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Uniqueness results in the inverse spectral Steklov problem

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Abstract

This paper is devoted to an inverse Steklov problem for a particular class of n -dimensional manifolds having the topology of a hollow sphere and equipped with a warped product metric. We prove that the knowledge of the Steklov spectrum determines uniquely the associated warping function up to a natural invariance.

Keywords. Inverse Calderón problem, Steklov spectrum, Weyl-Titchmarsh functions, Nevanlinna theorem, local Borg-Marchenko theorem.

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1 Introduction

1.1 The Calderón and Steklov problems.

Let (M, g) be a smooth compact manifold of dimension $n \geq 2$ with smooth boundary ∂M . We consider the Dirichlet problem

$$\begin{cases} -\Delta_g u = \lambda u & \text{in } M \\ u = \psi & \text{on } \partial M, \end{cases} \quad (1)$$

where $\psi \in H^{1/2}(\partial M)$ and $\lambda \in \mathbb{R}$ is assumed to lie outside the Dirichlet spectrum $\sigma(-\Delta_g)$ of the Laplace-Beltrami operator $-\Delta_g$. In a local coordinate system $(x^i)_{i=1,\dots,n}$, $-\Delta_g$ has the expression

$$-\Delta_g = - \sum_{1 \leq i, j \leq n} \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \partial_j),$$

where we have set $|g| = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$.

If $\lambda \notin \sigma(-\Delta_g)$, the Dirichlet problem (1) has a unique solution $u \in H^1(M)$, and we can define the so-called *Dirichlet-to-Neumann* (DN) operator as the map

$$\begin{aligned} \Lambda_g(\lambda) : H^{1/2}(\partial M) &\rightarrow H^{-1/2}(\partial M) \\ \psi &\mapsto \frac{\partial u}{\partial \nu} \Big|_{\partial M}, \end{aligned}$$

where ∂_ν is the unit normal derivative with respect to the unit outer normal vector on ∂M . This normal derivative has to be understood in the weak sense by :

$$\forall (\psi, \phi) \in H^{1/2}(\partial M)^2 : \langle \Lambda_g(\lambda) \psi, \phi \rangle = \int_M \langle du, dv \rangle_g d\text{Vol}_g + \lambda \int_M uv d\text{Vol}_g,$$

where u is the unique weak solution of the Dirichlet problem (1), and where v is any element of $H^1(M)$ such that $v|_{\partial M} = \phi$. When ψ is sufficiently smooth, this definition coincides with the usual one in local coordinates, that is

$$\partial_\nu u = \nu^i \partial_i u. \quad (2)$$

The anisotropic Calderón problem can be initially stated as: *does the knowledge of the DN map $\Lambda_g(\lambda)$ at a fixed frequency λ determine uniquely the metric g ?*

Due to a number of gauge invariances, it is well-known that the answer to the above question is negative in general. An observation made by Luc Tartar ([10], p.2) leads to the equality :

$$\Lambda_g(\lambda) = \Lambda_{\psi^*g}(\lambda),$$

where $\psi : M \rightarrow M$ is any smooth diffeomorphism which is equal to the identity on the boundary, (here ψ^*g is the pullback of g by ψ). Moreover, in dimension $n = 2$ and for $\lambda = 0$, there is one more gauge invariance. Indeed, thanks to the conformal invariance of the Laplacian, for every positive function c , we have

$$\Delta_{cg} = \frac{1}{c} \Delta_g.$$

Consequently, the solutions of the Dirichlet problem (1) associated to the metrics g and cg are the same when $\lambda = 0$. Moreover, if we assume that $c \equiv 1$ on the boundary, the unit outer normal vectors on ∂M are also the same for both metrics. Therefore,

$$\Lambda_{cg}(0) = \Lambda_g(0),$$

and it is not possible to determine uniquely the metric from the DN map. Hence, the appropriate question to adress is :

Assume $n \geq 3$ (resp. $n = 2$ and $\lambda \neq 0$). If $\Lambda_g(\lambda) = \Lambda_{\tilde{g}}(\lambda)$, is there a smooth diffeomorphism $\psi : M \rightarrow M$ with $\psi|_{\partial M} = id$ and $\psi^*g = \tilde{g}$?

This problem is still largely open. Some special cases have been answered positively (see [17] and [20] for a presentation of the latest developments) but the general case seems to be very difficult to tackle. However, the question becomes simpler if we assume that (M, g) and (M, \tilde{g}) belong to the same conformal class. Precisely :

For $c \in C^\infty(M)$, assume $\Lambda_g(\lambda) = \Lambda_{cg}(\lambda)$. Then, is there a smooth diffeomorphism $\psi : M \rightarrow M$ such that $\psi|_{\partial M} = id$ and $\psi^*(g) = cg$?

Actually, we can precise the above question thanks to the following result due to Lionheart ([11]) : any diffeomorphism $\psi : M \rightarrow M$ which satisfies $\psi^*(cg) = g$ and $\psi|_{\partial M} = id$ must be the identity. Then, the anisotropic Calderón inverse problem within the same conformal class can be replaced by

If $\Lambda_g(\lambda) = \Lambda_{cg}(\lambda)$, is it true that $c = 1$?

Some deep results have been obtained in [6] for conformally transversally anisotropic manifolds (M, g) of dimension $n \geq 3$, i.e for manifolds

$$M \subset \mathbb{R} \times K, \quad g = c(x, y_K)(dx^2 + g_K)$$

where (K, g_K) is a smooth compact manifold of dimension $n-1$. Under some geometrical conditions on K , such as *simplicity* (a compact manifold K is said *simple* if any two points of K are connected by a unique geodesic and if its boundary is strictly convex) the conformal factor c is entirely determined by the DN map at frequency $\lambda = 0$.

In the same way, for a class of manifolds $M = [0, 1] \times K$ which have the topology of a cylinder, various results of uniqueness (or non-uniqueness when the Dirichlet data and the Neumann data are measured on disjoint sets) have been obtained in ([3, 4]). The proofs make use of separation of variables and introduces a connection between the DN map and the Weyl-Titchmarsh functions associated to a separated ordinary differential equation corresponding to the horizontal variable of the cylinder. This connection allows to use some nice results from complex analysis.

In this paper, we are interested in a relative inverse problem within the conformal class of certain warped product manifold $M = [0, 1] \times \mathbb{S}^{n-1}$ (where \mathbb{S}^{n-1} is the $(n-1)$ -sphere) but under a weaker assumption :

does the spectrum of the DN map characterize the conformal factor ?

We recall that $\Lambda_g(\lambda)$ is an elliptic pseudodifferential operator of order 1 and is self-adjoint on $L^2(\partial M, dS_g)$ where dS_g is the metric induced by g on the boundary ∂M . Thus, the DN operator $\Lambda_g(\lambda)$ has a discrete spectrum denoted $\sigma(\Lambda_g(\lambda))$ accumulating at infinity

$$\sigma(\Lambda_g(\lambda)) = \{\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \rightarrow +\infty\}.$$

This spectrum is called the *Steklov spectrum* (see [9], p.2). In the particular case where $\lambda = 0$, using the Green's formula, we see that the DN operator is positive and we have $\lambda_0 = 0$. The properties of the Steklov spectrum are highly sensitive to the smoothness of the boundary ∂M . For example (see [8]): if ∂M is smooth, the eigenvalues satisfy the Weyl formula for the Steklov spectrum:

$$\lambda_j = 2\pi \left(\frac{j}{\text{Vol}(\mathbb{B}^{n-1})\text{Vol}(\partial M)} \right)^{\frac{1}{n-1}} + O(1) \quad (3)$$

A much more refined asymptotic holds if M is a smooth surface, involving the lengths of the connected components of M , with a rate of decay of $O(j^{-\infty})$ (see [7]), but this formula fails for

polygons ([8]). In some particular cases, when the boundary ∂M is just C^1 , it is also possible to find a one term asymptotic of the Steklov spectrum counted with multiplicity (see [1]). For more specific domains with Lipschitz boundary, a recent paper ([7]) proves a two-term asymptotic formula for cuboids, i.e domains defined, for $n \geq 3$, as

$$M = (-a_1, a_1) \times \dots \times (-a_n, a_n) \in \mathbb{R}^n.$$

The formula for the counting function $N(\lambda)$ of Steklov eigenvalues is the following ($C_1, C_2 \in \mathbb{R}$):

$$N(\lambda) = C_1 \text{Vol}_{n-1}(\partial M) \lambda^{n-1} + C_2 \text{Vol}_{n-2}(\partial^2 M) \lambda^{n-2} + O(\lambda^n),$$

where $\partial^2 M$ denotes the union of all the $n - 2$ dimensional facets of M , $\eta = \frac{2}{3}$ if $n = 3$ and $\eta = n - 2 - \frac{1}{n-1}$ if $n \geq 4$. As a corollary, the authors deduce also that, for a rectangle, the Steklov spectrum determines its side lengths: in other words they are Steklov spectrum invariants. Moreover, it is well-known that some other geometrical quantities of the boundary of a Riemannian manifold surface are Steklov spectrum invariants. Let us mention for instance (see [7, 8] for details and reference therein):

- a) the dimension of the manifold and the volume of its boundary.
- b) When $\dim M \geq 3$, the integral of the mean curvature on ∂M .
- c) When $\dim M = 2$, the number and the lengths of the connected components of ∂M .

1.2 The main result

In this section, we give the main results of the paper. Let $M = [0, 1] \times \mathbb{S}^{n-1}$ be a manifold equipped with the metric

$$g = f(x)(dx^2 + g_{\mathbb{S}}), \quad (4)$$

where \mathbb{S}^{n-1} is the unit sphere in \mathbb{R}^n , $g_{\mathbb{S}}$ is the metric induced by the euclidean metric on \mathbb{R}^n , and where the conformal factor f is a smooth positive function of the variable $x \in [0, 1]$ only. Note that the manifold M has two boundaries $\Gamma_0 = \{0\} \times \mathbb{S}^{n-1}$ and $\Gamma_1 = \{1\} \times \mathbb{S}^{n-1}$. Thus, the DN map will be shown to have the structure of a matrix operator defined on $H^{1/2}(\mathbb{S}^{n-1}) \oplus H^{1/2}(\mathbb{S}^{n-1})$.

In this setting, we want to answer the following question : does the Steklov spectrum determine uniquely the warping function $f(x)$? This question has been answered positively in [5] when $K = (0, 1] \times \mathbb{S}^{n-1}$ equipped with the metric (4). The difference between this case and the one we are studying here is that the boundary of K is connected whereas that of M is made of two connected components. Due to a natural gauge invariance, we emphasize it is hopeless to recover the metric from the spectrum data. Indeed, let $\psi : M \rightarrow M$ be a smooth diffeomorphism. We have ([9]):

$$\Lambda_{\psi^*g}(\lambda) = \varphi^* \circ \Lambda_g(\lambda) \circ \varphi^{*-1},$$

where $\varphi := \psi|_{\partial M}$ and where $\varphi^* : C^\infty(\partial M) \rightarrow C^\infty(\partial M)$ is the application defined by $\varphi^*h := h \circ \varphi$. As a by-product, one has:

$$\sigma(\Lambda_{\psi^*g}(\lambda)) = \sigma(\Lambda_g(\lambda)).$$

Thus, if we can find a diffeomorphism ψ preserving the warped product structure of the manifold M given by (4), we are able to find a counterexample to uniqueness from the knowledge of the Steklov spectrum. For instance, consider the map

$$\psi : (x, y) \in [0, 1] \times \mathbb{S}^{n-1} \mapsto (1 - x, y) \in [0, 1] \times \mathbb{S}^{n-1}.$$

A straightforward computation gives $\psi^*g = f(1 - x)(dx^2 + g_{\mathbb{S}})$. Thus, the above discussion shows that $\Lambda_g(\lambda)$ and $\Lambda_{\psi^*g}(\lambda)$ have the same Steklov spectrum. Now, we can reformulate more precisely our initial question. Let g and \tilde{g} be two Riemannian metrics given by (4) with conformal factor $f(x)$, (resp. $\tilde{f}(x)$). Assume that $\Lambda_g(\lambda)$ and $\Lambda_{\tilde{g}}(\lambda)$ are Steklov isospectral. Then, is it true that :

$$f(x) = \tilde{f}(x) \text{ or } f(x) = \tilde{f}(1-x)?$$

In what follows, we answer positively this question in dimension $n = 2$ with $\lambda \neq 0$, and in dimension $n \geq 3$ for any frequency λ with an additional hypothesis on the metrics on the boundary ∂M .

We choose the sphere \mathbb{S}^{n-1} as the transversal manifold of our cylinder for a purely technical reason. Indeed, we need a precise (in fact an exact) asymptotic of the eigenvalues of the Laplace Beltrami operator $\Delta_{g_{\mathbb{S}}}$ on $(\mathbb{S}^{n-1}, g_{\mathbb{S}})$ in order to get our uniqueness on the conformal factor f . Moreover, it is important to understand that we fix the transversal metric $g_{\mathbb{S}}$ in our inverse problem. Otherwise, it is known that we could find two non isometric transversal metrics such that the associated Riemannian manifolds are isospectral. This would lead to Steklov isospectral cylinders. Similarly, when $\dim(M) \geq 3$, it is known (see [7, 12]) that we can find Steklov isospectral manifolds $M = [0, L] \times K$ and $\tilde{M} = [0, L] \times \tilde{K}$ (with K and \tilde{K} not connected) such that the areas of the connected components of M and \tilde{M} are not the same.

Our main result is the following:

Theorem 1.1. *Let $M = [0, 1] \times \mathbb{S}^{n-1}$ be a smooth Riemannian manifold equipped with the metric*

$$g = f(x)(dx^2 + g_{\mathbb{S}}) \quad , \quad (\text{resp. } \tilde{g} = \tilde{f}(x)(dx^2 + g_{\mathbb{S}})),$$

and let λ be a frequency not belonging to the Dirichlet spectrum of $-\Delta_g$ and $-\Delta_{\tilde{g}}$ on M . Then,

1. *For $n = 2$ and $\lambda \neq 0$,*

$$(\sigma(\Lambda_g(\lambda)) = \sigma(\Lambda_{\tilde{g}}(\lambda))) \Leftrightarrow (f = \tilde{f} \quad \text{or} \quad f = \tilde{f} \circ \eta)$$

where $\eta(x) = 1 - x$ for all $x \in [0, 1]$.

2. *For $n \geq 3$, and if moreover*

$$f, \tilde{f} \in \mathcal{C}_b := \left\{ f \in C^\infty([0, 1]), \left| \frac{f'(k)}{f(k)} \right| \leq \frac{1}{n-2}, k = 0 \text{ and } 1 \right\},$$

$$(\sigma(\Lambda_g(\lambda)) = \sigma(\Lambda_{\tilde{g}}(\lambda))) \Leftrightarrow (f = \tilde{f} \quad \text{or} \quad f = \tilde{f} \circ \eta)$$

Let us explain briefly the outline of the proof. In both cases $n = 2$ or $n \geq 3$, the proof consists in four steps. We emphasize that in the first three steps, we do not use explicitly that the transversal manifold is the unit sphere \mathbb{S}^{n-1} .

Step 1 : we follow the same approach as in [3]. Since the manifold M has the topology of a cylinder and is equipped with a warped product metric, we can use separation of variables and write the solution u of the Dirichlet problem (1) as

$$u(x, y) = \sum_{m=0}^{+\infty} u_m(x) Y_m(y),$$

reducing this problem to a countable family of Sturm-Liouville equations with boundary conditions :

$$\begin{cases} -v_m'' + qv_m = -\mu_m v_m, & \text{on }]0, 1[\\ v_m(0) = f^{\frac{n-2}{4}}(0)\psi_m^0, \quad v_m(1) = f^{\frac{n-2}{4}}(1)\psi_m^1, \end{cases} \quad \forall m \in \mathbb{N} \quad (5)$$

where

$$q = \frac{(f^{\frac{n-2}{4}})''}{f^{\frac{n-2}{4}}} - \lambda f \quad \text{and} \quad v_m = f^{\frac{n-2}{4}} u_m, \quad \forall m \in \mathbb{N}.$$

The sequence (Y_m) refers to an orthonormal basis of eigenvectors of the Laplace-Beltrami operator $-\Delta_{g_S}$ on the unit sphere.

Step 2 : we use the above decomposition to write the DN operator as an infinite matrix which is block diagonal. More precisely, let us introduce the basis $\mathfrak{B} = (\{e_m^1, e_m^2\})_{m \in \mathbb{N}}$, where $e_m^1 = (Y_m, 0)$ and $e_m^2 = (0, Y_m)$. For each $m \in \mathbb{N}$, we denote $\Lambda_g^m(\lambda)$ the restricted operator of $\Lambda_g(\lambda)$ on the subspace spanned by $\{e_m^1, e_m^2\}$. We can write $\Lambda_g(\lambda)$ in the basis \mathfrak{B} as the infinite matrix:

$$[\Lambda_g]_{\mathfrak{B}} = \begin{pmatrix} \Lambda_g^1(\lambda) & 0 & 0 & \cdots \\ 0 & \Lambda_g^2(\lambda) & 0 & \cdots \\ 0 & 0 & \Lambda_g^3(\lambda) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Each $(2, 2)$ matrix $\Lambda_g^m(\lambda)$ has a simple interpretation involving the so-called Weyl-Titchmarsh theory associated to the Sturm-Liouville equation (5). More precisely, if we denote

$$\sigma(-\Delta_{g_S}) = \{0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_m \leq \dots \rightarrow +\infty\},$$

the spectrum of $-\Delta_{g_S}$ and $h := f^{n-2}$, we shall see that :

$$\Lambda_g^m(\lambda) = \begin{pmatrix} -\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{1}{4\sqrt{f(0)}} \frac{h'(0)}{h(0)} & -\frac{1}{\sqrt{f(0)}} \frac{h^{1/4}(1)}{h^{1/4}(0)} \frac{1}{\Delta(\mu_m)} \\ -\frac{1}{\sqrt{f(1)}} \frac{h^{1/4}(0)}{h^{1/4}(1)} \frac{1}{\Delta(\mu_m)} & -\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{1}{4\sqrt{f(1)}} \frac{h'(1)}{h(1)} \end{pmatrix}.$$

where $M(z)$, $N(z)$ are the Weyl-Titchmarsh functions and $\Delta(z)$ is the characteristic function associated to (5). In particular, the trace (respectively the determinant) of these operators $\Lambda_g^m(\lambda)$ are meromorphic functions evaluated in μ_m . Moreover, one can prove that the Steklov Spectrum is made of two subsequences $(\lambda^-(\mu_m))_{m \in \mathbb{N}}$ and $(\lambda^+(\mu_m))_{m \in \mathbb{N}}$ satisfying the following asymptotic expansion

$$\begin{cases} \lambda^-(\mu_m) = -\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{1}{4\sqrt{f(1)}} \frac{h'(1)}{h(1)} + O(\sqrt{\mu_m} e^{-\sqrt{\mu_m}}) \\ \lambda^+(\mu_m) = -\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{1}{4\sqrt{f(0)}} \frac{h'(0)}{h(0)} + O(\sqrt{\mu_m} e^{-\sqrt{\mu_m}}) \end{cases}$$

Step 3 : we prove that the knowledge of the trace and the determinant of all restricted operators $\Lambda_g^m(\lambda)$, for m large enough, characterizes f up to an involution: there is $m_0 \in \mathbb{N}$ such that :

$$\begin{aligned} (\text{Tr } \Lambda_g^m(\lambda) = \text{Tr } \Lambda_{\tilde{g}}^m(\lambda) \text{ and } \det \Lambda_g^m(\lambda) = \det \Lambda_{\tilde{g}}^m(\lambda) \text{ for all } m \geq m_0) \\ \Leftrightarrow (f = \tilde{f} \text{ or } f = \tilde{f} \circ \eta.) \end{aligned}$$

Note that the third step requires to know the asymptotic expansion of the Steklov spectrum.

Step 4 : in this step, we use explicitly that the transversal manifold is the unit sphere \mathbb{S}^{n-1} equipped with the metric induced by the euclidean metric on \mathbb{R}^n . From the equality between the sets $\sigma(\Lambda_g(\lambda)) = (\lambda^\pm(\mu_m))_{m \in \mathbb{N}}$ and $\sigma(\Lambda_{\tilde{g}}(\lambda)) = (\tilde{\lambda}^\pm(\mu_m))_{m \in \mathbb{N}}$, we want to deduce the equalities

$$\begin{cases} \lambda^-(\mu_m) = \tilde{\lambda}^-(\mu_m) \\ \lambda^-(\mu_m) = \tilde{\lambda}^+(\mu_m) \end{cases} \quad \text{or} \quad \begin{cases} \lambda^-(\mu_m) = \tilde{\lambda}^+(\mu_m) \\ \lambda^+(\mu_m) = \tilde{\lambda}^-(\mu_m), \end{cases}$$

for integers m belonging to a set \mathcal{L} satisfying the Müntz conditions $\sum_{m \in \mathcal{L}} \frac{1}{m} = +\infty$. Clearly, for every $\lambda^\pm(\mu_m)$ in $\sigma(\Lambda_g(\lambda))$, there is $\tilde{\lambda}^-(\mu_\ell)$ or $\tilde{\lambda}^+(\mu_\ell)$ in $\sigma(\Lambda_{\tilde{g}}(\lambda))$ such that

$$\lambda^\pm(\mu_m) = \tilde{\lambda}^-(\mu_\ell) \quad \text{or} \quad \lambda^\pm(\mu_m) = \tilde{\lambda}^+(\mu_\ell).$$

The assumption $\left| \frac{f'(k)}{f(k)} \right| \leq \frac{1}{n-2}$ for $k \in \{0, 1\}$ is used here to ensure that $m = \ell$ if m is large enough. This step leads to distinguish the cases $f(0) = f(1)$ and $f(0) \neq f(1)$. Then we prove that $\text{Tr } \Lambda_g^m(\lambda) = \text{Tr } \Lambda_{\tilde{g}}^m(\lambda)$ and $\det \Lambda_g^m(\lambda) = \det \Lambda_{\tilde{g}}^m(\lambda)$ for all m large enough. The result follows from the Step 3.

2 Reduction to ordinary differential equations

In this section, (K, g_K) is an arbitrary closed manifold of dimension $n-1$ and $M = [0, 1] \times K$ is equipped with the metric $g = f(x)(dx^2 + g_K)$.

2.1 The separation of variables

The boundary ∂M of the manifold M has two distinct connected components

$$\Gamma_0 = \{0\} \times K \text{ and } \Gamma_1 = \{1\} \times K,$$

so we can decompose $H^1(\partial M)$ as the direct sum :

$$H^{1/2}(\partial M) = H^{1/2}(\Gamma_0) \oplus H^{1/2}(\Gamma_1).$$

Each element ψ of $H^{1/2}(\partial M)$ can be written as

$$\psi = \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix}, \quad \psi^0 \in H^{1/2}(\Gamma_0) \text{ and } \psi^1 \in H^{1/2}(\Gamma_1).$$

The Laplacian $-\Delta_{g_K}$ is a self-adjoint operator on $L^2(K)$ and has pure point spectrum (μ_m) , with $\mu_0 = 0 < \mu_1 \leq \mu_2 \leq \dots \mu_m \rightarrow +\infty$. We denote $(Y_m)_{m \in \mathbb{N}}$ the associated orthonormal Hilbert basis of the eigenvectors.

Now, we decompose ψ^0 and ψ^1 as:

$$\psi^0 = \sum_{m \in \mathbb{N}} \psi_m^0 Y_m, \quad \psi^1 = \sum_{m \in \mathbb{N}} \psi_m^1 Y_m.$$

Then, we are looking for the unique solution $u(x, y)$ of the Dirichlet problem (1) in the form:

$$u(x, y) = \sum_{m=0}^{+\infty} u_m(x) Y_m(y).$$

Proposition 2.1. *The equation (1) is equivalent to the following countable system of Sturm-Liouville equations :*

$$\begin{cases} -v_m'' + qv_m = -\mu_m v_m, & \text{on }]0, 1[\\ v_m(0) = f^{\frac{n-2}{4}}(0) \psi_m^0, \quad v_m(1) = f^{\frac{n-2}{4}}(1) \psi_m^1, \end{cases} \quad \forall m \in \mathbb{N}, \quad (6)$$

where

$$q = \frac{(f^{\frac{n-2}{4}})''}{f^{\frac{n-2}{4}}} - \lambda f \quad \text{and} \quad v_m = f^{\frac{n-2}{4}} u_m, \quad \forall m \in \mathbb{N}.$$

Proof. The transformation law for the Laplacian operator under conformal change of metric gives

$$-\Delta_g = f^{-\frac{n+2}{4}}(-\Delta_{g_0} + q_f)f^{\frac{n-2}{4}},$$

where $g_0 = dx^2 + g_K$ and $q_f =: \frac{(f^{\frac{n-2}{4}})''}{f^{\frac{n-2}{4}}}$. Thus, we have :

$$\begin{aligned} -\Delta_g u(x, y) = \lambda u(x, y) &\Leftrightarrow f^{-\frac{n+2}{4}} \left(-\frac{\partial^2}{\partial x^2} - \Delta_{g_K} + q_f \right) f^{\frac{n-2}{4}} u(x, y) = \lambda u(x, y) \\ &\Leftrightarrow \left(-\frac{\partial^2}{\partial x^2} - \Delta_{g_K} + q_f \right) v(x, y) = \lambda f(x) v(x, y) \end{aligned}$$

thanks to the change of variable $v = f^{\frac{n-2}{4}} u$. Writing v as

$$v(x, y) = \sum_{m=0}^{+\infty} v_m(x) Y_m(y),$$

we get :

$$\begin{aligned} -\Delta_g u(x, y) = \lambda u(x, y) &\Leftrightarrow \sum_{m=0}^{+\infty} \left(-\frac{\partial^2}{\partial x^2} - \Delta_{g_K} + q_f \right) v_m(x) Y_m = \sum_{m=0}^{+\infty} \lambda f(x) v_m(x) Y_m \\ &\Leftrightarrow \sum_{m=0}^{+\infty} \left(-v_m''(x) + \mu_m v_m(x) + q_f(x) v_m(x) \right) Y_m \\ &= \sum_{m=0}^{+\infty} \lambda f(x) v_m(x) Y_m \\ &\Leftrightarrow \forall m \in \mathbb{N}, -v_m''(x) + q(x) v_m(x) = -\mu_m v_m(x), \end{aligned}$$

where $q =: q_f - \lambda f$. Finally, v_m satisfies the above boundary conditions. \square

It turns out that this family of Sturm-Liouville equations fits into the so-called Weyl-Titchmarsh theory which we recall in the next section for the convenience of the reader.

2.2 The Weyl-Titchmarsh functions

Consider the differential equation:

$$-u'' + qu = -zu, \quad z \in \mathbb{C}. \quad (7)$$

Let $\{c_0, s_0\}$ and $\{c_1, s_1\}$ be the two fundamental systems of solutions of (7) with boundary conditions

$$\begin{cases} c_0(0) = 1, & c_0'(0) = 0 \\ c_1(1) = 1, & c_1'(1) = 0 \end{cases} \quad \text{and} \quad \begin{cases} s_0(0) = 0, & s_0'(0) = 1 \\ s_1(1) = 0, & s_1'(1) = 1. \end{cases} \quad (8)$$

Since the wronskian $W(f, g) = fg' - f'g$ of two solutions f and g of (7) depends only on the parameter z , we can define the following holomorphic functions:

Definition 2.2. The *characteristic function* $\Delta(z)$ of the equation (7) is defined for every $z \in \mathbb{C}$ by :

$$\Delta(z) := W(s_0, s_1) = s_0(1) = -s_1(0).$$

We also set $D(z) := W(c_0, s_1) = c_0(1)$ and $E(z) := W(c_1, s_0) = c_1(0)$.

Now, we recall the three following results, (see for instance [3] and [13] for details).

Proposition 2.3. Set $\Pi^+ = \{z \in \mathbb{C}, \Re(z) > 0\}$ the half-right plane of the complex plane. The functions Δ , D and E are analytic on \mathbb{C} and have on Π^+ the following asymptotics as $|z| \rightarrow +\infty$:

$$\begin{aligned}\Delta(z) &= \frac{\sinh(\sqrt{z})}{\sqrt{z}} + O\left(\frac{e^{|\Re(\sqrt{z})|}}{z}\right), & D(z) &= \cosh(\sqrt{z}) + O\left(\frac{e^{|\Re(\sqrt{z})|}}{\sqrt{z}}\right), \\ E(z) &= \cosh(\sqrt{z}) + O\left(\frac{e^{|\Re(\sqrt{z})|}}{\sqrt{z}}\right).\end{aligned}$$

where \sqrt{z} is the principal square root of z . In particular, $\Delta(z)$, $D(z)$ and $E(z)$ are entire functions of order $\frac{1}{2}$.

Proposition 2.4. The roots (α_j) of the function $z \mapsto \Delta(z)$ are real and simple. they are the opposite of the eigenvalues of the operator $-\frac{d^2}{dx^2} + q =: H$ on $L^2((0,1), dx)$ with Dirichlet boundary conditions.

Proposition 2.5. For all $z \in \mathbb{C}$, $\Delta(z)$ can be written as an infinite product. There is a constant $C \in \mathbb{R}$ such that :

$$\Delta(z) = C \prod_{k=0}^{\infty} \left(1 - \frac{z}{\alpha_k}\right).$$

Proof. As a consequence of Proposition 2.3 and Hadamard's factorization theorem, we can write $\Delta(z)$ as the infinite product :

$$\Delta(z) = Cz^p \prod_{k=0}^{\infty} \left(1 - \frac{z}{\alpha_k}\right),$$

with $p \in \{0,1\}$. In order to prove that $z = 0$ is not a root of Δ , we use the same argument as in [3] (Remark 3.1 p.19). If $\Delta(0) = 0$, it follows from Proposition 2.4, that there is an eigenfunction u_0 associated to the eigenvalue 0 for the operator $H = -\frac{d^2}{dx^2} + q$. But, from Proposition 2.1, the function $u := u_0 Y_0$ is then a nontrivial solution of the Dirichlet problem :

$$\begin{cases} -\Delta_g u = \lambda u & \text{in } M \\ u = 0 & \text{on } \partial M, \end{cases}$$

which is not possible since $\lambda \notin \sigma(-\Delta_g)$. □

Remark 1. As a by-product, we see that $\Delta(z)$ is uniquely determined by its (simple) roots (up to a multiplicative constant).

Now, consider the Weyl-Titchmarsh solutions ψ and ϕ of (7) having the form :

$$\psi(x) = c_0(x) + M(z)s_0(x), \quad \phi(x) = c_1(x) - N(z)s_1(x)$$

and satisfying the Dirichlet boundary condition at $x = 1$ and $x = 0$ respectively. $M(z)$, (resp. $N(z)$) are called the *Weyl-Titchmarsh functions* associated to (7) and, using Wronskian identities, we have:

Proposition 2.6. The Weyl-Titchmarsh functions M and N can be written as :

$$\forall z \in \mathbb{C}, \quad M(z) = -\frac{D(z)}{\Delta(z)}, \quad N(z) = -\frac{E(z)}{\Delta(z)}.$$

Proof. Since $\psi(1) = 0$ and $\phi(0) = 0$, we have :

$$M(z) = -\frac{c_0(1, z)}{s_0(1, z)} = -\frac{D(z)}{\Delta(z)}, \quad N(z) = \frac{c_1(0, z)}{s_1(0, z)} = -\frac{E(z)}{\Delta(z)}.$$

□

The four previous propositions are the key points to solve our uniqueness result. We will see that the Steklov spectrum can be expressed in terms of the Weyl-Titchmarsh and characteristic functions defined above. We will take advantage of the holomorphic properties of $\Delta(z)$, $D(z)$ and $E(z)$ and we will use to the Nevanlinna theorem, (see the next section for details).

2.3 Link between the DN map and the Weyl-Titchmarsh functions

First, we remark that, thanks to separation of variables, $\Lambda_g(\lambda)$ leaves invariant each subspace spanned by $\{(Y_m, 0), (0, Y_m)\}$. Indeed, if u is the solution of (1), we have for each $\psi \in H^{1/2}(\partial M)$:

$$\Lambda_g(\lambda)\psi = \Lambda_g(\lambda) \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} = \begin{pmatrix} (\partial_\nu u)|_{\Gamma_0} \\ (\partial_\nu u)|_{\Gamma_1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{f(0)}}(\partial_x u)|_{x=0} \\ \frac{1}{\sqrt{f(1)}}(\partial_x u)|_{x=1} \end{pmatrix}$$

Consequently, for every $m \in \mathbb{N}$:

$$\Lambda_g(\lambda) \begin{pmatrix} \psi_m^0 \\ \psi_m^1 \end{pmatrix} \otimes Y_m = \begin{pmatrix} -\frac{1}{\sqrt{f(0)}}u'_m(0) \\ \frac{1}{\sqrt{f(1)}}u'_m(1) \end{pmatrix} \otimes Y_m.$$

Its restriction on each space spanned by $(1, 0) \otimes Y_m$ and $(0, 1) \otimes Y_m$ is denoted $\Lambda_g^m(\lambda)$. We can write $\Lambda_g^m(\lambda)$ the 2×2 matrix

Set :

$$\begin{pmatrix} L^m(\lambda) & T_R^m(\lambda) \\ T_L^m(\lambda) & R^m(\lambda) \end{pmatrix}$$

and we have by definition:

$$\begin{pmatrix} L^m(\lambda) & T_R^m(\lambda) \\ T_L^m(\lambda) & R^m(\lambda) \end{pmatrix} \begin{pmatrix} \psi_m^0 \\ \psi_m^1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{f(0)}}u'_m(0) \\ \frac{1}{\sqrt{f(1)}}u'_m(1) \end{pmatrix}.$$

The full Steklov spectrum is then equal to the union of the eigenvalues of each operator $\Lambda_g^m(\lambda)$. In the next Proposition, we express the restricted operator $\Lambda_g^m(\lambda)$ in terms of the Weyl-Titchmarsh functions.

Proposition 2.7. *For all $m \in \mathbb{N}$, we have :*

$$\Lambda_g^m(\lambda) = \begin{pmatrix} -\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{\ln'(h)(0)}{4\sqrt{f(0)}} & -\frac{1}{\sqrt{f(0)}} \frac{h^{1/4}(1)}{h^{1/4}(0)} \frac{1}{\Delta(\mu_m)} \\ -\frac{1}{\sqrt{f(1)}} \frac{h^{1/4}(0)}{h^{1/4}(1)} \frac{1}{\Delta(\mu_m)} & -\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{\ln'(h)(1)}{4\sqrt{f(1)}} \end{pmatrix}$$

where $h =: f^{n-2}$

Proof. Recall that v_m is the m -th Fourier coefficient of $v = h^{1/4}u$ with respect to Y_m . We have :

$$\begin{aligned}\Lambda_g^m(\lambda) \begin{pmatrix} \psi_m^0 \\ \psi_m^1 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{\sqrt{f(0)}} u'_m(0) \\ \frac{1}{\sqrt{f(1)}} u'_m(1) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{v'_m(0)}{h^{1/4}(0)\sqrt{f(0)}} + \frac{1}{4\sqrt{f(0)}} \frac{h'(0)}{h(0)} u_m(0) \\ \frac{v'_m(1)}{h^{1/4}(1)\sqrt{f(1)}} - \frac{1}{4\sqrt{f(1)}} \frac{h'(1)}{h(1)} u_m(1) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{v'_m(0)}{h^{1/4}(0)\sqrt{f(0)}} \\ \frac{v'_m(1)}{h^{1/4}(1)\sqrt{f(1)}} \end{pmatrix} + \begin{pmatrix} \frac{1}{4\sqrt{f(0)}} \frac{h'(0)}{h(0)} \psi_m^0 \\ -\frac{1}{4\sqrt{f(1)}} \frac{h'(1)}{h(1)} \psi_m^1 \end{pmatrix}\end{aligned}$$

It remains to find a simple expression of the first term of the (RHS). Since v_m is a solution of (6), it can be written as a linear combination of the fundamental solutions $\{c_0, s_0\}$ or $\{c_1, s_1\}$ defined in (8), i.e there exists $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$ such that :

$$v_m = \alpha c_0 + \beta s_0 = \gamma c_1 + \delta s_1.$$

Consequently:

$$\begin{pmatrix} v_m(0) \\ v_m(1) \end{pmatrix} = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} \gamma c_1(0) + \delta s_1(0) \\ \alpha c_0(1) + \beta s_0(1) \end{pmatrix}$$

The second equality can be rewritten as:

$$\begin{pmatrix} \alpha - \gamma c_1(0) \\ \gamma - \alpha c_0(1) \end{pmatrix} = \begin{pmatrix} \delta s_1(0) \\ \beta s_0(1) \end{pmatrix},$$

which is equivalent to :

$$\begin{pmatrix} 1 & -c_1(0) \\ -c_0(1) & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} \delta s_1(0) \\ \beta s_0(1) \end{pmatrix}.$$

As

$$\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} h^{1/4}(0)u_m(0) \\ h^{1/4}(1)u_m(1) \end{pmatrix} = \begin{pmatrix} h^{1/4}(0)\psi_m^0 \\ h^{1/4}(1)\psi_m^1 \end{pmatrix}$$

we obtain:

$$\begin{pmatrix} \frac{h^{1/4}(0)}{s_1(0)} & -\frac{h^{1/4}(1)c_1(0)}{s_1(0)} \\ -\frac{h^{1/4}(0)c_0(1)}{s_0(1)} & \frac{h^{1/4}(1)}{s_0(1)} \end{pmatrix} \begin{pmatrix} \psi_m^0 \\ \psi_m^1 \end{pmatrix} = \begin{pmatrix} \delta \\ \beta \end{pmatrix}.$$

But $\delta = v'_m(1)$ and $\beta = v'_m(0)$. Thus:

$$\begin{aligned}
\Lambda_g^m(\lambda) \begin{pmatrix} \psi_m^0 \\ \psi_m^1 \end{pmatrix} &= \begin{pmatrix} -\frac{v'_m(0)}{h^{1/4}(0)\sqrt{f(0)}} \\ \frac{v'_m(1)}{h^{1/4}(1)\sqrt{f(1)}} \end{pmatrix} + \begin{pmatrix} \frac{1}{4\sqrt{f(0)}} \frac{h'(0)}{h(0)} \psi_m^0 \\ -\frac{1}{4\sqrt{f(1)}} \frac{h'(1)}{h(1)} \psi_m^1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{f(0)}} \frac{c_0(1)}{s_0(1)} & -\frac{1}{\sqrt{f(0)}} \frac{h^{1/4}(1)}{h^{1/4}(0)s_0(1)} \\ \frac{1}{\sqrt{f(1)}} \frac{h^{1/4}(0)}{h^{1/4}(1)s_1(0)} & -\frac{1}{\sqrt{f(1)}} \frac{c_1(0)}{s_1(0)} \end{pmatrix} \begin{pmatrix} \psi_m^0 \\ \psi_m^1 \end{pmatrix} + \begin{pmatrix} \frac{1}{4\sqrt{f(0)}} \frac{h'(0)}{h(0)} \psi_m^0 \\ -\frac{1}{4\sqrt{f(1)}} \frac{h'(1)}{h(1)} \psi_m^1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{f(0)}} \frac{c_0(1)}{s_0(1)} + \frac{1}{4\sqrt{f(0)}} \frac{h'(0)}{h(0)} & -\frac{1}{\sqrt{f(0)}} \frac{h^{1/4}(1)}{h^{1/4}(0)s_0(1)} \\ \frac{1}{\sqrt{f(1)}} \frac{h^{1/4}(0)}{h^{1/4}(1)s_1(0)} & -\frac{1}{\sqrt{f(1)}} \frac{c_1(0)}{s_1(0)} - \frac{1}{4\sqrt{f(1)}} \frac{h'(1)}{h(1)} \end{pmatrix} \begin{pmatrix} \psi_m^0 \\ \psi_m^1 \end{pmatrix}
\end{aligned}$$

Recalling that:

$$M(\mu_m) = -\frac{c_0(1)}{s_0(1)}, \quad N(\mu_m) = \frac{c_1(0)}{s_1(0)} \quad \text{et} \quad \Delta(\mu_m) = -s_1(0) = s_0(1),$$

we get finally:

$$\Lambda_g^m(\lambda) = \begin{pmatrix} -\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{1}{4\sqrt{f(0)}} \frac{h'(0)}{h(0)} & -\frac{1}{\sqrt{f(0)}} \frac{h^{1/4}(1)}{h^{1/4}(0)} \frac{1}{\Delta(\mu_m)} \\ -\frac{1}{\sqrt{f(1)}} \frac{h^{1/4}(0)}{h^{1/4}(1)} \frac{1}{\Delta(\mu_m)} & -\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{1}{4\sqrt{f(1)}} \frac{h'(1)}{h(1)} \end{pmatrix}.$$

□

3 A characterisation by the trace and the determinant

In this section, (K, g_K) is still an arbitrary closed manifold of dimension $n-1$ and $M = [0, 1] \times K$ is equipped with the metric $g = f(x)(dx^2 + g_K)$. We have the following result:

Proposition 3.1. *Assume that, for every $m \in \mathbb{N}$, we have :*

$$\det(\Lambda_g^m(\lambda)) = \det(\Lambda_{\tilde{g}}^m(\lambda)) \quad \text{and} \quad \text{Tr}(\Lambda_g^m(\lambda)) = \text{Tr}(\Lambda_{\tilde{g}}^m(\lambda)).$$

Then :

$$f = \tilde{f} \quad \text{or} \quad f = \tilde{f} \circ \eta$$

where, for all $x \in [0, 1]$, $\eta(x) = 1 - x$.

Remark 2. This Proposition is still true if the equalities about the trace and the determinant of $\Lambda_g^m(\lambda)$ are satisfied for $m \geq m_0$, with $m_0 \in \mathbb{N}$.

In order to prove this proposition, let us calculate the eigenvalues of the operator $\Lambda_g^m(\lambda)$ and their asymptotics.

Lemma 3.2. $\Lambda_g^m(\lambda)$ has two eigenvalues $\lambda^-(\mu_m)$ and $\lambda^+(\mu_m)$ whose asymptotics are given by :

$$\begin{cases} \lambda^-(\mu_m) \underset{m \rightarrow \infty}{=} \frac{\sqrt{\mu_m}}{\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right) \\ \lambda^+(\mu_m) \underset{m \rightarrow \infty}{=} \frac{\sqrt{\mu_m}}{\sqrt{f(0)}} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right). \end{cases}$$

Remark 3. If $f(0) < f(1)$, we have $\lambda^-(\mu_m) < \lambda^+(\mu_m)$. This assumption will be made, without loss of generality, each time that $f(0) \neq f(1)$. The notations $\lambda^-(\mu_m)$ and $\lambda^+(\mu_m)$ refer to that choice.

Remark 4. The Weyl's law for the eigenvalues of the Laplace-Beltrami operator gives the following asymptotic :

$$\mu_m = 4\pi^2 \left(\text{Vol}(\mathbb{B}^{n-1}) \text{Vol}(K) \right)^{-\frac{2}{n-1}} m^{\frac{2}{n-1}} + O(1),$$

so, by replacing it in Lemma 3.2, one has :

$$\begin{aligned} \lambda^-(\mu_m) &\underset{m \rightarrow \infty}{=} \frac{\sqrt{\mu_m}}{\sqrt{f(1)}} + O(1) \\ &= \frac{2\pi}{\sqrt{f(1)}} \left(\text{Vol}(\mathbb{B}^{n-1}) \text{Vol}(K) \right)^{-\frac{1}{n-1}} m^{\frac{1}{n-1}} + O(1). \end{aligned}$$

The boundary component Γ_1 consists in copy of K equipped with the metric $\gamma_1 = f(1)g_K$. It follows that we have $\text{Vol}(\Gamma_1) = \int_{\Gamma_1} d\text{Vol}_{\gamma_1} = f(1)^{\frac{n-1}{2}} \text{Vol}(K)$, hence $\frac{1}{\sqrt{f(1)}} = \frac{\text{Vol}(K)^{\frac{1}{n-1}}}{\text{Vol}(\Gamma_1)^{\frac{1}{n-1}}}$. Consequently :

$$\lambda^-(\mu_m) = 2\pi \left(\frac{m}{\text{Vol}(\mathbb{B}^{n-1}) \text{Vol}(\Gamma_1)} \right)^{\frac{1}{n-1}} + O(1).$$

In other words, an asymptotic of $\lambda^-(\mu_m)$ is exactly given by the Weyl's law restricted to the connected component boundary Γ_1 . In the same way, one can prove :

$$\lambda^+(\mu_m) = 2\pi \left(\frac{m}{\text{Vol}(\mathbb{B}^{n-1}) \text{Vol}(\Gamma_0)} \right)^{\frac{1}{n-1}} + O(1).$$

We recognize again the Weyl's law, restricted to the connected component boundary Γ_0 .

Remark 5. The equalities proved in Lemma 3.2 highlight the link that exists between the Steklov spectrum and the spectrum of $-\Delta_{g_S}$. Let us denote the eigenvalues of the Laplace-Beltrami operator on $(\mathbb{S}, f(0)g_S)$ by

$$\mu_0^{(0)} \leq \mu_1^{(0)} \leq \mu_2^{(0)} \leq \dots \rightarrow +\infty$$

with $\mu_m^{(0)} = \frac{\mu_m}{f(0)}$ for $m \in \mathbb{N}$. Lemma 3.2 implies in particular that there is a constant $C_f^{(0)} > 0$ only depending on the conformal factor f at $x = 0$ such that

$$|\lambda^+(\mu_m) - \sqrt{\mu_m^{(0)}}| \leq C_f^{(0)}$$

This can be related to results obtained in [14] (Theorem 1.7 p.2) where it is proved that, for a bounded domain $\partial\Omega \subset \mathbb{R}^n$ with boundary of class C^2 which has only one boundary component, there is a bound $C_\Omega > 0$ depending on Ω such that

$$|\lambda_m - \sqrt{\mu_m}| \leq C_\Omega \quad \forall m \in \mathbb{N}.$$

where λ_m and μ_m are respectively the m^{th} eigenvalue of the DN map and the the m^{th} eigenvalue of the Laplace-Beltrami operator on the boundary. One can notice that, in our case, $C_f^{(0)}$ only depends on the metric on the boundary component $(\mathbb{S}^{n-1}, f(0)g_S)$. This fact can be compared to a recent result (see [2]) where it is proved that the previous bound C_Ω can be chosen uniformly with respect to a class of manifolds \mathcal{M} satisfying some geometrical conditions only in a neighborhood of the boundary (Theorem 3, p.3).

In the same way, Lemma 3.2 implies also the existence of $C_f^{(1)} > 0$ only depending on the conformal factor f at $x = 1$ such that

$$|\lambda^-(\mu_m) - \sqrt{\mu_m^{(1)}}| \leq C_f^{(1)}$$

where $\mu_m^{(1)} = \frac{\mu_m}{f(1)}$ is the m^{th} eigenvalue of the Laplace-Beltrami operator on $(\mathbb{S}, f(1)g_S)$.

Let us prove Lemma 3.2.

Proof. We distinguish two cases :

- Assume $f(0) \neq f(1)$ (for instance $f(0) < f(1)$).

The characteristic polynomial $P(X)$ of $\Lambda_g^m(\lambda)$ is :

$$P(X) = X^2 - \text{Tr}(\Lambda_g^m(\lambda))X + \det(\Lambda_g^m(\lambda)).$$

To simplify the notation, we set :

$$C_0 = \frac{\ln(h)'(0)}{4\sqrt{f(0)}}, \quad C_1 = \frac{\ln(h)'(1)}{4\sqrt{f(1)}}.$$

Thanks to Propositions 2.3 and 2.7, for m large enough, $\text{Tr}(\Lambda_g^m(\lambda))$ and $\det(\Lambda_g^m(\lambda))$ satisfy:

$$\begin{cases} \text{Tr}(\Lambda_g^m(\lambda)) = -\frac{M(\mu_m)}{\sqrt{f(0)}} - \frac{N(\mu_m)}{\sqrt{f(1)}} + C_0 - C_1. \\ \det(\Lambda_g^m) = \left(-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0\right) \left(-\frac{N(\mu_m)}{\sqrt{f(1)}} - C_1\right) + O(\mu_m e^{-2\sqrt{\mu_m}}). \end{cases}$$

The asymptotics of the discriminant δ of $P(X)$ depending on $M(\mu_m)$ and $N(\mu_m)$ can thus be written :

$$\begin{aligned} \delta &= \left(-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 - \frac{N(\mu_m)}{\sqrt{f(1)}} - C_1\right)^2 - 4\left(-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0\right) \left(-\frac{N(\mu_m)}{\sqrt{f(1)}} - C_1\right) \\ &\quad + O(\mu_m e^{-2\sqrt{\mu_m}}). \\ &= \left(-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 + \frac{N(\mu_m)}{\sqrt{f(1)}} + C_1\right)^2 + O(\mu_m e^{-2\sqrt{\mu_m}}). \end{aligned}$$

Now, let us recall the result obtained by Simon in [19] :

Theorem 3.3. $M(z^2)$ has the following asymptotic expansion :

$$\forall A \in \mathbb{N}, \quad -M(z^2) \underset{z \rightarrow \infty}{=} z + \sum_{j=0}^A \frac{\beta_j(0)}{z^{j+1}} + o\left(\frac{1}{z^{A+1}}\right)$$

$$\text{where, for every } x \in [0, 1], \beta_j(x) \text{ is defined by : } \begin{cases} \beta_0(x) = \frac{1}{2}q(x) \\ \beta_{j+1}(x) = \frac{1}{2}\beta_j'(x) + \frac{1}{2}\sum_{l=0}^j \beta_l(x)\beta_{j-l}(x). \end{cases}$$

Of course, by symmetry, one has immediately:

Corollary 3.4. $N(z^2)$ has the following asymptotic expansion :

$$\forall A \in \mathbb{N}, \quad -N(z^2) \underset{z \rightarrow \infty}{=} z + \sum_{j=0}^A \frac{\gamma_j(0)}{z^{j+1}} + o\left(\frac{1}{z^{A+1}}\right)$$

$$\text{where, for all } x \in [0, 1], \gamma_j(x) \text{ is defined by : } \begin{cases} \gamma_0(x) = \frac{1}{2}q(1-x) \\ \gamma_{j+1}(x) = \frac{1}{2}\gamma_j'(x) + \frac{1}{2}\sum_{l=0}^j \gamma_l(x)\gamma_{j-l}(x). \end{cases}$$

We deduce from Theorem 3.3 and Corollary 3.4:

$$-\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{N(\mu_m)}{\sqrt{f(1)}} = \underbrace{\left(\frac{1}{\sqrt{f(0)}} - \frac{1}{\sqrt{f(1)}} \right)}_{>0} \sqrt{\mu_m} + O\left(\frac{1}{\sqrt{\mu_m}}\right).$$

Thus, recalling that

$$\delta = \left(-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 + \frac{N(\mu_m)}{\sqrt{f(1)}} + C_1 \right)^2 + O(\mu_m e^{-2\sqrt{\mu_m}}).$$

we obtain:

$$\begin{aligned} \sqrt{\delta} &= \left(\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 + C_1 \right) \sqrt{1 + O\left(e^{-2\sqrt{\mu_m}}\right)} \\ &= \frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 + C_1 + O(\sqrt{\mu_m} e^{-2\sqrt{\mu_m}}). \end{aligned}$$

Hence :

$$\begin{cases} \lambda^-(\mu_m) = \frac{1}{2} \left[\left(-\frac{M(\mu_m)}{\sqrt{f(0)}} - \frac{N(\mu_m)}{\sqrt{f(1)}} + C_0 - C_1 \right) - \sqrt{\delta} \right] \\ \lambda^+(\mu_m) = \frac{1}{2} \left[\left(-\frac{M(\mu_m)}{\sqrt{f(0)}} - \frac{N(\mu_m)}{\sqrt{f(1)}} + C_0 - C_1 \right) + \sqrt{\delta} \right], \end{cases}$$

and therefore, substituting C_1 and C_2 by their values and $M(\mu_m)$ and $N(\mu_m)$ by their asymptotics, we get :

$$\begin{cases} \lambda^-(\mu_m) = \frac{\sqrt{\mu_m}}{\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right) \\ \lambda^+(\mu_m) = \frac{\sqrt{\mu_m}}{\sqrt{f(0)}} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right). \end{cases}$$

- Assume now $f(0) = f(1)$. In this case, the restricted DN map

$$\Lambda_g^m(\lambda) = \begin{pmatrix} -\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}} & -\frac{1}{\sqrt{f(0)}} \frac{1}{\Delta(\mu_m)} \\ -\frac{1}{\sqrt{f(0)}} \frac{1}{\Delta(\mu_m)} & -\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} \end{pmatrix}$$

is a symmetric matrix and we can use the well-known result:

Lemma 3.5. *Let H be a Hilbert space, $A \in \mathcal{L}(H)$ be a selfadjoint operator. Let $\epsilon > 0$. Assume there exists $\lambda_0 \in \mathbb{R}$ and $u_0 \in H$ a unit vector such that $\|(A - \lambda_0 Id)u_0\| \leq \epsilon$. Then there exists an element λ in the spectrum of A such that $|\lambda - \lambda_0| \leq \epsilon$.*

We apply this theorem with $A = \Lambda_g^m(\lambda)$, $\lambda_0 = -\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}}$ and $U_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

We have :

$$A - \lambda_0 I_2 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{f(0)}} \frac{1}{\Delta(\mu_m)} \\ -\frac{1}{\sqrt{f(0)}} \frac{1}{\Delta(\mu_m)} & -\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} - \lambda_0 \end{pmatrix}$$

Hence :

$$(A - \lambda_0 I_2)U_0 = -\frac{1}{\sqrt{f(0)}} \begin{pmatrix} 0 \\ \frac{1}{\Delta(\mu_m)} \end{pmatrix} = O(\sqrt{\mu_m} e^{-\sqrt{\mu_m}}).$$

Lemma 3.5 gives $\lambda_m^+ \in \sigma(\Lambda_g^m(\lambda))$ such that :

$$\begin{aligned} \lambda_m^+ &= -\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}} + O(\sqrt{\mu_m} e^{-\sqrt{\mu_m}}) \\ &= \frac{\sqrt{\mu_m}}{\sqrt{f(0)}} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right) \end{aligned}$$

from Theorem 3.3. The second eigenvalue λ_m^2 (which is distinct from λ_m^1 otherwise $\Lambda_g^m(\lambda)$ would be a homothety) can be deduced from the first one :

$$\begin{aligned} \lambda_m^- &= \text{Tr}(\Lambda_g^m(\lambda)) - \lambda_m^+ \\ &= -\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} + O(\sqrt{\mu_m} e^{-\sqrt{\mu_m}}) \\ &= \frac{\sqrt{\mu_m}}{\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right). \end{aligned}$$

□

Lemma 3.6. *Under the hypothesis of Proposition 3.1, we have the following alternative :*

$$\begin{cases} f(0) = \tilde{f}(0) \\ f(1) = \tilde{f}(1) \end{cases} \quad \text{or} \quad \begin{cases} f(0) = \tilde{f}(1) \\ f(1) = \tilde{f}(0). \end{cases}$$

Proof. We begin with:

$$\frac{\text{Tr}(\Lambda_g^m(\lambda))}{\sqrt{\mu_m}} = \frac{\text{Tr}(\Lambda_g^m(\lambda))}{\sqrt{\mu_m}}, \quad \forall m \in \mathbb{N}.$$

Thus, it follows from Lemma 3.2 that

$$\frac{1}{\sqrt{f(0)}} + \frac{1}{\sqrt{f(1)}} = \frac{1}{\sqrt{\tilde{f}(0)}} + \frac{1}{\sqrt{\tilde{f}(1)}}.$$

In the same way, thanks to the relations:

$$\frac{\det(\Lambda_g^m(\lambda))}{\mu_m} = \frac{\det(\Lambda_g^m(\lambda))}{\mu_m} \quad \forall m \in \mathbb{N},$$

we get :

$$\frac{1}{\sqrt{f(0)f(1)}} = \frac{1}{\sqrt{\tilde{f}(0)\tilde{f}(1)}},$$

and the proof is complete. □

Case 1 : $f(0) = \tilde{f}(0)$ et $f(1) = \tilde{f}(1)$.

Lemma 3.7. *Under the hypotheses of Proposition 3.1, we have:*

$$\Delta(z) = \tilde{\Delta}(z), \quad \forall z \in \mathbb{C}. \quad (9)$$

Proof. We write the equality $\text{Tr}(\Lambda_g^m(\lambda)) = \text{Tr}(\Lambda_{\tilde{g}}^m(\lambda))$ as follows : for all $\mu_m \in \sigma(-\Delta_{g_K})$,

$$\begin{aligned} \left(-\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}} \right) + \left(-\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} \right) &= \left(-\frac{\tilde{M}(\mu_m)}{\sqrt{f(0)}} + \frac{\ln(\tilde{h})'(0)}{4\sqrt{f(0)}} \right) \\ &+ \left(-\frac{\tilde{N}(\mu_m)}{\sqrt{f(1)}} - \frac{\ln(\tilde{h})'(1)}{4\sqrt{f(1)}} \right). \end{aligned}$$

Thus, using Theorem 3.3, one gets when $m \rightarrow +\infty$,

$$\frac{\ln(h)'(0)}{4\sqrt{f(0)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} = \frac{\ln(\tilde{h})'(0)}{4\sqrt{f(0)}} - \frac{\ln(\tilde{h})'(1)}{4\sqrt{f(1)}}. \quad (10)$$

It follows that

$$\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{N(\mu_m)}{\sqrt{f(1)}} = \frac{\tilde{M}(\mu_m)}{\sqrt{f(0)}} + \frac{\tilde{N}(\mu_m)}{\sqrt{f(1)}},$$

or equivalently

$$\tilde{\Delta}(\mu_m) \left(\frac{D(\mu_m)}{\sqrt{f(0)}} + \frac{E(\mu_m)}{\sqrt{f(1)}} \right) = \Delta(\mu_m) \left(\frac{\tilde{D}(\mu_m)}{\sqrt{f(0)}} + \frac{\tilde{E}(\mu_m)}{\sqrt{f(1)}} \right). \quad (11)$$

In the same way, using $\det(\Lambda_g^m(\lambda)) = \det(\Lambda_{\tilde{g}}^m(\lambda))$, we have:

$$\begin{aligned} \left(\frac{M(\mu_m)}{\sqrt{f(0)}} - \frac{\ln'(h)(0)}{4\sqrt{f(0)}} \right) \left(\frac{N(\mu_m)}{\sqrt{f(1)}} + \frac{\ln'(h)(1)}{4\sqrt{f(1)}} \right) - \frac{1}{\sqrt{f(0)f(1)}\Delta^2(\mu_m)} \\ = \left(\frac{\tilde{M}(\mu_m)}{\sqrt{f(0)}} - \frac{\ln'(\tilde{h})(0)}{4\sqrt{f(0)}} \right) \times \left(\frac{\tilde{N}(\mu_m)}{\sqrt{f(1)}} + \frac{\ln'(\tilde{h})(1)}{4\sqrt{f(1)}} \right) - \frac{1}{\sqrt{f(0)f(1)}\tilde{\Delta}^2(\mu_m)}. \end{aligned}$$

Multiplying on both sides by $\Delta^2(\mu_m)\tilde{\Delta}^2(\mu_m)$, we obtain:

$$\begin{aligned} \tilde{\Delta}^2(\mu_m) \left[\left(-\frac{D(\mu_m)}{\sqrt{f(0)}} - \Delta(\mu_m) \frac{\ln'(h)(0)}{4\sqrt{f(0)}} \right) \left(-\frac{E(\mu_m)}{\sqrt{f(1)}} + \Delta(\mu_m) \frac{\ln'(h)(1)}{4\sqrt{f(1)}} \right) - \frac{1}{\sqrt{f(0)f(1)}} \right] \\ = \Delta^2(\mu_m) \left[\left(-\frac{\tilde{D}(\mu_m)}{\sqrt{f(0)}} - \tilde{\Delta}(\mu_m) \frac{\ln'(\tilde{h})(0)}{4\sqrt{f(0)}} \right) \left(-\frac{\tilde{E}(\mu_m)}{\sqrt{f(1)}} + \tilde{\Delta}(\mu_m) \frac{\ln'(\tilde{h})(1)}{4\sqrt{f(1)}} \right) - \frac{1}{\sqrt{f(0)f(1)}} \right] \end{aligned} \quad (12)$$

Now, let us show that the equalities (11) and (12) can be analytically extended with respect to μ_m in the half-right complex plane. First, let us recall the definition of the so-called Nevanlinna class:

Definition 3.8. Set $\Pi^+ = \{z \in \mathbb{C}, \Re(z) > 0\}$ the half-right plane of the complex plane. The Nevanlinna class $\mathcal{N}(\Pi^+)$ is the set of analytic functions f on Π^+ such that

$$\sup_{0 < r < 1} \int_{-\pi}^{\pi} \ln^+ \left| f \left(\frac{1 - re^{i\theta}}{1 + re^{i\theta}} \right) \right| d\theta < \infty,$$

with:

$$\ln^+(x) = \begin{cases} \ln x & \text{if } \ln(x) \geq 0 \\ 0 & \text{if } \ln(x) < 0. \end{cases}$$

We have the following result [15]:

Proposition 3.9. Let $h \in H(\Pi^+)$ an analytic function on Π^+ , A and C two constants. Assume :

$$|h(z)| \leq Ce^{A\Re(z)}, \quad \forall z \in \Pi^+.$$

Then $h \in \mathcal{N}(\Pi^+)$.

Thus, thanks to the asymptotics of Proposition 2.3, the holomorphic functions defined by $\delta : z \mapsto \Delta(z^2)$, $d : z \mapsto D(z^2)$ and $e : z \mapsto E(z^2)$ belong to $\mathcal{N}(\Pi^+)$. Let us recall now a useful uniqueness theorem involving functions in the Nevanlinna class (see [15] for instance):

Theorem 3.10. *Let $h \in \mathcal{N}(\Pi^+)$ and $\mathcal{L} \subset \mathbb{R}_+$ be a countable set such that*

$$\sum_{\mu \in \mathcal{L}} \frac{1}{\mu} = +\infty.$$

Then :

$$(h(\mu) = 0, \forall \mu \in \mathcal{L}) \Rightarrow h \equiv 0 \text{ on } \Pi^+.$$

Now, by the Weyl law, (cf [16]), we get:

$$\mu_m \underset{m \rightarrow +\infty}{=} c(K)m^{\frac{2}{n-1}} + O(1)$$

where $c(K) = \frac{(2\pi)^2}{(\omega_1 \text{vol}(K))^{\frac{2}{n-1}}}$ and ω_1 is the volume of the unit ball in \mathbb{R}^{n-1} . Thus, for a fixed $T \in \mathbb{N}$, we have :

$$\frac{\mu_{(mT)^{n-1}}}{c(K)T^2} = m^2 + \frac{O(1)}{c(K)T^2}.$$

As a consequence, for m and T large enough, the real numbers $\mu_{(mT)^{n-1}}$ are always distinct. Now, we set :

$$\mathcal{L} = \{\sqrt{\mu_{(mT)^{n-1}}}, m \in \mathbb{N}\}.$$

Using $\sqrt{\mu_{(mT)^{n-1}}} \underset{m \rightarrow +\infty}{\sim} \sqrt{c(K)T}m$, one has:

$$\sum_{\mu \in \mathcal{L}} \frac{1}{\mu} = +\infty.$$

Thus, thanks to Theorem 3.10, the relations (11) and (12) are still true if one replaces μ_m by $z^2 \in \mathbb{C}$, then by z .

Now, let us prove that $\Delta(z) = \tilde{\Delta}(z)$ for any $z \in \mathbb{C}$. We recall that these functions are entire of order $\frac{1}{2}$ and their roots are simple. So, using Hadamard's factorization theorem (see Proposition 2.5), we deduce that these functions are entirely described by their roots (up to a multiplicative constant). Consequently, in order to prove that $\Delta = \tilde{\Delta}$, it is enough to show that their roots are the same.

Set $\mathcal{P} = \{\alpha_j \in \mathbb{C}, \Delta(\alpha_j) = 0\}$. Let α_k be in \mathcal{P} and let us show that $\tilde{\Delta}(\alpha_k) = 0$. By definition, $-\alpha_k$ is an eigenvalue of the Sturm Liouville operator $H = -\frac{d^2}{dx^2} + q$ with Dirichlet boundary conditions. Thus, from Proposition 2.4, α_k is real and since the potential q is real, the quantities $D(\alpha_k)$ and $E(\alpha_k)$ are also real. Using (11) and (12) with μ_m replaced with α_k , we obtain the following system:

$$\begin{cases} \left(\frac{D(\alpha_k)}{\sqrt{f(0)}} + \frac{E(\alpha_k)}{\sqrt{f(1)}} \right) \tilde{\Delta}(\alpha_k) = 0 \\ \left(\frac{D(\alpha_k)E(\alpha_k) - 1}{\sqrt{f(0)f(1)}} \right) \tilde{\Delta}(\alpha_k)^2 = 0. \end{cases}$$

To finish the proof, we distinguish two cases:

- If $\frac{D(\alpha_k)}{\sqrt{f(0)}} + \frac{E(\alpha_k)}{\sqrt{f(1)}} \neq 0$ then $\tilde{\Delta}(\alpha_k) = 0$.

- $$\left(\frac{D(\alpha_k)^2}{f(0)} + \frac{1}{\sqrt{f(0)f(1)}}\right)\tilde{\Delta}(\alpha_k)^2 = 0$$

$$\Delta(z), \tilde{\Delta}(z) \sim \frac{\sinh(\sqrt{z})}{\sqrt{z}},$$

As a consequence, we can simplify (12) (with μ_m replaced by z), and we get on \mathbb{C} :

$$\left\{ \begin{aligned} &\left(\frac{M(z)}{\sqrt{f(0)}} - \frac{\ln'(h)(0)}{4\sqrt{f(0)}} \right) + \left(\frac{N(z)}{\sqrt{f(1)}} + \frac{\ln'(h)(1)}{4\sqrt{f(1)}} \right) = \left(\frac{\tilde{M}(z)}{\sqrt{f(0)}} - \frac{\ln(\tilde{h})'(0)}{4\sqrt{f(0)}} \right) \\ &\hspace{20em} + \left(\frac{\tilde{N}(z)}{\sqrt{f(1)}} + \frac{\ln'(\tilde{h})(1)}{4\sqrt{f(1)}} \right) \\ &\left(\frac{M(z)}{\sqrt{f(0)}} - \frac{\ln'(h)(0)}{4\sqrt{f(0)}} \right) \left(\frac{N(z)}{\sqrt{f(1)}} + \frac{\ln'(h)(1)}{4\sqrt{f(1)}} \right) = \left(\frac{\tilde{M}(z)}{\sqrt{f(0)}} - \frac{\ln(\tilde{h})'(0)}{4\sqrt{f(0)}} \right) \\ &\hspace{20em} \times \left(\frac{\tilde{N}(z)}{\sqrt{f(1)}} + \frac{\ln'(\tilde{h})(1)}{4\sqrt{f(1)}} \right). \end{aligned} \right.$$

$$\begin{cases} \frac{M(z)}{\sqrt{f(0)}} - \frac{\ln'(h)(0)}{4\sqrt{f(0)}} = \frac{\tilde{M}(z)}{\sqrt{f(0)}} - \frac{\ln'(\tilde{h})'(0)}{4\sqrt{f(0)}} \\ \frac{N(z)}{\sqrt{f(1)}} + \frac{\ln'(h)(1)}{4\sqrt{f(1)}} = \frac{\tilde{N}(z)}{\sqrt{f(1)}} + \frac{\ln'(\tilde{h})(1)}{4\sqrt{f(1)}}, \end{cases}$$
$$\begin{cases} \frac{M(z)}{\sqrt{f(0)}} - \frac{\ln'(h)(0)}{4\sqrt{f(0)}} = \frac{\tilde{N}(z)}{\sqrt{f(1)}} + \frac{\ln'(\tilde{h})(1)}{4\sqrt{f(1)}} \\ \frac{N(z)}{\sqrt{f(1)}} + \frac{\ln'(h)(1)}{4\sqrt{f(1)}} = \frac{\tilde{M}(z)}{\sqrt{f(0)}} - \frac{\ln(\tilde{h})'(0)}{4\sqrt{f(0)}}. \end{cases}$$

- $$\frac{\ln'(h)(0)}{4\sqrt{f(0)}} = \frac{\ln(\tilde{h})'(0)}{4\sqrt{f(0)}}$$

$$M(z) = \tilde{M}(z) \text{ for all } z \in \mathbb{C} \setminus \mathbb{R}.$$

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$$q = \tilde{q} \quad \text{sur } [0, 1]. \quad (13)$$

Recall that :

$$q = \frac{(h^{1/4})''}{h^{1/4}} - \lambda f$$

where we have set $h = f^{n-2}$ (respectively \tilde{q} and $\tilde{h} = \tilde{f}^{n-2}$). In particular, in the 2 dimensional case, we get immediately $f = \tilde{f}$ since $\lambda \neq 0$. In dimension $n \geq 3$, we see that f and \tilde{f} same verify the same ODE with $f(0) = \tilde{f}(0)$. Moreover, the relation

$$\frac{\ln'(h)(0)}{4\sqrt{f(0)}} = \frac{\ln(\tilde{h})'(0)}{4\sqrt{\tilde{f}(0)}},$$

implies :

$$f'(0) = \tilde{f}'(0).$$

Thus, the Cauchy-Lipschitz's theorem says that $f = \tilde{f}$.

- Secondly, assume there exists $z_0 \in \mathbb{C}$ satisfying :

$$\frac{M(z_0)}{\sqrt{f(0)}} - \frac{\ln'(h)(0)}{4\sqrt{f(0)}} \neq \frac{\tilde{M}(z_0)}{\sqrt{f(0)}} - \frac{\ln(\tilde{h})'(0)}{4\sqrt{\tilde{f}(0)}}.$$

By a standard continuity argument, there is a ball B of center z_0 such that :

$$\frac{M(z)}{\sqrt{f(0)}} - \frac{\ln'(h)(0)}{4\sqrt{f(0)}} \neq \frac{\tilde{M}(z)}{\sqrt{f(0)}} - \frac{\ln(\tilde{h})'(0)}{4\sqrt{\tilde{f}(0)}}, \quad \forall z \in B.$$

Then, necessarily, we have:

$$\frac{M(z)}{\sqrt{f(0)}} - \frac{\ln'(h)(0)}{4\sqrt{f(0)}} = \frac{\tilde{N}(z)}{\sqrt{f(1)}} + \frac{\ln(\tilde{h})'(1)}{4\sqrt{\tilde{f}(1)}}, \quad \forall z \in B. \quad (14)$$

Then, using the analytic continuation principle, the previous equality is true for every $z \in \mathbb{C} \setminus \mathcal{P}$, where \mathcal{P} is the set of roots of $\Delta(z)$. Thanks to the asymptotics of $M(z)$ and $\tilde{N}(z)$, one gets :

$$f(0) = f(1) \quad (= \tilde{f}(1)) \quad \text{and} \quad \frac{\ln'(h)(0)}{4\sqrt{f(0)}} = -\frac{\ln(\tilde{h})'(1)}{4\sqrt{\tilde{f}(1)}}.$$

Hence, simplifying in (14), we obtain

$$M(z) = \tilde{N}(z) \quad \text{for all } z \in \mathbb{C} \setminus \mathbb{R}.$$

By symmetry, \tilde{N} has the same role as \tilde{M} for the potential $x \mapsto \tilde{q}(1-x)$. Now, it follows, from the Borg-Marchenko's theorem that:

$$q(x) = \tilde{q}(1-x) \quad \forall x \in [0, 1],$$

and as previously, one gets:

$$f = \tilde{f} \circ \eta.$$

Case 2 : $f(0) = \tilde{f}(1)$ and $f(1) = \tilde{f}(0)$.

The proof is identical interchanging the roles of M and N .

4 Uniqueness results on the trace and the determinant

In this section, we assume that $(K, g_K) = (\mathbb{S}^{n-1}, g_{\mathbb{S}})$, and our main result is the following:

Proposition 4.1. *Assume that $\sigma(\Lambda_g(\lambda)) = \sigma(\Lambda_{\tilde{g}}(\lambda))$. Then, there is $m_0 \in \mathbb{N}$ such that :*

$$\forall m \in \mathbb{N}, m \geq m_0 \Rightarrow \det(\Lambda_g^m(\lambda)) = \det(\Lambda_{\tilde{g}}^m(\lambda)) \text{ and } \text{Tr}(\Lambda_g^m(\lambda)) = \text{Tr}(\Lambda_{\tilde{g}}^m(\lambda)).$$

Before giving the proof of this proposition, let us begin by the following lemma:

Lemma 4.2. *Under the hypothesis $\sigma(\Lambda_g(\lambda)) = \sigma(\Lambda_{\tilde{g}}(\lambda))$, we have the alternative :*

$$\begin{cases} f(0) = \tilde{f}(0) \\ f(1) = \tilde{f}(1) \end{cases} \quad \text{or} \quad \begin{cases} f(0) = \tilde{f}(1) \\ f(1) = \tilde{f}(0). \end{cases}$$

Proof. First, let us define the set:

$$\Sigma(\Lambda_g(\lambda)) = \{\lambda^\pm(\kappa_m), \quad m \in \mathbb{N}\}$$

where κ_m is the m -th eigenvalue of the usual Laplace-Beltrami operator on the sphere \mathbb{S}^{n-1} and counted without multiplicity. Thanks to our hypothesis, one has obviously

$$\Sigma(\Lambda_g(\lambda)) = \Sigma(\Lambda_{\tilde{g}}(\lambda)).$$

When $K = \mathbb{S}^{n-1}$, we have an explicit formula for κ_m (see for instance [18]) :

$$\kappa_m = m(m + n - 2), \quad \forall m \in \mathbb{N}.$$

The proof involves two steps.

Step 1. First we have :

$$f(0)^{\frac{n-1}{2}} + f(1)^{\frac{n-1}{2}} = \tilde{f}(0)^{\frac{n-1}{2}} + \tilde{f}(1)^{\frac{n-1}{2}}. \quad (15)$$

Indeed, it is known that the Steklov spectrum determines the volume of the boundary of M : this is an immediate consequence of the Weyl's law (3) for Steklov eigenvalues (see [8]). We have $\partial M = \Gamma_0 \cup \Gamma_1$ where, for $i \in \{0, 1\}$, Γ_i is the sphere \mathbb{S}^{n-1} equipped with the metric $\gamma_i = f(i)g_{\mathbb{S}}$. Hence :

$$\begin{aligned} \text{Vol}(\partial M) = \text{Vol}(\partial \tilde{M}) &\Leftrightarrow \int_{\Gamma_0} d\text{Vol}_{\gamma_0} + \int_{\Gamma_1} d\text{Vol}_{\gamma_1} = \int_{\Gamma_0} d\text{Vol}_{\tilde{\gamma}_0} + \int_{\Gamma_1} d\text{Vol}_{\tilde{\gamma}_1} \\ &\Leftrightarrow \left(f(0)^{\frac{n-1}{2}} + f(1)^{\frac{n-1}{2}} \right) \text{Vol}(\mathbb{S}^{n-1}) = \left(\tilde{f}(0)^{\frac{n-1}{2}} + \tilde{f}(1)^{\frac{n-1}{2}} \right) \text{Vol}(\mathbb{S}^{n-1}), \end{aligned}$$

and this proves the claim.

Step 2. We show that : $f(0) \in \{\tilde{f}(0), \tilde{f}(1)\}$.

Assume this statement is false. Without loss of generality, assume that $f(0) < \min\{\tilde{f}(0), \tilde{f}(1)\}$. Then the equality (15) implies : $f(1) > \max\{\tilde{f}(0), \tilde{f}(1)\}$.

Our strategy is the following : we prove that one of the elements of $\Sigma(\Lambda_g(\lambda))$ is not in $\Sigma(\Lambda_{\tilde{g}}(\lambda))$ and this shall give a contradiction.

Let ε and A two positive numbers. Set :

$$\alpha_A = \inf\{|\lambda^+(\kappa_n) - \lambda^-(\kappa_m)| \mid m, n \geq A\}.$$

We claim that $\alpha_A \leq \frac{1}{2\sqrt{f(1)}} + \varepsilon$ for a sufficiently large A .

Indeed, let $n \in \mathbb{N}$ and set

$$m = \max \left\{ j \in \mathbb{N} \mid \lambda^-(\kappa_j) < \lambda^+(\kappa_n) - \frac{1}{2\sqrt{f(1)}} - \frac{\varepsilon}{2} \right\}$$

By definition of m , $\lambda^-(\kappa_{m+1}) \geq \lambda^+(\kappa_n) - \frac{1}{2\sqrt{f(1)}} - \varepsilon$. But :

$$\begin{aligned} \lambda^-(\kappa_{m+1}) &= \lambda^-(\kappa_m) + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\ &< \lambda^+(\kappa_n) - \frac{1}{2\sqrt{f(1)}} - \frac{\varepsilon}{2} + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\ &< \lambda^+(\kappa_n) + \frac{1}{2\sqrt{f(1)}} + \frac{\varepsilon}{2} + O\left(\frac{1}{m}\right) \end{aligned}$$

Hence

$$\alpha_A \leq |\lambda^+(\kappa_n) - \lambda^-(\kappa_{m+1})| \leq \frac{1}{2\sqrt{f(1)}} + \varepsilon, \quad \text{for } m, n \geq A \text{ sufficiently large.}$$

From the above, we can now choose $(m, n) \in \mathbb{N}^2$ such that :

$$-\frac{1}{2\sqrt{f(1)}} - \varepsilon \leq \lambda^+(\kappa_n) - \lambda^-(\kappa_m) \leq \frac{1}{2\sqrt{f(1)}} + \varepsilon. \quad (16)$$

The equality of the sets $\Sigma(\Lambda_g(\lambda))$ and $\Sigma(\Lambda_{\tilde{g}}(\lambda))$ gives two elements $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of $\Sigma(\Lambda_{\tilde{g}}(\lambda))$ such that $\{\lambda^-(\kappa_m), \lambda^+(\kappa_n)\} = \{\tilde{\lambda}_1, \tilde{\lambda}_2\}$. Remark that if $\lambda^-(\kappa_m) \neq \lambda^+(\kappa_n)$ then the elements $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ can not both belong to the same sequence $(\tilde{\lambda}^-(\kappa_\ell))$ or $(\tilde{\lambda}^+(\kappa_p))$ because, for indexes m and n large enough, one would get :

$$|\tilde{\lambda}_1 - \tilde{\lambda}_2| \geq \min \left\{ \frac{1}{\sqrt{\tilde{f}(0)}}, \frac{1}{\sqrt{\tilde{f}(1)}} \right\} - \varepsilon$$

and, by choosing ε small enough, this would contradict (16). Consequently, we have the existence of $(\ell, p) \in \mathbb{N}^2$ such that

$$\{\lambda^-(\kappa_m), \lambda^+(\kappa_n)\} = \{\tilde{\lambda}^-(\kappa_\ell), \tilde{\lambda}^+(\kappa_p)\}.$$

Let us assume first that

$$\begin{cases} \lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_\ell) \\ \lambda^+(\kappa_n) = \tilde{\lambda}^+(\kappa_p). \end{cases} \quad (17)$$

* Case 1 : $\lambda^-(\kappa_m) \leq \lambda^+(\kappa_n)$. Then $\lambda^-(\kappa_{m+1})$ is not equal to any element of $\Sigma(\Lambda_{\tilde{g}}(\lambda))$. Indeed we have on one hand :

$$\begin{aligned} \lambda^-(\kappa_{m+1}) &= \lambda^-(\kappa_m) + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\ &= \tilde{\lambda}^-(\kappa_\ell) + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\ &= \tilde{\lambda}^-(\kappa_{\ell+1}) + \underbrace{\left[\frac{1}{\sqrt{f(1)}} - \frac{1}{\sqrt{\tilde{f}(1)}} \right]}_{<0} + O\left(\frac{1}{m}\right) + O\left(\frac{1}{\ell}\right) \end{aligned}$$

Hence, for m and ℓ large enough : $\tilde{\lambda}^-(\kappa_\ell) < \lambda^-(\kappa_{m+1}) < \tilde{\lambda}^-(\kappa_{\ell+1})$. On the other hand :

$$\begin{aligned}
\lambda^-(\kappa_{m+1}) &= \lambda^-(\kappa_m) + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\
&\leq \lambda^+(\kappa_n) + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\
&= \tilde{\lambda}^+(\kappa_p) + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\
&= \tilde{\lambda}^+(\kappa_{p+1}) + \underbrace{\left[\frac{1}{\sqrt{f(1)}} - \frac{1}{\sqrt{\tilde{f}(0)}} \right]}_{<0} + O\left(\frac{1}{m}\right) + O\left(\frac{1}{p}\right).
\end{aligned}$$

Moreover, from (16) and (17) :

$$\begin{aligned}
\lambda^-(\kappa_{m+1}) &= \lambda^-(\kappa_m) + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\
&= \lambda^-(\kappa_m) + \frac{1}{2\sqrt{f(1)}} + \varepsilon + O\left(\frac{1}{m}\right) + \frac{1}{2\sqrt{f(1)}} - \varepsilon \\
&\geq \tilde{\lambda}^+(\kappa_p) + \frac{1}{2\sqrt{f(1)}} - 2\varepsilon
\end{aligned}$$

As a result, for m and p large enough : $\tilde{\lambda}^+(\kappa_p) < \lambda^-(\kappa_{m+1}) < \tilde{\lambda}^+(\kappa_{p+1})$

The sequences $(\tilde{\lambda}^-(\kappa_m))$ and $(\tilde{\lambda}^+(\kappa_m))$ being strictly increasing (at least for m large enough), none of the elements of $\Sigma(\Lambda_{\tilde{g}}(\lambda))$ can be equal to $\lambda^-(\kappa_{m+1})$. This refutes $\Sigma(\Lambda_{\tilde{g}}(\lambda)) = \Sigma(\Lambda_g(\lambda))$.

* Case 2 : $\lambda^-(\kappa_m) > \lambda^+(\kappa_n)$. With similar arguments, one can prove :

$$\max(\tilde{\lambda}^-(\kappa_{\ell-1}), \tilde{\lambda}^+(\kappa_{p-1})) < \lambda^-(\kappa_{m-1}) < \min(\tilde{\lambda}^-(\kappa_{\ell}), \tilde{\lambda}^+(\kappa_p)).$$

Now, if we assume that

$$\begin{cases} \lambda^-(\kappa_m) = \tilde{\lambda}^+(\kappa_p) \\ \lambda^+(\kappa_n) = \tilde{\lambda}^-(\kappa_{\ell}) \end{cases}$$

one can also prove, by interchanging the roles of $\tilde{f}(0)$ and $\tilde{f}(1)$, that $\lambda^-(\kappa_{m+1})$ or $\lambda^-(\kappa_{m-1})$ does not belong to the set $\Sigma(\Lambda_{\tilde{g}}(\lambda))$.

Thus $f(0) \in \{\tilde{f}(0), \tilde{f}(1)\}$. Associated to the equality (15), this gives the wanted conclusion. \square

From now on, we assume that $f(0) = \tilde{f}(0)$ and $f(1) = \tilde{f}(1)$, since the case $f(0) = \tilde{f}(1)$ and $f(1) = \tilde{f}(0)$ is obtained by substituting the roles of $\tilde{\lambda}^-(\kappa_m)$ and $\tilde{\lambda}^+(\kappa_m)$. First, let us begin by a simple case:

4.1 The case $\mathbf{f}(0) = \mathbf{f}(1)$

Without loss of generality, we can assume that $f(0) = f(1) = 1$. Thanks to Lemma 3.2, $\Lambda_g^m(\lambda)$ has two eigenvalues $\lambda^-(\mu_m)$ and $\lambda^+(\mu_m)$ whose asymptotics are given by :

$$\begin{cases} \lambda^-(\mu_m) = \sqrt{\mu_m} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right) \\ \lambda^+(\mu_m) = \sqrt{\mu_m} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right). \end{cases}$$

We recall that :

$$\sqrt{\kappa_m} = m + \frac{n-2}{2} + O\left(\frac{1}{m}\right),$$

and for $m \in \mathbb{N}$, we set $V_m = \left\{ \lambda^-(\kappa_m), \lambda^+(\kappa_m) \right\}$ and $\tilde{V}_m := \left\{ \tilde{\lambda}^-(\kappa_m), \tilde{\lambda}^+(\kappa_m) \right\}$.

• The two dimensional case

In this case, the eigenvalues $\lambda^\pm(\kappa_m)$ have the following asymptotics :

$$\begin{cases} \lambda^-(\kappa_m) = m + O\left(\frac{1}{m}\right) \\ \lambda^+(\kappa_m) = m + O\left(\frac{1}{m}\right). \end{cases}$$

For m large enough, the sets V_m and \tilde{V}_m are both included in the interval $\left[m - \frac{1}{4}, m + \frac{1}{4}\right]$. In particular $V_m \cap \tilde{V}_{m'} = \emptyset$ if $m \neq m'$. The equality $\Sigma(\Lambda_g(\lambda)) = \Sigma(\Lambda_{\tilde{g}}(\lambda))$ leads to the equalities $V_m = \tilde{V}_m$ if m is greater than some index m_0 . Consequently, there is m_0 such that, for $m \geq m_0$:

$$\begin{cases} \lambda^-(\kappa_m) + \lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m) + \tilde{\lambda}^+(\kappa_m) \\ \lambda^-(\kappa_m)\lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m)\tilde{\lambda}^+(\kappa_m). \end{cases}$$

Of course, the previous equalities are still true when κ_m is replaced by μ_m . Thus, we have proved:

$$\forall m \in \mathbb{N}, m \geq m_0 \Rightarrow \text{Tr}(\Lambda_g^m) = \text{Tr}(\Lambda_{\tilde{g}}^m) \text{ and } \det(\Lambda_g^m) = \det(\Lambda_{\tilde{g}}^m).$$

• The $n \geq 3$ dimensional case

In this case, we use the following asymptotics of the eigenvalues $\lambda^\pm(\kappa_m)$:

$$\begin{cases} \lambda^-(\kappa_m) = m + \frac{n-2}{2} - \frac{h'(1)}{4} + O\left(\frac{1}{m}\right) \\ \lambda^+(\kappa_m) = m + \frac{n-2}{2} + \frac{h'(0)}{4} + O\left(\frac{1}{m}\right). \end{cases}$$

Contrary to the two dimensional case, we can not conclude that the sets V_m and \tilde{V}_m are equal. This is due to the presence of the constants $h'(0)$ and $h'(1)$ in these asymptotics. We have the following Proposition:

Proposition 4.3. *There is $\mathcal{L} \subset \mathbb{N}$ such that $\sum_{m \in \mathcal{L}} \frac{1}{m} = +\infty$ satisfying :*

$$\forall m \in \mathcal{L}, \begin{cases} \lambda^-(\kappa_m) + \lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m) + \tilde{\lambda}^+(\kappa_m) \\ \lambda^-(\kappa_m)\lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m)\tilde{\lambda}^+(\kappa_m) \end{cases}$$

Proof. We start with the following Lemma:

Lemma 4.4. *There is $\mathcal{L}_1 \subset \mathbb{N}$ such that $\sum_{m \in \mathcal{L}_1} \frac{1}{m} = +\infty$ satisfying :*

$$\left(\lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m), \forall m \in \mathcal{L}_1 \right) \quad \text{or} \quad \left(\lambda^-(\kappa_m) = \tilde{\lambda}^+(\kappa_m), \forall m \in \mathcal{L}_1 \right)$$

Proof. Since $\Sigma(\Lambda_g(\lambda)) = \Sigma(\Lambda_{\tilde{g}}(\lambda))$, we have the inclusion :

$$\{\lambda^-(\kappa_m), m \in \mathbb{N}\} \subset \{\tilde{\lambda}^-(\kappa_m), m \in \mathbb{N}\} \cup \{\tilde{\lambda}^+(\kappa_m), m \in \mathbb{N}\}.$$

Thus, there exists a sequence of integers (a_m) such that $\sum_{m \in \mathbb{N}} \frac{1}{a_m} = +\infty$ and :

$$\{\lambda^-(\kappa_{a_m}), m \in \mathbb{N}\} \subset \{\tilde{\lambda}^-(\kappa_m), m \in \mathbb{N}\} \quad \text{or} \quad \{\lambda^-(\kappa_{a_m}), m \in \mathbb{N}\} \subset \{\tilde{\lambda}^+(\kappa_m), m \in \mathbb{N}\}.$$

For instance, let us study the first case (since the second case is similar). We can find another sequence of integers (\tilde{a}_m) such that :

$$\lambda^-(\kappa_{a_m}) = \tilde{\lambda}^-(\kappa_{\tilde{a}_m}), \quad \forall m \in \mathbb{N}.$$

Thanks to Lemma 3.2, one obtains :

$$a_m - \tilde{a}_m = \frac{\tilde{h}'(1)}{4} - \frac{h'(1)}{4} + O\left(\frac{1}{a_m}\right).$$

Therefore, the sequence of integers $(a_m - \tilde{a}_m)$ converges to the integer $\frac{\tilde{h}'(1)}{4} - \frac{h'(1)}{4} = \frac{n-2}{4} \frac{f'(1)}{f(1)} - \frac{n-2}{4} \frac{\tilde{f}'(1)}{\tilde{f}(1)}$.

Recalling that $f, \tilde{f} \in \mathcal{C}_b = \left\{ f \in C^\infty([0, 1]), \left| \frac{f'(i)}{f(i)} \right| \leq \frac{1}{n-2}, i \in \{0, 1\} \right\}$, we get:

$$\left| \frac{n-2}{4} \frac{f'(1)}{f(1)} - \frac{n-2}{4} \frac{\tilde{f}'(1)}{\tilde{f}(1)} \right| \leq \frac{1}{2}.$$

As this quantity must be an integer, we have proved that, for $m \geq m_0$:

$$a_m = \tilde{a}_m.$$

We conclude the proof of the Lemma setting $\mathcal{L}_1 = \{a_m, m \geq m_0\}$. □

Now, we can finish the proof of the Proposition. For instance, assume that:

$$\lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m), \quad \forall m \in \mathcal{L}_1.$$

Repeating the previous argument with $\lambda^+(\kappa_m)$, $m \in \mathcal{L}_1$, we have the inclusion:

$$\{\lambda^+(\kappa_m), m \in \mathcal{L}_1\} \subset \{\tilde{\lambda}^-(\kappa_m), m \in \mathbb{N}\} \cup \{\tilde{\lambda}^+(\kappa_m), m \in \mathbb{N}\}$$

There is a sequence of integers, denoted $(b_m) \in \mathcal{L}_1$, such that $\sum_{m \in \mathbb{N}} \frac{1}{b_m} = +\infty$ and :

$$\{\lambda^+(\kappa_{b_m}), m \in \mathbb{N}\} \subset \{\tilde{\lambda}^-(\kappa_m), m \in \mathbb{N}\} \quad \text{or} \quad \{\lambda^+(\kappa_{b_m}), m \in \mathbb{N}\} \subset \{\tilde{\lambda}^+(\kappa_m), m \in \mathbb{N}\}.$$

- In the second case, there is a sequence (\tilde{b}_m) such that :

$$\lambda^+(\kappa_{b_m}) = \tilde{\lambda}^+(\kappa_{\tilde{b}_m})$$

which implies as previously that $b_m = \tilde{b}_m$. Then, setting $\mathcal{L} = \{b_m, m \in \mathbb{N}\}$ we have for all $m \in \mathcal{L}$:

$$\begin{cases} \lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m) \\ \lambda^+(\kappa_m) = \tilde{\lambda}^+(\kappa_m) \end{cases}$$

- In the first case, we still could show that :

$$\lambda^+(\kappa_{b_m}) = \tilde{\lambda}^-(\kappa_{b_m}) \quad (= \lambda^-(\kappa_{b_m}))$$

But this is not possible because $\Lambda_g^{b_m}(\lambda)$ would then be an homothety, which contradicts the matrix expression of $\Lambda_g^{b_m}(\lambda)$.

Hence the proof of Proposition 4.3 is complete. □

We recall that \mathcal{P} is the set of roots of $\Delta(z)$. For every $z \in \mathbb{C} \setminus \mathcal{P}$, let us set

$$B(z) = -\frac{D(z) + E(z)}{\Delta(z)} + C_0 - C_1 \tag{18}$$

$$C(z) = \left(-\frac{D(z)}{\Delta(z)} + C_0 \right) \left(-\frac{E(z)}{\Delta(z)} - C_1 \right) - \frac{1}{\sqrt{f(0)f(1)\Delta(z)^2}}, \tag{19}$$

where

$$C_0 = \frac{\ln(h)'(0)}{4\sqrt{f(0)}}, \quad C_1 = \frac{\ln(h)'(1)}{4\sqrt{f(1)}}.$$

For every m in \mathbb{N} , we have (cf proof of Lemma 3.2):

$$B(\kappa_m) = \lambda^-(\kappa_m) + \lambda^+(\kappa_m) \quad \text{and} \quad C(\kappa_m) = \lambda^-(\kappa_m)\lambda^+(\kappa_m).$$

We introduce the functions g_1 and g_2 on Π^+ as follows:

$$\begin{aligned} g_1(z) &= \Delta(z^2)\tilde{\Delta}(z^2)[B(z^2) - \tilde{B}(z^2)] \\ g_2(z) &= \Delta(z^2)^2\tilde{\Delta}(z^2)^2[C(z^2) - \tilde{C}(z^2)]. \end{aligned}$$

We claim that g_1 and g_2 are identically zero. Indeed :

- g_1, g_2 are holomorphic on Π^+ .
- $g_1, g_2 \in \mathcal{N}(\Pi^+)$ thanks to the estimates of Proposition 2.3.
- Thanks to Proposition 4.3, we have $g_1(\sqrt{\kappa_m}) = g_2(\sqrt{\kappa_m}) = 0$ for every $m \in \mathcal{L}$. As $\sqrt{\kappa_m} \sim m$, one has $\sum_{m \in \mathcal{L}} \frac{1}{\sqrt{\kappa_m}} = +\infty$. Thus, we can conclude, by Nevanlinna's theorem :

$$g_1 \equiv g_2 \equiv 0 \quad \text{on } \Pi^+.$$

In particular, for every m in \mathbb{N} :

$$\begin{cases} \lambda^-(\kappa_m) + \lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m) + \tilde{\lambda}^+(\kappa_m) \\ \lambda^-(\kappa_m)\lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m)\tilde{\lambda}^+(\kappa_m) \end{cases}$$

and, by replacing κ_m by μ_m we have for every m in \mathbb{N} :

$$\text{Tr}(\Lambda_g^m) = \text{Tr}(\Lambda_{\tilde{g}}^m) \quad \text{and} \quad \det(\Lambda_g^m) = \det(\Lambda_{\tilde{g}}^m).$$

4.2 The case $f(0) \neq f(1)$

Without loss of generality, assume that $f(0) < f(1)$. Thanks to the asymptotics:

$$\sqrt{\kappa_m} = m + \frac{n-2}{2} + O\left(\frac{1}{m}\right)$$

and Lemma 3.2, we have :

$$\begin{cases} \lambda^-(\kappa_m) = \frac{m}{\sqrt{f(1)}} + \frac{n-2}{2\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\ \lambda^+(\kappa_m) = \frac{m}{\sqrt{f(0)}} + \frac{n-2}{2\sqrt{f(0)}} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}} + O\left(\frac{1}{m}\right). \end{cases}$$

Since $f(0) \neq f(1)$, the proof is a little more delicate. As previously, we have to find a subset $\mathcal{L} \subset \mathbb{N}$ such that $\sum_{m \in \mathcal{L}} \frac{1}{m} = +\infty$ and such that $\lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m)$ or $\lambda^+(\kappa_m) = \tilde{\lambda}^+(\kappa_m)$ for all $m \in \mathcal{L}$.

We have the following Proposition:

Proposition 4.5. *There is $\mathcal{L} \subset \mathbb{N}$ satisfying $\sum_{m \in \mathcal{L}} \frac{1}{m} = +\infty$ and such that :*

$$\{\lambda^-(\kappa_m), m \in \mathcal{L}\} \subset \{\tilde{\lambda}^-(\kappa_m), m \in \mathbb{N}\}$$

Proof. If the subset of the elements of $\{\lambda^-(\kappa_m), m \in \mathbb{N}\}$ belonging to $\{\tilde{\lambda}^+(\kappa_m), m \in \mathbb{N}\}$ is finite, the proposition is true. If not, there exist two strictly increasing sequences $\varphi, \psi : \mathbb{N} \rightarrow \mathbb{N}$ such that:

$$\lambda^-(\kappa_{\psi(m)}) = \tilde{\lambda}^+(\kappa_{\varphi(m)}). \quad (20)$$

We emphasize that the functions φ and ψ are built so that an integer $m \in \mathbb{N}$ which is not in the image of ψ (respectively in that of φ) satisfies $\lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_n)$ for some $n \in \mathbb{N}$ (respectively $\lambda^+(\kappa_n) = \tilde{\lambda}^+(\kappa_m)$ for some $n \in \mathbb{N}$).

Replacing $\lambda^+(\kappa_{\varphi(m)})$ and $\tilde{\lambda}^-(\kappa_{\psi(m)})$ by their asymptotics in the equality (20), we have :

$$\frac{\varphi(m)}{\sqrt{f(0)}} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}} + \frac{n-2}{2\sqrt{f(0)}} + O\left(\frac{1}{\varphi(m)}\right) = \frac{\psi(m)}{\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} + \frac{n-2}{2\sqrt{f(1)}} + O\left(\frac{1}{\psi(m)}\right).$$

Setting $C = -\frac{\ln(h)'(1)}{4\sqrt{f(1)}} - \frac{\ln(h)'(0)}{4\sqrt{f(0)}} + \frac{n-2}{2\sqrt{f(1)}} - \frac{n-2}{2\sqrt{f(0)}}$ and noticing that the previous equality implies that $O\left(\frac{1}{\psi(m)}\right) = O\left(\frac{1}{\varphi(m)}\right)$, one can write (20) as follows:

$$\frac{\varphi(m)}{\sqrt{f(0)}} = \frac{\psi(m)}{\sqrt{f(1)}} + C + O\left(\frac{1}{\varphi(m)}\right). \quad (21)$$

Now, we have the following Lemma:

Lemma 4.6. *There exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, $\psi(m+1) \geq \psi(m) + 2$.*

Proof. Set $A = \frac{\sqrt{f(1)}}{\sqrt{f(0)}} > 1$ and $C' = -\sqrt{f(1)}C$. It follows from (21) that:

$$\psi(m) = A\varphi(m) + C' + O\left(\frac{1}{\varphi(m)}\right).$$

Assume $\psi(m+1) = \psi(m) + 1$. Then :

$$\begin{aligned}
\psi(m) + 1 &= \psi(m+1) = A\varphi(m+1) + C' + O\left(\frac{1}{\varphi(m)}\right) \\
&\geq A(\varphi(m) + 1) + C' + O\left(\frac{1}{\varphi(m)}\right) \\
&= A\varphi(m) + C' + A + O\left(\frac{1}{\varphi(m)}\right) \\
&= \psi(m) + A + O\left(\frac{1}{\varphi(m)}\right).
\end{aligned}$$

Thus, we get:

$$1 \geq A + O\left(\frac{1}{\varphi(m)}\right)$$

which is clearly false for m large enough. \square

Consequently, the range of ψ doesn't contain two consecutive integers. We deduce from this the following Lemma :

Lemma 4.7. *Denote a_1, a_2, \dots the sequence of all integers that are not in the range of ψ . There exists $C > 0$ such that this sequence satisfies :*

$$a_m \leq 2m + C. \quad (22)$$

Proof. We set $p_0 = \psi(m_0)$ and $C := a_{p_0} - 2p_0$. Clearly, (22) is true for $m = p_0$. Now, assume that (22) is satisfied for a fixed $m \geq p_0$. Then :

- either $a_m + 1$ is not in the range of ψ and so : $a_{m+1} = a_m + 1$,
- or $a_m + 1$ is in the range of ψ . Then $a_m + 2$ is not in the image of ψ since it does not contain consecutive integers and consequently $a_{m+1} = a_m + 2$.

In both cases, we get $a_{m+1} \leq 2(m+1) + C$. Then, the proof of the Lemma follows by a standard induction argument. \square

In particular, Lemma 4.7 shows that :

$$\sum_{n \in \mathbb{N}} \frac{1}{a_n} = +\infty.$$

Thus, setting $\mathcal{L} = \{a_n, n \in \mathbb{N}\}$, we have :

$$\{\lambda^-(\kappa_m), m \in \mathcal{L}\} \subset \{\tilde{\lambda}^-(\kappa_m), m \in \mathbb{N}\}$$

which concludes the proof of the Proposition. \square

As previously, we have to treat differently the case of the dimension $n = 2$ and $n \geq 3$.

The two dimensional case:

Lemma 4.8. *For all $m \in \mathcal{L}$, one has:*

$$\lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m).$$

Proof. By construction, for each element m of \mathcal{L} , there exists $\ell(m) \in \mathbb{N}$ such that :

$$\lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_{\ell(m)}).$$

Using Lemma 3.2, we get:

$$m = \ell(m) + O\left(\frac{1}{m}\right) + O\left(\frac{1}{\ell(m)}\right),$$

which implies $m = \ell(m)$ for m large enough. □

Let us consider again the functions B and C defined in (18) and (19). Setting

$$R(z) = B(z)^2 - 4C(z),$$

we get:

$$\lambda^\pm(\kappa_m) = \frac{1}{2} \left(B(\kappa_m) \pm \sqrt{R(\kappa_m)} \right).$$

We can define on Π^+ the function $\lambda^-(z)$ by $\lambda^-(z) = \frac{1}{2} \left(B(z) - \sqrt{R(z)} \right)$.

Lemma 4.9. *For every $t \in \mathbb{R}_+$ large enough, we have the alternative :*

$$\lambda^-(t) = \tilde{\lambda}^-(t) \quad \text{or} \quad \lambda^+(t) = \tilde{\lambda}^+(t).$$

Proof. From Lemma 4.8, there is $\mathcal{L} \subset \mathbb{N}$ satisfying $\sum_{m \in \mathcal{L}} \frac{1}{m} = +\infty$ and such that

$$\forall m \in \mathcal{L}, \quad B(\kappa_m) - \sqrt{R(\kappa_m)} = \tilde{B}(\kappa_m) - \sqrt{\tilde{R}(\kappa_m)}.$$

One has

$$\left(B(\kappa_m) - \tilde{B}(\kappa_m) \right)^2 = \left(\sqrt{R(\kappa_m)} - \sqrt{\tilde{R}(\kappa_m)} \right)^2,$$

and then

$$\underbrace{\left[\left(B(\kappa_m) - \tilde{B}(\kappa_m) \right)^2 - R(\kappa_m) - \tilde{R}(\kappa_m) \right]^2}_{:=g_1(\kappa_m)} - 4R(\kappa_m)\tilde{R}(\kappa_m) = 0.$$

For $z \in \Pi^+$, let us define:

$$g_2(z) = \Delta(z^2)^4 \tilde{\Delta}(z^2)^4 g_1(z^2).$$

Then g_2 is identically zero. Indeed :

- g_2 is holomorphic on Π^+ .
- $g_2 \in \mathcal{N}(\Pi^+)$ thanks to the estimates of Proposition 2.3.
- We have $g_2(\sqrt{\kappa_m}) = 0$ for every $m \in \mathcal{L}$, which enable us to conclude, by Nevanlinna's theorem, that

$$g_2 \equiv 0 \text{ on } \Pi^+.$$

We have obtained:

$$\forall z \in \Pi^+, \quad \left[\left(B(z^2) - \tilde{B}(z^2) \right)^2 - R(z^2) - \tilde{R}(z^2) \right]^2 - 4R(z^2)\tilde{R}(z^2) = 0.$$

It easily follows that for $z \in \Pi^+$, we have four alternatives:

$$\lambda^-(z^2) = \tilde{\lambda}^-(z^2), \quad \lambda^+(z^2) = \tilde{\lambda}^-(z^2), \quad \lambda^-(z^2) = \tilde{\lambda}^+(z^2) \quad \text{or} \quad \lambda^+(z^2) = \tilde{\lambda}^+(z^2).$$

We can note that the asymptotics of Lemma 3.2, involving $\lambda^\pm(\kappa_m)$, can be extended on \mathbb{R}_+ for $\lambda^\pm(t)$ (the proof remains exactly the same). Now, if a real sequence of positive numbers (t_m) satisfies, for instance, $\lambda^+(t_m^2) = \tilde{\lambda}^-(t_m^2)$ with $t_m \rightarrow +\infty$, then, using Lemma 3.2 with t_m^2 instead of κ_m we get $f(0) = f(1)$, and that contradicts our hypothesis. We use the same argument to exclude the third case when z is a positive real number large enough. Finally, we have for t positive large enough :

$$\lambda^-(t) = \tilde{\lambda}^-(t) \quad \text{or} \quad \lambda^+(t) = \tilde{\lambda}^+(t).$$

□

Case 1 : Assume now there exists $x \in \mathbb{R}_+$ as large as we want such that $\lambda^-(x) \neq \tilde{\lambda}^-(x)$. By a standard continuity argument, we can find an interval I centered in x such that :

$$\forall t \in I, \quad \lambda^-(t) \neq \tilde{\lambda}^-(t),$$

which implies necessarily that $\forall t \in I, \quad \lambda^+(t) = \tilde{\lambda}^+(t)$. Moreover, there is $L > 0$ such that the real function

$$t \mapsto \lambda^+(t) - \tilde{\lambda}^+(t)$$

is analytic on the interval $[L, +\infty[$. Thus, $\forall t \geq L$, one has $\lambda^+(t) = \tilde{\lambda}^+(t)$ by the analytic continuation principle. One deduces there exists $m_0 \in \mathbb{N}$ such that, for $m \in \mathcal{L}$ and $m \geq m_0$:

$$\lambda^+(\kappa_m) = \tilde{\lambda}^+(\kappa_m) \quad \text{and} \quad \lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m).$$

Without loss of generality, assume that $\min \mathcal{L}$ is greater than m_0 . As a by product, one gets:

$$\forall m \in \mathcal{L}, \quad \begin{cases} \lambda^-(\kappa_m) + \lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m) + \tilde{\lambda}^+(\kappa_m) \\ \lambda^-(\kappa_m)\lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m)\tilde{\lambda}^+(\kappa_m) \end{cases}$$

Hence, we have proved the same result as the one of Proposition 4.3. We can deduce similarly that both previous equalities are in fact true for every $m \in \mathbb{N}$, i.e

$$\forall m \in \mathbb{N}, \quad \text{Tr}(\Lambda_g^m) = \text{Tr}(\tilde{\Lambda}_g^m) \quad \text{and} \quad \det(\Lambda_g^m) = \det(\tilde{\Lambda}_g^m).$$

We are thus brought back to the third section.

Case 2 : For all $t \in \mathbb{R}$, t large enough, we have $\lambda^-(t) = \tilde{\lambda}^-(t)$. In particular, for all $m \in \mathbb{N}$ large enough, $\lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m)$. Hence, for m large enough, one also has $\lambda^+(\kappa_m) = \tilde{\lambda}^+(\kappa_m)$. Indeed, let $m \in \mathbb{N}$. If $\lambda^+(\kappa_m) \in (\tilde{\lambda}^+(\kappa_\ell))$, there is $\ell(m) \in \mathbb{N}$ such that $\lambda^+(\kappa_m) = \tilde{\lambda}^+(\kappa_{\ell(m)})$. One can prove, as in the proof of Lemma 4.8 that $\ell(m) = m$. The same holds if we assume that $\tilde{\lambda}^+(\kappa_m) \in (\lambda^+(\kappa_\ell))$. The only other option is that there are integers p and ℓ such that

$$\begin{cases} \lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_p) \\ \tilde{\lambda}^+(\kappa_m) = \lambda^-(\kappa_\ell) \end{cases}$$

Then $\lambda^+(\kappa_m) - \tilde{\lambda}^+(\kappa_m) = \tilde{\lambda}^-(\kappa_p) - \lambda^-(\kappa_\ell)$, which implies

$$O\left(\frac{1}{m}\right) = \frac{p-\ell}{\sqrt{f(1)}} + O\left(\frac{1}{p}\right) + O\left(\frac{1}{\ell}\right).$$

Hence $p = \ell$ for m, p, ℓ large enough and so $\lambda^+(\kappa_m) = \tilde{\lambda}^+(\kappa_m)$. There is then $m_0 \in \mathbb{N}$ such that

$$\forall m \geq m_0, \quad \text{Tr}(\Lambda_g^m) = \text{Tr}(\tilde{\Lambda}_g^m) \quad \text{and} \quad \det(\Lambda_g^m) = \det(\tilde{\Lambda}_g^m).$$

We are brought back again to the third section (see Remark 2 p. 13).

The case of the dimension $n \geq 3$.

The following arguments are more or less immediate adaptations of the two dimensional case. We just refer to it for details.

Lemma 4.10. *There exists $\mathcal{L} \subset \mathbb{N}$ such that $\sum_{m \in \mathcal{L}} \frac{1}{m} = +\infty$ and :*

$$\tilde{\lambda}^-(\kappa_m) = \lambda^-(\kappa_m), \quad \forall m \in \mathcal{L}.$$

Proof. By definition of the sequence (a_n) defined in Lemma 4.7, there is another sequence of integers (\tilde{a}_n) such that :

$$\lambda^-(\kappa_{a_n}) = \tilde{\lambda}^-(\kappa_{\tilde{a}_n})$$

As in the proof of Lemma 4.4, we show that $a_m = \tilde{a}_m$ and we set $\mathcal{L} = \{a_m, m \in \mathbb{N}\}$. \square

We have the following Lemma (the proof is identical to the two dimensional case).

Lemma 4.11. *For all $t \in \mathbb{R}_+$ large enough, we have the alternative :*

$$\lambda^-(t) = \tilde{\lambda}^-(t), \quad \lambda^+(t) = \tilde{\lambda}^+(t).$$

We deduce then, for all $m \in \mathcal{L}$:

$$\lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m) \quad \text{and} \quad \lambda^+(\kappa_m) = \tilde{\lambda}^+(\kappa_m).$$

It follows that :

$$\forall m \in \mathbb{N}, \quad \text{Tr}(\Lambda_g^m) = \text{Tr}(\Lambda_{\tilde{g}}^m) \text{ and } \det(\Lambda_g^m) = \det(\Lambda_{\tilde{g}}^m).$$

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