{A} = \sum_{n=2}^{\infty}[b_{n-2}]_{\mu-1}x^n = \sum_{n=0}^{\infty}\varepsilon(n;0,1)[b_{n-2}]_{\mu-1}x^n \quad (58)$$

Eq.56 then reads as:

$$\begin{aligned}\sum_{n=0}^{\infty}[b_n]_\mu x^n &= \frac{2\mu-1+\nu}{\mu}\sum_{n=0}^{\infty}[b_n]_{\mu-1}x^n - \frac{\mu-1+\nu}{\mu}\sum_{n=0}^{\infty}[b_n]_{\mu-2}x^n \\ &\quad -\frac{2s^2}{\mu}\sum_{n=0}^{\infty}\varepsilon(n;0,1)[b_{n-2}]_{\mu-1}x^n\end{aligned} \quad (59)$$

We then obtain a recurrence relation for the b_n-coefficients:

$$[b_n]_\mu = \frac{2\mu-1+\nu}{\mu}[b_n]_{\mu-1} - \frac{\mu-1+\nu}{\mu}[b_n]_{\mu-2} - \frac{2s^2}{\mu}\varepsilon(n;0,1)[b_{n-2}]_{\mu-1} \quad (60)$$

As an example, let us evaluate $[b_n]_2$ knowing $[b_n]_1$ and $[b_n]_0$. For $\mu = 2$, Eq.60 becomes:

$$[b_n]_2 = \frac{\nu+3}{2}[b_n]_1 - \frac{\nu+1}{2}[b_n]_0 - s^2\varepsilon(n;0,1)[b_{n-2}]_1 \quad (61)$$

This may be evaluated by using the expressions for $[b_n]_0$ and $[b_n]_1$. For the last term, $[b_{n-2}]_1$ involves $\varepsilon(n-2;0,1,...,\nu)$ and $\varepsilon(n-2;0,1,...,\nu+2)$. The evaluation of this last term then relies on the obvious expression:

$$\varepsilon(n;0,1)\varepsilon(n-2;0,1,...,k) = \varepsilon(n;0,1)\varepsilon(n;2,3,...,k+2) = \varepsilon(n;0,1,...,k+2) \tag{62}$$

We then obtain:

$$\begin{aligned}[b_n]_2 &= G_\nu[\varepsilon(n;0,1,...,\nu)\frac{(\nu+1)(\nu+2)}{2}E(1) - 2s^2\varepsilon(n;0,1,...,\nu+2)(\nu+2)E(3) \\ &\quad +2s^4\varepsilon(n;0,1,...,\nu+4)E(5)]\end{aligned} \tag{63}$$

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Once the coefficients b_n are known, we readily know the coefficients b_{n-2m} and obtain new expressions for the coefficients c_n using Eq.3. These new expressions are equated with the ones we have obtained previously, e.g. Eqs.35-43. This leads to:

$$\begin{aligned}c_n &= (-i)^n(2n+1)[g_{n,TM}^{\nu+1}]_\mu P_n^{\nu+1}(0) \\ &= (n+\frac{1}{2})\sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_\mu, \text{ for } n \geqslant \nu+1, (n-\nu) \text{ odd}, \nu \geqslant 0\end{aligned} \tag{64}$$

$$\begin{aligned}c_n &= (-i)^n(2n+1)[g_{n,TM}^{\nu-1}]_\mu P_n^{\nu-1}(0) \\ &= (n+\frac{1}{2})\sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_\mu, \text{ for } n \geqslant \nu-1, (n-\nu) \text{ odd}, \nu > 0\end{aligned} \tag{65}$$

$$\begin{aligned}c_n &= (-i)^n(2n+1)[g_{n,TM}^{-1}]_\mu P_n^1(0) \\ &= (n+\frac{1}{2})\sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_\mu, \text{ for } n \geqslant +1, n \text{ odd}\end{aligned} \tag{66}$$

Next, from Eqs.8 and 11, we have:

$$P_n^m(0) = \frac{2^m}{\sqrt{\pi}} \frac{(-1)^{\frac{n+m}{2}}}{(\frac{n-m}{2})!} \Gamma(\frac{n+m+1}{2}),\ (n-m) \text{ even} \tag{67}$$

allowing one to evaluate the expressions for the P_n^m's in Eqs.64-66, and to obtain explicit expressions for the TM-BSCs $g_{n,TM}^m$, $(n-m)$ even. They read as:

$$\begin{aligned} [g_{n,TM}^{\nu+1}]_\mu &= \frac{i^n\sqrt{\pi}}{2^{\nu+2}} \frac{(-1)^{\frac{n+\nu+1}{2}}}{\Gamma(\frac{n+\nu}{2}+1)} (\frac{n-\nu-1}{2})! \sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m} \frac{\Gamma(\frac{1}{2}+n-m)}{m!} [b_{n-2m}(\mu)] \\ \text{for } n &\geqslant \nu+1, (n-\nu) \text{ odd}, \nu \geqslant 0 \end{aligned} \tag{68}$$

$$\begin{aligned} [g_{n,TM}^{\nu-1}]_\mu &= \frac{i^n\sqrt{\pi}}{2^{\nu}} \frac{(-1)^{\frac{n+\nu-1}{2}}}{\Gamma(\frac{n+\nu}{2})} (\frac{n-\nu+1}{2})! \sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m} \frac{\Gamma(\frac{1}{2}+n-m)}{m!} [b_{n-2m}(\mu)] \\ \text{for } n &\geqslant \nu-1, (n-\nu) \text{ odd}, \nu > 0 \end{aligned} \tag{69}$$

$$\begin{aligned} [g_{n,TM}^{-1}]_\mu &= \frac{i^n\sqrt{\pi}}{4} \frac{(-1)^{\frac{n+1}{2}}}{\Gamma(\frac{n}{2}+1)} (\frac{n-1}{2})! \sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m} \frac{\Gamma(\frac{1}{2}+n-m)}{m!} [b_{n-2m}(\mu)] \\ \text{for } n &\geqslant +1, n \text{ odd} \end{aligned} \tag{70}$$

3.2 TM-BSCS, $(n-m)$ odd.

The procedure for this case, namely $(n-m)$ odd, being quite similar to the one used for the previous case, we shall be content with a few intermediary relations aiming to help the reader who would like to check the derivations, and with the final results. Instead of Eq.7, we now deal with Eq.13 which requires us to evaluate the derivative $[\partial E_r/\partial \cos\theta]_{\theta=\pi/2}$. After a few pages of careful computations, we obtain, from Eq.17:

$$\begin{aligned} [\frac{\partial E_r(\cos\theta)}{\partial \cos\theta}]_{\theta=\pi/2} &= \frac{E_0}{k} e^{i\nu\varphi} \cos\varphi \frac{1}{w_0} (\frac{r\sqrt{2}}{w_0})^\nu L_\mu^\nu (\frac{2r^2}{w_0^2}) \exp(\frac{-r^2}{w_0^2}) \\ &\quad \{\frac{-2ir^3}{kw_0^4} + ikr[\frac{2}{k^2w_0^2}(2\mu+\nu+1) - 1]\} \end{aligned} \tag{71}$$

Afterward, instead of Eq.33, we obtain:

$$x^{1/2} g(x)[\delta_{m,\nu+1} + \delta_{m,\nu-1}] \tag{72}$$
$$= \sum_{n=|m|}^{\infty} (-i)^n (2n+1) g^m_{n,TM} [\frac{dP_n^{|m|}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2} J_{n+1/2}(x),\ (n-m) \text{ odd}$$

with the function $g(x)$ now given by:

$$g(x) = \frac{i}{\sqrt{2\pi}} xs(xs\sqrt{2})^\nu L^\nu_\mu(2x^2s^2) \exp(-x^2s^2)\{-2ix^3s^4 + ix[2s^2(2\mu+\nu+1)-1]\} \tag{73}$$

To account for the modulus of m in Eq.72, still with $\nu \geq 0$, we again have to distinguish three cases: (i) $m = \nu + 1$, $\forall \nu \geq 0$ (ii) $m = \nu - 1$, $\nu > 0$ (iii) $m = \nu - 1$, $\nu = 0$ i.e. $m = -1$. Furthermore, still adding a subscript μ to conveniently decorate various quantities, we have:

(i) $m = \nu + 1$, $\forall \nu \geq 0$

$$x^{1/2}[g(x)]_\mu = \sum_{n=\nu+1}^{\infty} (-i)^n (2n+1) [g^{\nu+1}_{n,TM}]_\mu [\frac{dP_n^{\nu+1}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2} J_{n+1/2}(x),\ (n-\nu) \text{ even} \tag{74}$$

$$c_n = (-i)^n (2n+1) [g^{\nu+1}_{n,TM}]_\mu [\frac{dP_n^{\nu+1}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2},\ n \geq \nu+1,\ n-\nu \text{ even} \tag{75}$$

$$c_n = 0, \text{ otherwise} \tag{76}$$

(ii) $m = \nu - 1$, $\nu > 0$

$$x^{1/2}[g(x)]_\mu = \sum_{n=\nu-1}^{\infty} (-i)^n (2n+1) [g^{\nu-1}_{n,TM}]_\mu [\frac{dP_n^{\nu-1}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2} J_{n+1/2}(x),\ (n-\nu) \text{ even} \tag{77}$$

$$c_n = (-i)^n(2n+1)[g^{\nu-1}_{n,TM}]_\mu[\frac{dP^{\nu-1}_n(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2},\ n \geq \nu-1,\ n-\nu \text{ even} \quad (78)$$

$$c_n = 0, \text{ otherwise} \quad (79)$$

(iii) $m = \nu - 1$, $\nu = 0$

$$x^{1/2}[g(x)]_\mu = \sum_{n=2}^{\infty}(-i)^n(2n+1)[g^{-1}_{n,TM}]_\mu[\frac{dP^1_n(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2}J_{n+1/2}(x),\ n \text{ even} \quad (80)$$

$$c_n = (-i)^n(2n+1)[g^{-1}_{n,TM}]_\mu[\frac{dP^1_n(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2},\ n \geq 2,\ n \text{ even} \quad (81)$$

$$c_n = 0, \text{ otherwise} \quad (82)$$

Next, instead of Eq.44, we rewrite $[g(x)]_\mu$ under the form:

$$[g(x)]_\mu = L^\nu_\mu(2x^2s^2)\exp(-x^2s^2)[H_v x^{\nu+4} + K_\nu x^{\nu+2}] \quad (83)$$

in which:

$$H_v = -2is^4G_\nu \quad (84)$$

$$K_\nu = i[2s^2(2\mu+\nu+1)-1]G_\nu \quad (85)$$

$$G_\nu = \frac{is}{\sqrt{2\pi}}(s\sqrt{2})^\nu \quad (86)$$

Let us first again consider the cases $\mu = 0$ and 1 to be used to start a recurrence. Instead of 51, we now have:

$$[g(x)]_0 = H_v \sum_{n=\nu+4}^{\infty} F(2)x^n + K_v \sum_{n=\nu+2}^{\infty} F(1)x^n \tag{87}$$

$$[b_n]_0 = \varepsilon(n;0,1,...,\nu+3)H_\nu F(2) + \varepsilon(n;0,1,...,\nu+1)K_\nu F(1) \tag{88}$$

in which we introduced the notation:

$$F(u) = \frac{(-s^2)^{\frac{n-\nu}{2}-u}}{(\frac{n-\nu}{2}-u)!} \tag{89}$$

Similarly, for $\mu = 1$:

$$[g(x)]_1 = (1+\nu)[g(x)]_0 - 2s^2\{x^2 g(x)_0\} \tag{90}$$

$$\begin{aligned} [b_n]_1 \quad = \quad & \varepsilon(n;0,1,...,\nu+1)(1+\nu)K_\nu F(1) \\ & +\varepsilon(n;0,1,...,\nu+3)[(1+\nu)H_\nu - 2s^2 K_\nu]F(2) \\ & -2s^2\varepsilon(n;0,1,...,\nu+5)H_\nu F(3) \end{aligned} \tag{91}$$

For the general case, Eqs.56 and 60 are still valid, as conveniently repeated below:

$$[g(x)]_\mu = \frac{2\mu-1+\nu}{\mu}[g(x)]_{\mu-1} - \frac{\mu-1+\nu}{\mu}[g(x)]_{\mu-2} - \frac{2s^2}{\mu}\{x^2[g(x)]_{\mu-1}\} \tag{92}$$

$$[b_n]_\mu = \frac{2\mu-1+\nu}{\mu}[b_n]_{\mu-1} - \frac{\mu-1+\nu}{\mu}[b_n]_{\mu-2} - \frac{2s^2}{\mu}\varepsilon(n;0,1)[b_{n-2}]_{\mu-1} \tag{93}$$

which, however, lead to expressions for the coefficients $[b_n]_\mu$ which are different from the ones for the case $(n-m)$ even, due to the different expressions for $[b_n]_0$ and $[b_n]_1$ which are used to initialize the recurrence process. As an example, instead of Eq.63, we now have:

$$
\begin{aligned}
[b_n]_2 &= 2s^4\varepsilon(n;0,1,...,\nu+7)H_\nu F(4) \\
&\quad -2s^2\varepsilon(n;0,1,...,\nu+5)[(\nu+2)H_\nu - s^2K_\nu]F(3) \\
&\quad +(\nu+2)\varepsilon(n;0,1,...,\nu+3)[\frac{\nu+1}{2}H_\nu - 2s^2K_\nu]F(2) \\
&\quad +\frac{(\nu+1)(\nu+2)}{2}\varepsilon(n;0,1,...,\nu+1)K_\nu F(1)
\end{aligned} \tag{94}
$$

Thereafter, instead of Eqs.64-66, we obtain:

$$
\begin{aligned}
c_n &= (-i)^n(2n+1)[g^{\nu+1}_{n,TM}]_\mu[\frac{dP_n^{\nu+1}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2} \\
&= (n+\frac{1}{2})\sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_\mu, \text{ for } n \geqslant \nu+1, (n-\nu) \text{ even}, \nu \geqslant 0
\end{aligned} \tag{95}
$$

$$
\begin{aligned}
c_n &= (-i)^n(2n+1)[g^{\nu-1}_{n,TM}]_\mu[\frac{dP_n^{\nu-1}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2} \\
&= (n+\frac{1}{2})\sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_\mu, \text{ for } n \geqslant \nu-1, (n-\nu) \text{ even}, \nu > 0
\end{aligned} \tag{96}
$$

$$
\begin{aligned}
c_n &= (-i)^n(2n+1)[g^{-1}_{n,TM}]_\mu[\frac{dP_n^{1}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2} \\
&= (n+\frac{1}{2})\sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_\mu, \text{ for } n \geqslant +2, n \text{ even}
\end{aligned} \tag{97}
$$

Next, from Eqs.15 and 11, we establish:

$$
[\frac{dP_n^{\nu+1}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2} = \frac{(-1)^{\frac{n+\nu}{2}}2^{\nu+2}}{\sqrt{\pi}(\frac{n-\nu}{2}-1)!}\Gamma(\frac{n+\nu}{2}+\frac{3}{2}) \tag{98}
$$

$$
[\frac{dP_n^{\nu-1}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2} = \frac{(-1)^{\frac{n+\nu}{2}-1}2^{\nu}}{\sqrt{\pi}(\frac{n-\nu}{2})!}\Gamma(\frac{n+\nu}{2}+\frac{1}{2}) \tag{99}
$$

$$[\frac{dP_n^1(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2} = \frac{4(-1)^{\frac{n}{2}}}{\sqrt{\pi}(\frac{n}{2}-1)!}\Gamma(\frac{n}{2}+\frac{3}{2}) \tag{100}$$

and eventually obtain:

$$\begin{aligned}[g_{n,TM}^{\nu+1}]_\mu &= \frac{i^n\sqrt{\pi}}{2^{\nu+3}}\frac{(-1)^{\frac{n+\nu}{2}}}{\Gamma(\frac{n+\nu}{2}+\frac{3}{2})}(\frac{n-\nu}{2}-1)! \\ &\quad \sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_\mu, \\ \text{for } n &\geqslant \nu+1, (n-\nu) \text{ even}, \nu \geqslant 0\end{aligned} \tag{101}$$

$$\begin{aligned}[g_{n,TM}^{\nu-1}]_\mu &= \frac{i^n\sqrt{\pi}}{2^{\nu+1}}\frac{(-1)^{\frac{n+\nu}{2}-1}}{\Gamma(\frac{n+\nu}{2}+\frac{1}{2})}(\frac{n-\nu}{2})! \sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_\mu \\ \text{for } n &\geqslant \nu-1, (n-\nu) \text{ even}, \nu > 0\end{aligned} \tag{102}$$

$$\begin{aligned}[g_{n,TM}^{-1}]_\mu &= \frac{i^n\sqrt{\pi}}{8}\frac{(-1)^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+\frac{3}{2})}(\frac{n}{2}-1)! \sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_\mu \\ \text{for } n &\geqslant +2, n \text{ even}\end{aligned} \tag{103}$$

3.3 TE-BSCs.

To deal with the TE-BSCs, we have to use H_r in Eqs.7 and 16 instead of E_r. Computations then run in the same way but for the replacement of a $\cos\varphi$ by a $\sin\varphi$, see Eqs.26 and 29. This implies that, in Eqs.33 and 72, we simply have to change $[\delta_{m,\nu+1}+\delta_{m,\nu-1}]$ to $[\delta_{m,\nu+1}-\delta_{m,\nu-1}]/i$. Then, without redoing the computations, we obtain the following results relating the TM- and the TE-BSCs, valid whatever the parity of $(n-m)$:

$$g_{n,TM}^m = ig_{n,TE}^m,\ m = \nu+1, \forall\nu \geq 0 \tag{104}$$

$$g_{n,TM}^m = -ig_{n,TE}^m,\ m = \nu-1, \nu > 0 \tag{105}$$

$$g_{n,TM}^{-1} = -ig_{n,TE}^{-1},\ m = \nu-1, \nu = 0 \tag{106}$$

4 Conclusion.

For use in some analytical or semi-analytical light scattering theories, like GLMTs or EBCM, the electromagnetic fields may be encoded in a set of coefficients named beam shape coefficients (BSCs). In the case of Laguerre-Gauss beams (either freely propagating or focused by a lens), the usual quadrature technique which allows one to evaluate the BSCs has to be carried out numerically, leading to time-consuming evaluations. Localized approximations, which are traditionally used to speed-up the computations of the BSCs by orders of magnitude, unfortunately have a limited domain of validity for Laguerre-Gauss beams and have to be considered with care and suspicion. Therefore, borrowing another technique from the arsenal of methods available to evaluate BSCs, the present paper establishes rigorous expressions to the evaluation of BSCs using finite series. It must be noted that, although the procedure used is rigorous, it is, in the present paper, applied to a paraxial approximation of Laguerre-Gauss beams freely propagating, i.e. using a beam description which does not perfectly satisfies Maxwell's equations. Relying on our experience concerning paraxial Gaussian beams, either using finite series ([3], pp. 164-171) or quadratures [8], the non-Maxwellian character of the paraxial beam should be reflected by a blowing-up of BSCs values for high-order partial waves. These faulty BSCs could be set to zero or they would have no consequence because they would appear for partial waves which, in any case, convey vanishing amplitudes. Once this is implemented, the BSCs obtained define a beam which perfectly satisfies Maxwell's equations, resulting from the internal coherency of the theory based on the fact that it uses a basis of regular VSWFs, independently of the particular technique used to evaluate the BSCs. In the present case, however, the finite series procedure allows one to transform a non-Maxwellian beam to a Maxwellian beam, illustrating the remodeling process involved in the evaluation of BSCs using finite series. Beams generated by the BSCs evaluated by finite series therefore automatically provide a Maxwellian description of Laguerre-Gauss beams freely propagating. Subsequent papers will be devoted to the algorithmic implementation of the formulas obtained in this paper, and later on, to the application of the finite series technique to the case of Laguerre-Gauss beams focused by a lens.

It is furthermore to be noted that the derivations of BSCs in the present paper have been carried in spherical coordinates for parallel illumination, in on-axis situations. BSCs for other kinds of curvilinear systems of coordinates can afterward be obtained from the BSCs in spherical coordinates by using an extrinsic method, see [40] for the definition of intrinsic and extrinsic methods, and the introduction of [41] for a review of extrinsic methods in spheroidal and cylindrical coordinates. BSCs for oblique illumination may be deduced from the BSCs for parallel illumination by using rules of transformations of BSCs under rotation of coordinate systems [42], [43], [44], [45], [46]. BSCs for off-axis situations can be obtained from the BSCs for on-axis situations by using translation theorems [47] or by introducing the off-axis expressions of the fields

right at the beginning of the algebraic treatment, as actually done in the case of Gaussian beams, see [20]. The last procedure however requires an extra-algebraic work. It is indeed a defect of the finite series technique that each modification of the field expressions requires an extra-algebraic work before implementation in computer programs, although it has been stated that the "whole process may be in principle carried out in an automatic way by using a formal computation procedure which would furthermore generate FORTRAN sources" ([3], p. 121).

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