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Uniqueness results in the inverse spectral Steklov problem

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Abstract

This paper is devoted to an inverse Steklov problem for a particular class of n -dimensional manifolds having the topology of a hollow sphere and equipped with a warped product metric. We prove that the knowledge of the Steklov spectrum determines uniquely the associated warping function up to a natural invariance.

Keywords. Inverse Calderón problem, Steklov spectrum, Weyl-Titchmarsh functions, Nevanlinna theorem, local Borg-Marchenko theorem.

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1 Introduction

1.1 The Calderón and Steklov problems.

Let (M, g) be a smooth compact manifold of dimension $n \geq 2$ with smooth boundary ∂M . We consider the Dirichlet problem

$$\begin{cases} -\Delta_g u = \lambda u & \text{in } M \\ u = \psi & \text{on } \partial M, \end{cases} \quad (1)$$

where $\psi \in H^{1/2}(\partial M)$ and $\lambda \in \mathbb{R}$ is assumed to lie outside the Dirichlet spectrum $\sigma(-\Delta_g)$ of the Laplace-Beltrami operator $-\Delta_g$. In a local coordinate system $(x^i)_{i=1, \dots, n}$, $-\Delta_g$ has the expression

$$-\Delta_g = - \sum_{1 \leq i, j \leq n} \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \partial_j),$$

where we have set $|g| = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$.

If $\lambda \notin \sigma(-\Delta_g)$, the Dirichlet problem (1) has a unique solution $u \in H^1(M)$, and we can define the so-called *Dirichlet-to-Neumann* (DN) operator as the map

$$\begin{aligned} \Lambda_g(\lambda) &: H^{1/2}(\partial M) \rightarrow H^{-1/2}(\partial M) \\ \psi &\mapsto \left. \frac{\partial u}{\partial \nu} \right|_{\partial M}, \end{aligned}$$

where ∂_ν is the unit normal derivative with respect to the unit outer normal vector on ∂M . This normal derivative has to be understood in the weak sense by :

$$\forall (\psi, \phi) \in H^{1/2}(\partial M)^2 : \langle \Lambda_g(\lambda) \psi, \phi \rangle = \int_M \langle du, dv \rangle_g d\text{Vol}_g + \lambda \int_M uv d\text{Vol}_g,$$

where u is the unique weak solution of the Dirichlet problem (1), and where v is any element of $H^1(M)$ such that $v|_{\partial M} = \phi$. When ψ is sufficiently smooth, this definition coincides with the usual one in local coordinates, that is

$$\partial_\nu u = \nu^i \partial_i u. \quad (2)$$

The anisotropic Calderón problem can be initially stated as: *does the knowledge of the DN map $\Lambda_g(\lambda)$ at a fixed frequency λ determine uniquely the metric g ?*

Due to a number of gauge invariances, it is well-known that the answer to the above question is negative in general. An observation made by Luc Tartar ([8], p.2) leads to the equality :

$$\Lambda_g(\lambda) = \Lambda_{\psi^*g}(\lambda),$$

where $\psi : M \rightarrow M$ is any smooth diffeomorphism which is equal to the identity on the boundary, (here ψ^*g is the pullback of g by ψ). Moreover, in dimension $n = 2$ and for $\lambda = 0$, there is one more gauge invariance. Indeed, thanks to the conformal invariance of the Laplacian, for every positive function c , we have

$$\Delta_{cg} = \frac{1}{c} \Delta_g.$$

Consequently, the solutions of the Dirichlet problem (1) associated to the metrics g and cg are the same when $\lambda = 0$. Moreover, if we assume that $c \equiv 1$ on the boundary, the unit outer normal vectors on ∂M are also the same for both metrics. Therefore,

$$\Lambda_{cg}(0) = \Lambda_g(0),$$

and it is not possible to determine uniquely the metric from the DN map. Hence, the appropriate question to address is :

Assume $n \geq 3$ (resp. $n = 2$ and $\lambda \neq 0$). If $\Lambda_g(\lambda) = \Lambda_{\tilde{g}}(\lambda)$, is there a smooth diffeomorphism $\psi : M \rightarrow M$ with $\psi|_{\partial M} = id$ and $\psi^*g = \tilde{g}$?

This problem is still largely open. Some special cases have been answered positively (see [16] and [13] for a presentation of the latest developments) but the general case seems to be very difficult to tackle. However, the question becomes simpler if we assume that (M, g) and (M, \tilde{g}) belong to the same conformal class. Precisely :

For $c \in C^\infty(M)$, assume $\Lambda_g(\lambda) = \Lambda_{cg}(\lambda)$. Then, is there a smooth diffeomorphism $\psi : M \rightarrow M$ such that $\psi|_{\partial M} = id$ and $\psi^*(g) = cg$?

Actually, we can precise the above question thanks to the following result due to Lionheart ([9]) : any diffeomorphism $\psi : M \rightarrow M$ which satisfies $\psi^*(cg) = g$ and $\psi|_{\partial M} = id$ must be the identity. Then, the anisotropic Calderón inverse problem within the same conformal class can be replaced by

If $\Lambda_g(\lambda) = \Lambda_{cg}(\lambda)$, is it true that $c = 1$?

Some deep results have been obtained in [4] for conformally transversally anisotropic manifolds (M, g) of dimension $n \geq 3$, i.e for manifolds

$$M \subset \subset \mathbb{R} \times K, \quad g = c(x, y_K)(dx^2 + g_K)$$

where (K, g_K) is a smooth compact manifold of dimension $n-1$. Under some geometrical conditions on K , such as *simplicity* (a compact manifold K is said *simple* if any two points of K are connected by a unique geodesic and if its boundary is strictly convex) the conformal factor c is entirely determined by the DN map at frequency $\lambda = 0$.

In the same way, for a class of manifolds $M = [0, 1] \times K$ which have the topology of a cylinder, various results of uniqueness (or non-uniqueness when the Dirichlet data and the Neumann data are measured on disjoint sets) have been obtained in ([2, 3]). The proofs make use of separation of variables and introduces a connection between the DN map and the Weyl-Titchmarsh functions associated to a separated ordinary differential equation corresponding to the horizontal variable of the cylinder. This connection allows to use some nice results from complex analysis.

In this paper, we are interested in a relative inverse problem within the conformal class of certain warped product manifold $M = [0, 1] \times \mathbb{S}^{n-1}$ (where \mathbb{R}^{n-1} is the $(n-1)$ -sphere) but under a weaker assumption :

does the spectrum of the DN map characterize the conformal factor ?

We recall that $\Lambda_g(\lambda)$ is a pseudodifferential operator of order 1 and is self-adjoint on $L^2(\partial M, dS_g)$ where dS_g is the metric induced by g on the boundary ∂M . Thus, the DN operator $\Lambda_g(\lambda)$ has a discrete spectrum denoted $\sigma(\Lambda_g(\lambda))$ accumulating at infinity

$$\sigma(\Lambda_g(\lambda)) = \{\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \rightarrow +\infty\}.$$

This spectrum is called the *Steklov spectrum* (see [7], p.2). In the particular case where $\lambda = 0$, using the Green's formula, we see that the DN operator is positive and we have $\lambda_0 = 0$. The properties of the Steklov spectrum are highly sensitive to the smoothness of the boundary ∂M . For example (see [6]): if ∂M is smooth, the eigenvalues satisfy the Weyl formula for the Steklov spectrum:

$$\lambda_j = 2\pi \left(\frac{j}{\text{Vol}(\mathbb{B}^{n-1})\text{Vol}(\partial M)} \right)^{\frac{1}{n-1}} + O(1) \quad (3)$$

A much more refined asymptotic holds if M is a smooth surface, involving the lengths of the connected components of M , with a rate of decay of $O(j^{-\infty})$ (see [5]), but this formula fails for polygons ([6]). Moreover, no asymptotic is known yet for Lipschitz domains. It is also well-known that some geometrical quantities of the boundary of a Riemannian manifold surface are Steklov spectrum invariants. Let us mention for instance (see [5, 6] for details and reference therein):

- a) the dimension of the manifold and the volume of its boundary.
- b) When $\dim M \geq 3$, the integral of the mean curvature on ∂M .
- c) When $\dim M = 2$, the number and the lengths of the connected components of ∂M .

1.2 The main result

In this section, we give the main results of the paper. Let $M = [0, 1] \times \mathbb{S}^{n-1}$ be a manifold equipped with the metric

$$g = f(x)(dx^2 + g_{\mathbb{S}}), \quad (4)$$

where \mathbb{S}^{n-1} is the unit sphere in \mathbb{R}^n , $g_{\mathbb{S}}$ is the metric induced by the euclidean metric on \mathbb{R}^n , and where the conformal factor f is a smooth positive function of the variable $x \in [0, 1]$ only. Note that the manifold M has two boundaries $\Gamma_0 = \{0\} \times \mathbb{S}^{n-1}$ and $\Gamma_1 = \{1\} \times \mathbb{S}^{n-1}$. Thus, the DN map will be shown to have the structure of a matrix operator defined on $H^{1/2}(\mathbb{S}^{n-1}) \oplus H^{1/2}(\mathbb{S}^{n-1})$.

In this setting, we want to answer the following question : does the Steklov spectrum determine uniquely the warping function $f(x)$? This question has been answered positively in [1] when $K = (0, 1] \times \mathbb{S}^{n-1}$ equipped with the metric (4). The difference between this case and the one we are studying here is that the boundary of K is connected whereas that of M is made of two connected components. Due to a natural gauge invariance, we emphasize it is hopeless to recover the metric from the spectrum data. Indeed, let $\psi : M \rightarrow M$ be a smooth diffeomorphism. We have ([7]):

$$\Lambda_{\psi^*g}(\lambda) = \varphi^* \circ \Lambda_g(\lambda) \circ \varphi^{*-1},$$

where $\varphi := \psi|_{\partial M}$ and where $\varphi^* : C^\infty(\partial M) \rightarrow C^\infty(\partial M)$ is the application defined by $\varphi^*h := h \circ \varphi$. As a by-product, one has:

$$\sigma(\Lambda_{\psi^*g}(\lambda)) = \sigma(\Lambda_g(\lambda)).$$

Thus, if we can find a diffeomorphism ψ preserving the warped product structure of the manifold M given by (4), we are able to find a counterexample to uniqueness from the knowledge of the Steklov spectrum. For instance, consider the map

$$\psi : (x, y) \in [0, 1] \times \mathbb{S}^{n-1} \mapsto (1 - x, y) \in [0, 1] \times \mathbb{S}^{n-1}.$$

A straightforward calculus gives $\psi^*g = f(1-x)(dx^2 + g_{\mathbb{S}})$. Thus, the above discussion shows that $\Lambda_g(\lambda)$ and $\Lambda_{\psi^*g}(\lambda)$ have the same Steklov spectrum. Now, we can reformulate more precisely our initial question. Let g and \tilde{g} be two Riemannian metrics given by (4) with conformal factor $f(x)$, (resp. $\tilde{f}(x)$). Assume that $\Lambda_g(\lambda)$ and $\Lambda_{\tilde{g}}(\lambda)$ are Steklov isospectral. Then, is it true that :

$$f(x) = \tilde{f}(x) \text{ or } f(x) = \tilde{f}(1-x)?$$

In what follows, we answer positively this question in dimension $n = 2$ with $\lambda \neq 0$, and in dimension $n \geq 3$ for any frequency λ (with an additional hypothesis on the metrics on the boundary ∂M). Our main result is the following:

Theorem 1.1. *Let $M = [0, 1] \times \mathbb{S}^{n-1}$ be a smooth Riemannian manifold equipped with the metric*

$$g = f(x)(dx^2 + g_{\mathbb{S}}) \quad , \quad (\text{resp. } \tilde{g} = \tilde{f}(x)(dx^2 + g_{\mathbb{S}})),$$

and let λ be a frequency not belonging to the Dirichlet spectrum of $-\Delta_g$ and $-\Delta_{\tilde{g}}$ on M . Then,

1. *For $n = 2$ and $\lambda \neq 0$,*

$$(\sigma(\Lambda_g(\lambda)) = \sigma(\Lambda_{\tilde{g}}(\lambda))) \Leftrightarrow (f = \tilde{f} \quad \text{or} \quad f = \tilde{f} \circ \eta)$$

where $\eta(x) = 1 - x$ for all $x \in [0, 1]$.

2. For $n \geq 3$, and if moreover

$$f, \tilde{f} \in \mathcal{C}_b := \left\{ f \in C^\infty([0, 1]), \left| \frac{f'(k)}{f(k)} \right| \leq \frac{1}{n-2}, k = 0 \text{ and } 1 \right\},$$

$$(\sigma(\Lambda_g(\lambda)) = \sigma(\Lambda_{\tilde{g}}(\lambda))) \Leftrightarrow (f = \tilde{f} \text{ or } f = \tilde{f} \circ \eta)$$

Let us explain briefly the outline of the proof. In both cases $n = 2$ or $n \geq 3$, the proof consists in four steps. We emphasize that in the first three steps, we do not use explicitly that the transversal manifold is the unit sphere \mathbb{S}^{n-1} .

Step 1 : we follow the same approach as in [2]. Since the manifold M has the topology of a cylinder and is equipped with a warped product metric, we can use separation of variables and write the solution u of the Dirichlet problem (1) as

$$u(x, y) = \sum_{m=0}^{+\infty} u_m(x) Y_m(y),$$

reducing this problem to a countable family of Sturm-Liouville equations with boundary conditions :

$$\begin{cases} -v_m'' + qv_m = -\mu_m v_m, & \text{sur }]0, 1[\\ v_m(0) = f^{\frac{n-2}{4}}(0)\psi_m^0, & v_m(1) = f^{\frac{n-2}{4}}(1)\psi_m^1, \end{cases} \quad \forall m \in \mathbb{N} \quad (5)$$

where

$$q = \frac{(f^{\frac{n-2}{4}})''}{f^{\frac{n-2}{4}}} - \lambda f \quad \text{and} \quad v_m = f^{\frac{n-2}{4}} u_m, \quad \forall m \in \mathbb{N}.$$

The sequence (Y_m) refers to an orthonormal basis of eigenvectors of the Laplace-Beltrami operator $-\Delta_{g_S}$ on the unit sphere.

Step 2 : we use the above decomposition to write the DN operator as an infinite matrix which is block diagonal. More precisely, let us introduce the basis $\mathfrak{B} = (\{e_m^1, e_m^2\})_{m \in \mathbb{N}}$, where $e_m^1 = (Y_m, 0)$ and $e_m^2 = (0, Y_m)$. For each $m \in \mathbb{N}$, we denote $\Lambda_g^m(\lambda)$ the restricted operator of $\Lambda_g(\lambda)$ on the subspace spanned by $\{e_m^1, e_m^2\}$. We can write $\Lambda_g(\lambda)$ in the basis \mathfrak{B} as the infinite matrix:

$$[\Lambda_g]_{\mathfrak{B}} = \begin{pmatrix} \Lambda_g^1(\lambda) & 0 & 0 & \cdots \\ 0 & \Lambda_g^2(\lambda) & 0 & \cdots \\ 0 & 0 & \Lambda_g^3(\lambda) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Each $(2, 2)$ matrix $\Lambda_g^m(\lambda)$ has a simple interpretation involving the so-called Weyl-Titchmarsh theory associated to the Sturm-Liouville equation (5). More precisely, if we denote

$$\sigma(-\Delta_{g_S}) = \{0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_m \leq \dots \rightarrow +\infty\},$$

the spectrum of $-\Delta_{g_S}$, we shall see that :

$$\Lambda_g^m(\lambda) = \begin{pmatrix} -\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{1}{4\sqrt{f(0)}} \frac{h'(0)}{h(0)} & -\frac{1}{\sqrt{f(0)}} \frac{h^{1/4}(1)}{h^{1/4}(0)} \frac{1}{\Delta(\mu_m)} \\ -\frac{1}{\sqrt{f(1)}} \frac{h^{1/4}(0)}{h^{1/4}(1)} \frac{1}{\Delta(\mu_m)} & -\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{1}{4\sqrt{f(1)}} \frac{h'(1)}{h(1)} \end{pmatrix}.$$

where $M(z)$, $N(z)$ and $\Delta(z)$ are respectively the Weyl-Titchmarsh and the characteristic functions associated to (5). In particular, the trace (respectively the determinant) of these operators $\Lambda_g^m(\lambda)$ are meromorphic functions evaluated in μ_m .

Step 3 : we prove that the knowledge of the trace and the determinant of all restricted operators $\Lambda_g^m(\lambda)$, for m large enough, characterizes f up to an involution: there is $m_0 \in \mathbb{N}$ such that :

$$\begin{aligned} (\text{Tr } \Lambda_g^m(\lambda) = \text{Tr } \Lambda_{\tilde{g}}^m(\lambda) \text{ and } \det \Lambda_g^m(\lambda) = \det \Lambda_{\tilde{g}}^m(\lambda) \text{ for all } m \geq m_0) \\ \Leftrightarrow (f = \tilde{f} \text{ or } f = \tilde{f} \circ \eta.) \end{aligned}$$

Note that the third step requires to know the asymptotic expansion of the Steklov spectrum.

Step 4 : in this step, we use explicitly that the transversal manifold is the unit sphere \mathbb{S}^{n-1} equipped with the metric induced by the euclidean metric on \mathbb{R}^n . The equality between $\sigma(\Lambda_g(\lambda)) = \{\lambda_0 \leq \lambda_1 \leq \lambda_2, \dots\}$ and $\sigma(\Lambda_{\tilde{g}}(\lambda)) = \{\tilde{\lambda}_0 \leq \tilde{\lambda}_1 \leq \tilde{\lambda}_2, \dots\}$ implies that for every $m \in \mathbb{N}$, there is $p \in \mathbb{N}$ such that $\lambda_m = \tilde{\lambda}_p$. The assumption $\left| \frac{f'(k)}{f(k)} \right| \leq \frac{1}{n-2}$ for $k \in \{0, 1\}$ is used here to ensure that $m = p$ for m large enough. Then we prove that $\text{Tr } \Lambda_g^m(\lambda) = \text{Tr } \Lambda_{\tilde{g}}^m(\lambda)$ and $\det \Lambda_g^m(\lambda) = \det \Lambda_{\tilde{g}}^m(\lambda)$ for all m large enough. This step leads to distinguish the cases $f(0) = f(1)$ and $f(0) \neq f(1)$. Then, the result follows from the Step 3.

2 Reduction to ordinary differential equations

2.1 The separation of variables

The boundary ∂M of the manifold M has two distinct connected components

$$\Gamma_0 = \{0\} \times K \text{ and } \Gamma_1 = \{1\} \times K,$$

so we can decompose $H^1(\partial M)$ as the direct sum :

$$H^{1/2}(\partial M) = H^{1/2}(\Gamma_0) \oplus H^{1/2}(\Gamma_1).$$

Each element ψ of $H^{1/2}(\partial M)$ can be written as

$$\psi = \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix}, \quad \psi^0 \in H^{1/2}(\Gamma_0) \text{ and } \psi^1 \in H^{1/2}(\Gamma_1).$$

The Laplacian $-\Delta_{g_S}$ is a self-adjoint operator on $L^2(\mathbb{S}^{n-1})$ and has pure point spectrum (μ_m) , with $\mu_0 = 0 < \mu_1 \leq \mu_2 \leq \dots \mu_m \rightarrow +\infty$. We denote $(Y_m)_{m \in \mathbb{N}}$ the associated orthonormal Hilbert basis of the eigenvectors.

Now, we decompose ψ^0 and ψ^1 as:

$$\psi^0 = \sum_{m \in \mathbb{Z}} \psi_m^0 Y_m, \quad \psi^1 = \sum_{m \in \mathbb{Z}} \psi_m^1 Y_m.$$

Then, we are looking for the unique solution $u(x, y)$ of the Dirichlet problem (1) in the form:

$$u(x, y) = \sum_{m=0}^{+\infty} u_m(x) Y_m(y).$$

Proposition 2.1. *The equation (1) is equivalent to the following countable system of Sturm-Liouville equations :*

$$\begin{cases} -v_m'' + qv_m = -\mu_m v_m, & \text{sur }]0, 1[\\ v_m(0) = f^{\frac{n-2}{4}}(0)\psi_m^0, & v_m(1) = f^{\frac{n-2}{4}}(1)\psi_m^1, \end{cases} \quad \forall m \in \mathbb{N}, \quad (6)$$

where

$$q = \frac{(f^{\frac{n-2}{4}})''}{f^{\frac{n-2}{4}}} - \lambda f \quad \text{and} \quad v_m = f^{\frac{n-2}{4}} u_m, \quad \forall m \in \mathbb{N}.$$

Proof. The transformation law for the Laplacian operator under conformal change of metric gives

$$-\Delta_g = f^{-\frac{n+2}{4}}(-\Delta_{g_0} + q_f)f^{\frac{n-2}{4}},$$

where $g_0 = dx^2 + g_S$ and $q_f = \frac{(f^{\frac{n-2}{4}})''}{f^{\frac{n-2}{4}}}$. Thus, we have :

$$\begin{aligned} -\Delta_g u(x, y) = \lambda u(x, y) &\Leftrightarrow f^{-\frac{n+2}{4}} \left(-\frac{\partial^2}{\partial x^2} - \Delta_{g_K} + q_f \right) f^{\frac{n-2}{4}} u(x, y) = \lambda u(x, y) \\ &\Leftrightarrow \left(-\frac{\partial^2}{\partial x^2} - \Delta_{g_K} + q_f \right) v(x, y) = \lambda f(x) v(x, y) \end{aligned}$$

thanks to the change of variable $v = f^{\frac{n-2}{4}} u$. Writing v as

$$v(x, y) = \sum_{m=0}^{+\infty} v_m(x) Y_m(y),$$

we get :

$$\begin{aligned} -\Delta_g u(x, y) = \lambda u(x, y) &\Leftrightarrow \sum_{m=0}^{+\infty} \left(-\frac{\partial^2}{\partial x^2} - \Delta_{g_K} + q_f \right) v_m(x) Y_m = \sum_{m=0}^{+\infty} \lambda f(x) v_m(x) Y_m \\ &\Leftrightarrow \sum_{m=0}^{+\infty} \left(-v_m''(x) + \mu_m v_m(x) + q_f(x) v_m(x) \right) Y_m \\ &= \sum_{m=0}^{+\infty} \lambda f(x) v_m(x) Y_m \\ &\Leftrightarrow \forall m \in \mathbb{N}, \quad -v_m''(x) + q(x) v_m(x) = -\mu_m v_m(x), \end{aligned}$$

where $q =: q_f - \lambda f$. Finally, v_m satisfies the above boundary conditions. \square

It turns out that this family of Sturm-Liouville equations fits into the so-called Weyl-Titchmarsh theory which we recall in the next section for the convenience of the reader.

2.2 The Weyl-Titchmarsh functions

Consider the differential equation:

$$-u'' + qu = -zu, \quad z \in \mathbb{C}. \quad (7)$$

Let $\{c_0, s_0\}$ and $\{c_1, s_1\}$ be the two fundamental systems of solutions of (7) with boundary conditions

$$\begin{cases} c_0(0) = 1, & c_0'(0) = 0 \\ c_1(1) = 1, & c_1'(1) = 0 \end{cases} \quad \text{and} \quad \begin{cases} s_0(0) = 0, & s_0'(0) = 1 \\ s_1(1) = 0, & s_1'(1) = 1. \end{cases} \quad (8)$$

Since the wronskian $W(f, g) = fg' - f'g$ of two solutions f and g of (7) depends only on the parameter z , we can define the following holomorphic functions:

Definition 2.2. The *characteristic function* $\Delta(z)$ of the equation (7) is defined for every $z \in \mathbb{C}$ by :

$$\Delta(z) := W(s_0, s_1) = s_0(1) = -s_1(0).$$

We also set $D(z) := W(c_0, s_1) = c_0(1)$ and $E(z) := W(c_1, s_0) = c_1(0)$.

Now, we recall the three following results, (see for instance [2] and [10] for details).

Proposition 2.3. *The functions Δ , D and E are analytic on \mathbb{C} and have on Π^+ the following asymptotics on $\mathbb{C} \setminus \mathbb{R}_-$:*

$$\begin{aligned} \Delta(z) &= \frac{\sinh(\sqrt{z})}{\sqrt{z}} + O\left(\frac{e^{|\Re(\sqrt{z})|}}{\sqrt{z}}\right), & D(z) &= \cosh(\sqrt{z}) + O\left(\frac{e^{|\Re(\sqrt{z})|}}{\sqrt{z}}\right), \\ E(z) &= \cosh(\sqrt{z}) + O\left(\frac{e^{|\Re(\sqrt{z})|}}{\sqrt{z}}\right). \end{aligned}$$

where \sqrt{z} is the principal square root of z . In particular, $\Delta(z)$, $D(z)$ and $E(z)$ are entire functions of order $\frac{1}{2}$.

Proposition 2.4. *The roots (α_j) of the function $z \mapsto \Delta(z)$ are real and simple. they are the eigenvalues of the operator $-\frac{d^2}{dx^2} + q =: H$ on $L^2((0, 1), dx)$ with Dirichlet boundary conditions.*

Proposition 2.5. *For all $z \in \mathbb{C}$, $\Delta(z)$ can be written as an infinite product. There is a constant $C \in \mathbb{R}$ such that :*

$$\Delta(z) = C \prod_{k=0}^{\infty} \left(1 - \frac{z}{\alpha_k}\right).$$

Proof. As a consequence of Proposition 2.3 and Hadamard's factorization theorem, we can write $\Delta(z)$ as the infinite product :

$$\Delta(z) = Cz^p \prod_{k=0}^{\infty} \left(1 - \frac{z}{\alpha_k}\right),$$

with $p \in \{0, 1\}$. In order to prove that $z = 0$ is not a root of Δ , we use the same argument as in [2] (Remark 3.1 p.19). If $\Delta(0) = 0$, it follows from Proposition 2.4, there is a eigenfunction u_0 associated to the eigenvalue 0 for the operator $H = -\frac{d^2}{dx^2} + q$. But, from Proposition 2.1, the function $u := u_0 Y_0$ is then a nontrivial solution of the Dirichlet problem :

$$\begin{cases} -\Delta_g u = \lambda u & \text{dans } M \\ u = 0 & \text{sur } \partial M, \end{cases}$$

which is not possible since $\lambda \notin \sigma(-\Delta_g)$. □

Remark 1. As a by-product, we see that $\Delta(z)$ is uniquely determined by its (simple) roots (up to a multiplicative constant).

Now, consider the Weyl-Titchmarsh solutions ψ and ϕ of (7) having the form :

$$\psi(x) = c_0(x) + M(z)s_0(x), \quad \phi(x) = c_1(x) - N(z)s_1(x)$$

and satisfying the Dirichlet boundary condition at $x = 1$ and $x = 0$ respectively. $M(z)$, (resp. $N(z)$) are called the *Weyl-Titchmarsh functions* associated to (7) and, using Wronskian identities, we have:

Proposition 2.6. *The Weyl-Titchmarsh functions M and N can be written as :*

$$\forall z \in \mathbb{C}, \quad M(z) = -\frac{D(z)}{\Delta(z)}, \quad N(z) = -\frac{E(z)}{\Delta(z)}.$$

Proof. Since $\psi(1) = 0$ and $\phi(0) = 0$, we have :

$$M(z) = -\frac{c_0(1, z)}{s_0(1, z)} = -\frac{D(z)}{\Delta(z)}, \quad N(z) = \frac{c_1(0, z)}{s_1(0, z)} = -\frac{D(z)}{\Delta(z)}.$$

□

The four previous propositions are the key points to solve our uniqueness result. We will see that the Steklov spectrum can be expressed in terms of the Weyl-Titchmarsh and characteristic functions defined above. We will take advantage of the holomorphic properties of $\Delta(z)$, $D(z)$ and $E(z)$ and we will use to the Nevanlinna theorem, (see the next section for details).

2.3 Link between the DN map and the Weyl-Titchmarsh functions

First, we remark that, thanks to separation of variables, $\Lambda_g(\lambda)$ leaves invariant each subspace spanned by $\{(Y_m, 0), (0, Y_m)\}$. Indeed, if u is the solution of (1), we have for each $\psi \in H^{1/2}(\partial M)$:

$$\Lambda_g(\lambda)\psi = \Lambda_g(\lambda) \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} = \begin{pmatrix} (\partial_\nu u)|_{\Gamma_0} \\ (\partial_\nu u)|_{\Gamma_1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{f(0)}}(\partial_x u)|_{x=0} \\ \frac{1}{\sqrt{f(1)}}(\partial_x u)|_{x=1} \end{pmatrix}$$

Consequently, for every $m \in \mathbb{N}$:

$$\Lambda_g(\lambda) \begin{pmatrix} \psi_m^0 \\ \psi_m^1 \end{pmatrix} \otimes Y_m = \begin{pmatrix} -\frac{1}{\sqrt{f(0)}}u'_m(0) \\ \frac{1}{\sqrt{f(1)}}u'_m(1) \end{pmatrix} \otimes Y_m.$$

Its restriction on each space spanned by $(1, 0) \otimes Y_m$ and $(0, 1) \otimes Y_m$ is denoted $\Lambda_g^m(\lambda)$. We can write $\Lambda_g^m(\lambda)$ the 2×2 matrix

Set :

$$\begin{pmatrix} L^m(\lambda) & T_R^m(\lambda) \\ T_L^m(\lambda) & R^m(\lambda) \end{pmatrix}$$

and we have by definition:

$$\begin{pmatrix} L^m(\lambda) & T_R^m(\lambda) \\ T_L^m(\lambda) & R^m(\lambda) \end{pmatrix} \begin{pmatrix} \psi_m^0 \\ \psi_m^1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{f(0)}}u'_m(0) \\ \frac{1}{\sqrt{f(1)}}u'_m(1) \end{pmatrix}.$$

The full Steklov spectrum is then equal to the union of the eigenvalues of each operator $\Lambda_g^m(\lambda)$. In the next Proposition, we express the restricted operator $\Lambda_g^m(\lambda)$ in terms of the Weyl-Titchmarsh functions.

Proposition 2.7. *For all $m \in \mathbb{N}$, we have :*

$$\Lambda_g^m(\lambda) = \begin{pmatrix} -\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{\ln'(h)(0)}{4\sqrt{f(0)}} & -\frac{1}{\sqrt{f(0)}} \frac{h^{1/4}(1)}{h^{1/4}(0)} \frac{1}{\Delta(\mu_m)} \\ -\frac{1}{\sqrt{f(1)}} \frac{h^{1/4}(0)}{h^{1/4}(1)} \frac{1}{\Delta(\mu_m)} & -\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{\ln'(h)(1)}{4\sqrt{f(1)}} \end{pmatrix}$$

where $h =: f^{n-2}$

Proof. Recall that v_m is the m -th Fourier coefficient of $v = h^{1/4}u$ with respect to Y_m . We have :

$$\begin{aligned}\Lambda_g^m(\lambda) \begin{pmatrix} \psi_m^0 \\ \psi_m^1 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{\sqrt{f(0)}}u'_m(0) \\ \frac{1}{\sqrt{f(1)}}u'_m(1) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{v'_m(0)}{h^{1/4}(0)\sqrt{f(0)}} + \frac{1}{4\sqrt{f(0)}}\frac{h'(0)}{h(0)}u_m(0) \\ \frac{v'_m(1)}{h^{1/4}(1)\sqrt{f(1)}} - \frac{1}{4\sqrt{f(1)}}\frac{h'(1)}{h(1)}u_m(1) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{v'_m(0)}{h^{1/4}(0)\sqrt{f(0)}} \\ \frac{v'_m(1)}{h^{1/4}(1)\sqrt{f(1)}} \end{pmatrix} + \begin{pmatrix} \frac{1}{4\sqrt{f(0)}}\frac{h'(0)}{h(0)}\psi_m^0 \\ -\frac{1}{4\sqrt{f(1)}}\frac{h'(1)}{h(1)}\psi_m^1 \end{pmatrix}\end{aligned}$$

It remains to find a simple expression of the first term of the (RHS). Since v_m is a solution of (6), it can be written as a linear combination of the fundamental solutions $\{c_0, s_0\}$ or $\{c_1, s_1\}$ defined in (8), i.e there exists $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$ such that :

$$v_m = \alpha c_0 + \beta s_0 = \gamma c_1 + \delta s_1.$$

Consequently:

$$\begin{pmatrix} v_m(0) \\ v_m(1) \end{pmatrix} = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} \gamma c_1(0) + \delta s_1(0) \\ \alpha c_0(1) + \beta s_0(1) \end{pmatrix}$$

The second equality can be rewritten as:

$$\begin{pmatrix} \alpha - \gamma c_1(0) \\ \gamma - \alpha c_0(1) \end{pmatrix} = \begin{pmatrix} \delta s_1(0) \\ \beta s_0(1) \end{pmatrix},$$

which is equivalent to :

$$\begin{pmatrix} 1 & -c_1(0) \\ -c_0(1) & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} \delta s_1(0) \\ \beta s_0(1) \end{pmatrix}.$$

As

$$\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} h^{1/4}(0)u_m(0) \\ h^{1/4}(1)u_m(1) \end{pmatrix} = \begin{pmatrix} h^{1/4}(0)\psi_m^0 \\ h^{1/4}(1)\psi_m^1 \end{pmatrix}$$

we obtain:

$$\begin{pmatrix} \frac{h^{1/4}(0)}{s_1(0)} & -\frac{h^{1/4}(1)c_1(0)}{s_1(0)} \\ -\frac{h^{1/4}(0)c_0(1)}{s_0(1)} & \frac{h^{1/4}(1)}{s_0(1)} \end{pmatrix} \begin{pmatrix} \psi_m^0 \\ \psi_m^1 \end{pmatrix} = \begin{pmatrix} \delta \\ \beta \end{pmatrix}.$$

But $\delta = v'_m(1)$ and $\beta = v'_m(0)$. Thus:

$$\begin{aligned}
\Lambda_g^m(\lambda) \begin{pmatrix} \psi_m^0 \\ \psi_m^1 \end{pmatrix} &= \begin{pmatrix} -\frac{v'_m(0)}{h^{1/4}(0)\sqrt{f(0)}} \\ \frac{v'_m(1)}{h^{1/4}(1)\sqrt{f(1)}} \end{pmatrix} + \begin{pmatrix} \frac{1}{4\sqrt{f(0)}} \frac{h'(0)}{h(0)} \psi_m^0 \\ -\frac{1}{4\sqrt{f(1)}} \frac{h'(1)}{h(1)} \psi_m^1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{f(0)}} \frac{c_0(1)}{s_0(1)} & -\frac{1}{\sqrt{f(0)}} \frac{h^{1/4}(1)}{h^{1/4}(0)s_0(1)} \\ \frac{1}{\sqrt{f(1)}} \frac{h^{1/4}(0)}{h^{1/4}(1)s_1(0)} & -\frac{1}{\sqrt{f(1)}} \frac{c_1(0)}{s_1(0)} \end{pmatrix} \begin{pmatrix} \psi_m^0 \\ \psi_m^1 \end{pmatrix} + \begin{pmatrix} \frac{1}{4\sqrt{f(0)}} \frac{h'(0)}{h(0)} \psi_m^0 \\ -\frac{1}{4\sqrt{f(1)}} \frac{h'(1)}{h(1)} \psi_m^1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{f(0)}} \frac{c_0(1)}{s_0(1)} + \frac{1}{4\sqrt{f(0)}} \frac{h'(0)}{h(0)} & -\frac{1}{\sqrt{f(0)}} \frac{h^{1/4}(1)}{h^{1/4}(0)s_0(1)} \\ \frac{1}{\sqrt{f(1)}} \frac{h^{1/4}(0)}{h^{1/4}(1)s_1(0)} & -\frac{1}{\sqrt{f(1)}} \frac{c_1(0)}{s_1(0)} - \frac{1}{4\sqrt{f(1)}} \frac{h'(1)}{h(1)} \end{pmatrix} \begin{pmatrix} \psi_m^0 \\ \psi_m^1 \end{pmatrix}
\end{aligned}$$

Recalling that:

$$M(\mu_m) = -\frac{c_0(1)}{s_0(1)}, \quad N(\mu_m) = \frac{c_1(0)}{s_1(0)} \quad \text{et} \quad \Delta(\mu_m^2) = -s_1(0) = s_0(1),$$

we get finally:

$$\Lambda_g^m(\lambda) = \begin{pmatrix} -\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{1}{4\sqrt{f(0)}} \frac{h'(0)}{h(0)} & -\frac{1}{\sqrt{f(0)}} \frac{h^{1/4}(1)}{h^{1/4}(0)} \frac{1}{\Delta(\mu_m)} \\ -\frac{1}{\sqrt{f(1)}} \frac{h^{1/4}(0)}{h^{1/4}(1)} \frac{1}{\Delta(\mu_m)} & -\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{1}{4\sqrt{f(1)}} \frac{h'(1)}{h(1)} \end{pmatrix}.$$

□

3 A characterization by the trace and the determinant

We have the following result:

Proposition 3.1. *Assume that, for every $m \in \mathbb{N}$, we have :*

$$\det(\Lambda_g^m(\lambda)) = \det(\Lambda_g^m(\lambda)) \quad \text{and} \quad \text{Tr}(\Lambda_g^m(\lambda)) = \text{Tr}(\Lambda_g^m(\lambda)).$$

Then :

$$f = \tilde{f} \quad \text{or} \quad f = \tilde{f} \circ \eta$$

where, for all $x \in [0, 1]$, $\eta(x) = 1 - x$.

Remark 2. This Proposition is still true if the equalities about the trace and the determinant of $\Lambda_g^m(\lambda)$ are satisfied for $m \geq m_0$, with $m_0 \in \mathbb{N}$.

In order to prove this proposition, let us calculate the eigenvalues of the operator $\Lambda_g^m(\lambda)$ and their asymptotics.

Lemma 3.2. $\Lambda_g^m(\lambda)$ has two eigenvalues $\lambda^-(\mu_m)$ and $\lambda^+(\mu_m)$ whose asymptotics are given by :

$$\begin{cases} \lambda^-(\mu_m) \underset{m \rightarrow \infty}{=} \frac{\sqrt{\mu_m}}{\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right) \\ \lambda^+(\mu_m) \underset{m \rightarrow \infty}{=} \frac{\sqrt{\mu_m}}{\sqrt{f(0)}} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right). \end{cases}$$

Remark 3. If $f(0) < f(1)$, we have $\lambda^-(\mu_m) < \lambda^+(\mu_m)$. This assumption will be made, without loss of generality, each time that $f(0) \neq f(1)$. The notations $\lambda^-(\mu_m)$ and $\lambda^+(\mu_m)$ refer to that choice.

Remark 4. The Weyl's law for the eigenvalues of the Laplace-Beltrami operator gives the following asymptotic :

$$\mu_m = 4\pi^2 \left(\text{Vol}(\mathbb{B}^{n-1}) \text{Vol}(\mathbb{S}^{n-1}) \right)^{-\frac{2}{n-1}} m^{\frac{2}{n-1}} + O(1),$$

so, by replacing it in Lemma 3.2, one has :

$$\begin{aligned} \lambda^-(\mu_m) &\underset{m \rightarrow \infty}{=} \frac{\sqrt{\mu_m}}{\sqrt{f(1)}} + O(1) \\ &= \frac{2\pi}{\sqrt{f(1)}} \left(\text{Vol}(\mathbb{B}^{n-1}) \text{Vol}(\mathbb{S}^{n-1}) \right)^{-\frac{1}{n-1}} m^{\frac{1}{n-1}} + O(1). \end{aligned}$$

The boundary component Γ_1 is the sphere equipped with the metric $\gamma_1 = f(1)g_{\mathbb{S}}$. It follows that we have $\text{Vol}(\Gamma_1) = \int_{\Gamma_1} d\text{Vol}_{\gamma_1} = f(1)^{\frac{n-1}{2}} \text{Vol}(\mathbb{S}^{n-1})$, hence $\frac{1}{\sqrt{f(1)}} = \frac{\text{Vol}(\mathbb{S}^{n-1})^{\frac{1}{n-1}}}{\text{Vol}(\Gamma_1)^{\frac{1}{n-1}}}$. Consequently :

$$\lambda^-(\mu_m) = 2\pi \left(\frac{m}{\text{Vol}(\mathbb{B}^{n-1}) \text{Vol}(\Gamma_1)} \right)^{\frac{1}{n-1}} + O(1).$$

In other words, an asymptotic of $\lambda^-(\mu_m)$ is exactly given by the Weyl's law restricted to the connected component boundary Γ_1 . In the same way, one can prove :

$$\lambda^+(\mu_m) = 2\pi \left(\frac{m}{\text{Vol}(\mathbb{B}^{n-1}) \text{Vol}(\Gamma_0)} \right)^{\frac{1}{n-1}} + O(1).$$

We recognize again the Weyl's law, restricted to the connected component boundary Γ_0 .

Let us prove Lemma 3.2.

Proof. We distinguish two cases :

- Assume $f(0) \neq f(1)$ (for instance $f(0) < f(1)$).

The characteristic polynomial $P(X)$ of $\Lambda_g^m(\lambda)$ is :

$$P(X) = X^2 - \text{Tr}(\Lambda_g^m(\lambda))X + \det(\Lambda_g^m(\lambda)).$$

To simplify the notation, we set :

$$C_0 = \frac{\ln(h)'(0)}{4\sqrt{f(0)}}, \quad C_1 = \frac{\ln(h)'(1)}{4\sqrt{f(1)}}.$$

Thanks to Propositions 2.3 and 2.7, for m large enough, $\text{Tr}(\Lambda_g^m(\lambda))$ and $\det(\Lambda_g^m(\lambda))$ satisfy:

$$\begin{cases} \text{Tr}(\Lambda_g^m(\lambda)) = -\frac{M(\mu_m)}{\sqrt{f(0)}} - \frac{N(\mu_m)}{\sqrt{f(1)}} + C_0 - C_1. \\ \det(\Lambda_g^m(\lambda)) = \left(-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 \right) \left(-\frac{N(\mu_m)}{\sqrt{f(1)}} - C_1 \right) + O(e^{-2\mu_m}). \end{cases}$$

The asymptotics of the discriminant δ of $P(X)$ depending on $M(\mu_m)$ and $N(\mu_m)$ can thus be written :

$$\begin{aligned}\delta &= \left(-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 - \frac{N(\mu_m)}{\sqrt{f(1)}} - C_1 \right)^2 - 4 \left(-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 \right) \left(-\frac{N(\mu_m)}{\sqrt{f(1)}} - C_1 \right) \\ &\quad + O(e^{-2\mu_m}) \\ &= \left(-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 + \frac{N(\mu_m)}{\sqrt{f(1)}} + C_1 \right)^2 + O(e^{-2\mu_m}).\end{aligned}$$

Now, let us recall the result obtained by Simon in [15] :

Theorem 3.3. $M(z^2)$ has the following asymptotic expansion :

$$\forall A \in \mathbb{N}, \quad -M(z^2) \underset{z \rightarrow \infty}{=} z + \sum_{j=0}^A \frac{\beta_j(0)}{z^{j+1}} + o\left(\frac{1}{z^{A+1}}\right)$$

$$\text{where, for every } x \in [0, 1], \beta_j(x) \text{ is defined by : } \begin{cases} \beta_0(x) = \frac{1}{2}q(x) \\ \beta_{j+1}(x) = \frac{1}{2}\beta_j'(x) + \frac{1}{2}\sum_{l=0}^j \beta_l(x)\beta_{j-l}(x). \end{cases}$$

Of course, by symmetry, one has immediately:

Corollary 3.4. $N(z^2)$ has the following asymptotic expansion :

$$\forall A \in \mathbb{N}, \quad -N(z^2) \underset{z \rightarrow \infty}{=} z + \sum_{j=0}^A \frac{\gamma_j(0)}{z^{j+1}} + o\left(\frac{1}{z^{A+1}}\right)$$

$$\text{where, for all } x \in [0, 1], \gamma_j(x) \text{ is defined by : } \begin{cases} \gamma_0(x) = \frac{1}{2}q(1-x) \\ \gamma_{j+1}(x) = \frac{1}{2}\gamma_j'(x) + \frac{1}{2}\sum_{l=0}^j \gamma_l(x)\gamma_{j-l}(x). \end{cases}$$

We deduce from Theorem 3.3 and Corollary 3.4:

$$-\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{N(\mu_m)}{\sqrt{f(1)}} = \underbrace{\left(\frac{1}{\sqrt{f(0)}} - \frac{1}{\sqrt{f(1)}} \right)}_{>0} \sqrt{\mu_m} + O\left(\frac{1}{\sqrt{\mu_m}}\right).$$

Thus, recalling that

$$\delta = \left(-\frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 + \frac{N(\mu_m)}{\sqrt{f(1)}} + C_1 \right)^2 + O(e^{-2\mu_m})$$

we obtain:

$$\begin{aligned}\sqrt{\delta} &= \left(\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 + C_1 \right) \sqrt{1 + O\left(\frac{e^{-2\mu_m}}{\mu_m}\right)} \\ &= \frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{M(\mu_m)}{\sqrt{f(0)}} + C_0 + C_1 + O\left(\frac{e^{-2\mu_m}}{\sqrt{\mu_m}}\right).\end{aligned}$$

Hence :

$$\begin{cases} \lambda^-(\mu_m) = \frac{1}{2} \left[\left(-\frac{M(\mu_m)}{\sqrt{f(0)}} - \frac{N(\mu_m)}{\sqrt{f(1)}} + C_0 - C_1 \right) - \sqrt{\delta} + O(e^{-2\mu_m}) \right] \\ \lambda^+(\mu_m) = \frac{1}{2} \left[\left(-\frac{M(\mu_m)}{\sqrt{f(0)}} - \frac{N(\mu_m)}{\sqrt{f(1)}} + C_0 - C_1 \right) + \sqrt{\delta} + O(e^{-2\mu_m}) \right], \end{cases}$$

and therefore, substituting C_1 and C_2 by their values and $M(\mu_m)$ and $N(\mu_m)$ by their asymptotics, we get :

$$\begin{cases} \lambda^-(\mu_m) = \frac{\sqrt{\mu_m}}{\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right) \\ \lambda^+(\mu_m) = \frac{\sqrt{\mu_m}}{\sqrt{f(0)}} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right). \end{cases}$$

- Assume now $f(0) = f(1)$. In this case, the restricted DN map

$$\Lambda_g^m(\lambda) = \begin{pmatrix} -\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}} & -\frac{1}{\sqrt{f(0)}} \frac{1}{\Delta(\mu_m)} \\ -\frac{1}{\sqrt{f(0)}} \frac{1}{\Delta(\mu_m)} & -\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} \end{pmatrix}$$

is a symmetric matrix and we can use the well-known result:

Lemma 3.5. *Let H be a Hilbert space, $A \in \mathcal{L}(H)$ be a selfadjoint operator. Let $\epsilon > 0$. Assume there exists $\lambda_0 \in \mathbb{R}$ and $u_0 \in H$ a unit vector such that $\|(A - \lambda_0 Id)u_0\| \leq \epsilon$. Then there exists an element λ in the spectrum of A such that $|\lambda - \lambda_0| \leq \epsilon$.*

We apply this theorem with $A = \Lambda_g^m(\lambda)$, $\lambda_0 = -\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}}$ and $U_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

We have :

$$A - \lambda_0 I_2 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{f(0)}} \frac{1}{\Delta(\mu_m)} \\ -\frac{1}{\sqrt{f(0)}} \frac{1}{\Delta(\mu_m)} & -\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} - \lambda_0 \end{pmatrix}$$

Hence :

$$(A - \lambda_0 I_2)U_0 = -\frac{1}{\sqrt{f(0)}} \begin{pmatrix} 0 \\ -\frac{1}{\Delta(\mu_m)} \end{pmatrix} = O(\sqrt{\mu_m} e^{\sqrt{\mu_m}}).$$

Theorem 3.5 gives $\lambda_m^+ \in \sigma(\Lambda_g^m(\lambda))$ such that :

$$\begin{aligned} \lambda_m^+ &= -\frac{M(\mu_m)}{\sqrt{f(0)}} - \frac{\ln(h)'(0)}{4\sqrt{f(0)}} + O(\sqrt{\mu_m} e^{-\sqrt{\mu_m}}) \\ &= -\frac{\sqrt{\mu_m}}{\sqrt{f(0)}} - \frac{\ln(h)'(0)}{4\sqrt{f(0)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right) \end{aligned}$$

from Theorem 3.3. The second eigenvalue λ_m^2 (which is distinct from λ_m^1 otherwise $\Lambda_g^m(\lambda)$ would be a homothety) can be deduced from the first one :

$$\begin{aligned}\lambda_m^- &= \text{Tr}(\Lambda_g^m(\lambda)) - \lambda_m^+ \\ &= -\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} + O(\sqrt{\mu_m}e^{-\sqrt{\mu_m}}) \\ &= \frac{\sqrt{\mu_m}}{\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right).\end{aligned}$$

□

Lemma 3.6. *Under the hypothesis of Proposition 3.1, we have the following alternative :*

$$\begin{cases} f(0) = \tilde{f}(0) \\ f(1) = \tilde{f}(1) \end{cases} \quad \text{or} \quad \begin{cases} f(0) = \tilde{f}(1) \\ f(1) = \tilde{f}(0). \end{cases}$$

Proof. We begin with:

$$\frac{\text{Tr}(\Lambda_g^m(\lambda))}{\sqrt{\mu_m}} = \frac{\text{Tr}(\Lambda_{\tilde{g}}^m(\lambda))}{\sqrt{\mu_m}}, \quad \forall m \in \mathbb{N}.$$

Thus, it follows from Lemma 3.2 that

$$\frac{1}{\sqrt{f(0)}} + \frac{1}{\sqrt{f(1)}} = \frac{1}{\sqrt{\tilde{f}(0)}} + \frac{1}{\sqrt{\tilde{f}(1)}}.$$

In the same way, thanks to the relations:

$$\frac{\det(\Lambda_g^m(\lambda))}{\mu_m} = \frac{\det(\Lambda_{\tilde{g}}^m(\lambda))}{\mu_m} \quad \forall m \in \mathbb{N},$$

we get :

$$\frac{1}{\sqrt{f(0)f(1)}} = \frac{1}{\sqrt{\tilde{f}(0)\tilde{f}(1)}},$$

and the proof is complete. □

Case 1 : $f(0) = \tilde{f}(0)$ et $f(1) = \tilde{f}(1)$.

Lemma 3.7. *Under the hypotheses of Proposition 3.1, we have:*

$$\Delta(z) = \tilde{\Delta}(z), \quad \forall z \in \mathbb{C}. \quad (9)$$

Proof. We write the equality $\text{Tr}(\Lambda_g^m(\lambda)) = \text{Tr}(\Lambda_{\tilde{g}}^m(\lambda))$ as follows : for all $\mu_m \in \sigma(-\Delta_{gK})$,

$$\begin{aligned}\left(-\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}}\right) + \left(-\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}}\right) &= \left(-\frac{\tilde{M}(\mu_m)}{\sqrt{f(0)}} + \frac{\ln(\tilde{h})'(0)}{4\sqrt{f(0)}}\right) \\ &\quad + \left(-\frac{\tilde{N}(\mu_m)}{\sqrt{f(1)}} - \frac{\ln(\tilde{h})'(1)}{4\sqrt{f(1)}}\right).\end{aligned}$$

Thus, using Theorem 3.3, one gets when $m \rightarrow +\infty$,

$$\frac{\ln(h)'(0)}{4\sqrt{f(0)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} = \frac{\ln(\tilde{h})'(0)}{4\sqrt{f(0)}} - \frac{\ln(\tilde{h})'(1)}{4\sqrt{f(1)}}. \quad (10)$$

It follows that

$$\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{N(\mu_m)}{\sqrt{f(1)}} = \frac{\tilde{M}(\mu_m)}{\sqrt{f(0)}} + \frac{\tilde{N}(\mu_m)}{\sqrt{f(1)}},$$

or equivalently

$$\tilde{\Delta}(\mu_m) \left(\frac{D(\mu_m)}{\sqrt{f(0)}} + \frac{E(\mu_m)}{\sqrt{f(1)}} \right) = \Delta(\mu_m) \left(\frac{\tilde{D}(\mu_m)}{\sqrt{f(0)}} + \frac{\tilde{E}(\mu_m)}{\sqrt{f(1)}} \right). \quad (11)$$

In the same way, using $\det(\Lambda_g^m(\lambda)) = \det(\tilde{\Lambda}_g^m(\lambda))$, we have:

$$\begin{aligned} & \left(\frac{M(\mu_m)}{\sqrt{f(0)}} + \frac{\ln'(h)(0)}{4\sqrt{f(0)}} \right) \left(\frac{N(\mu_m)}{\sqrt{f(1)}} - \frac{\ln'(h)(1)}{4\sqrt{f(1)}} \right) - \frac{1}{\sqrt{f(0)f(1)}\Delta^2(\mu_m)} \\ &= \left(\frac{\tilde{M}(\mu_m)}{\sqrt{f(0)}} + \frac{\ln(\tilde{h})'(0)}{4\sqrt{f(0)}} \right) \times \left(\frac{\tilde{N}(\mu_m)}{\sqrt{f(1)}} - \frac{\ln'(\tilde{h})(1)}{4\sqrt{f(1)}} \right) - \frac{1}{\sqrt{f(0)f(1)}\tilde{\Delta}^2(\mu_m)}. \end{aligned}$$

Multiplying on both sides by $\Delta^2(\mu_m)\tilde{\Delta}^2(\mu_m)$, we obtain :

$$\begin{aligned} & \tilde{\Delta}^2(\mu_m) \left[\left(\frac{D(\mu_m)}{\sqrt{f(0)}} + \Delta(\mu_m) \frac{\ln'(h)(0)}{4\sqrt{f(0)}} \right) \left(\frac{E(\mu_m)}{\sqrt{f(1)}} - \Delta(\mu_m) \frac{\ln'(h)(1)}{4\sqrt{f(1)}} \right) - \frac{1}{\sqrt{f(0)f(1)}} \right] \\ &= \Delta^2(\mu_m) \left[\left(\frac{\tilde{D}(\mu_m)}{\sqrt{f(0)}} + \tilde{\Delta}(\mu_m) \frac{\ln(\tilde{h})'(0)}{4\sqrt{f(0)}} \right) \times \left(\frac{\tilde{E}(\mu_m)}{\sqrt{f(1)}} - \tilde{\Delta}(\mu_m) \frac{\ln'(\tilde{h})(1)}{4\sqrt{f(1)}} \right) - \frac{1}{\sqrt{f(0)f(1)}} \right] \end{aligned} \quad (12)$$

Now, let us show that the equalities (11) and (12) can be analytically extended with respect to μ_m in the half-right complex plane. First, let us recall the definition of the so-called Nevanlinna class:

Definition 3.8. Set $\Pi^+ = \{z \in \mathbb{C}, \Re(z) > 0\}$ the half-right plane of the complex plane. The Nevanlinna class $\mathcal{N}(\Pi^+)$ is the set of analytic functions f on Π^+ such that

$$\sup_{0 < r < 1} \int_{-\pi}^{\pi} \ln^+ \left| f \left(\frac{1 - re^{i\theta}}{1 + re^{i\theta}} \right) \right| d\theta < \infty,$$

with:

$$\ln^+(x) = \begin{cases} \ln x & \text{if } \ln(x) \geq 0 \\ 0 & \text{if } \ln(x) < 0. \end{cases}$$

We have the following result [11]:

Proposition 3.9. Let $h \in H(\Pi^+)$ an analytic function on Π^+ , A and C two constants. Assume :

$$|h(z)| \leq Ce^{A\Re(z)}, \quad \forall z \in \Pi^+.$$

Then $h \in \mathcal{N}(\Pi^+)$.

Thus, thanks to the asymptotics of Proposition 2.3, the holomorphic functions defined by $\delta : z \mapsto \Delta(z^2)$, $d : z \mapsto D(z^2)$ and $e : z \mapsto E(z^2)$ belong to $\mathcal{N}(\Pi^+)$. Let us recall now a useful uniqueness theorem involving functions in the Nevanlinna class (see [11] for instance):

Theorem 3.10. Let $h \in \mathcal{N}(\Pi^+)$ and $\mathcal{L} \subset \mathbb{R}_+$ be a countable set such that

$$\sum_{\mu \in \mathcal{L}} \frac{1}{\mu} = +\infty.$$

Then :

$$(h(\mu) = 0, \forall \mu \in \mathcal{L}) \Rightarrow h \equiv 0 \text{ sur } \Pi^+.$$

Now, by the Weyl law, (cf [12]), we get:

$$\mu_m \underset{m \rightarrow +\infty}{=} c(K)m^{\frac{2}{n-1}} + O(1)$$

where $c(K) = \frac{\sqrt{2\pi}}{(\omega_1 \text{vol}(K))^{\frac{1}{n-1}}}$ and ω_1 is the volume of the unit ball in \mathbb{R}^{n-1} . Thus, for a fixed $T \in \mathbb{N}$, we have :

$$\frac{\mu_{(mT)^{n-1}}}{c(K)T} = m^2 + \frac{O(1)}{c(K)T}.$$

As a consequence, for m and T large enough, the real numbers $\mu_{(mT)^{n-1}}$ are always distinct. Now, we set :

$$\mathcal{L} = \{\sqrt{\mu_{(mT)^{n-1}}}, m \in \mathbb{N}\}.$$

Using $\sqrt{\mu_{(mT)^{n-1}}} \underset{m \rightarrow +\infty}{\sim} \sqrt{c(K)T}m$, one has:

$$\sum_{\mu \in \mathcal{L}} \frac{1}{\mu} = +\infty.$$

Thus, thanks to Theorem 3.10, the relations (11) and (12) are still true if one replaces the integers m by $z \in \mathbb{C}$.

Now, let us prove that $\Delta(z) = \tilde{\Delta}(z)$ for any $z \in \mathbb{C}$. We recall that these functions are entire of order $\frac{1}{2}$ and their roots are simple. So, using Hadamard's factorization theorem (see Proposition 2.5), we deduce that these functions are entirely described by their roots (up to a multiplicative constant). Consequently, in order to prove that $\Delta = \tilde{\Delta}$, it is enough to show that their roots are the same.

Set $\mathcal{P} = \{\alpha_j \in \mathbb{C}, \Delta(\alpha_j) = 0\}$. Let α_k be in \mathcal{P} and let us show that $\tilde{\Delta}(\alpha_k) = 0$. By definition, $-\alpha_k$ is an eigenvalue of the Sturm Liouville operator $H = -\frac{d^2}{dx^2} + q$ with Dirichlet boundary conditions. Thus, from Proposition 2.4, α_k is real and since the potential q is real, the quantities $D(\alpha_k)$ and $E(\alpha_k)$ are also real. Using (11) and (12) with μ_m replaced with α_k , we obtain the following system:

$$\begin{cases} \left(\frac{D(\alpha_k)}{\sqrt{f(0)}} + \frac{E(\alpha_k)}{\sqrt{f(1)}} \right) \tilde{\Delta}(\alpha_k) = 0 \\ \left(\frac{D(\alpha_k)E(\alpha_k) - 1}{\sqrt{f(0)f(1)}} \right) \tilde{\Delta}(\alpha_k)^2 = 0. \end{cases}$$

To finish the proof, we distinguish two cases:

- If $\frac{D(\alpha_k)}{\sqrt{f(0)}} - \frac{E(\alpha_k)}{\sqrt{f(1)}} \neq 0$ then $\tilde{\Delta}(\alpha_k) = 0$.
- Otherwise $\frac{D(\alpha_k)}{\sqrt{f(0)}} = -\frac{E(\alpha_k)}{\sqrt{f(1)}}$. Then, substituting in the second equality, we have:

$$\left(\frac{D(\alpha_k)^2}{f(0)} + \frac{1}{\sqrt{f(0)f(1)}} \right) \tilde{\Delta}(\alpha_k)^2 = 0$$

Since $D(\alpha_k) \in \mathbb{R}$, we conclude that $\tilde{\Delta}(\alpha_k) = 0$. By a standard symmetric argument, it follows Δ and $\tilde{\Delta}$ have the same zeros. So, there exists C such that $\Delta = C\tilde{\Delta}$. But, as $\Delta(z)$ and $\tilde{\Delta}(z)$ have the same asymptotics :

$$\Delta(z), \tilde{\Delta}(z) \sim \frac{\sinh(\sqrt{z})}{\sqrt{z}},$$

we deduce that $C = 1$ and the proof is complete. \square

Remark 5. It is interesting to note that $\Delta(z) = \tilde{\Delta}(z)$ for all $z \in \mathbb{C}$ implies that (in fact is equivalent to) q and \tilde{q} are isospectral.

As a consequence, we can simplify (12) (with μ_m replaced by z), and we get on \mathbb{C} :

$$\left\{ \begin{array}{l} \left(\frac{M(z)}{\sqrt{f(0)}} + \frac{\ln'(h)(0)}{4\sqrt{f(0)}} \right) + \left(\frac{N(z)}{\sqrt{f(1)}} - \frac{\ln'(h)(1)}{4\sqrt{f(1)}} \right) = \left(\frac{\tilde{M}(z)}{\sqrt{f(0)}} + \frac{\ln(\tilde{h})'(0)}{4\sqrt{f(0)}} \right) \\ \hspace{20em} + \left(\frac{\tilde{N}(z)}{\sqrt{f(1)}} - \frac{\ln(\tilde{h})(1)}{4\sqrt{f(1)}} \right) \\ \left(\frac{M(z)}{\sqrt{f(0)}} + \frac{\ln'(h)(0)}{4\sqrt{f(0)}} \right) \left(\frac{N(z)}{\sqrt{f(1)}} - \frac{\ln'(h)(1)}{4\sqrt{f(1)}} \right) = \left(\frac{\tilde{M}(z)}{\sqrt{f(0)}} + \frac{\ln(\tilde{h})'(0)}{4\sqrt{f(0)}} \right) \\ \hspace{20em} \times \left(\frac{\tilde{N}(z)}{\sqrt{f(1)}} - \frac{\ln(\tilde{h})(1)}{4\sqrt{f(1)}} \right). \end{array} \right.$$

Thus, for each $z \in \mathbb{C}$, we have the following alternative:

$$\left\{ \begin{array}{l} \frac{M(z)}{\sqrt{f(0)}} + \frac{\ln'(h)(0)}{4\sqrt{f(0)}} = \frac{\tilde{M}(z)}{\sqrt{f(0)}} + \frac{\ln(\tilde{h})'(0)}{4\sqrt{f(0)}} \\ \frac{N(z)}{\sqrt{f(1)}} - \frac{\ln'(h)(1)}{4\sqrt{f(1)}} = \frac{\tilde{N}(z)}{\sqrt{f(1)}} - \frac{\ln(\tilde{h})(1)}{4\sqrt{f(1)}}, \end{array} \right.$$

or

$$\left\{ \begin{array}{l} \frac{M(z)}{\sqrt{f(0)}} + \frac{\ln'(h)(0)}{4\sqrt{f(0)}} = \frac{\tilde{N}(z)}{\sqrt{f(1)}} - \frac{\ln(\tilde{h})(1)}{4\sqrt{f(1)}} \\ \frac{N(z)}{\sqrt{f(1)}} - \frac{\ln'(h)(1)}{4\sqrt{f(1)}} = \frac{\tilde{M}(z)}{\sqrt{f(0)}} + \frac{\ln(\tilde{h})'(0)}{4\sqrt{f(0)}}. \end{array} \right.$$

• As a first step, assume that the first system is satisfied for all z in \mathbb{C} . Then, using the asymptotics of $M(z)$, (resp. $\tilde{M}(z)$), we get:

$$\frac{\ln'(h)(0)}{4\sqrt{f(0)}} = \frac{\ln(\tilde{h})'(0)}{4\sqrt{f(0)}}$$

and therefore :

$$M(z) = \tilde{M}(z) \text{ for all } z \in \mathbb{C} \setminus \mathbb{R}.$$

Thus, thanks to the Borg-Marchenko's theorem (see [15] for instance), we deduce:

$$q = \tilde{q} \quad \text{sur } [0, 1]. \quad (13)$$

Recall that :

$$q = \frac{(h^{1/4})''}{h^{1/4}} - \lambda f$$

where we have set $h = f^{n-2}$ (respectively \tilde{q} and $\tilde{h} = \tilde{f}^{n-2}$). In particular, in the 2 dimensional case, we get immediately $f = \tilde{f}$ since $\lambda \neq 0$. In dimension $n \geq 3$, we see that f and \tilde{f} same verify the same ODE with $f(0) = \tilde{f}(0)$. Moreover, the relation

$$\frac{\ln'(h)(0)}{4\sqrt{f(0)}} = \frac{\ln(\tilde{h})'(0)}{4\sqrt{\tilde{f}(0)}},$$

implies :

$$f'(0) = \tilde{f}'(0).$$

Thus, the Cauchy-Lipschitz's theorem says that $f = \tilde{f}$.

- Secondly, assume there exists $z_0 \in \mathbb{C}$ satisfying :

$$\frac{M(z_0)}{\sqrt{f(0)}} + \frac{\ln'(h)(0)}{4\sqrt{f(0)}} \neq \frac{\tilde{M}(z_0)}{\sqrt{f(0)}} + \frac{\ln(\tilde{h})'(0)}{4\sqrt{\tilde{f}(0)}}.$$

By a standard continuity argument, there is a ball B of center z_0 such that :

$$\frac{M(z)}{\sqrt{f(0)}} + \frac{\ln'(h)(0)}{4\sqrt{f(0)}} \neq \frac{\tilde{M}(z)}{\sqrt{f(0)}} + \frac{\ln(\tilde{h})'(0)}{4\sqrt{\tilde{f}(0)}}, \quad \forall z \in B.$$

Then, necessarily, we have:

$$\frac{M(z)}{\sqrt{f(0)}} + \frac{\ln'(h)(0)}{4\sqrt{f(0)}} = \frac{\tilde{N}(z)}{\sqrt{f(1)}} - \frac{\ln(\tilde{h})'(1)}{4\sqrt{\tilde{f}(1)}}, \quad \forall z \in B. \quad (14)$$

Then, using the analytic continuation principle, the previous equality is true for every $\mathbb{C} \setminus \mathcal{P}$, where \mathcal{P} is the set of roots of $\Delta(z)$. Thanks to the asymptotics of $M(z)$ and $\tilde{N}(z)$, one gets :

$$f(0) = f(1) \quad (= \tilde{f}(1)) \quad \text{and} \quad \frac{\ln'(h)(0)}{4\sqrt{f(0)}} = -\frac{\ln(\tilde{h})'(1)}{4\sqrt{\tilde{f}(1)}}.$$

Hence, simplifying in (14), we obtain

$$M(z) = \tilde{N}(z) \text{ for all } z \in \mathbb{C} \setminus \mathbb{R}.$$

By symmetry, \tilde{N} has the same role as \tilde{M} for the potential $x \mapsto \tilde{q}(1-x)$. Now, it follows, from the Borg-Marchenko's theorem that:

$$q(x) = \tilde{q}(1-x) \quad \forall x \in [0, 1],$$

and as previously, one gets:

$$f = \tilde{f} \circ \eta.$$

Case 2 : $f(0) = \tilde{f}(1)$ and $f(1) = \tilde{f}(0)$.

The proof is identical interchanging the roles of M and N .

4 Uniqueness results on the trace and the determinant

In this section, we assume that $K = \mathbb{S}^{n-1}$, and our main result is the following:

Proposition 4.1. *Assume that $\sigma(\Lambda_g(\lambda)) = \sigma(\Lambda_{\tilde{g}}(\lambda))$. Then, there is $m_0 \in \mathbb{N}$ such that :*

$$\forall m \in \mathbb{N}, m \geq m_0 \Rightarrow \det(\Lambda_g^m(\lambda)) = \det(\Lambda_{\tilde{g}}^m(\lambda)) \text{ and } \text{Tr}(\Lambda_g^m(\lambda)) = \text{Tr}(\Lambda_{\tilde{g}}^m(\lambda)).$$

Before giving the proof of this proposition, let us begin by the following lemma:

Lemma 4.2. *Under the hypothesis $\sigma(\Lambda_g(\lambda)) = \sigma(\Lambda_{\tilde{g}}(\lambda))$, we have the alternative :*

$$\begin{cases} f(0) = \tilde{f}(0) \\ f(1) = \tilde{f}(1) \end{cases} \quad \text{or} \quad \begin{cases} f(0) = \tilde{f}(1) \\ f(1) = \tilde{f}(0). \end{cases}$$

Proof. First, let us define the set:

$$\Sigma(\Lambda_g(\lambda)) = \{\lambda^\pm(\kappa_m), \quad m \in \mathbb{N}\}$$

where κ_m is the m -th eigenvalue of the usual Laplace-Beltrami operator on the sphere \mathbb{S}^{n-1} and counted without multiplicity. Thanks to our hypothesis, one has obviously

$$\Sigma(\Lambda_g(\lambda)) = \Sigma(\Lambda_{\tilde{g}}(\lambda)).$$

When $K = \mathbb{S}^{n-1}$, we have an explicit formula for κ_m (see for instance [14]) :

$$\kappa_m = m(m + n - 2), \quad \forall m \in \mathbb{N}.$$

The proof involves two steps :

- First we have :

$$f(0)^{\frac{n-1}{2}} + f(1)^{\frac{n-1}{2}} = \tilde{f}(0)^{\frac{n-1}{2}} + \tilde{f}(1)^{\frac{n-1}{2}}. \quad (15)$$

Indeed, it is known that the Steklov spectrum determines the volume of the boundary of M : this is an immediate consequence of the Weyl's law (3) for Steklov eigenvalues (see [6]). We have $\partial M = \Gamma_0 \cup \Gamma_1$ where, for $i \in \{0, 1\}$, Γ_i is the sphere \mathbb{S}^{n-1} equipped with the metric $\gamma_i = f(i)g_{\mathbb{S}}$. Hence :

$$\begin{aligned} \text{Vol}(\partial M) = \text{Vol}(\partial \tilde{M}) &\Leftrightarrow \int_{\Gamma_0} d\text{Vol}_{\gamma_0} + \int_{\Gamma_1} d\text{Vol}_{\gamma_1} = \int_{\Gamma_0} d\text{Vol}_{\tilde{\gamma}_0} + \int_{\Gamma_1} d\text{Vol}_{\tilde{\gamma}_1} \\ &\Leftrightarrow \left(f(0)^{\frac{n-1}{2}} + f(1)^{\frac{n-1}{2}} \right) \text{Vol}(\mathbb{S}^{n-1}) = \left(\tilde{f}(0)^{\frac{n-1}{2}} + \tilde{f}(1)^{\frac{n-1}{2}} \right) \text{Vol}(\mathbb{S}^{n-1}), \end{aligned}$$

and this proves the claim.

- Now, we show that : $f(0) \in \{\tilde{f}(0), \tilde{f}(1)\}$.

Assume this statement is false. Without loss of generality, assume that $f(0) < \min\{\tilde{f}(0), \tilde{f}(1)\}$. Then the equality (15) implies : $f(1) > \max\{\tilde{f}(0), \tilde{f}(1)\}$.

Our strategy is the following : we prove that one of the elements of $\Sigma(\Lambda_g(\lambda))$ is not in $\Sigma(\Lambda_{\tilde{g}}(\lambda))$ and this shall give a contradiction.

Let ε and A two positive numbers. Set :

$$\alpha_A = \inf\{|\lambda^+(\kappa_n) - \lambda^-(\kappa_m)| \mid m, n \geq A\}.$$

We claim that $\alpha_A \leq \frac{1}{2\sqrt{f(1)}} + \varepsilon$ for a sufficiently large A .

Indeed, let $n \in \mathbb{N}$ and set

$$m = \max \left\{ j \in \mathbb{N} \mid \lambda^-(\kappa_j) < \lambda^+(\kappa_n) - \frac{1}{2\sqrt{f(1)}} - \frac{\varepsilon}{2} \right\}$$

By definition of m , $\lambda^-(\kappa_{m+1}) \geq \lambda^+(\kappa_n) - \frac{1}{2\sqrt{f(1)}} - \varepsilon$. But :

$$\begin{aligned}\lambda^-(\kappa_{m+1}) &= \lambda^-(\kappa_m) + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\ &< \lambda^+(\kappa_n) - \frac{1}{2\sqrt{f(1)}} - \frac{\varepsilon}{2} + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\ &< \lambda^+(\kappa_n) + \frac{1}{2\sqrt{f(1)}} + \frac{\varepsilon}{2} + O\left(\frac{1}{m}\right)\end{aligned}$$

Hence

$$\alpha_A \leq |\lambda^+(\kappa_n) - \lambda^-(\kappa_{m+1})| \leq \frac{1}{2\sqrt{f(1)}} + \varepsilon, \quad \text{for } m, n \geq A \text{ sufficiently large.}$$

From the above, we can now choose $(m, n) \in \mathbb{N}^2$ such that :

$$-\frac{1}{2\sqrt{f(1)}} - \varepsilon \leq \lambda^+(\kappa_n) - \lambda^-(\kappa_m) \leq \frac{1}{2\sqrt{f(1)}} + \varepsilon. \quad (16)$$

The equality of the sets $\Sigma(\Lambda_g(\lambda))$ and $\Sigma(\Lambda_{\tilde{g}}(\lambda))$ gives two elements $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of $\Sigma(\Lambda_{\tilde{g}}(\lambda))$ such that $\{\lambda^-(\kappa_m), \lambda^+(\kappa_n)\} = \{\tilde{\lambda}_1, \tilde{\lambda}_2\}$. Remark that if $\lambda^-(\kappa_m) \neq \lambda^+(\kappa_n)$ then the elements $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ can not both belong to the same sequence $(\tilde{\lambda}^-(\kappa_\ell))$ or $(\tilde{\lambda}^+(\kappa_p))$ because, for indexes m and n large enough, one would get :

$$|\tilde{\lambda}_1 - \tilde{\lambda}_2| \geq \min\left\{\frac{1}{\sqrt{\tilde{f}(0)}}, \frac{1}{\sqrt{\tilde{f}(1)}}\right\} - \varepsilon$$

and, by choosing ε small enough, this would contradict (16). Consequently, we have the existence of $(\ell, p) \in \mathbb{N}^2$ such that, for example :

$$\begin{cases} \lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_\ell) \\ \lambda^+(\kappa_n) = \tilde{\lambda}^+(\kappa_p). \end{cases} \quad (17)$$

* Case 1 : $\lambda^-(\kappa_m) \leq \lambda^+(\kappa_n)$. Then $\lambda^-(\kappa_{m+1})$ is not equal to any element of $\Sigma_{\tilde{g}}(\lambda)$. Ideed we have on one hand :

$$\begin{aligned}\lambda^-(\kappa_{m+1}) &= \lambda^-(\kappa_m) + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\ &= \tilde{\lambda}^-(\kappa_\ell) + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\ &= \tilde{\lambda}^-(\kappa_{\ell+1}) + \underbrace{\left[\frac{1}{\sqrt{f(1)}} - \frac{1}{\sqrt{\tilde{f}(0)}}\right]}_{<0} + O\left(\frac{1}{m}\right) + O\left(\frac{1}{\ell}\right)\end{aligned}$$

Hence, for m and ℓ large enough : $\tilde{\lambda}^-(\kappa_\ell) < \lambda^-(\kappa_{m+1}) < \tilde{\lambda}^-(\kappa_{\ell+1})$. On the other hand :

$$\begin{aligned}
\lambda^-(\kappa_{m+1}) &= \lambda^-(\kappa_m) + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\
&\leq \lambda^+(\kappa_n) + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\
&= \tilde{\lambda}^+(\kappa_p) + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\
&= \tilde{\lambda}^+(\kappa_{p+1}) + \underbrace{\left[\frac{1}{\sqrt{f(1)}} - \frac{1}{\sqrt{\tilde{f}(1)}} \right]}_{<0} + O\left(\frac{1}{m}\right) + O\left(\frac{1}{p}\right).
\end{aligned}$$

Moreover, from (16) and (17) :

$$\begin{aligned}
\lambda^-(\kappa_{m+1}) &= \lambda^-(\kappa_m) + \frac{1}{\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\
&= \lambda^-(\kappa_m) + \frac{1}{2\sqrt{f(1)}} + \varepsilon + O\left(\frac{1}{m}\right) + \frac{1}{2\sqrt{f(1)}} - \varepsilon \\
&\geq \tilde{\lambda}^+(\kappa_p) + \frac{1}{2\sqrt{f(1)}} - 2\varepsilon
\end{aligned}$$

As a result, for m and p large enough : $\tilde{\lambda}^+(\kappa_p) < \lambda^-(\kappa_{m+1}) < \tilde{\lambda}^+(\kappa_{p+1})$

The sequences $(\tilde{\lambda}^-(\kappa_m))$ and $(\tilde{\lambda}^+(\kappa_m))$ being strictly increasing (at least for m large enough), none of the elements of $\Sigma_{\tilde{g}}(\lambda)$ can be equal to $\lambda^-(\kappa_{m+1})$. This refutes $\Sigma_{\tilde{g}}(\lambda) = \Sigma_g(\lambda)$.

* Case 2 : $\lambda^-(\kappa_m) > \lambda^+(\kappa_n)$. With similar arguments, one can prove :

$$\max(\tilde{\lambda}^-(\kappa_{\ell-1}), \tilde{\lambda}^+(\kappa_{p-1})) < \lambda^-(\kappa_{m-1}) < \min(\tilde{\lambda}^-(\kappa_{\ell}), \tilde{\lambda}^+(\kappa_p)).$$

Thus $f(0) \in \{\tilde{f}(0), \tilde{f}(1)\}$. Associated to the equality (15), this gives the wanted conclusion. \square

From now on, we assume that $f(0) = \tilde{f}(0)$ and $f(1) = \tilde{f}(1)$, since the case $f(0) = \tilde{f}(1)$ and $f(1) = \tilde{f}(0)$ is obtained by substituting the roles of $\tilde{\lambda}^-(\kappa_m)$ and $\tilde{\lambda}^+(\kappa_m)$. First, let us begin by a simple case:

4.1 The case $\mathbf{f(0) = f(1)}$

Without loss of generality, we can assume that $f(0) = f(1) = 1$. Thanks to Lemma 3.2, $\Lambda_g^m(\lambda)$ has two eigenvalues $\lambda^-(\mu_m)$ and $\lambda^+(\mu_m)$ whose asymptotics are given by :

$$\begin{cases} \lambda^-(\mu_m) = \sqrt{\mu_m} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right) \\ \lambda^+(\mu_m) = \sqrt{\mu_m} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}} + O\left(\frac{1}{\sqrt{\mu_m}}\right). \end{cases}$$

We recall that :

$$\sqrt{\kappa_m} = m + \frac{n-2}{2} + O\left(\frac{1}{m}\right),$$

and for $m \in \mathbb{N}$, we set $V_m = \left\{ \lambda^-(\kappa_m), \lambda^+(\kappa_m) \right\}$ and $\tilde{V}_m := \left\{ \tilde{\lambda}^-(\kappa_m), \tilde{\lambda}^+(\kappa_m) \right\}$.

• **The two dimensional case**

In this case, the eigenvalues $\lambda^\pm(\kappa_m)$ have the following asymptotics :

$$\begin{cases} \lambda^-(\kappa_m) = m + O\left(\frac{1}{m}\right) \\ \lambda^+(\kappa_m) = m + O\left(\frac{1}{m}\right). \end{cases}$$

For m large enough, the sets V_m and \tilde{V}_m are both included in the interval $\left[m - \frac{1}{4}, m + \frac{1}{4}\right]$. In particular $V_m \cap \tilde{V}_{m'} = \emptyset$ if $m \neq m'$. The equality $\Sigma(\Lambda_g(\lambda)) = \Sigma(\Lambda_{\tilde{g}}(\lambda))$ leads to the equalities $V_m = \tilde{V}_m$ if m is greater than some index m_0 . Consequently, there is m_0 such that, for $m \geq m_0$:

$$\begin{cases} \lambda^-(\kappa_m) + \lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m) + \tilde{\lambda}^+(\kappa_m) \\ \lambda^-(\kappa_m)\lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m)\tilde{\lambda}^+(\kappa_m). \end{cases}$$

Of course, the previous equalities are still true when κ_m is replaced by μ_m . Thus, we have proved:

$$\forall m \in \mathbb{N}, m \geq m_0 \Rightarrow \text{Tr}(\Lambda_g^m) = \text{Tr}(\Lambda_{\tilde{g}}^m) \text{ and } \det(\Lambda_g^m) = \det(\Lambda_{\tilde{g}}^m).$$

• **The $n \geq 3$ dimensional case**

In this case, we use the following asymptotics of the eigenvalues $\lambda^\pm(\kappa_m)$:

$$\begin{cases} \lambda^-(\kappa_m) = m + \frac{n-2}{2} - \frac{h'(1)}{4} + O\left(\frac{1}{m}\right) \\ \lambda^+(\kappa_m) = m + \frac{n-2}{2} + \frac{h'(0)}{4} + O\left(\frac{1}{m}\right). \end{cases}$$

Contrary to the two dimensional case, we can not conclude that the sets V_m and \tilde{V}_m are equal. This is due to the presence of the constants $h'(0)$ and $h'(1)$ in these asymptotics. We have the following Proposition:

Proposition 4.3. *There is $\mathcal{L} \subset \mathbb{N}$ such that $\sum_{m \in \mathcal{L}} \frac{1}{m} = +\infty$ satisfying :*

$$\forall m \in \mathcal{L}, \begin{cases} \lambda^-(\kappa_m) + \lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m) + \tilde{\lambda}^+(\kappa_m) \\ \lambda^-(\kappa_m)\lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m)\tilde{\lambda}^+(\kappa_m) \end{cases}$$

Proof. We start with the following Lemma:

Lemma 4.4. *There is $\mathcal{L}_1 \subset \mathbb{N}$ such that $\sum_{m \in \mathcal{L}_1} \frac{1}{m} = +\infty$ satisfying :*

$$\left(\lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m), \forall m \in \mathcal{L}_1 \right) \quad \text{or} \quad \left(\lambda^-(\kappa_m) = \tilde{\lambda}^+(\kappa_m), \forall m \in \mathcal{L}_1 \right)$$

Proof. Since $\Sigma(\Lambda_g(\lambda)) = \Sigma(\Lambda_{\tilde{g}}(\lambda))$, we have the inclusion :

$$\{\lambda^-(\kappa_m), m \in \mathbb{N}\} \subset \{\tilde{\lambda}^-(\kappa_m), m \in \mathbb{N}\} \cup \{\tilde{\lambda}^+(\kappa_m), m \in \mathbb{N}\}.$$

Thus, there exists a sequence of integers (a_m) such that $\sum_{m \in \mathbb{N}} \frac{1}{a_m} = +\infty$ and :

$$\{\lambda^-(\kappa_{a_m}), m \in \mathbb{N}\} \subset \{\tilde{\lambda}^-(\kappa_m), m \in \mathbb{N}\} \quad \text{or} \quad \{\lambda^-(\kappa_{a_m}), m \in \mathbb{N}\} \subset \{\tilde{\lambda}^+(\kappa_m), m \in \mathbb{N}\}.$$

For instance, let us study the first case (since the second case is similar). We can find another sequence of integers (\tilde{a}_m) such that :

$$\lambda^-(\kappa_{a_m}) = \tilde{\lambda}^-(\kappa_{\tilde{a}_m}), \quad \forall m \in \mathbb{N}.$$

Thanks to Lemma 3.2, one obtains :

$$a_m - \tilde{a}_m = \frac{\tilde{h}'(1)}{4} - \frac{h'(1)}{4} + O\left(\frac{1}{a_m}\right).$$

Therefore, the sequence of integers $(a_m - \tilde{a}_m)$ converges to the integer $\frac{\tilde{h}'(1)}{4} - \frac{h'(1)}{4} = \frac{n-2}{4} \frac{f'(1)}{f(1)} - \frac{n-2}{4} \frac{\tilde{f}'(1)}{\tilde{f}(1)}$.

Recalling that $f, \tilde{f} \in \mathcal{C}_b = \left\{ f \in C^\infty([0, 1]), \left| \frac{f'(i)}{f(i)} \right| \leq \frac{1}{n-2}, i \in \{0, 1\} \right\}$, we get:

$$\left| \frac{n-2}{4} \frac{f'(1)}{f(1)} - \frac{n-2}{4} \frac{\tilde{f}'(1)}{\tilde{f}(1)} \right| \leq \frac{1}{2}.$$

As this quantity must be an integer, we have proved that, for $m \geq m_0$:

$$a_m = \tilde{a}_m.$$

We conclude the proof of the Lemma setting $\mathcal{L}_1 = \{a_m, m \geq m_0\}$. □

Now, we can finish the proof of the Proposition. For instance, assume that:

$$\lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m), \quad \forall m \in \mathcal{L}_1.$$

Repeating the previous argument with $\lambda^+(\kappa_m)$, $m \in \mathcal{L}_1$, we have the inclusion:

$$\{\lambda^+(\kappa_m), m \in \mathcal{L}_1\} \subset \{\tilde{\lambda}^-(\kappa_m), m \in \mathbb{N}\} \cup \{\tilde{\lambda}^+(\kappa_m), m \in \mathbb{N}\}$$

There is a sequence of integers, denoted $(b_m) \in \mathcal{L}_1$, such that $\sum_{m \in \mathbb{N}} \frac{1}{b_m} = +\infty$ and :

$$\{\lambda^+(\kappa_{b_m}), m \in \mathbb{N}\} \subset \{\tilde{\lambda}^-(\kappa_m), m \in \mathbb{N}\} \quad \text{or} \quad \{\lambda^+(\kappa_{b_m}), m \in \mathbb{N}\} \subset \{\tilde{\lambda}^+(\kappa_m), m \in \mathbb{N}\}.$$

- In the second case, there is a sequence (\tilde{b}_m) such that :

$$\lambda^+(\kappa_{b_m}) = \tilde{\lambda}^+(\kappa_{\tilde{b}_m})$$

which implies as previously that $b_m = \tilde{b}_m$. Then, setting $\mathcal{L}_2 = \{b_m, m \in \mathbb{N}\}$ we have for all $m \in \mathcal{L}_2$:

$$\begin{cases} \lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m) \\ \lambda^+(\kappa_m) = \tilde{\lambda}^+(\kappa_m) \end{cases}$$

- In the first case, we still show that :

$$\lambda^+(\kappa_{b_m}) = \tilde{\lambda}^-(\kappa_{b_m}) \quad (= \lambda^-(\kappa_{b_m}))$$

Additionally, using the same arguments with

$$\{\tilde{\lambda}^+(\kappa_{b_m}), m \in \mathbb{N}\} \subset \{\lambda^-(\kappa_m), m \in \mathbb{N}\} \cup \{\lambda^+(\kappa_m), m \in \mathbb{N}\},$$

we prove that there is a subsequence $(b_{\varphi(m)})$ such that $\sum_{m \in \mathbb{N}} \frac{1}{b_{\varphi(m)}} = +\infty$ and : $\left(\tilde{\lambda}^+(\kappa_{b_{\varphi(m)}}) = \lambda^+(\kappa_{b_{\varphi(m)}}), \forall m \in \mathbb{N}\right)$ or $\left(\tilde{\lambda}^+(\kappa_{b_{\varphi(m)}}) = \lambda^-(\kappa_{b_{\varphi(m)}}), \forall m \in \mathbb{N}\right)$. But as $\lambda^-(\kappa_{b_{\varphi(m)}}) = \lambda^+(\kappa_{b_{\varphi(m)}})$, the conclusion follows on $\mathcal{L}_3 = \{b_{\varphi(m)}, m \in \mathbb{N}\}$.

To sum up, we have proved there exists a set $\mathcal{L} := \mathcal{L}_3 \subset \mathbb{N}$ such that $\sum_{m \in \mathcal{L}} \frac{1}{m} = +\infty$ and for all $m \in \mathcal{L}$:

$$\begin{cases} \lambda^-(\kappa_m) + \lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m) + \tilde{\lambda}^+(\kappa_m) \\ \lambda^-(\kappa_m)\lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m)\tilde{\lambda}^+(\kappa_m). \end{cases}$$

□

Now, let us denote $\kappa_z = z(z + n - 2)$. For every $z \in \Pi^+$, we set $B(\kappa_z) := \lambda^-(\kappa_z) + \lambda^+(\kappa_z)$ and $C(\kappa_z) := \lambda^-(\kappa_z)\lambda^+(\kappa_z)$. One has:

$$B(\kappa_z) = -\frac{D(\kappa_z) + E(\kappa_z)}{\Delta(\kappa_z)} + C_1 - C_2 \quad (18)$$

$$C(\kappa_z) = \left(-\frac{D(\kappa_z)}{\Delta(\kappa_z)} + C_1\right) \left(-\frac{E(\kappa_z)}{\Delta(\kappa_z)} - C_2\right) - \frac{1}{\Delta(\kappa_z)^2}, \quad (19)$$

where C_1 and C_2 are suitable constants. We introduce two functions g_1 and g_2 on Π^+ as follow:

$$\begin{aligned} g_1(z) &= \Delta(\kappa_z)\tilde{\Delta}(\kappa_z)(B(\kappa_z) - \tilde{B}(\kappa_z)) \\ g_2(z) &= \Delta(\kappa_z)^2\tilde{\Delta}(\kappa_z)^2(C(\kappa_z) - \tilde{C}(\kappa_z)). \end{aligned}$$

We claim that g_1 and g_2 are identically zero. Indeed :

- g_1, g_2 are holomorphic on Π^+ .
- $g_1, g_2 \in \mathcal{N}(\Pi^+)$ thanks to the estimates of Proposition 2.3.
- We have $g_1(m) = g_2(m) = 0$ for every $m \in \mathcal{L}$, which enable us to conclude, by Nevanlinna's theorem :

$$g_1 \equiv g_2 \equiv 0 \text{ sur } \Pi^+.$$

In particular, for every $m \in \mathbb{N}$:

$$\begin{cases} \lambda^-(\kappa_m) + \lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m)\tilde{\lambda}^+(\kappa_m) \\ \lambda^-(\kappa_m)\lambda^+(\kappa_m) = \tilde{\lambda}^-(\kappa_m)\tilde{\lambda}^+(\kappa_m) \end{cases}$$

and, by replacing κ_m by μ_m we have for every $m \geq m_0$:

$$\text{Tr}(\Lambda_g^m) = \text{Tr}(\Lambda_{\tilde{g}}^m) \text{ and } \det(\Lambda_g^m) = \det(\Lambda_{\tilde{g}}^m).$$

4.2 The case $f(0) \neq f(1)$

Without loss of generality, assume that $f(0) < f(1)$. Thanks to the asymptotics:

$$\sqrt{\kappa_m} = m + \frac{n-2}{2} + O\left(\frac{1}{m}\right)$$

and Lemma 3.2, we have :

$$\begin{cases} \lambda^-(\kappa_m) = \frac{m}{\sqrt{f(1)}} + \frac{n-2}{2\sqrt{f(1)}} - \frac{\ln(h)'(1)}{4\sqrt{f(1)}} + O\left(\frac{1}{m}\right) \\ \lambda^+(\kappa_m) = \frac{m}{\sqrt{f(0)}} + \frac{n-2}{2\sqrt{f(0)}} + \frac{\ln(h)'(0)}{4\sqrt{f(0)}} + O\left(\frac{1}{m}\right). \end{cases}$$

Since $f(0) \neq f(1)$, the proof is a little more delicate. As previously, we have to find a subset $\mathcal{L} \subset \mathbb{N}$ such that $\sum_{m \in \mathcal{L}} \frac{1}{m} = +\infty$ and such that $\lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m)$ or $\lambda^+(\kappa_m) = \tilde{\lambda}^+(\kappa_m)$ for all $m \in \mathcal{L}$.

We have the following Proposition:

Proposition 4.5. *There is $\mathcal{L} \subset \mathbb{N}$ satisfying $\sum_{m \in \mathcal{L}} \frac{1}{m} = +\infty$ and such that :*

$$\{\lambda^-(\kappa_m), m \in \mathcal{L}\} \subset \{\tilde{\lambda}^-(\kappa_m), m \in \mathbb{N}\}$$

Proof. If the subset of the elements of $\{\lambda^-(\kappa_m), m \in \mathbb{N}\}$ belonging to $\{\tilde{\lambda}^+(\kappa_m), m \in \mathbb{N}\}$ is finite, the proposition is true. If not, there exist two strictly increasing sequences $\varphi, \psi : \mathbb{N} \rightarrow \mathbb{N}$ such that:

$$\lambda^-(\kappa_{\psi(m)}) = \tilde{\lambda}^+(\kappa_{\varphi(m)}). \quad (20)$$

We emphasize that the functions φ and ψ are built so that an integer $m \in \mathbb{N}$ which is not in the image of ψ (respectively in that of φ) satisfies $\lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_n)$ for some $n \in \mathbb{N}$ (respectively $\lambda^+(\kappa_n) = \tilde{\lambda}^+(\kappa_m)$ for some $n \in \mathbb{N}$).

Replacing $\lambda^+(\kappa_{\varphi(m)})$ and $\tilde{\lambda}^-(\kappa_{\psi(m)})$ by their asymptotics in the equality (20), we have :

$$\frac{\varphi(m)}{\sqrt{f(0)}} = \frac{\psi(m)}{\sqrt{f(1)}} + C + O\left(\frac{1}{\varphi(m)}\right) \quad (21)$$

where $C = -\frac{\ln(h)'(1)}{4\sqrt{f(1)}} - \frac{\ln(h)'(0)}{4\sqrt{f(0)}} + \frac{n-2}{2\sqrt{f(1)}} - \frac{n-2}{2\sqrt{f(0)}}$.

Now, we have the following Lemma:

Lemma 4.6. *There exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, $\psi(m+1) \geq \psi(m) + 2$.*

Proof. Set $A = \frac{\sqrt{f(1)}}{\sqrt{f(0)}} > 1$ and $C' = -\sqrt{f(1)}C$. It follows from (21) that:

$$\psi(m) = A\varphi(m) + C' + O\left(\frac{1}{\varphi(m)}\right).$$

Assume $\psi(m+1) = \psi(m) + 1$. Then :

$$\begin{aligned} \psi(m) + 1 = \psi(m+1) &= A\varphi(m+1) + C' + O\left(\frac{1}{\varphi(m)}\right) \\ &\geq A(\varphi(m) + 1) + C' + O\left(\frac{1}{\varphi(m)}\right) \\ &= A\varphi(m) + C' + A + O\left(\frac{1}{\varphi(m)}\right) \\ &= \psi(m) + A + O\left(\frac{1}{\varphi(m)}\right). \end{aligned}$$

Thus, we get:

$$1 \geq A + O\left(\frac{1}{\varphi(m)}\right)$$

which is clearly false for m large enough. \square

Consequently, the range of ψ doesn't contain two consecutive integers. We deduce from this the following Lemma :

Lemma 4.7. *Denote a_1, a_2, \dots the sequence of all integers that are not in the range of ψ . There exists $C > 0$ such that this sequence satisfies :*

$$a_m \leq 2m + C. \quad (22)$$

Proof. We set $p_0 = \psi(m_0)$ and $C := a_{p_0} - 2p_0$. Clearly, (22) true for $m = p_0$. Now, assume that (22) is satisfied for a fixed $m \geq p_0$. Then :

- either $a_m + 1$ is not in the range of ψ and so : $a_{m+1} = a_m + 1$,
- or $a_m + 1$ is in the range of ψ . Then $a_m + 2$ is not in the image of ψ since it does not contain consecutive integers and consequently $a_{m+1} = a_m + 2$.

In both cases, we get $a_{m+1} \leq 2(m+1) + C$. Then, the proof of the Lemma follows by a standard induction argument. \square

In particular, Lemma 4.7 shows that :

$$\sum_{n \in \mathbb{N}} \frac{1}{a_n} = +\infty.$$

Thus, setting $\mathcal{L} = \{a_n, n \in \mathbb{N}\}$, we have :

$$\{\lambda^-(\kappa_m), m \in \mathcal{L}\} \subset \{\tilde{\lambda}^-(\kappa_m), m \in \mathbb{N}\}$$

which concludes the proof of the Proposition. \square

As previously, we have to treat differently the case of the dimension $n = 2$ and $n \geq 3$.

The two dimensional case:

Lemma 4.8. *For all $m \in \mathcal{L}$, one has:*

$$\lambda^+(m) = \tilde{\lambda}^+(m).$$

Proof. By construction, for each element m of \mathcal{L} , there exists $n(m) \in \mathbb{N}$ such that :

$$\lambda^+(m) = \tilde{\lambda}^+(\kappa_{n(m)}).$$

Using Lemma 3.2, we get:

$$m = n(m) + O\left(\frac{1}{m}\right) + O\left(\frac{1}{n(m)}\right),$$

which implies $m = n(m)$. for m large enough. \square

Lemma 4.9. *For every $z \in \Pi^+$ large enough, we have the alternative :*

$$\lambda_-(z) = \tilde{\lambda}^-(z) \text{ or } \lambda_+(z) = \tilde{\lambda}^+(z).$$

Proof. Recall that we have set :

$$B(\kappa_m) = \lambda^-(\kappa_m) + \lambda^+(\kappa_m), \text{ and } C(\kappa_m) = \lambda^-(\kappa_m)\lambda^+(\kappa_m).$$

For every $m \in \mathbb{N}$, the characteristic polynomial $P(X)$ associated to the eigenvalues $\lambda^\pm(\kappa_m)$ is given by:

$$P(X) = X^2 - B(\kappa_m)X + C(\kappa_m).$$

Setting $R(\kappa_m) = B(\kappa_m)^2 - 4C(\kappa_m)$, we get:

$$\lambda^\pm(\kappa_m) = \frac{1}{2} \left(B(\kappa_m) \pm \sqrt{R(\kappa_m)} \right).$$

In the same way, we define \tilde{B} and \tilde{R} such that :

$$\tilde{\lambda}^\pm(\kappa_m) = \frac{1}{2} \left(\tilde{B}(\kappa_m) \pm \sqrt{\tilde{R}(\kappa_m)} \right)$$

For $m \in \mathcal{L}$, we begin with the equality :

$$B(\kappa_m) - \sqrt{R(\kappa_m)} = \tilde{B}(\kappa_m) - \sqrt{\tilde{R}(\kappa_m)}.$$

One deduces :

$$\left(B(\kappa_m) - \tilde{B}(\kappa_m) \right)^2 = \left(\sqrt{R(\kappa_m)} - \sqrt{\tilde{R}(\kappa_m)} \right)^2,$$

and we have

$$\underbrace{\left[\left(B(\kappa_m) - \tilde{B}(\kappa_m) \right)^2 - R(\kappa_m) - \tilde{R}(\kappa_m) \right]^2}_{:=g_1(m)} - 4R(\kappa_m)\tilde{R}(\kappa_m) = 0.$$

For $z \in \Pi^+$, we set $\kappa_z = z(z+n-2)$, and we define:

$$g_2(z) = \Delta(\kappa_z)^4 \tilde{\Delta}(\kappa_z)^4 g_1(z).$$

Then g_2 is identically zero. Indeed :

- g_2 is holomorphic on Π^+ .
- $g_2 \in \mathcal{N}(\Pi^+)$ thanks to the estimates of Proposition 2.3.
- We have $g_2(m) = 0$ for every $m \in \mathcal{L}$, which enable us to conclude, by Nevanlinna's theorem :

$$g_2 \equiv 0 \equiv g_1 \text{ sur } \Pi^+.$$

In other words, we have obtained:

$$\left[\left(B(\kappa_z) - \tilde{B}(\kappa_z) \right)^2 - R(\kappa_z) - \tilde{R}(\kappa_z) \right]^2 - 4R(\kappa_z)\tilde{R}(\kappa_z) = 0.$$

It easily follows that for $z \in \Pi^+$, we have four alternatives:

$$\lambda^-(\kappa_z) = \tilde{\lambda}^-(\kappa_z), \quad \lambda^+(\kappa_z) = \tilde{\lambda}^-(\kappa_z), \quad \lambda^-(\kappa_z) = \tilde{\lambda}^+(\kappa_z) \quad \text{or} \quad \lambda^+(\kappa_z) = \tilde{\lambda}^+(\kappa_z).$$

If a sequence (z_n) satisfies, for instance, $\lambda^+(\kappa_{z_n}) = \tilde{\lambda}^-(\kappa_{z_n})$ with $|z_n| \rightarrow +\infty$, then, using Lemma 3.2 we get $f(0) = f(1)$, and that contradicts our hypothesis. We use the same argument to exclude the third case. Finally, by replacing κ_z by z , we have for $|z|$ large enough :

$$\lambda^-(z) = \tilde{\lambda}^-(z) \quad \text{or} \quad \lambda^+(z) = \tilde{\lambda}^+(z).$$

□

Assume now there exists $x \in \mathbb{R}$ large enough such that $\lambda^-(x) \neq \tilde{\lambda}^-(x)$. By a standard continuity argument, we can find an interval I centered in x such that :

$$\forall t \in I, \lambda^-(t) \neq \tilde{\lambda}^-(t),$$

which implies necessarily that $\forall t \in I, \lambda^+(t) = \tilde{\lambda}^+(t)$. Moreover, there is $L > 0$ such that the real function $t \mapsto \lambda^+(t) - \tilde{\lambda}^+(t)$ is analytic on the interval $[L, +\infty[$. Thus, $\forall t \geq L$, one has $\lambda^+(t) = \tilde{\lambda}^+(t)$ by the analytic continuation principle. One deduces there exists $m_0 \in \mathbb{N}$ such that, for m larger than m_0 :

$$\lambda^+(\kappa_m) = \tilde{\lambda}^+(\kappa_m), \quad \lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m).$$

As a by product, one gets:

$$\forall m \in \mathbb{N}, m \geq m_0 \Rightarrow \text{Tr}(\Lambda_g^m) = \text{Tr}(\tilde{\Lambda}_g^m) \text{ and } \det(\Lambda_g^m) = \det(\tilde{\Lambda}_g^m).$$

The case of the dimension $n \geq 3$.

The following arguments are more or less immediate adaptations of the two dimensional case. We just refer to it for details.

Lemma 4.10. *There exists $\mathcal{L} \subset \mathbb{N}$ such that $\sum_{m \in \mathcal{L}} \frac{1}{m} = +\infty$ and :*

$$\tilde{\lambda}^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m), \quad \forall m \in \mathcal{L}.$$

Proof. By definition of the sequence (a_n) defined in Lemma 4.7, there is another sequence of integers (\tilde{a}_n) such that :

$$\lambda^-(\kappa_{a_n}) = \tilde{\lambda}^-(\kappa_{\tilde{a}_n})$$

As in the proof of Lemma 4.4, we show that $a_m = \tilde{a}_m$ and we set $\mathcal{L} = \{a_m, m \in \mathcal{L}_p\}$. □

We have the following Lemma (the proof is identical to the two dimensional case).

Lemma 4.11. *For all $z \in \mathbb{C}$ such that $|z|$ is large enough, we have the alternative :*

$$\lambda^-(\kappa_z) = \tilde{\lambda}^-(\kappa_z), \quad \lambda^+(\kappa_z) = \tilde{\lambda}^+(\kappa_z).$$

We deduce then, for $m \geq m_0$:

$$\lambda^-(\kappa_m) = \tilde{\lambda}^-(\kappa_m) \quad \text{and} \quad \lambda^+(\kappa_m) = \tilde{\lambda}^+(\kappa_m).$$

It follows that :

$$\forall m \in \mathbb{N}, m \geq m_0 \Rightarrow \text{Tr}(\Lambda_g^m) = \text{Tr}(\tilde{\Lambda}_g^m) \text{ and } \det(\Lambda_g^m) = \det(\tilde{\Lambda}_g^m).$$

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