Flag versions of quiver Grassmannians for Dynkin quivers have no odd cohomology over Z

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Abstract

We prove that flag versions of quiver Grassmannians (also knows as Lusztig's fibers) for Dynkin quivers (types A, D, E) have no odd cohomology over $\mathbb{Z}$. Moreover, for types A and D we prove that these varieties have α-partitions into affine spaces. We also show that to prove the same statement for type E, it is enough to check this for indecomposable representations.

We also give a flag version of the result of Irelli-Esposito-Franzen-Reineke on rigid representations: we prove that flag versions of quiver Grassmannians for rigid representations have a diagonal decomposition. In particular, they have no odd cohomology over $\mathbb{Z}$.

Contents

1 Introduction

1.1 Motivation

The motivation of this paper comes from the study of KLR (Khovanov-Lauda-Rouqier) algebras in [11]. KLR algebras, also called quiver-Hecke algebras, were introduced by Khovanov-Lauda [9] and Rouquier [14] for categorification of quantum groups. Let Γ be a quiver without loops and let ν be a dimension vector for this quiver. Let $\mathbf{k}$ be a field. Then we can construct the KLR algebra R_ν over $\mathbf{k}$ associated with Γ and ν.

To motivate our reaserch, we recall some facts about the geometric study of KLR algebras. The reader can see [11] for more details.

A geometric construction of KLR algebras is given in [14] and [15] (over a field of characteristic zero). The positive characteristic case is done in [11]. Let us describe the idea of this geometric construction. There is a complex algebraic group G and complex algebraic varieties X and Y (depending on Γ and ν) with a G-action and a proper G-invariant map $\pi\colon X \to Y$ such that the KLR algebra R_ν is isomorphic to the extension algebra $\mathrm{Ext}^*_G(\pi_*\underline{\mathbf{k}}_X, \pi_*\underline{\mathbf{k}}_X)$. Here $\underline{\mathbf{k}}_X$ is the constant G-equivariant sheaf on X. The sheaf $\pi_*\underline{\mathbf{k}}_X$ is the pushforward of $\underline{\mathbf{k}}_X$, this pushforward is an element of the G-equivariant bounded derived category of sheaves of $\mathbf{k}$-vector spaces on Y. Abusing the terminology, we say sometimes "a sheaf" for "a complex of sheaves".

For understanding the algebra R_ν, it is important to understand the decomposition of the sheaf $\pi_*\underline{\mathbf{k}}_X$ into indecomposables. Now, assume that Γ is a Dynkin quiver. In this case the number of G-orbits in Y is finite. Assume first that the characteristic of $\mathbf{k}$ is zero. We can deduce from the decomposition theorem [1, Thm. 6.2.5] and from [13, Thm. 2.2] that the indecomposable direct summands of $\pi_*\underline{\mathbf{k}}_X$ up to shift are exactly the sheaves of the form $\mathrm{IC}(\mathcal{O})$, where $\mathcal{O}$ is a G-orbit in Y and $\mathrm{IC}(\mathcal{O})$ is the simple perverse sheaf associated with the orbit $\mathcal{O}$.

Now, we want to understand what happens if the characteristic of $\mathbf{k}$ is positive. In this case the decomposition theorem fails, so the theory of perverse sheaves does not help us to understand $\pi_*\underline{\mathbf{k}}_X$. However, Juteau-Mautner-Williamson [8] introduce a new tool for this: parity sheaves. For the variety Y as above, their construction yields some new sheaves $\mathcal{E}(\mathcal{O})$ on Y. It happens, that in characteristic zero we have $\mathrm{IC}(\mathcal{O}) = \mathcal{E}(\mathcal{O})$. However, in positive characteristics, the sheaves $\mathcal{E}(\mathcal{O})$ behave better than $\mathrm{IC}(\mathcal{O})$. In particular, [8]

gives a version of the decomposition theorem for parity sheaves that also works for positive characteristics. However, this version needs an extra-condition: the fibers of π must have no odd cohomology groups over $\mathbf{k}$. Note that the fibers of the map π are isomorphic to flag versions of quiver Grassmannians.

A version of the following conjecture is considered in [11].

Conjecture 1.1. *Assume that Γ is a Dynkin quiver. Then for each field $\mathbf{k}$ and each $y \in Y$ we have $H^{\mathrm{odd}}(\pi^{-1}(y), \mathbf{k}) = 0$.*

It is proved in [11] that if Conjecture 1.1 holds, then in any characteristic the indecomposable direct summands of $\pi_* \underline{\mathbf{k}}_X$ up to shift are exactly the sheaves of the form $\mathcal{E}(\mathcal{O})$. In particular, this yields a new basis in the quantum group $U_q^-(\mathfrak{g})$ for types A, D, E in terms of parity sheaves.

Conjecture 1.1 is proved in [11] only in type A. It remained open for types D and E.

Recently, Conjecture 1.1 was proved by McNamara [12]. The paper [12] studies algebras of the form $\mathrm{Ext}^*_G(\pi_* \underline{\mathbf{k}}_X, \pi_* \underline{\mathbf{k}}_X)$ for more general G, X and Y. This paper relates the condition $H^{\mathrm{odd}}(\pi^{-1}(y), \mathbf{k}) = 0$ with an algebraic property of the algebra $\mathrm{Ext}^*_G(\pi_* \underline{\mathbf{k}}_X, \pi_* \underline{\mathbf{k}}_X)$: it should be polynomial quasi-hereditary. On the other hand, KLR algebras R_ν for Dynkin quivers are known to be polynomial quasi-hereditary due to [2]. This allows to prove Conjecture 1.1.

As we see, McNamara's proof uses essentially the representation theory of KLR algebras. In the present paper we give a more direct proof of Conjecture 1.1. We don't use KLR algebras, we work directly with the geometric description of the fibers of π. These fibers are flag versions of quiver Grassmannians. Our result is stronger because we also show that the fibers have no odd cohomology over $\mathbb{Z}$. Moreover, in types A and D we also show that the fibers have α-partitions into affine spaces. Note that this result is also stronger than the similar result from [11] in type A.

There is an example of a situation where it is important to know that the fibers of π have no odd cohomology over a ring (not only over a field). In the Williamson's contre-exemple [16] in type A to the Kleshchev-Ram conjecture, he uses the modular system $(\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{F}_p)$. In particular, for this, it is important to know that we have $H^{\mathrm{odd}}(\pi^{-1}(y), \mathbb{Z}_p) = 0$. Now, it is possible to have a similar construction in types D and E.

See also [11, Sec. 3.10] for some other consequences of Conjecture 1.1. One more application of Conjecture 1.1 to Kato's reflection functors is given in [12, Sec. 7].

1.2 Results

The fibers of π have an explicit geometric description: they are flag versions of quiver Grassmannians for complete flag types. However, all the methods of the present paper work well for non-complete flags. So we will also allow non-complete flags below.

Let $\Gamma = (I, H)$ be a quiver, where I and H are the sets of its vertices and arrows respectively. Fix an increasing sequence of dimension vectors $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_d)$

and set $v = \mathbf{v}_d$. Let $\mathbf{V}$ be a representation of Γ of dimension v. A flag version of the quiver Grassmannian is the following variety:

$$\mathcal{F}_{\mathbf{v}}(\mathbf{V}) = \{\mathbf{V}^1 \subset \mathbf{V}^2 \subset \ldots \subset \mathbf{V}^d = \mathbf{V};\ \mathbf{V}^r \text{ is a subrepresentation of } \mathbf{V} \text{ of dimension } \mathbf{v}_r\}.$$

Quiver Grassmannians are special cases of $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ with $d = 2$. Note that the fibers $\pi^{-1}(y)$ discussed above are always of the form $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$.

The main results of the paper are the following two theorems.

Theorem 1.2. *Assume that Γ is a Dynkin quiver of type A or D. Then the variety $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ is either empty or has an α-partition into affine spaces.*

Theorem 1.3. *Assume that Γ is a Dynkin quiver (types A, D, E). Then we have $H^{\mathrm{odd}}(\mathcal{F}_{\mathbf{v}}(\mathbf{V}), \mathbb{Z}) = 0$.*

This paper is inspired by [7], where these theorems are proved for quiver Grassmannians. Moreover, for quiver Grassmannians the analogue of Theorem 1.2 also holds for type E and for affine quivers. The present paper gives generalizations of some results from [7] to flag versions. These generalizations are not straightforward, they need some new ideas that we explain below.

Let us first summarize the idea of the proof in [7] for usual quiver Grassmannians. Let $\mathbf{V}$ be a representation of Γ with dimension vector v and let v' be a dimension vector. The quiver Grassmannian is the following variety

$$\mathcal{G}_{v'}(\mathbf{V}) = \{\mathbf{V}' \subset \mathbf{V};\ \dim \mathbf{V}' = v',\ \mathbf{V}' \text{ is a subrepresentation of } \mathbf{V}\}.$$

Let $0 \to \mathbf{V} \to \mathbf{U} \to \mathbf{W} \to 0$ be a short exact sequence of representations of Γ. Set

$$\mathcal{G}_{v',w'} = \{\mathbf{U}' \in \mathcal{G}_{v'+w'}(\mathbf{U});\ \dim(\mathbf{U}' \cap \mathbf{V}) = v'\}.$$

The key step of their proof is the following proposition, see [7, Thm. 26] and its proof.

Proposition 1.4. *If $\mathrm{Ext}^1_\Gamma(\mathbf{W}, \mathbf{V}) = 0$, then*

$$\mathcal{G}_{v',w'}(\mathbf{U}) \to \mathcal{G}_{v'}(\mathbf{V}) \times \mathcal{G}_{w'}(\mathbf{W})$$

is a vector bundle.

For a Dynkin quiver, its indecomposable representations can be ordered in such a way that there are no extensions in one direction. So, Proposition 1.4 reduces the statement to indecomposable representations. This allows to prove Theorems 1.2 and 1.3 for quiver Grassmannians in [7].

Now, let us explain how the present paper generalizes this approach for flag versions. We need to find a generalization of Proposition 1.4 for flags. We start from the following observation. For each representation $\mathbf{V}$ of Γ and each $\mathbf{v}$, the variety $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ can be seen as a quiver Grassmannian for some bigger quiver $\widehat{\Gamma}_d$ and some representation $\Phi(\mathbf{V})$ of $\widehat{\Gamma}_d$. However, we cannot deduce our result directly from [7] because the quiver $\widehat{\Gamma}_d$ has no reason to be Dynkin.

One more obstruction is that the natural functor $\Phi\colon \mathrm{Rep}(\Gamma) \to \mathrm{Rep}(\widehat{\Gamma}_d)$ does not preserve extensions. To fix this difficulty, we introduce a full subcategory $\mathrm{Rep}^0(\widehat{\Gamma}_d)$ of $\mathrm{Rep}(\widehat{\Gamma}_d)$ such that the image of Φ is in $\mathrm{Rep}^0(\widehat{\Gamma}_d)$ and such that the functor $\Phi\colon \mathrm{Rep}(\Gamma) \to \mathrm{Rep}^0(\widehat{\Gamma}_d)$ preserves extensions. However, this creates a new obstruction: the category $\mathrm{Rep}^0(\Gamma)$ may have nonzero second extensions (while the category $\mathrm{Rep}(\widehat{\Gamma}_d)$ has no second extensions). The absence of second extensions in the category $\mathrm{Rep}(\Gamma)$ was an important point in the proof of Proposition 1.4. To overcome this problem, we prove some Ext-vanishing properties of the category $\mathrm{Rep}^0(\widehat{\Gamma}_d)$ in Section 2.5. This allows us to get an analogue of Proposition 1.4, see Proposition 2.17.

This allows us to reduce the proof of Theorems 1.2 and 1.3 to indecomposable representations. For types A and D, the indecomposable representations are very easy to describe. So Theorem 1.2 can be done by hand for indecomposables. Type E is more complicated. We don't know how to check Theorem 1.2 for indecomposables in this case. Note, however, that in type E the number of cases to check is finite (we have a finite number of indecomposable representations and a finite number of flag types for each indecomposable representation).

In type E, we manage to prove a weaker statement: Theorem 1.3. It is also enough to check it only for indecomposable representations. We do this in more generality. We prove the following result.

Theorem 1.5. *Let Γ be an arbitrary quiver. Assume that $\mathbf{V} \in \mathrm{Rep}(\Gamma)$ satisfies $\mathrm{Ext}^1_\Gamma(\mathbf{V}, \mathbf{V}) = 0$. Then $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ is either empty or has a diagonal decomposition. In particular, we have $H^{\mathrm{odd}}(\mathcal{F}_{\mathbf{v}}(\mathbf{V}), \mathbb{Z}) = 0$.*

The theorem above generalizes a similar result from [7] about quiver Grassmannians.

2 Flag versions of quiver Grassmannians

We assume that all quivers have a finite number of vertices and arrows. We also assume that all representations of quivers are finite dimensional and are over $\mathbb{C}$.

For a $\mathbb{C}$-algebra A we denote by $\mathrm{mod}(A)$ the category of finite dimensional representations of A.

For integers a and b such that $a < b$ we set $[a; b] = \{a, a+1, \dots, b-1, b\}$.

2.1 Quivers

Let $\Gamma = (I, H)$ be a quiver. We denote by I and H the sets of its vertices and arrows respectively. For each arrow $h \in H$ we write h' and h'' for its source and target respectively.

Definition 2.1. A *dimension vector* v for Γ is a collection of positive integers $(v_i)_{i\in I}$. A *representation* $\mathbf{V}$ of Γ is a collection of finite dimensional complex vector spaces $\mathbf{V}_i$ for $i \in I$ and a collection of linear maps $\mathbf{V}_{h'} \to \mathbf{V}_{h''}$ for each $h \in H$. We denote by $\mathrm{Rep}(\Gamma)$ the category of representations of Γ. We say that

v is the dimension vector of $\mathbf{V}$ if we have $\dim \mathbf{V}_i = v_i$ for each $i \in I$. In this case we can write $\dim \mathbf{V} = v$.

2.2 Extended quiver

Let $\Gamma = (I, H)$ be a quiver. Fix an integer $d > 0$.

Definition 2.2. The extended quiver $\widehat{\Gamma}_d = (\widehat{I}, \widehat{H})$ is the quiver obtained from Γ in the following way. The vertex set $\widehat{I}$ of $\widehat{\Gamma}_d$ is a union of d copies of the vetrex set I of Γ, i.e., we have $\widehat{I} = I \times [1; d]$. The quiver $\widehat{\Gamma}_d$ has the following arrows:

- an arrow $(i, r) \to (j, r)$ for each arrow $i \to j$ in Γ and each $r \in [1; d]$.
- an arrow $(i, r) \to (i, r + 1)$ for each $i \in I$ and $r \in [1; d - 1]$.

2.3 The functor Φ

Let $\mathrm{Rep}^0(\widehat{\Gamma}_d)$ be the full category of $\mathrm{Rep}(\widehat{\Gamma}_d)$ containing the objects $\mathbf{V} \in \mathrm{Rep}(\widehat{\Gamma}_d)$ that satisfy the following condition: for each arrow $i \to j$ in Γ and each $r \in [1; d - 1]$, the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{V}_{(i,r)} & \longrightarrow & \mathbf{V}_{(j,r)} \\ \downarrow & & \downarrow \\ \mathbf{V}_{(i,r+1)} & \longrightarrow & \mathbf{V}_{(j,r+1)} \end{array}$$

Fix $r \in [1; d]$. Consider the functor $\Phi\colon \mathrm{Rep}(\Gamma) \to \mathrm{Rep}^0(\widehat{\Gamma}_d)$ defined in the following way.

- For each $i \in I$ and $r \in [1; d]$, we have $\Phi(\mathbf{V})_{(i,r)} = \mathbf{V}_i$.
- For each arrow $i \to j$ in Γ and each $t \in [1; d]$, the map $\Phi(\mathbf{V})_{(i,r)} \to \Phi(\mathbf{V})_{(j,r)}$ is defined as $\mathbf{V}_i \to \mathbf{V}_j$.
- For each $i \in I$ and each $r \in [1; d - 1]$, the map $\Phi(\mathbf{V})_{(i,r)} \to \Phi(\mathbf{V})_{(i,r+1)}$ is $\mathrm{Id}_{\mathbf{V}_i}$.

Lemma 2.3. *The functor Φ is exact and fully faithful.*

Proof. It is obvious from the construction that the functor is exact. It is also clear that the functor Φ is injective on morphisms. Let us prove that it is also surjective on morphisms.

Consider a morphism $\phi \in \mathrm{Hom}_{\widehat{\Gamma}_d}(\Phi(\mathbf{V}), \Phi(\mathbf{W}))$. By definition, for each $i \in I$ and $r \in [1; d - 1]$, we have the following commutative diagram.

$$\begin{array}{ccc} \Phi(\mathbf{V})_{(i,r)} & \xrightarrow{\phi_{(i,r)}} & \Phi(\mathbf{W})_{(i,r)} \\ \downarrow & & \downarrow \\ \Phi(\mathbf{V})_{(i,r+1)} & \xrightarrow{\phi_{(i,r+1)}} & \Phi(\mathbf{W})_{(i,r+1)} \end{array}$$

After the identification $\mathbf{V}_i = \Phi(\mathbf{V})_{(i,r)} = \Phi(\mathbf{V})_{(i,r+1)}$ and $\mathbf{W}_i = \Phi(\mathbf{W})_{(i,r)} = \Phi(\mathbf{W})_{(i,r+1)}$, we get the diagram

$$\begin{array}{ccc} \mathbf{V}_i & \xrightarrow{\phi_{(i,r)}} & \mathbf{W}_i \\ {\scriptstyle \mathrm{Id}_{\mathbf{V}_i}}\downarrow & & {\scriptstyle \mathrm{Id}_{\mathbf{W}_i}}\downarrow \\ \mathbf{V}_i & \xrightarrow{\phi_{(i,r+1)}} & \mathbf{W}_i \end{array}$$

This shows that after the identification, the maps $\phi_{(i,r)}$ and $\phi_{(i,r+1)}$ are the same. This proves that ϕ is in the image of Φ. □

2.4 The functor Φ as a bimodule

The category $\mathrm{Rep}(\Gamma)$ is equivalent to $\mathrm{mod}(\mathbb{C}\Gamma)$, where $\mathbb{C}\Gamma$ is the path algebra for the quiver Γ. Similarly, we have an equivalence of categories $\mathrm{Rep}(\widehat{\Gamma}_d) \simeq \mathrm{mod}(\mathbb{C}\widehat{\Gamma}_d)$. The subcategory $\mathrm{Rep}^0(\widehat{\Gamma}_d)$ of $\mathrm{Rep}(\widehat{\Gamma}_d)$ is equivalent to $\mathrm{mod}(Q)$, where Q is the quotient of the path algebra $\mathbb{C}\widehat{\Gamma}_d$ by the ideal generated by the commutativity relations

$$\begin{array}{ccc} (i,r) & \longrightarrow & (j,r) \\ \downarrow & & \downarrow \\ (i,r+1) & \longrightarrow & (j,r+1) \end{array}$$

for each arrow $i \to j$ and each $r \in [1, d-1]$.

Let us give another description of Q. Let L_d by the quiver having d vertices $1, 2, \ldots, d$ and an arrow $r \to (r+1)$ for each $r \in [1; d-1]$. The set of vertices of the quiver $\widehat{\Gamma}_d$ can be considered as a direct product of sets of vertices of Γ and L_d. It is easy to see that we have an identification $Q = \mathbb{C}\Gamma \otimes \mathbb{C}L_d$. The algebra $\mathbb{C}\Gamma$ has idempotents e_i for $i \in I$, the algebra $\mathbb{C}L_d$ has idempotents e_r, $r \in [1; d]$, the algebra Q has idempotents $e_{(i,r)} = e_i e_r$.

Now, we describe the functor $\Phi\colon \mathrm{mod}(\mathbb{C}\Gamma) \to \mathrm{mod}(Q)$ in terms of bimodules. The following lemma is obvious.

Lemma 2.4. *For each $\mathbf{V} \in \mathrm{mod}(\mathbb{C}\Gamma)$, we have $\Phi(\mathbf{V}) = Qe_1 \otimes_{\mathbb{C}\Gamma} \mathbf{V}$.*

The description of the functor Φ above makes obvious the following.

Lemma 2.5. *The functor $\Phi\colon \mathrm{Rep}(\Gamma) \to \mathrm{Rep}^0(\widehat{\Gamma}_d)$ sends projective objects to projective objects.*

Let us write Ext^i_Γ for the ith extension functor in the category $\mathrm{Rep}(\Gamma)$. We will also often need the extension functor in the category $\mathrm{Rep}^0(\widehat{\Gamma}_d)$ (not in $\mathrm{Rep}(\widehat{\Gamma}_d)$). We will denote this extension functor Ext^i_Q.

Corollary 2.6. *For each $\mathbf{V}, \mathbf{W} \in \mathrm{Rep}(\Gamma)$ and each $i \geqslant 0$, we have an isomorphism*

$$\mathrm{Ext}^i_Q(\Phi(\mathbf{V}), \Phi(\mathbf{W})) \simeq \mathrm{Ext}^i_\Gamma(\mathbf{V}, \mathbf{W}).$$

Proof. This follows from Lemmas 2.3 and 2.5. □

2.5 Ext vanishing properties

For each representation $\mathbf{U} \in \mathrm{Rep}^0(\widehat{\Gamma}_d) = \mathrm{mod}(Q)$ and $r \in [1; d]$, set $\mathbf{U}_r = e_r\mathbf{U}$. We can consider $\mathbf{U}_r$ as a representation of Γ.

Lemma 2.7. *Assume* $\mathbf{V}, \mathbf{W} \in \mathrm{Rep}(\Gamma)$ *are such that* $\mathrm{Ext}^1_\Gamma(\mathbf{W}, \mathbf{V}) = 0$. *For each subrepresentation* $\mathbf{W}' \subset \Phi(\mathbf{W})$, *we have* $\mathrm{Ext}^1_Q(\mathbf{W}', \Phi(\mathbf{V})) = 0$.

Proof. Consider a short exact sequence $0 \to \Phi(\mathbf{V}) \to \mathbf{U} \to \mathbf{W}' \to 0$ in $\mathrm{Rep}^0(\widehat{\Gamma}_d)$. Let us show that it splits. For each $r \in [1; d]$, we have a short exact sequence in $\mathrm{Rep}(\Gamma)$.

$$0 \to \mathbf{V} \to \mathbf{U}_r \to \mathbf{W}'_r \to 0.$$

Moreover, this sequence splits because $\mathrm{Ext}^1_\Gamma(\mathbf{W}, \mathbf{V}) = 0$ and $\mathbf{W}'_r \subset \mathbf{W}$ implies $\mathrm{Ext}^1_\Gamma(\mathbf{W}'_r, \mathbf{V}) = 0$ (recall that the category $\mathrm{Rep}(\Gamma)$ has no second extensions). We have to show, the the split morphisms $s_r \colon \mathbf{W}'_r \to \mathbf{U}_r$ can be chosen in such a way that their direct sum for $r \in [1; d]$ gives a morphism $s \colon \mathbf{W}' \to \mathbf{U}$ in $\mathrm{Rep}^0(\widehat{\Gamma}_d)$.

For each $r \in [1; d-1]$ we have a commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{V} & \xrightarrow{i_r} & \mathbf{U}_r & \xrightarrow{\pi_r} & \mathbf{W}'_r & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{Id}_\mathbf{V}}\downarrow & & {\scriptstyle f_r}\downarrow & & {\scriptstyle g_r}\downarrow & & \\
0 & \longrightarrow & \mathbf{V} & \xrightarrow{i_{r+1}} & \mathbf{U}_{r+1} & \xrightarrow{\pi_{r+1}} & \mathbf{W}'_{r+1} & \longrightarrow & 0
\end{array}$$

The right vertical map g_r is injective and the left vertical map is the identity. Then it is clear that the middle vertical map f_r is injective. Let $s_{r+1} \colon \mathbf{W}'_{r+1} \to \mathbf{U}_{r+1}$ be a morphism in $\mathrm{Rep}(\Gamma)$ that splits the bottom short exact sequence. We claim that we have $s_{r+1}g_r(\mathbf{W}'_r) \subset f_r(\mathbf{U}_r)$. Indeed, fix $w_r \in \mathbf{W}'_r$. Let us show $s_{r+1}g_r(w_r) \in f_r(\mathbf{U}_r)$. Since π_r is surjective, we can find $u_r \in \mathbf{U}_r$ such that $\pi_r(u_r) = w_r$. Since we have $\pi_{r+1}f_r(u_r) = g_r(w_r)$ and $\pi_{r+1}s_{r+1}g_r(w_r) = g_r(w_r)$, the difference $s_{r+1}g_r(w_r) - f_r(u_r)$ is in the kernel of π_{r+1}. Then this difference is of the form $i_{r+1}(v)$ for some $v \in \mathbf{V}$. Then we get $s_{r+1}g_r(w_r) = f_r(u_r) + i_{r+1}(v) = f_r(u_r) + f_r i_r(v)$. This shows that $s_{r+1}g_r(w_r)$ is in the image of f_r.

This implies that s_{r+1} induces a morphism $s_r \colon \mathbf{W}'_r \to \mathbf{U}_r$ in $\mathrm{Rep}(\Gamma)$ such that the following diagram is commutative

$$\begin{array}{ccc}
\mathbf{U}_r & \xleftarrow{s_r} & \mathbf{W}'_r \\
{\scriptstyle f_r}\downarrow & & {\scriptstyle g_r}\downarrow \\
\mathbf{U}_{r+1} & \xleftarrow{s_{r+1}} & \mathbf{W}'_{r+1}
\end{array}$$

Now we use the construction above recursively in the following way. We fix a split morphism $s_d \colon \mathbf{W}'_d \to \mathbf{U}_d$. It yields, a split morphism $s_{d-1} \colon \mathbf{W}'_{d-1} \to \mathbf{U}_{d-1}$. It yields in its order a split morphism $s_{d-2} \colon \mathbf{W}'_{d-2} \to \mathbf{U}_{d-2}$, etc. Together, these split morphisms $s_1, s_2, \ldots, s_d$ yield a split morphism $s \colon \mathbf{W}' \to \mathbf{U}$ in $\mathrm{Rep}^0(\widehat{\Gamma}_d)$. □

It is well-known, that each module $\mathbf{W} \in \mathrm{Rep}(\Gamma)$ has a two-step projective resolution. In particular, this implies, that we have $\mathrm{Ext}^i_\Gamma(\mathbf{W}, \mathbf{V}) = 0$ for $i \geqslant 2$. The same is true for $\mathrm{Rep}(\widehat{\Gamma}_d)$, but not necessary for $\mathrm{Rep}^0(\widehat{\Gamma}_d)$. However, the category $\mathrm{Rep}^0(\widehat{\Gamma}_d)$ has a weaker property.

Lemma 2.8. *For each $i \geqslant 3$ and $\mathbf{V}, \mathbf{W} \in \mathrm{Rep}^0(\widehat{\Gamma}_d)$, we have $\mathrm{Ext}^i_Q(\mathbf{W}, \mathbf{V}) = 0$.*

Proof. As explained above, the algebras $\mathbb{C}\Gamma$ and $\mathbb{C}L_d$ have projective dimensions at most 1. Then the algebra $Q = \mathbb{C}\Gamma \otimes \mathbb{C}L_d$ has projective dimension at most 2. □

Lemma 2.9. *For each $\mathbf{W} \in \mathrm{Rep}(\Gamma)$, each $\mathbf{V}' \in \mathrm{Rep}^0(\widehat{\Gamma}_d)$ and $i \geqslant 2$, we have $\mathrm{Ext}^i_Q(\Phi(\mathbf{W}), \mathbf{V}') = 0$.*

Proof. As explained above, the representation $\mathbf{W}$ has a 2-step projective resolution $0 \to P_1 \to P_0 \to \mathbf{W}$. Applying the functor Φ, we get a 2-step projective resolution $0 \to \Phi(P_1) \to \Phi(P_0) \to \Phi(\mathbf{W})$. This implies the statement. □

Lemma 2.10. *For each $\mathbf{W} \in \mathrm{Rep}(\Gamma)$, each subrepresentation $\mathbf{W}' \subset \Phi(\mathbf{W})$, each $\mathbf{V}' \in \mathrm{Rep}^0(\widehat{\Gamma}_d)$ and $i \geqslant 2$, we have $\mathrm{Ext}^i_Q(\mathbf{W}', \mathbf{V}') = 0$.*

Proof. Apply the functor $\mathrm{Hom}_Q(\bullet, \mathbf{V}')$ to the short exact sequence

$$0 \to \mathbf{W}' \to \Phi(\mathbf{W}) \to \Phi(\mathbf{W})/\mathbf{W}' \to 0.$$

We get an exact sequence

$$\mathrm{Ext}^i_Q(\Phi(\mathbf{W}), \mathbf{V}') \to \mathrm{Ext}^i_Q(\mathbf{W}', \mathbf{V}') \to \mathrm{Ext}^{i+1}_Q(\Phi(\mathbf{W})/\mathbf{W}', \mathbf{V}').$$

The left term is zero by Lemma 2.9 and the right term is zero by Lemma 2.8. This implies that the middle term is also zero. □

Corollary 2.11. *Let $\mathbf{V}, \mathbf{W} \in \mathrm{Rep}(\Gamma)$ be such that $\mathrm{Ext}^1_\Gamma(\mathbf{W}, \mathbf{V}) = 0$. Then for each subrepresentations $\mathbf{V}' \subset \Phi(\mathbf{V})$, $\mathbf{W}' \subset \Phi(\mathbf{W})$ we have $\mathrm{Ext}^1_Q(\mathbf{W}', \Phi(\mathbf{V})/\mathbf{V}') = 0$.*

Proof. Apply the functor $\mathrm{Hom}_Q(\mathbf{W}', \bullet)$ to the short exact sequence

$$0 \to \mathbf{V}' \to \Phi(\mathbf{V}) \to \Phi(\mathbf{V})/\mathbf{V}' \to 0.$$

We get an exact sequence

$$\mathrm{Ext}^1_Q(\mathbf{W}', \Phi(\mathbf{V})) \to \mathrm{Ext}^1_Q(\mathbf{W}', \Phi(\mathbf{V})/\mathbf{V}') \to \mathrm{Ext}^2_Q(\mathbf{W}', \mathbf{V}').$$

The left term is zero by Lemma 2.7 and the right term is zero by Lemma 2.10. Then the middle term is also zero. □

2.6 Some exact sequences

Fix $\mathbf{W} \in \mathrm{Rep}^0(\widehat{\Gamma}_d)$.

Lemma 2.12. *There is a short exact sequence*

$$0 \to \bigoplus_{r\in[1;d-1]} Qe_{r+1} \otimes_{\mathbb{C}\Gamma} \mathbf{W}_r \to \bigoplus_{i\in I, r\in[1;d]} Qe_r \otimes_{\mathbb{C}\Gamma} \mathbf{W}_r \to \mathbf{W} \to 0.$$

Proof. The proof is very similar to [4, p.7]. □

Corollary 2.13. *For each $\mathbf{V}, \mathbf{W} \in \mathrm{Rep}^0(\widehat{\Gamma}_d)$, we have a long exact sequence*

$$0 \to \mathrm{Hom}_{\widehat{\Gamma}_d}(\mathbf{W}, \mathbf{V}) \to \bigoplus_{r\in[1;d]} \mathrm{Hom}_\Gamma(\mathbf{W}_r, \mathbf{V}_r) \to \bigoplus_{r\in[1;d-1]} \mathrm{Hom}_\Gamma(\mathbf{W}_r, \mathbf{V}_{r+1}) \to \mathrm{Ext}^1_Q(\mathbf{W}, \mathbf{V})$$

Proof. We apply the functor $\mathrm{Hom}_Q(\bullet, \mathbf{V})$ to the short exact sequence above. □

2.7 Flags

Let $\mathbf{V} = \bigoplus_{i\in I} \mathbf{V}_i$ be a representation of Γ with dimension vector $v = (v_i)_{i\in I}$.

Definition 2.14. A *flag type* of weight v is a d-tuple of dimension vectors $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_d)$, $\mathbf{v}_r = (\mathbf{v}_{r,i})_{i\in I}$ such that $\mathbf{v}_d = v$ and for each $i \in I$ we have $\mathbf{v}_{1,i} \leqslant \mathbf{v}_{2,i} \leqslant \ldots \leqslant \mathbf{v}_{d,i}$.

A *flag* in $\mathbf{V}$ of type $\mathbf{v}$ is a sequence of subrepresentations $\mathbf{V}^1 \subset \mathbf{V}^2 \subset \ldots \subset \mathbf{V}^d = \mathbf{V}$ of $\mathbf{V}$ such that $\dim \mathbf{V}^r = \mathbf{v}_r$.

Denote by $\mathcal{F}_\mathbf{v}(\mathbf{V})$ the set of all flags of type $\mathbf{v}$ in $\mathbf{V}$. This set has an obvious structure of a projective algebraic variety over $\mathbb{C}$.

2.8 Reduction to the indecomposable case

For two dimension vectors v and w we set $< w, v >= \sum_{i\in I} w_i \cdot v_i - \sum_{h\in H} w_{h'} \cdot v_{h''}$. We start from the following well-known lemma, see [4, §1].

Lemma 2.15. *Let $\mathbf{V}, \mathbf{W} \in \mathrm{Rep}(\Gamma)$ be two representations. Let v and w be their dimension vectors. Then we have an exact sequence*

$$0 \to \mathrm{Hom}_\Gamma(\mathbf{W}, \mathbf{V}) \to \bigoplus_{i\in I} \mathrm{Hom}(\mathbf{W}_i, \mathbf{V}_i) \to \bigoplus_{h\in H} \mathrm{Hom}(\mathbf{W}_{h'}, \mathbf{V}_{h''}) \to \mathrm{Ext}^1_\Gamma(\mathbf{W}, \mathbf{V}) \to 0.$$

In particular, we have

$$\dim \mathrm{Hom}_\Gamma(\mathbf{W}, \mathbf{V}) - \dim \mathrm{Ext}^1_\Gamma(\mathbf{W}, \mathbf{V}) =< w, v > .$$

Corollary 2.16. *If additionally we have $\mathrm{Ext}^1_\Gamma(\mathbf{V}, \mathbf{W}) = 0$, then we have a short exact sequence*

$$0 \to \mathrm{Hom}_\Gamma(\mathbf{W}, \mathbf{V}) \to \bigoplus_{i\in I} \mathrm{Hom}(\mathbf{W}_i, \mathbf{V}_i) \to \bigoplus_{h\in H} \mathrm{Hom}(\mathbf{W}_{h'}, \mathbf{V}_{h''}) \to 0 \qquad (1)$$

In particular, we have $\dim \mathrm{Hom}_\Gamma(\mathbf{W}, \mathbf{V}) =< w, v >$.

Let $\mathbf{V}$ and $\mathbf{W}$ be representations of Γ with dimension vectors v and w respectively. Let $\mathbf{v}$ and $\mathbf{w}$ be flag types (with respect to v and w respectively). Assume that $0 \to \mathbf{V} \to \mathbf{U} \xrightarrow{\pi} \mathbf{W} \to 0$ is a short exact sequence in $\mathrm{Rep}(\Gamma)$. For each flag ϕ in $\mathbf{U}$ we denote by $\phi \cap \mathbf{V}$ the flag in $\mathbf{V}$ obtained by the intersection of components of ϕ with $\mathbf{V}$ and we denote by $\pi(\phi)$ the flag in $\mathbf{W}$ obtained by the images in $\mathbf{W}$ of the components of ϕ. Set

$$\mathcal{F}_{\mathbf{v},\mathbf{w}}(\mathbf{U}) = \{\phi \in \mathcal{F}_{\mathbf{v}+\mathbf{w}}(\mathbf{U});\ \phi \cap V \in \mathcal{F}_{\mathbf{v}}(\mathbf{V})\}.$$

It is clear from the definition that if ϕ is in $\mathcal{F}_{\mathbf{v},\mathbf{w}}(\mathbf{U})$, then $\pi(\phi)$ is in $\mathcal{F}_{\mathbf{w}}(\mathbf{W})$.

For a flag type $\mathbf{v}$, we set $\overline{\mathbf{v}}_r = \mathbf{v}_r - \mathbf{v}_{r-1}$ for $r \in [2; d]$ and $\overline{\mathbf{v}}_1 = \mathbf{v}_1$. Similarly, we define $\overline{\mathbf{w}}_r$ for a flag type $\mathbf{w}$.

Proposition 2.17. *Assume that we have* $\mathrm{Ext}^1_\Gamma(\mathbf{W}, \mathbf{V}) = 0$. *Then*

$$\mathcal{F}_{\mathbf{v},\mathbf{w}}(\mathbf{U}) \to \mathcal{F}_{\mathbf{v}}(\mathbf{V}) \times \mathcal{F}_{\mathbf{w}}(\mathbf{W}), \qquad \phi \mapsto (\phi \cap \mathbf{V}, \pi(\phi))$$

is a vector bundle of rank

$$\sum_{r=1}^{d-1} \sum_{t=r+1}^{d} < \overline{\mathbf{v}}_r, \overline{\mathbf{w}}_t > .$$

Proof. Fix an isomorphism $\mathbf{U} \simeq \mathbf{V} \oplus \mathbf{W}$. This is possible by the assumption on Ext^1.

First, we want to understand the fibers of the given map. Fix $(\phi_{\mathbf{V}}, \phi_{\mathbf{W}}) \in \mathcal{F}_{\mathbf{v}}(\mathbf{V}) \times \mathcal{F}_{\mathbf{w}}(\mathbf{W})$,

$$\phi_{\mathbf{V}} = (\{0\} \subset \mathbf{V}^1 \subset \mathbf{V}^2 \subset \ldots \subset \mathbf{V}^d = \mathbf{V}), \qquad \phi_{\mathbf{W}} = (\{0\} \subset \mathbf{W}^1 \subset \mathbf{W}^2 \subset \ldots \subset \mathbf{W}^d = \mathbf{W}).$$

Let us describe the fiber of $(\phi_{\mathbf{V}}, \phi_{\mathbf{W}})$. For each $r \in [1; d]$, we want to construct a subrepresentation $\mathbf{U}^r$ of $\mathbf{U}$ such that $\mathbf{U}^r \cap \mathbf{V} = \mathbf{V}^r$ and $\pi(\mathbf{U}^r) = \mathbf{W}^r$. The choices of such a subrepresentation are parameterized by $\mathrm{Hom}_\Gamma(\mathbf{W}^r, \mathbf{V}/\mathbf{V}^r)$. Indeed, to each map $f \in \mathrm{Hom}_\Gamma(\mathbf{W}^r, \mathbf{V}/\mathbf{V}^r)$ we can associate a subrepresentation $\mathbf{U}^r \subset \mathbf{V} \oplus \mathbf{W}$ generated by the elements $(v, w) \in \mathbf{V} \oplus \mathbf{W}^r$ such that the image of v in $\mathbf{V}/\mathbf{V}^r$ is $f(w)$.

Moreover, for each $r \in [1; d-1]$, we must have $\mathbf{U}^r \subset \mathbf{U}^{r+1}$. This condition is equivalent to the commutativity of the following diagram

$$\begin{array}{ccc} \mathbf{W}^r & \longrightarrow & \mathbf{V}/\mathbf{V}^r \\ \downarrow & & \downarrow \\ \mathbf{W}^{r+1} & \longrightarrow & \mathbf{V}/\mathbf{V}^{r+1}. \end{array}$$

To sum up, a point of the fiber of $(\phi_{\mathbf{V}}, \phi_{\mathbf{W}})$ is described by a family of homomorphisms $\mathrm{Hom}_\Gamma(\mathbf{W}^r, \mathbf{V}/\mathbf{V}^r)$ for $r \in [1; d]$ such that for each $r \in [1; d-1]$ the diagram above commutes.

Now, we change the point of view to describe this fiber in a different way. We can consider the flag $\phi_{\mathbf{V}}$ as a subrepresentation $\mathbf{V}' = \bigoplus_{r \in [1;d], i \in I} \mathbf{V}'_{(r,i)}$

of $\Phi(\mathbf{V})$. The component $\mathbf{V}^r$ of the flag $\phi_{\mathbf{V}}$ is now considered as a part $\mathbf{V}'_r = \bigoplus_{i\in I} \mathbf{V}'_{(r,i)}$ of the representation $\mathbf{V}'$ of $\widehat{\Gamma}_d$. The flag dimension $\mathbf{v}$ can be considered as a dimension vector for the quiver $\widehat{\Gamma}_d$. Moreover, the representation $\mathbf{V}'$ of $\widehat{\Gamma}_d$ has dimension $\mathbf{v}$. Similarly, we consider the flag $\phi_{\mathbf{W}}$ as a subrepresentation $\mathbf{W}'$ of $\Phi(\mathbf{W})$ of dimension $\mathbf{w}$. Then the fiber of $(\phi_{\mathbf{V}}, \psi_{\mathbf{W}})$ is simply $\mathrm{Hom}_{\widehat{\Gamma}_d}(\mathbf{W}', \Phi(\mathbf{V})/\mathbf{V}')$.

We have an obvious inclusion of vector spaces $\mathrm{Hom}_{\widehat{\Gamma}_d}(\mathbf{W}', \Phi(\mathbf{V})/\mathbf{V}') \subset \bigoplus_{r\in[1;d]} \mathrm{Hom}_\Gamma(\mathbf{W}'_r, \mathbf{V}/\mathbf{V}'_r)$. Let us show that $\mathcal{F}_{\mathbf{v},\mathbf{w}}(\mathbf{U})$ is a subbundle of the vector bundle on $\mathcal{F}_{\mathbf{v}}(\mathbf{V}) \times \mathcal{F}_{\mathbf{w}}(\mathbf{W})$ with fiber $\bigoplus_{r\in[1;d]} \mathrm{Hom}_\Gamma(\mathbf{W}'_r, \mathbf{V}/\mathbf{V}'_r)$. For this, it is enough to present $\mathcal{F}_{\mathbf{v},\mathbf{w}}(\mathbf{U})$ as a kernel of a morphism of vector bundles of constant rank.

Indeed, since we assumed $\mathrm{Ext}^1_\Gamma(\mathbf{W}, \mathbf{V}) = 0$, Corollary 2.11 implies $\mathrm{Ext}^1_Q(\mathbf{W}', \Phi(\mathbf{V})/\mathbf{V}') = 0$. Then the exact sequence in Corollary 2.13 yields a short exact sequence

$$0 \to \mathrm{Hom}_{\widehat{\Gamma}_d}(\mathbf{W}', \Phi(\mathbf{V})/\mathbf{V}') \to \bigoplus_{r\in[1;d]} \mathrm{Hom}_\Gamma(\mathbf{W}'_r, \mathbf{V}/\mathbf{V}'_r) \to \bigoplus_{r\in[1;d-1]} \mathrm{Hom}_\Gamma(\mathbf{W}'_r, \mathbf{V}/\mathbf{V}'_{r+1}) \to 0 \tag{2}$$

This short exact sequence implies that $\mathcal{F}_{\mathbf{v},\mathbf{w}}(\mathbf{U})$ is a kernel of a surjective morphism of vector bundles. In particular, it is also a vector bundle. (Note that the vector bundles with fibers $\mathrm{Hom}_\Gamma(\mathbf{W}'_r, \mathbf{V}/\mathbf{V}'_r)$ and $\mathrm{Hom}_\Gamma(\mathbf{W}'_r, \mathbf{V}/\mathbf{V}'_{r+1})$ are defined via the same procedure. They are kernels of surjective morphisms of vector bundles coming from the short exact sequence (1). See also the proof of [7, Thm. 26].)

Let us calculate the rank of this vector bundle. It is clear from the short exact sequence above that it is equal to

$$\sum_{r=1}^{d} \dim \mathrm{Hom}_\Gamma(\mathbf{W}'_r, \mathbf{V}/\mathbf{V}'_r) - \sum_{r=1}^{d-1} \dim \mathrm{Hom}_\Gamma(\mathbf{W}'_r, \mathbf{V}/\mathbf{V}'_{r+1}).$$

By Corollary 2.16, this rank is equal to

$$\begin{aligned} \textstyle\sum_{r=1}^{d} < \mathbf{w}_r, v - \mathbf{v}_r > - \sum_{r=1}^{d-1} < \mathbf{w}_r, v - \mathbf{v}_{r+1} > &= \textstyle\sum_{r=1}^{d-1} < \mathbf{w}_r, \overline{\mathbf{v}}_{r+1} > \\ &= \textstyle\sum_{r=1}^{d-1} \sum_{t=r+1}^{d} < \overline{\mathbf{w}}_r, \overline{\mathbf{v}}_t > . \end{aligned}$$

□

2.9 α-partition

Definition 2.18. Let X be a complex algebraic variety. We say that X admits an α-partition into affine spaces if X has a finite partition $X = X_1 \coprod X_2 \coprod \ldots \coprod X_n$ such that

1. for each $1 \leqslant r \leqslant n$, the union $X_1 \coprod \ldots \coprod X_r$ is closed,
2. each X_r is isomorphic to an affine space.

Proposition 2.19. *Let Γ be a Dynkin quiver. Let $\mathbf{V}$ be a representation of Γ. Let us decompose it in a direct sum of indecomposable representations*

$$\mathbf{V} = \mathbf{V}_1^{\oplus n_1} \oplus \ldots \oplus \mathbf{V}_k^{\oplus n_k}.$$

Assume that for each $r \in [1;k]$ and each flag type $\mathbf{v}^r$ of weight $(\dim \mathbf{V}_r)$, the variety $\mathcal{F}_{\mathbf{v}^r}(\mathbf{V}_r)$ is either empty or has an α-partition into affine spaces. Then for each flag type of weight $(\dim \mathbf{V})$, the variety $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ is either empty or has an α-partition into affine spaces.

Proof. We can assume that $\mathbf{V}_1, \ldots, \mathbf{V}_k$ are ordered in such a way that $\mathrm{Ext}^1(\mathbf{V}_r, \mathbf{V}_t) = 0$ if $r \leqslant t$. Then the statement follows easily from Proposition 2.17 by induction. □

2.10 Types A and D

The goal of this section is to prove the following theorem.

Theorem 2.20. *Assume that Γ is a Dynkin quiver of type A or D. Then the variety $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ is either empty or has an α-partition into affine spaces.*

Proof. By Proposition 2.19, it is enough to prove the statement for indecomposable representations. This is done in two lemmas below. □

Lemma 2.21. *Assume that Γ is a Dynkin quiver of type A and that $\mathbf{V}$ is an indecomposable representation. Then the variety $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ is either empty or is a singleton.*

Proof. The statement follows from the fact that for each $i \in I$ the dimension of $\mathbf{V}_i$ is 0 or 1. □

Lemma 2.22. *Assume that Γ is a Dynkin quiver of type D. Then the variety $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ is either empty or is a singleton or is a direct product of some copies of $\mathbb{P}^1_{\mathbb{C}}$.*

Proof. For each $i \in I$ the dimension of $\mathbf{V}_i$ is 0, 1 or 2. Then it is clear that the variety $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ is naturally included to a direct product of $\mathbb{P}^1_{\mathbb{C}}$. Indeed, a point of $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ is given by a choice of 1-dimensional subspaces $\mathbf{V}'_i$ inside of some 2-dimensional $\mathbf{V}_i$'s, these choices must satisfy some list of conditions. We may have the following types of conditions.

- For a given i with $\dim \mathbf{V}_i = 2$, we may have a condition that the 1-dimensional subspace $\mathbf{V}'_i$ of $\mathbf{V}_i$ is equal to a fixed 1-dimensional subspace (i.e., this condition imposes some choice of $\mathbf{V}'_i$ inside of some two-dimensional $\mathbf{V}_i$).
- For some arrow $i \to j$, the map $\mathbf{V}_i \to \mathbf{V}_j$ may be an isomorphism and $\dim \mathbf{V}_i = \dim \mathbf{V}_j = 2$. We may have a condition that the map $\mathbf{V}_i \to \mathbf{V}_j$ sends $\mathbf{V}'_i$ to $\mathbf{V}'_j$.

- We may have an impossible condition, implying that $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ is empty.

It is clear that the conditions above just reduce the number of $\mathbb{P}^1_{\mathbb{C}}$ in the direct product (or make the variety empty). This proves the statement. □

Remark 2.23. It is clear from the proof of Theorem 2.20 that for proving an analogue of this theorem for type E, it is enough to prove it for indecomposable representations.

We are going to prove a weaker statement in type E: the variety $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ has no odd cohomology over $\mathbb{Z}$. The same argument as above shows that for proving this statement, it is enough to check this only for indecomposable representations.

3 Rigid representations

The goal of this section is to show that the variety $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ has no odd cohomology over $\mathbb{Z}$. It is enough to check this statement only for indecomposable representations, see Remark 2.23. In fact, we are going to prove this fore more general class of representations. In is well-known that for each indecomposable representation $\mathbf{V}$ of a Dynkin quiver we have $\mathrm{Ext}^1_\Gamma(\mathbf{V},\mathbf{V}) = 0$, see for example the proof of [4, Thm. 1]. We will prove that if some representation of a quiver satisfies this condition, then the variety $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ has no odd cohomology over $\mathbf{V}$. Moreover, we will prove that in this case the variety $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ has a diagonal decomposition if it is not empty.

3.1 Rigid representations

Let Γ be an arbitrary quiver.

Definition 3.1. We say that a representation $\mathbf{V}$ of Γ is *rigid* if we have $\mathrm{Ext}^1_\Gamma(\mathbf{V},\mathbf{V}) = 0$.

Choose a vertex $i \in I$. For each flag type $\mathbf{v}$, the d-tuple $\mathbf{v}_i = (\mathbf{v}_{i,1}, \mathbf{v}_{i,2}, \dots, \mathbf{v}_{i,d})$ can be seen as a flag type for a quiver with one vertex. The vector space $\mathbf{V}_i$ can be seen as a representation of this quiver. The variety $\mathcal{F}_{\mathbf{v}_i}(\mathbf{V}_i)$ is a usual (non-complete) flag variety. The variety $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ is obviously included to a direct product of usual (non-complete) flag varieties in the following way:

$$\mathcal{F}_{\mathbf{v}}(\mathbf{V}) \subset \prod_{i\in I} \mathcal{F}_{\mathbf{v}_i}(\mathbf{V}_i).$$

Set $\mathrm{Rep}_v(\Gamma) = \bigoplus_{h\in H} \mathrm{Hom}(\mathbb{C}^{v_{h'}}, \mathbb{C}^{v_{h''}})$. We can see an element $X \in \mathrm{Rep}_v(\Gamma)$ as a representation of Γ with dimension vector v.

Let $\mathcal{Q}$ be the subvariety of $\mathrm{Rep}_v(\Gamma) \times \prod_{i\in I} \mathcal{F}_{\mathbf{v}_i}(\mathbb{C}^{v_i})$ given by

$$\mathcal{Q} = \{(X,\phi);\ X \in \mathrm{Rep}_v(\Gamma),\ \phi \in \mathcal{F}_{\mathbf{v}}(X)\}.$$

Lemma 3.2. *Assume that* $\mathbf{V}$ *is rigid. Let* $\mathbf{v}$ *be a flag type such that* $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ *is not empty. Then the variety* $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ *is a smooth projective variety of dimension* $\sum_{r<t} < \overline{\mathbf{v}}_r, \overline{\mathbf{v}}_t >$.

Proof. The proof is similar to [3, Cor. 4].

The map

$$\mathcal{Q} \to \prod_{i\in I} \mathcal{F}_{\mathbf{v}_i}(\mathbb{C}^{v_i}), \qquad (X,\phi) \to \phi$$

is clearly a vector bundle of rank

$$\sum_{h\in H}\sum_{r=1}^{d} \overline{\mathbf{v}}_{h',r}\cdot \mathbf{v}_{h'',r} = \sum_{h\in H}\sum_{r\geqslant t} \overline{\mathbf{v}}_{h',r}\cdot \overline{\mathbf{v}}_{h'',t}.$$

On the other hand, the map

$$\pi\colon \mathcal{Q} \to \mathrm{Rep}_v(\Gamma), \qquad (X,\phi) \to X$$

is proper. If some representation $X \in \mathrm{Rep}_v(\Gamma)$ is isomorphic to $\mathbf{V}$, then the fiber $\pi^{-1}(X)$ is isomorphic to $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$. Since $\mathbf{V}$ is rigid, the subset of representations in $\mathrm{Rep}_v(\Gamma)$ isomorphic to $\mathbf{V}$ is open, see [4, Lem. 1]. On the other hand, since we assumed that $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ is not empty, we see that the image of π is dense. Since the image of π is closed, the map π is surjective. Then the generic fiber of π is smooth and has dimension $\dim \mathcal{Q} - \dim \mathrm{Rep}_v(\Gamma)$.

This implies that $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ is a smooth projective variety of dimension

$$\begin{array}{ll} & \sum_{h\in H}\sum_{r\geqslant t} \overline{\mathbf{v}}_{h',r}\cdot \overline{\mathbf{v}}_{h'',t} + \dim \prod_{i\in I} \mathcal{F}_{\mathbf{v}_i}(\mathbb{C}^{v_i}) - \dim \mathrm{Rep}_v(\Gamma) \\ = & \sum_{h\in H}\sum_{r\geqslant t} \overline{\mathbf{v}}_{h',r}\cdot \overline{\mathbf{v}}_{h'',t} + \sum_{i\in I}\sum_{r<t} \overline{\mathbf{v}}_{i,r}\cdot \overline{\mathbf{v}}_{i,t} - \sum_{h\in H}\sum_{r,t} \overline{\mathbf{v}}_{h',r}\cdot \overline{\mathbf{v}}_{h'',t} \\ = & \sum_{r<t} < \overline{\mathbf{v}}_r, \overline{\mathbf{v}}_t > . \end{array}$$

□

3.2 Diagonal decomposition

In is proved in [7, Thm. 36] that quiver Grassmannians associated with rigid representations have zero odd cohomology groups over $\mathbb{Z}$. The goal of this section is to prove the same statement for flag generalizations of quiver Grassmannians. In fact, [7, Thm. 36] proves a stronger property, that they call property (S), see [7, Def. 11]. The key point of their proof is that to check property (S), it is enough to construct a diagonal decomposition.

Let X be a smooth complete complex variety. We denote $A^*(X)$ the Chow ring of X. Let $\Delta \subset X\times X$ be the diagonal. Denote by $\pi_1, \pi_2\colon X\times X \to X$ the two natural projections.

Definition 3.3. We say that X has a *diagonal decomposition* if the class $[\Delta]$ of the diagonal $\Delta \subset X\times X$ in $A^*(X\times X)$ has the following decomposition

$$[\Delta] = \sum_{j\in J} \pi_1^*\alpha_j \cdot \pi_2^*\beta_j, \tag{3}$$

where J is a finite set and $\alpha_j, \beta_j \in A^*(X)$.

It is proved in [5, Thm. 2.1] that if X has a diagonal decomposition, then it satisfies property (S). Then, in particular, we have $H^{\mathrm{odd}}(X,\mathbb{Z})=0$.

For a vector bundle $\mathcal{E}$ on X we denote by $c_i(\mathcal{E})$ its ith Chern class, $c_i(\mathcal{E})\in A^i(X)$. Denote by $c(\mathcal{E},t)$ the Chern polynomial $c(\mathcal{E},t)=\sum_i c_i(\mathcal{E})t^i$.

Theorem 3.4. *Assume that $\mathbf{V}\in\mathrm{Rep}(\Gamma)$ is rigid and that $\mathbf{v}$ is a flag type. Then $\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ is either empty or has a diagonal decomposition. In particular, we have $H^{\mathrm{odd}}(\mathcal{F}_{\mathbf{v}}(\mathbf{V}),\mathbb{Z})=0$.*

Proof. The proof is similar to [7, Thm. 3.6]. Similarly to their proof, we are going to construct a vector bundle on $\mathcal{F}_{\mathbf{v}}(\mathbf{V})\times\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ of rank $(\dim\mathcal{F}_{\mathbf{v}}(\mathbf{V}))$ that has a section that is zero exactly on the diagonal. We manage to construct such a bundle due to Proposition 2.17.

Indeed, since $\mathrm{Ext}^1_\Gamma(\mathbf{V},\mathbf{V})=0$, we can apply Proposition 2.17 directly. We obtain a vector bundle $\mathcal{E}$ of the desired dimension, its fiber over $(\mathbf{V}',\mathbf{V}'')\in\mathcal{F}_{\mathbf{v}}(\mathbf{V})\times\mathcal{F}_{\mathbf{v}}(\mathbf{V})$ is $\mathrm{Hom}_{\widehat{\Gamma}_d}(\mathbf{V}',\Phi(\mathbf{V})/\mathbf{V}'')$. As above, we see $\mathbf{V}'$ and $\mathbf{V}''$ as representations of $\widehat{\Gamma}_d$ that are subrepresentations of $\Phi(\mathbf{V})$. The composition $\mathbf{V}'\to\Phi(\mathbf{V})\to\Phi(\mathbf{V})/\mathbf{V}''$ yields a section is this vector bundle. This section is zero exactly on the diagonal. Then by [6, Prop. 14.1, Ex. 14.1.1], we can describe the class of the diagonal in term of the top Chern class of the bundle: $[\Delta]=c_{\mathrm{top}}(\mathcal{E})$.

Let $D\subset A^*(\mathcal{F}_{\mathbf{v}}(\mathbf{V})\times\mathcal{F}_{\mathbf{v}}(\mathbf{V}))$ be the vector subspace formed by the elements of the form as in the right hand side of (3) for $X=\mathcal{F}_{\mathbf{v}}(\mathbf{V})$. It is easy to see that D is a subring.

We want to prove that $[\Delta]\in D$. Let us prove that all the coefficients of the Chern polynomial $c(\mathcal{E},t)$ are in D. Recall from (2) that $\mathcal{E}$ is a part of a short exact sequence of vector bundles $0\to\mathcal{E}\to\mathcal{F}\to\mathcal{G}\to 0$, where

$$\mathcal{F}=\bigoplus_{r\in[1;d]}\mathcal{F}_r,\qquad \mathcal{F}_r \text{ has fiber } \mathrm{Hom}_\Gamma(\mathbf{V}'_r,\mathbf{V}/\mathbf{V}''_r)$$

and

$$\mathcal{G}=\bigoplus_{r\in[1;d-1]}\mathcal{G}_r,\qquad \mathcal{G}_r \text{ has fiber } \mathrm{Hom}_\Gamma(\mathbf{V}'_r,\mathbf{V}/\mathbf{V}''_{r+1}).$$

So we have $c(\mathcal{E},t)=c(\mathcal{F},t)\cdot c(\mathcal{G},t)^{-1}$. In particular, it is enough to prove that the coefficients of $c(\mathcal{F}_r,t)$ and $c(\mathcal{G}_r,t)$ are in D.

Let us show this for $\mathcal{F}_r$, the proof for $\mathcal{G}_r$ is similar. By (1), the bundle $\mathcal{F}_r$ is a part of a short exact sequence

$$0\to\mathcal{F}_r\to\mathcal{X}_r\to\mathcal{Y}_r\to 0,$$

where

$$\mathcal{X}_r=\bigoplus_{i\in I}\mathcal{X}_{i,r},\qquad \mathcal{X}_{i,r} \text{ has fiber } \mathrm{Hom}(\mathbf{V}'_{i,r},\mathbf{V}_i/\mathbf{V}''_{i,r})$$

and

$$\mathcal{Y}_r=\bigoplus_{h\in H}\mathcal{Y}_{h,r},\qquad \mathcal{Y}_{h,r} \text{ has fiber } \mathrm{Hom}(\mathbf{V}'_{h',r},\mathbf{V}_i/\mathbf{V}''_{h'',r}).$$

Then we have $c(\mathcal{F}_r, t) = c(\mathcal{X}_r, t) \cdot c(\mathcal{Y}_r, t)^{-1}$. So it is enough to prove that the coefficients of $c(\mathcal{X}_{i,r}, t)$ and $c(\mathcal{Y}_{h,r}, t)$ are in D. This is clear from [10], [6, Ex. 14.5.2]. □

3.3 Type E

Theorem 3.5. *Assume that Γ is a Dynkin quiver (types A, D, E). Then we have $H^{\mathrm{odd}}(\mathcal{F}_{\mathbf{v}}(\mathbf{V}), \mathbb{Z}) = 0$.*

Proof. For types A and D the statement already follows from Theorem 2.20.

Let us prove the statement for type E. As explained in Remark 2.23, it is enough to check the statement only for indecomposable representations. Since indecomposable representations are rigid, the statement follows from Theorem 3.4. □

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