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# $\Gamma$ -CONVERGENCE OF NONCONVEX INTEGRALS DEFINED ON SOBOLEV FUNCTIONS AND VECTOR MEASURES

OMAR ANZA HAFSA, JEAN-PHILIPPE MANDALLENA, AND HAMDI ZORGATI

ABSTRACT. We study the  $\Gamma$ -convergence of nonconvex integral functionals defined on the product space of Sobolev functions and vector measures. We prove an integral representation result of the  $\Gamma$ -limit by assuming abstract conditions on the behavior of minimization problems on small balls associated with the integral functionals. We apply the result to prove new relaxation and homogenization theorems with additional vector measure variable.

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## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^N$  be a bounded open set with Lipschitz boundary. Let  $p > 1$ ,  $m \in \mathbb{N}^*$  and  $l \in \mathbb{N}^*$ . Let  $\mathcal{M}(\Omega; \mathbb{R}^l)$  be the space of the vector Radon measures and  $\mathcal{O}(\Omega)$  be the set of all nonempty open subsets of  $\Omega$ . For each  $\varepsilon > 0$  we consider  $I_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$  defined by

$$I_\varepsilon(u, \nu; O) := \begin{cases} \int_O f_\varepsilon \left( x, u(x), \nabla u(x), \frac{d\nu}{d\mathcal{L}_N}(x) \right) d\mathcal{L}_N(x) & \text{if } \nu \ll \mathcal{L}_N \\ \infty & \text{otherwise,} \end{cases}$$

where  $\mathcal{L}_N$  is the Lebesgue measure on  $\Omega$ , and the integrands  $f_\varepsilon : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m \times N} \times \mathbb{R}^l \rightarrow [0, \infty[$  are Borel measurable and satisfy the following coercivity and growth conditions

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(H<sub>1</sub>) there exist  $C, c > 0$  such that for every  $\varepsilon > 0$  and every  $(x, u, \xi, v) \in \Omega \times \mathbb{R}^m \times \mathbb{M} \times \mathbb{R}^l$

$$c(|\xi|^p + |v|) \leq f_\varepsilon(x, u, \xi, v) \leq C(1 + |u|^p + |\xi|^p + |v|).$$

This double dependency in the framework of  $\Gamma$ -convergence appeared naturally in several works in the mathematical analysis of solid mechanics (see for instance [BJ99]). In [FKP94] the chemical composition, as additional variable, was considered in the minimizing energy represented by an  $L^\infty$  vector field. The additional variable can be connected to the displacement or deformation variable via a condition, for instance, in [LDZ06] an  $L^p$  Cosserat vector emerged, through a dimension reduction process, for martensitic thin films, representing the bending moment. In [BZZ08], this Cosserat vector turns out to be a vector measure due to the linear growth of the energy, which induce a double dependency with a bounded variation deformation (see also [BFM03, BFM09]).

In this work, we are interested by problems where the two different variables  $u$  and  $\nu$  are not connected. The analysis of this kind of problems, by the methods of  $\Gamma$ -convergence and relaxation, was studied in [CRZ10, CRZ11, RZ13, CZ16, CZ17] in the case where the additional variable lies in  $L^p$  ( $p \geq 1$ ) space while the variable  $u$  belongs to Sobolev or  $BV$  spaces.

Our goal is to provide an integral representation of the  $\Gamma$ -limit of  $\{I_\varepsilon\}_{\varepsilon > 0}$  when  $\varepsilon \rightarrow 0$  with respect to the following convergence: the sequence  $\{(u_n, \nu_n)\}_n \subset W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  converges to  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  and we write  $(u_n, \nu_n) \rightarrow (u, \nu)$

$$\text{if } u_n \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \quad \text{and} \quad \nu_n \xrightarrow{*} \nu \text{ in } \mathcal{M}(\Omega; \mathbb{R}^l) \text{ as } n \rightarrow \infty.$$

The integral representation of the  $\Gamma$ -limit of  $\{I_\varepsilon\}_{\varepsilon > 0}$  is studied via “local-global” conditions on the minimization problems associated with  $I_\varepsilon$

$$\frac{m_\varepsilon(u, \nu; B_\rho(x))}{\sigma(B_\rho(x))} := \frac{1}{\sigma(B_\rho(x))} \inf \left\{ I_\varepsilon(w, \lambda; B_\rho(x)) : \lambda(B_\rho(x)) = \nu(B_\rho(x)), \right. \\ \left. \lambda \ll \mathcal{L}_N \text{ and } w \in u + W_0^{1,p}(B_\rho(x); \mathbb{R}^m) \right\}, \quad (1.1)$$

where  $\sigma \in \{\mathcal{L}_N, |\nu^s|\}$  (see (2.1) for the definition of  $m_\varepsilon$ ). To show at the same time the integral representation and the  $\Gamma$ -convergence of  $\{I_\varepsilon\}_{\varepsilon > 0}$ , we impose conditions on the behavior of (1.1) when  $\varepsilon \rightarrow 0$  around small balls centered at  $x$ . Roughly, when  $\sigma = \mathcal{L}_N$  the successive limits  $\varepsilon \rightarrow 0$  and then  $\rho \rightarrow 0$  of (1.1) give rise to the limit integrand of the integral representation of the  $\Gamma$ -limit of  $\{I_\varepsilon\}_\varepsilon$  with respect to the Lebesgue measure. In the same way, when  $\sigma = |\nu^s|$  (here  $\nu^s$  is the singular part of  $\nu$  in the Lebesgue decomposition of  $\nu$  with respect to  $\mathcal{L}_N$ ) we find the limit integrand of the singular part.

This approach is partly inspired by the works of [DMM86], [BFM98] and [BB00] (see also [AHM16, AHM18, AHM17, AHCM17]). It allows not to assume neither quasiconvexity nor convexity assumptions on the initial integrands. Moreover, it seems to us that this procedure gives in a more natural way the formula for the limit integrands. Nevertheless, we have to highlight that this method is heavily based on the coercivity conditions, which is not the case for the direct method of integral representation for  $\Gamma$ -limits (see for instance [But89, DM93, BD98]).

**1.1. Plan of the paper.** In §2 we introduce general conditions (H<sub>2</sub>) and (H<sub>3</sub>) on local minimization problems  $m_\varepsilon$  associated with the integrals  $\{I_\varepsilon\}_{\varepsilon > 0}$  and we state the main result of the paper Theorem 2.1. The proof is based on two propositions (Proposition 2.1 and Proposition 2.2). The first one provides global bounds under integral form for the  $\Gamma$ -liminf and  $\Gamma$ -limsup. The second proposition is of a local nature and allows to refine the formulas for the limit integrands. At the end of this section, we give the proof of

the main result which is almost a direct consequence of the two propositions and the conditions (H<sub>2</sub>) and (H<sub>3</sub>).

The section §3 is devoted to some applications of the main result Theorem 2.1. In §3.1 and §3.2, we state two relaxation theorems with additional vector measures variable. Relaxation means that  $f_\varepsilon \equiv f$  is constant with respect to  $\varepsilon$ . By assuming some natural conditions, we prove a relaxation theorem which is truly local when  $p > N$ . While the situation is different when  $p \leq N$ , indeed, the complement of the set of the Lebesgue points of  $u$  can be of non zero  $\nu^s$  measure, which leads to a nonlocal dependence with respect to  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  of the singular part. In §3.2, a second relaxation theorem is established under stronger assumption on the integrand allowing to eliminate the variable  $u$  in the singular part of the relaxed integral. In §3.3 and §3.4, we state two homogenization theorems. The first one deals with integrands of the form  $W(x, \frac{x}{\varepsilon}, \xi, v)$  which are Borel measurable and periodic with respect to the second variable. The second homogenization theorem is concerned with integrands of the form  $W(\frac{x}{\varepsilon}, \frac{u}{\varepsilon}, \xi, \frac{v}{\varepsilon})$  which are periodic with respect to all the variables except  $\xi$ . In this case, we find that the  $\Gamma$ -limit depends only on the gradient variable (see Theorem 3.4).

The section §4 is devoted to the proofs of Proposition 2.1 and Proposition 2.2.

In §5 we prove the relaxation theorems. The proofs consist mainly in showing that conditions (H<sub>2</sub>) and (H<sub>3</sub>) are satisfied.

In §6 we prove the homogenization theorems. We verify the general conditions using mainly subadditive theorems.

In §7, we prove some auxiliary results which are needed in the proofs of Proposition 2.1 and Proposition 2.2. The first part is concerned with results on the behavior of the local minimization problems  $m_\varepsilon$  with respect the measure variable. In the second part, we prove some results we need for establishing the bound under integral form of the  $\Gamma$ -limsup. The last subsection §7.3 is devoted to the proof of a differentiation result of the Vitali envelopes for subadditive and dominated set function defined on open sets.

**1.2. Notation.** We denote by  $\mathcal{M}_+(\Omega)$  the set of all positive Radon measures on  $\Omega$  and by  $\mathcal{M}(\Omega; \mathbb{R}^l)$  the space of all  $\mathbb{R}^l$ -valued Radon measures on  $\Omega$  with  $l \in \mathbb{N}^*$ . Let  $\mu \in \mathcal{M}_+(\Omega)$ . If  $w \in L^1_\mu(\Omega; \mathbb{R}^l)$  then  $w\mu \in \mathcal{M}(\Omega; \mathbb{R}^l)$  where

$$w\mu(B) := \int_B w d\mu \quad \text{for all Borel set } B \subset \Omega.$$

In particular, for a Borel set  $A \subset \Omega$  we denote by  $\mathbb{1}_A \mu$  the measure defined by

$$\mathbb{1}_A \mu(B) = \int_B \mathbb{1}_A d\mu \quad \text{for all Borel set } B \subset \Omega.$$

When we write  $\nu \ll \mu$  for  $\mu \in \mathcal{M}_+(\Omega)$  and  $\nu \in \mathcal{M}(\Omega; \mathbb{R}^l)$  we mean that the total variation  $|\nu|$  of  $\nu$  is absolutely continuous with respect to  $\mu$ , i.e.  $|\nu| \ll \mu$ .

The Lebesgue decomposition of  $\lambda \in \mathcal{M}(\Omega; \mathbb{R}^l)$  with respect to  $\mu$  is

$$\lambda = \frac{d\lambda}{d\mu}(\cdot) \mu + \lambda^s$$

where  $\frac{d\lambda}{d\mu}(\cdot) \in L^1_\mu(\Omega; \mathbb{R}^l)$  is given by  $\frac{d\lambda}{d\mu}(x) = \lim_{\rho \rightarrow 0} \frac{\lambda(B_\rho(x))}{\mu(B_\rho(x))}$   $\mu$ -a.e. in  $\Omega$ , and where  $\lambda^s$  and  $\mu$  are mutually singular, which is denoted by  $\lambda^s \perp \mu$ , and means that there exists a Borel set  $X \subset \Omega$  such that  $\mu(\Omega \setminus X) = 0$  and  $|\lambda^s|(X) = 0$ .

- We denote by  $R_\mu^x := \{\rho \in ]0, \infty[ : \mu(\partial B_\rho(x)) > 0\}$ .
- We denote by  $\mathbb{Y} := ]0, 1[^N$  and  $Y := ]-1, 1[^N$ .

- We denote by  $\mathbb{S}^l$  the unit sphere centered at 0 of  $\mathbb{R}^l$ , i.e.  $\mathbb{S}^l := \{v \in \mathbb{R}^l : |v| = 1\}$ .

For every  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  we set

- $u_x(\cdot) := u(x) + \nabla u(x)(\cdot - x)$  for all  $x \in \Omega$ .
- $u_{x,\rho} := \int_{B_\rho(x)} u(z) d\mathcal{L}_N(z) = \frac{1}{\mathcal{L}_N(B_\rho(x))} \int_{B_\rho(x)} u(z) d\mathcal{L}_N(z)$  for all  $x \in \Omega$  and all  $\rho > 0$ .
- $\mathbb{L}_u := \left\{ x \in \Omega : \lim_{\rho \rightarrow 0} u_{x,\rho} \text{ exists} \right\}$ .

For every  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ ,  $x \in \Omega$  and every  $\nu \in \mathcal{M}(\Omega; \mathbb{R}^l)$  with  $\nu = \frac{d\nu}{d\mathcal{L}_N} \mathcal{L}_N + \nu^s$ , we denote by

$$u_{x,\rho}^\sigma(\cdot) := \begin{cases} u_x(\cdot) & \text{if } \sigma = \mathcal{L}_N \\ u_{x,\rho} & \text{if } \sigma = |\nu^s|. \end{cases}$$

For each open set  $O \subset \Omega$ , we denote by  $\mathcal{A}_0(O) \subset W_0^{1,p}(O; \mathbb{R}^m) \times L^1(O; \mathbb{R}^l)$  the set of the all pairs  $(\varphi, \psi)$  such that  $\varphi \in W_0^{1,p}(O; \mathbb{R}^m)$  and  $\psi \in L^1(O; \mathbb{R}^l)$  with  $\int_O \psi d\mathcal{L}_N = 0$ , i.e.

$$\mathcal{A}_0(O) := \left\{ (\varphi, \psi) \in W_0^{1,p}(O; \mathbb{R}^m) \times L^1(O; \mathbb{R}^l) : \int_O \psi(x) d\mathcal{L}_N(x) = 0 \right\}.$$

## 2. MAIN RESULT

**2.1.  $\Gamma$ -convergence theorem.** The  $\Gamma$ -convergence is studied through the behavior of local minimization problems associated with the integrals. Let  $\varepsilon > 0$ . Let  $m_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$  be defined by

$$m_\varepsilon(u, \nu; O) := \inf \left\{ I_\varepsilon(w, \lambda; O) : \mathcal{M}(\Omega; \mathbb{R}^l) \ni \lambda \ll \mathcal{L}_N, \lambda(O) = \nu(O), \right. \\ \left. \text{and } w \in u + W_0^{1,p}(O; \mathbb{R}^m) \right\}, \quad (2.1)$$

where  $W_0^{1,p}(O; \mathbb{R}^m) := \{\phi \in W^{1,p}(\Omega; \mathbb{R}^m) : \phi = 0 \text{ on } \Omega \setminus O\}$ . We consider the following two “local-global” conditions on  $\{m_\varepsilon\}_\varepsilon$ :

(H<sub>2</sub>) for every  $O \in \mathcal{O}(\Omega)$ ,  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  and  $v \in L^1(\Omega; \mathbb{R}^l)$  it holds

$$\overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x, v(x) \mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} = \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x, v(x) \mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \quad \mathcal{L}_N\text{-a.e. in } O;$$

(H<sub>3</sub>) for every  $O \in \mathcal{O}(\Omega)$ ,  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ ,  $\nu \in \mathcal{M}(\Omega; \mathbb{R}^l)$  and  $w \in L^1_{|\nu^s|}(\Omega; \mathbb{R}^l)$ , it holds

$$\overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}, w(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} = \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}, w(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} \quad |\nu^s|\text{-a.e. in } O.$$

The condition (H<sub>2</sub>) expresses the independence of the  $\Gamma$ -limit with respect to sequences  $\varepsilon \rightarrow 0$ . The condition (H<sub>3</sub>) expresses simultaneously the independence of the  $\Gamma$ -limit with respect to  $\varepsilon \rightarrow 0$  and the independence of the limit behavior of the average of the “local minimization problems” on small balls around points in a set with zero Lebesgue measure. We will see in §3 that (H<sub>3</sub>) is fulfilled when we assume the existence of a recession function, with respect to the measure variable, associated with the integrands.

*Remark 2.1.* Note that  $(H_3)$  is equivalent to write that the limits with respect to  $\rho$  exist

$$\lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} = \lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} \quad |\nu^s|\text{-a.e. in } O.$$

Let  $O \in \mathcal{O}(\Omega)$ . We say that the sequence  $\{(u_\varepsilon, \nu_\varepsilon)\}_{\varepsilon > 0} \subset W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  converges to  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(O; \mathbb{R}^l)$  if  $u_\varepsilon \rightarrow u$  in  $L^p(\Omega; \mathbb{R}^m)$  and  $\nu_\varepsilon \xrightarrow{*} \nu$  in  $\mathcal{M}(O; \mathbb{R}^l)$ , and we write

$$(u_\varepsilon, \nu_\varepsilon) \rightarrow (u, \nu) \quad \text{in } L^p(\Omega; \mathbb{R}^m) \times \mathcal{M}(O; \mathbb{R}^l).$$

For every  $(u, \nu, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l) \times \mathcal{O}(\Omega)$  we set

$$I_+(u, \nu; O) := \inf \left\{ \overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon, \nu_\varepsilon; O) : (u_\varepsilon, \nu_\varepsilon) \rightarrow (u, \nu) \text{ in } L^p(\Omega; \mathbb{R}^m) \times \mathcal{M}(O; \mathbb{R}^l) \right\};$$

$$I_-(u, \nu; O) := \inf \left\{ \underline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon, \nu_\varepsilon; O) : (u_\varepsilon, \nu_\varepsilon) \rightarrow (u, \nu) \text{ in } L^p(\Omega; \mathbb{R}^m) \times \mathcal{M}(O; \mathbb{R}^l) \right\}.$$

For each  $O \in \mathcal{O}(\Omega)$ , if  $I_-(\cdot, \cdot; O) = I_+(\cdot, \cdot; O)$  then we say that  $\{I_\varepsilon(\cdot, \cdot; O)\}_{\varepsilon > 0}$   $\Gamma$ -converges to  $I_0(\cdot, \cdot; O) := I_-(\cdot, \cdot; O) = I_+(\cdot, \cdot; O)$ . (For more details on the theory of  $\Gamma$ -convergence we refer to [DM93])

Now, we state the main result of the paper.

**Theorem 2.1.** *Assume that  $(H_1)$ ,  $(H_2)$  and  $(H_3)$  hold. For every  $O \in \mathcal{O}(\Omega)$  the family of integral functionals  $\{I_\varepsilon(\cdot, \cdot; O)\}_\varepsilon$   $\Gamma$ -converges to  $I_0(\cdot, \cdot; O)$  at each  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  where*

$$I_0(u, \nu; O) = \int_O f_0 \left( x, u(x), \nabla u(x), \frac{d\nu}{d\mathcal{L}_N}(x) \right) d\mathcal{L}_N(x) + \int_O f_{0,u}^s \left( x, \frac{d\nu}{d|\nu^s|}(x) \right) d|\nu^s|(x),$$

where  $f_0 : \Omega \times \mathbb{R}^m \times \mathbb{M} \times \mathbb{R}^l \rightarrow [0, \infty[$  is defined by

$$f_0(x, u, \xi, v) := \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u + \xi(\cdot - x), v\mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))},$$

and where, for each  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  the integrand  $f_{0,u}^s : \Omega \times \mathbb{S}^l \rightarrow [0, \infty[$  is defined by

$$f_{0,u}^s(x, w) := \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( \int_{B_\rho(x)} u d\mathcal{L}_N, w|\nu^s|; B_\rho(x) \right)}{|\nu^s|(B_\rho(x))}.$$

*Remark 2.2.* Note that  $f_{0,u}^s$  does not depend on the pointwise value  $u(x)$ , but on the limit behavior of the average of  $u$  at  $x$  on small balls. It is possible, for instance when  $u$  is not regular (i.e. when  $p \leq N$ ), that  $\int_{B_\rho(x)} u(z) d\mathcal{L}_N(z)$  does not converge when  $\rho \rightarrow 0$  since  $x$  belongs to a negligible set for the Lebesgue measure. This naturally leads to consider the set  $\mathbb{L}_u$  of points  $x$  where that limit exists (see §3.1).

**2.2. Global bounds for  $I_+$ ,  $I_-$  and local bounds for  $m_\varepsilon$ .** The proof of Theorem 2.1 is based on the following two propositions below. The Proposition 2.1 provides lower (resp. upper) bound of the limit  $I_-$  (resp.  $I_+$ ) under integral form. The lower bound for  $I_-$  is obtained by using the “localization method” (see [FM92]), which consists in analyzing the weak limit (of a converging subsequence) of the sequence of measures

$$\left\{ f_\varepsilon \left( \cdot, u_\varepsilon, \nabla u_\varepsilon, \frac{d\nu_\varepsilon}{d\mathcal{L}_N} \right) \mathcal{L}_N \right\}_{\varepsilon>0}.$$

The upper bound for  $I_+$  is also obtained through a localization technique but using, among others ingredients, the differentiability of the lower Vitali envelope of the open set function  $\overline{\lim}_{\varepsilon \rightarrow 0} m_\varepsilon(u, \nu; \cdot)$ , see §7.3 and Lemma 7.6.

**Proposition 2.1** (Bounds for  $I_-$  and  $I_+$ ). *Let  $O \in \mathcal{O}(\Omega)$ .*

(i) *For every  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$ , there exists a sequence  $\{(u_\varepsilon, \nu_\varepsilon)\}_{\varepsilon>0} \subset W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  such that*

$$(u_\varepsilon, \nu_\varepsilon) \rightarrow (u, \nu) \text{ in } L^p(\Omega; \mathbb{R}^m) \times \mathcal{M}(O; \mathbb{R}^l), \quad \sup_{\varepsilon>0} I_\varepsilon(u_\varepsilon, \nu_\varepsilon; O) < \infty$$

and

$$I_-(u, \nu; O) \geq \sum_{\sigma \in \{\mathcal{L}_N, |\nu^s|\}} \int_O \overline{\lim}_{t \rightarrow 1^-} \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{I_\varepsilon(u_\varepsilon, \nu_\varepsilon; B_{t\rho}(x))}{\sigma(B_\rho(x))} d\sigma(x).$$

(ii) *Assume that*

(H'<sub>1</sub>) *there exists  $c > 0$  such that for every  $\varepsilon > 0$  and every  $(x, u, \xi, v) \in \Omega \times \mathbb{R}^m \times \mathbb{M} \times \mathbb{R}^l$*

$$c(|\xi|^p + |v|) \leq f_\varepsilon(x, u, \xi, v);$$

(H'<sub>2</sub>) *for every  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  there exists  $\alpha_{u,\nu} \in \mathcal{M}_+(\Omega)$  with  $\alpha_{u,\nu} \ll \mu := \mathcal{L}_N + |\nu^s|$  such that*

$$\sup_{\varepsilon>0} m_\varepsilon(u, \nu; U) \leq \alpha_{u,\nu}(U) \quad \text{for all } U \in \mathcal{O}(\Omega).$$

*Then we have, for every  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$*

$$I_+(u, \nu; O) \leq \sum_{\sigma \in \{\mathcal{L}_N, |\nu^s|\}} \int_O \lim_{R_\mu^\# \# \rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u, \nu; B_\rho(x))}{\sigma(B_\rho(x))} d\sigma(x).$$

*Remark 2.3.* Note that the condition (H<sub>1</sub>) implies (H'<sub>1</sub>) and (H'<sub>2</sub>). Indeed, for each pair  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  we set  $\alpha_{u,\nu} := C \left( 1 + |u|^p + |\nabla u|^p + \frac{|\nu|(\Omega)}{\mathcal{L}_N(\Omega)} \right) \mathcal{L}_N$ . Hence, for every  $U \in \mathcal{O}(\Omega)$  we have  $\sup_{\varepsilon>0} m_\varepsilon(u, \nu; U) \leq \alpha_{u,\nu}(U)$ .

The second lemma below gives lower and upper bounds for the limit integrands. It consists in “locally replacing” the arguments  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  (and  $(u_\varepsilon, \nu_\varepsilon) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$ ) of the integral representations obtained in Proposition 2.1, by local approximations on “small” balls centered at  $x$ . In case  $x$  belongs to the support of the Lebesgue measure, we consider the affine tangent map of  $u$  at  $x$ , as local approximation of  $u$ , and the absolute continuous part of  $\nu$  at  $x$  for  $\nu$ , i.e.

$$(u, \nu) \quad \text{“locally replaced” by} \quad \left( u_x(\cdot), \frac{d\nu}{d\mathcal{L}_N}(x) \mathcal{L}_N \right).$$

When  $x$  belongs to the support of the singular part  $\nu^s$ , we consider the mean value of  $u$  on balls centered at  $x$ , and the derivative of  $\nu$  with respect to the total variation  $|\nu^s|$  at  $x$  as local approximations, i.e.

$$(u, \nu) \quad \text{“locally replaced” by} \quad \left( u_{x,\rho}, \frac{d\nu}{d|\nu^s|}(x) |\nu^s| \right)$$

for “small”  $\rho > 0$ .

**Proposition 2.2** (Local bounds). *Let  $O \in \mathcal{O}(\Omega)$ . Assume that (H<sub>1</sub>) holds. Then*

(i) *for every pair  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  and every sequence  $\{(u_\varepsilon, \nu_\varepsilon)\}_{\varepsilon>0} \subset W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  satisfying*

$$(u_\varepsilon, \nu_\varepsilon) \rightarrow (u, \nu) \text{ in } L^p(\Omega; \mathbb{R}^m) \times \mathcal{M}(O; \mathbb{R}^l) \quad \text{and} \quad \sup_{\varepsilon>0} I_\varepsilon(u_\varepsilon, \nu_\varepsilon; O) < \infty,$$

*we have*

$$\overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_x, \frac{d\nu}{d\mathcal{L}_N}(x) \mathcal{L}_N; B_\rho(x) \right)}{\mathcal{L}_N(B_\rho(x))} \leq \overline{\lim}_{t \rightarrow 1^-} \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{I_\varepsilon(u_\varepsilon, \nu_\varepsilon; B_{t\rho}(x))}{\mathcal{L}_N(B_\rho(x))} \quad \mathcal{L}_N \text{-a.e. in } O, \quad (2.2)$$

*and*

$$\overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_{x,\rho}, \frac{d\nu}{d|\nu^s|}(x) |\nu^s|; B_\rho(x) \right)}{|\nu^s|(B_\rho(x))} \leq \overline{\lim}_{t \rightarrow 1^-} \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{I_\varepsilon(u_\varepsilon, \nu_\varepsilon; B_{t\rho}(x))}{|\nu^s|(B_\rho(x))} \quad |\nu^s| \text{-a.e. in } O; \quad (2.3)$$

(ii) *for every  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  we have*

$$\lim_{R_\mu^+ \ni \rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u, \nu; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \leq \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_x, \frac{d\nu}{d\mathcal{L}_N}(x) \mathcal{L}_N; B_\rho(x) \right)}{\mathcal{L}_N(B_\rho(x))} \quad \mathcal{L}_N \text{-a.e. in } O, \quad (2.4)$$

*and*

$$\lim_{R_\mu^+ \ni \rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u, \nu; B_\rho(x))}{|\nu^s|(B_\rho(x))} \leq \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_{x,\rho}, \frac{d\nu}{d|\nu^s|}(x) |\nu^s|; B_\rho(x) \right)}{|\nu^s|(B_\rho(x))} \quad |\nu^s| \text{-a.e. in } O. \quad (2.5)$$

Using both Propositions 2.1 and 2.2 and the conditions (H<sub>2</sub>) and (H<sub>3</sub>), we see (§2.3) that these global and local inequalities give at the same time the formulas for the limit integrands and lead to the  $\Gamma$ -convergence of  $\{I_\varepsilon\}_{\varepsilon>0}$ . In fact, the assumptions (H<sub>2</sub>) and (H<sub>3</sub>) allows us to bridge the difference between the lower and the upper bounds. In §3.1, §3.2, §3.3 and §3.4 we will exhibit explicit conditions on the initial integrands which entail (H<sub>2</sub>) and (H<sub>3</sub>).

A consequence of Propositions 2.1 and 2.2 is that the limit integrands are measurable. The proof of the following corollary is similar to that of Theorem 2.1 and is a consequence of Lemma 7.6.



**Corollary 2.1.** *Let  $(u, \nu, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l) \times \mathcal{O}(\Omega)$  and let  $\sigma \in \{\mathcal{L}_N, |\nu^s|\}$ . If  $(H_1)$ ,  $(H_2)$  and  $(H_3)$  hold then for  $\sigma$ -a.e.  $x \in O$  the following limit exists*

$$\lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_{x,\rho}^\sigma, \frac{d\nu}{d\sigma}(x) \sigma; B_\rho(x) \right)}{\sigma(B_\rho(x))} =: \Psi_\sigma(x).$$

Moreover,  $\Psi_\sigma$  is  $\sigma$ -measurable.

**2.3. Proof of Theorem 2.1.** Let  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  and  $O \in \mathcal{O}(\Omega)$ . Combining Proposition 2.1, Corollary 2.1, Proposition 2.2 and conditions  $(H_2)$  and  $(H_3)$ , we can write, for  $\mathcal{L}_N$ -a.e.  $x \in O$ , that

$$\begin{aligned} \underline{\lim}_{\rho \rightarrow 0} \frac{I_-(u, \nu; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} &\geq \overline{\lim}_{\rho \rightarrow 0} \underline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_x, \frac{d\nu}{d\mathcal{L}_N}(x) \mathcal{L}_N; B_\rho(x) \right)}{\mathcal{L}_N(B_\rho(x))} \\ &\geq \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_x, \frac{d\nu}{d\mathcal{L}_N}(x) \mathcal{L}_N; B_\rho(x) \right)}{\mathcal{L}_N(B_\rho(x))} \\ &\geq \lim_{R_\mu^x \not\rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u, \nu; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \geq \lim_{\rho \rightarrow 0} \frac{I_+(u, \nu; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))}. \end{aligned}$$

It follows, since  $I_-(u, \nu; \cdot) \leq I_+(u, \nu; \cdot)$ , that for  $\mathcal{L}_N$ -a.e.  $x \in O$  we have

$$\begin{aligned} \lim_{\rho \rightarrow 0} \frac{I_-(u, \nu; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} &= \lim_{\rho \rightarrow 0} \frac{I_+(u, \nu; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \\ &= \lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_x, \frac{d\nu}{d\mathcal{L}_N}(x) \mathcal{L}_N; B_\rho(x) \right)}{\mathcal{L}_N(B_\rho(x))} \\ &= \lim_{R_\mu^x \not\rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u, \nu; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \\ &= \overline{\lim}_{\rho \rightarrow 0} \underline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_x, \frac{d\nu}{d\mathcal{L}_N}(x) \mathcal{L}_N; B_\rho(x) \right)}{\mathcal{L}_N(B_\rho(x))}. \end{aligned}$$

(Note that all these functions are measurable because  $x \mapsto \lim_{R_\mu^x \not\rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u, \nu; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))}$  is measurable by Lemma 7.6 and Remarks 7.2 (iii)). Similarly, we have for  $|\nu^s|$ -a.e.  $x \in O$

$$\begin{aligned} \lim_{\rho \rightarrow 0} \frac{I_-(u, \nu; B_\rho(x))}{|\nu^s|(B_\rho(x))} &= \lim_{\rho \rightarrow 0} \frac{I_+(u, \nu; B_\rho(x))}{|\nu^s|(B_\rho(x))} = \lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_{x,\rho}, \frac{d\nu}{d|\nu^s|}(x) |\nu^s|; B_\rho(x) \right)}{|\nu^s|(B_\rho(x))} \\ &= \lim_{R_\mu^x \not\rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u, \nu; B_\rho(x))}{|\nu^s|(B_\rho(x))} \\ &= \lim_{\rho \rightarrow 0} \underline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_{x,\rho}, \frac{d\nu}{d|\nu^s|}(x) |\nu^s|; B_\rho(x) \right)}{|\nu^s|(B_\rho(x))}. \end{aligned}$$

Using Corollary 2.1 and Proposition 2.1 we can write

$$\begin{aligned} I_-(u, \nu; O) &\geq \sum_{\sigma \in \{\mathcal{L}_N, |\nu^s|\}} \int_O \lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_{x, \rho}^\sigma, \frac{d\nu}{d\sigma}(x) \sigma; B_\rho(x) \right)}{\sigma(B_\rho(x))} d\sigma(x) \\ &= \sum_{\sigma \in \{\mathcal{L}_N, |\nu^s|\}} \int_O \lim_{R_\mu^x \not\rightarrow \rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u, \nu; B_\rho(x))}{\sigma(B_\rho(x))} d\sigma(x) \geq I_+(u, \nu; O), \end{aligned}$$

which completes the proof. ■

**Proof of Corollary 2.1.** Let  $O \in \mathcal{O}(\Omega)$ . Let  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$ . By Lemma 2.1 (i) there exists  $\{(u_\varepsilon, \nu_\varepsilon)\}_{\varepsilon > 0} \subset W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  such that  $(u_\varepsilon, \nu_\varepsilon) \rightarrow (u, \nu)$  in  $L^p(\Omega; \mathbb{R}^m) \times \mathcal{M}(O; \mathbb{R}^l)$ ,  $\sup_{\varepsilon > 0} I_\varepsilon(u_\varepsilon, \nu_\varepsilon; O) < \infty$  and satisfying for  $\mathcal{L}_N$ -a.e.  $x \in O$

$$\underline{\lim}_{\rho \rightarrow 0} \frac{I_-(u, \nu; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \geq \overline{\lim}_{t \rightarrow 1^-} \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{I_\varepsilon(u_\varepsilon, \nu_\varepsilon; B_{t\rho}(x))}{\mathcal{L}_N(B_\rho(x))}.$$

It follows by using Propositions 2.1 and 2.2 that for  $\mathcal{L}_N$ -a.e.  $x \in O$

$$\begin{aligned} &\underline{\lim}_{\rho \rightarrow 0} \frac{I_-(u, \nu; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \\ &\geq \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_x, \frac{d\nu}{d\mathcal{L}_N}(x) \mathcal{L}_N; B_\rho(x) \right)}{\mathcal{L}_N(B_\rho(x))} = \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_x, \frac{d\nu}{d\mathcal{L}_N}(x) \mathcal{L}_N; B_\rho(x) \right)}{\mathcal{L}_N(B_\rho(x))} \\ &\geq \underline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_x, \frac{d\nu}{d\mathcal{L}_N}(x) \mathcal{L}_N; B_\rho(x) \right)}{\mathcal{L}_N(B_\rho(x))} \\ &\geq \lim_{D_\mu^x \not\rightarrow \rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u, \nu; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \geq \overline{\lim}_{\rho \rightarrow 0} \frac{I_+(u, \nu; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))}. \end{aligned}$$

Similarly, using again Propositions 2.1 and 2.2 we have for  $|\nu^s|$ -a.e.  $x \in O$

$$\begin{aligned} \underline{\lim}_{\rho \rightarrow 0} \frac{I_-(u, \nu; B_\rho(x))}{|\nu^s|(B_\rho(x))} &\geq \overline{\lim}_{t \rightarrow 1^-} \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{I_\varepsilon(u_\varepsilon, \nu_\varepsilon; B_{t\rho}(x))}{|\nu^s|(B_\rho(x))} \\ &\geq \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_{x, \rho}, \frac{d\nu}{d|\nu^s|}(x) |\nu^s|; B_\rho(x) \right)}{|\nu^s|(B_\rho(x))} \\ &\geq \underline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_{x, \rho}, \frac{d\nu}{d|\nu^s|}(x) |\nu^s|; B_\rho(x) \right)}{|\nu^s|(B_\rho(x))} \\ &= \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_{x, \rho}, \frac{d\nu}{d|\nu^s|}(x) |\nu^s|; B_\rho(x) \right)}{|\nu^s|(B_\rho(x))} \\ &\geq \lim_{D_\mu^x \not\rightarrow \rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u, \nu; B_\rho(x))}{|\nu^s|(B_\rho(x))} \\ &\geq \overline{\lim}_{\rho \rightarrow 0} \frac{I_+(u, \nu; B_\rho(x))}{|\nu^s|(B_\rho(x))}. \end{aligned}$$

Therefore we have for every  $\sigma \in \{\mathcal{L}_N, |\nu^s|\}$  and for  $\sigma$ -a.e.  $x \in O$

$$\lim_{\rho \rightarrow 0} \frac{I_-(u, \nu; B_\rho(x))}{\sigma(B_\rho(x))} = \lim_{\rho \rightarrow 0} \frac{I_+(u, \nu; B_\rho(x))}{\sigma(B_\rho(x))} = \Psi_\sigma(x) = \lim_{D_\mu^s \not\rightarrow \rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u, \nu; B_\rho(x))}{\sigma(B_\rho(x))}. \quad (2.6)$$

But, by Lemma 7.6 and Remarks 7.2 (iii) the last function in (2.6) is  $\sigma$ -measurable and thus the function  $\Psi_\sigma$  is also  $\sigma$ -measurable. ■

### 3. APPLICATION TO RELAXATION AND HOMOGENIZATION THEOREMS

**3.1. Relaxation theorem with singular part depending on  $u$ .** Let  $f : \Omega \times \mathbb{R}^m \times \mathbb{M} \times \mathbb{R}^l \rightarrow [0, \infty[$  be a Borel measurable integrand. We consider the integral functional  $I : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$  defined by

$$I(u, \nu; O) := \begin{cases} \int_O f\left(x, u(x), \nabla u(x), \frac{d\nu}{d\mathcal{L}_N}(x)\right) d\mathcal{L}_N(x) & \text{if } \nu \ll \mathcal{L}_N \\ \infty & \text{otherwise.} \end{cases}$$

We define its “relaxed” by

$$\bar{I}(u, \nu; O) := \inf \left\{ \liminf_{n \rightarrow \infty} I(u_n, \nu_n; O) : u_n \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \text{ and } \nu_n \xrightarrow{*} \nu \text{ in } \mathcal{M}(O; \mathbb{R}^l) \right\}$$

for all  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  and all  $O \in \mathcal{O}(\Omega)$ .

Consider the following conditions:

( $\mathcal{R}_1$ ) there exist  $C, c > 0$  such that for every  $(x, u, \xi, v) \in \Omega \times \mathbb{R}^m \times \mathbb{M} \times \mathbb{R}^l$  we have

$$c(|\xi|^p + |v|) \leq f(x, u, \xi, v) \leq C(1 + |u|^p + |\xi|^p + |v|);$$

( $\mathcal{R}_2$ ) there exist  $C_1 > 0, q > 0$  and  $q' > 0$  such that for every  $(x, x', u, u', \xi, v) \in \Omega \times \Omega \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{M} \times \mathbb{R}^l$

$$|f(x, u, \xi, v) - f(x', u', \xi, v)| \leq C_1 \left( |u - u'|^q + |x - x'|^{q'} \right) (1 + |\xi|^p + |v|);$$

( $\mathcal{R}_3$ ) there exist  $T > 0, \beta > 0$  and  $r \in ]0, 1[$  such that for every  $t > T$ , every  $(x, u, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{M}$  and every  $w \in \mathbb{S}^l$  we have

$$\left| \frac{f(x, u, \xi, tw)}{t} - f^\infty(x, u, \xi, w) \right| \leq \frac{\beta}{t^r}, \quad \text{where} \quad f^\infty(x, u, \xi, w) := \overline{\lim}_{t \rightarrow \infty} \frac{f(x, u, \xi, tw)}{t}.$$

For every  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  we consider the set  $\mathbb{L}_u$  of points  $x \in \Omega$  where  $u_{x,\rho}$  admits a limit as  $\rho \rightarrow 0$ , i.e.

$$\mathbb{L}_u := \left\{ x \in \Omega : \lim_{\rho \rightarrow 0} u_{x,\rho} \text{ exists} \right\}.$$

**Theorem 3.1.** *Assume that ( $\mathcal{R}_1$ ), ( $\mathcal{R}_2$ ) and ( $\mathcal{R}_3$ ) hold. Assume that for every  $O \in \mathcal{O}(\Omega)$ , and every  $(u, v) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^1_{|\nu^s|}(\Omega; \mathbb{S}^l)$*

$$\liminf_{\rho \rightarrow 0} \frac{m(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} = \overline{\lim}_{\rho \rightarrow 0} \frac{m(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} \quad |\nu^s|\text{-a.e. in } O \setminus \mathbb{L}_u.$$

Then for every  $O \in \mathcal{O}(\Omega)$  and every  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  we have

$$\begin{aligned} \bar{I}(u, \nu; O) &= \int_O \mathcal{Q}_0 f \left( x, u(x), \nabla u(x), \frac{d\nu}{d\mathcal{L}_N}(x) \right) d\mathcal{L}_N(x) \\ &\quad + \int_{O \cap \mathbb{L}_u} \mathcal{Q}_0^s f^\infty \left( x, \bar{u}(x), \frac{d\nu}{d|\nu^s|}(x) \right) d|\nu^s|(x) \\ &\quad + \int_{O \setminus \mathbb{L}_u} f_{0,u}^s \left( x, \frac{d\nu}{d|\nu^s|}(x) \right) d|\nu^s|(x), \end{aligned}$$

where  $\bar{u}(x) = \lim_{\rho \rightarrow 0} u_{x,\rho}$  for all  $x \in \mathbb{L}_u$ , and for every  $(x, u, \xi, v, w) \in \Omega \times \mathbb{R}^m \times \mathbb{M} \times \mathbb{R}^l \times \mathbb{S}^l$

$$\mathcal{Q}_0 f(x, u, \xi, v) := \inf \left\{ \int_Y f(x, u, \xi + \nabla \varphi(y), v + \psi(y)) d\mathcal{L}_N(y) : \right. \\ \left. \varphi \in W_0^{1,p}(Y; \mathbb{R}^m) \text{ and } \psi \in L^1(Y; \mathbb{R}^l), \int_Y \psi(y) d\mathcal{L}_N(y) = 0 \right\}, \quad (3.1)$$

$$\mathcal{Q}_0^s f^\infty(x, u, w) := \inf \left\{ \int_Y f^\infty(x, u, \nabla \varphi(y), w + \psi(y)) d\mathcal{L}_N(y) : \right. \\ \left. \varphi \in W_0^{1,p}(Y; \mathbb{R}^m) \text{ and } \psi \in L^1(Y; \mathbb{S}^l), \int_Y \psi(y) d\mathcal{L}_N(y) = 0 \right\}, \quad (3.2)$$

and

$$f_{0,u}^s \left( x, \frac{d\nu}{d|\nu^s|}(x) \right) := \lim_{\rho \rightarrow 0} \frac{m \left( u_{x,\rho}, \frac{d\nu}{d|\nu^s|}(x) | \nu^s |; B_\rho(x) \right)}{|\nu^s|(B_\rho(x))} \quad |\nu^s|\text{-a.e. in } O \setminus \mathbb{L}_u.$$

*Remark 3.1.* Formula giving  $\mathcal{Q}_0 f$  is not new, see [FKP94, LDR00]. Observe that there is simultaneously a “quasiconvexification” with respect to the gradient variable  $\xi$  and a “convexification” with respect to  $v$  the absolute part with respect to  $\mathcal{L}_N$  of the measure variable, it was called quasiconvex-convex envelope of  $f$  by [CRZ11]. We can also note that there is no contribution of the gradient of  $u$  to the formula giving  $\mathcal{Q}_0^s f^\infty$ , this due to the fact that a local approximation of  $u$ , around balls with radius  $\rho$  centered at point  $x$  of the support of  $\nu^s$ , is  $u_{x,\rho}$  whose gradient is zero. The main difficulty to extend the method and so Theorem 2.1 to  $BV(\Omega; \mathbb{R}^m)$ , is to find the “good local approximation” for  $D^s u$  the singular part in the Lebesgue decomposition of the distributional derivative of  $u \in BV(\Omega; \mathbb{R}^m)$ .

If we assume that  $p > N$  and  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  then  $\Omega \setminus \mathbb{L}_u = \emptyset$  and  $\bar{u}(x) = u(x)$  for all  $x \in \Omega$ .

**Corollary 3.1.** *Assume that  $p > N$ . Assume that  $(\mathcal{R}_1)$ ,  $(\mathcal{R}_2)$  and  $(\mathcal{R}_3)$  hold. Then for every  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  and  $O \in \mathcal{O}(\Omega)$ , we have*

$$\begin{aligned} \bar{I}(u, \nu; O) &= \int_O \mathcal{Q}_0 f \left( x, u(x), \nabla u(x), \frac{d\nu}{d\mathcal{L}_N}(x) \right) d\mathcal{L}_N(x) \\ &\quad + \int_O \mathcal{Q}_0^s f^\infty \left( x, u(x), \frac{d\nu}{d|\nu^s|}(x) \right) d|\nu^s|(x), \end{aligned}$$

where  $\mathcal{Q}_0 f$  is given by (3.1) and  $\mathcal{Q}_0^s f$  by (3.2).

*Remark 3.2.* We denote by  $\mathcal{H}^k$  the  $k$ -Hausdorff measure on  $\mathbb{R}^N$  with  $k \in \mathbb{R}_+$ . Assume that  $p \leq N$ . Then for every  $O \in \mathcal{O}(\Omega)$  and for every  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  satisfying  $|\nu^s| \ll \mathcal{H}^k$  for some  $k > N - p$ , we have

$$\begin{aligned} \bar{I}(u, \nu; O) &= \int_O \mathcal{Q}_0 f \left( x, u(x), \nabla u(x), \frac{d\nu}{d\mathcal{L}_N}(x) \right) d\mathcal{L}_N(x) \\ &\quad + \int_O \mathcal{Q}_0^s f^\infty \left( x, u(x), \frac{d\nu}{d|\nu^s|}(x) \right) d|\nu^s|(x) \end{aligned}$$

since  $|\nu^s|(\Omega \setminus \mathbb{L}_u) = 0$  by Federer-Ziemer theorem [FZ73].

**3.2. Relaxation theorem with singular part not depending on  $u$ .** Consider the condition:

$(\mathcal{R}'_2)$  there exist  $C_2 > 0$  and  $q' > 0$  such that for every  $(x, x', u, u', \xi, v) \in \Omega \times \Omega \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{M} \times \mathbb{R}^l$

$$|f(x, u, \xi, v) - f(x', u', \xi, v)| \leq C_2 \left( |u - u'|^p + |x - x'|^{q'} \right).$$

*Remark 3.3.* Let  $W : \mathbb{M} \times \mathbb{R}^l \rightarrow [0, \infty[$  and  $h : \mathbb{R}^m \rightarrow [0, \infty[$  be Borel measurable functions such that for some  $C > 0$  we have

$$|h(u) - h(u')| \leq C|u - u'|^p \quad \text{for all } (u, u') \in \mathbb{R}^m \times \mathbb{R}^m.$$

Then the condition  $(\mathcal{R}'_2)$  is satisfied for integrand of the form  $f(x, u, \xi, v) = W(\xi, v) + h(u)$  for all  $(x, u, \xi, v) \in \Omega \times \mathbb{R}^m \times \mathbb{M} \times \mathbb{R}^l$ . While  $(\mathcal{R}_2)$  is satisfied for integrand of the form  $f(x, u, \xi, v) = W(\xi, v)h(u)$  for all  $(x, u, \xi, v) \in \Omega \times \mathbb{R}^m \times \mathbb{M} \times \mathbb{R}^l$ .

The variable  $u$  does not appear in the singular part when  $(\mathcal{R}_2)$  is replaced by  $(\mathcal{R}'_2)$ .

**Theorem 3.2.** Assume that  $(\mathcal{R}_1)$ ,  $(\mathcal{R}'_2)$  and  $(\mathcal{R}_3)$  hold. Then we have for every  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  and every  $O \in \mathcal{O}(\Omega)$

$$\begin{aligned} \bar{I}(u, \nu; O) &= \int_O \mathcal{Q}_0 f \left( x, u(x), \nabla u(x), \frac{d\nu}{d\mathcal{L}_N}(x) \right) d\mathcal{L}_N(x) \\ &\quad + \int_O \mathcal{Q}_0^s f^\infty \left( x, \frac{d\nu}{d|\nu^s|}(x) \right) d|\nu^s|(x), \end{aligned}$$

where  $\mathcal{Q}_0 f$  is given by (3.1) and for every  $(x, w) \in \Omega \times \mathbb{S}^l$

$$\begin{aligned} \mathcal{Q}_0^s f^\infty(x, w) &:= \inf \left\{ \int_Y f^\infty(x, 0, \nabla \varphi(y), w + \psi(y)) d\mathcal{L}_N(y) : \right. \\ &\quad \left. \varphi \in W_0^{1,p}(Y; \mathbb{R}^m) \text{ and } \psi \in L^1(Y; \mathbb{R}^l), \int_Y \psi(y) d\mathcal{L}_N(y) = 0 \right\}. \end{aligned}$$

**3.3. Homogenization theorem with additional vector measure variable.** Let  $W : \Omega \times \mathbb{R}^N \times \mathbb{M} \times \mathbb{R}^l \rightarrow [0, \infty[$  be a Borel measurable integrand 1-periodic with respect to the second variable, i.e.

$(\mathcal{H}_0)$   $W(x, y + z, \xi, v) = W(x, y, \xi, v)$  for all  $z \in \mathbb{Z}^N$  and all  $(x, y, \xi, v) \in \Omega \times \mathbb{R}^N \times \mathbb{M} \times \mathbb{R}^l$ .

For every  $\varepsilon > 0$ , we consider the functionals  $I_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$  defined by

$$I_\varepsilon(u, \nu; O) := \begin{cases} \int_O W \left( x, \frac{x}{\varepsilon}, \nabla u(x), \frac{d\nu}{d\mathcal{L}_N}(x) \right) d\mathcal{L}_N(x) & \text{if } \nu \ll \mathcal{L}_N \\ \infty & \text{otherwise.} \end{cases}$$

Consider the following conditions on  $W$ :

( $\mathcal{H}_1$ ) there exist  $C, c > 0$  such that for every  $(x, y, \xi, v) \in \Omega \times \mathbb{R}^N \times \mathbb{M} \times \mathbb{R}^l$  we have

$$c(|\xi|^p + |v|) \leq W(x, y, \xi, v) \leq C(1 + |\xi|^p + |v|);$$

( $\mathcal{H}_2$ ) there exist  $C_1 > 0$  and  $q' > 0$  such that for every  $(x, x', y, \xi, v) \in \Omega \times \Omega \times \mathbb{R}^N \times \mathbb{M} \times \mathbb{R}^l$  we have

$$|W(x, y, \xi, v) - W(x', y, \xi, v)| \leq C_1|x - x'|^{q'}(1 + |\xi|^p + |v|);$$

( $\mathcal{H}_3$ ) there exist  $T > 0$ ,  $\beta > 0$  and  $r \in ]0, 1[$  such that for every  $t > T$ , every  $(x, y, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{M}$  and every  $v \in \mathbb{S}^l$ , we have

$$\left| \frac{W(x, y, \xi, tv)}{t} - W^\infty(x, y, \xi, v) \right| \leq \frac{\beta}{t^r}, \quad \text{where } W^\infty(x, y, \xi, v) := \overline{\lim}_{t \rightarrow \infty} \frac{W(x, y, \xi, tv)}{t}.$$

We have the following homogenization theorem.

**Theorem 3.3.** *Assume that ( $\mathcal{H}_0$ ), ( $\mathcal{H}_1$ ), ( $\mathcal{H}_2$ ) and ( $\mathcal{H}_3$ ) hold. For every  $O \in \mathcal{O}(\Omega)$  the family  $\{I_\varepsilon(\cdot, \cdot; O)\}_\varepsilon$   $\Gamma$ -converges at each  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  to  $I_0(\cdot, \cdot; O)$  given by*

$$I_0(u, \nu; O) = \int_O \mathcal{H}_0 W \left( x, \nabla u(x), \frac{d\nu}{d\mathcal{L}_N}(x) \right) d\mathcal{L}_N(x) + \int_O \mathcal{H}_0^s W^\infty \left( x, \frac{d\nu}{d|\nu^s|}(x) \right) d|\nu^s|(x),$$

where for every  $(x, \xi, v, w) \in \Omega \times \mathbb{M} \times \mathbb{R}^l \times \mathbb{S}^l$

$$\mathcal{H}_0 W(x, \xi, v) := \inf_{k \in \mathbb{N}^*} \inf_{k\mathbb{Y}} \left\{ \int_{k\mathbb{Y}} W(x, y, \xi + \nabla \varphi(y), v + \psi(y)) d\mathcal{L}_N(y) : \varphi \in W_0^{1,p}(k\mathbb{Y}; \mathbb{R}^m) \text{ and } \psi \in L^1(k\mathbb{Y}; \mathbb{R}^l), \int_{k\mathbb{Y}} \psi(y) d\mathcal{L}_N(y) = 0 \right\},$$

$$\mathcal{H}_0^s W^\infty(x, w) := \inf_{k \in \mathbb{N}^*} \inf_{k\mathbb{Y}} \left\{ \int_{k\mathbb{Y}} W^\infty(x, y, \nabla \varphi(y), w + \psi(y)) d\mathcal{L}_N(y) : \varphi \in W_0^{1,p}(k\mathbb{Y}; \mathbb{R}^m) \text{ and } \psi \in L^1(k\mathbb{Y}; \mathbb{R}^l), \int_{k\mathbb{Y}} \psi(y) d\mathcal{L}_N(y) = 0 \right\}.$$

*Remark 3.4.* We could assume dependence on  $u$  for  $W$ , in this case, we have to modify the hypothesis ( $\mathcal{H}_1$ ), ( $\mathcal{H}_2$ ) and ( $\mathcal{H}_3$ ) making them similar to those of the relaxation theorems, but we choose not to overload the presentation of the proof of Theorem 3.3. Nevertheless, we can note that Theorem 3.3 is an extension of the homogenization result of [CRZ11].

**3.4. Homogenization theorem with additional periodic vector measure variable.** Let  $W : \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{M} \times \mathbb{R}^l \rightarrow [0, \infty[$  be a Borel measurable integrand 1-periodic with respect to all variables except to the variable  $\xi \in \mathbb{M}$ , i.e.

( $\mathcal{H}'_0$ ) for every  $(z, z', \tau) \in \mathbb{Z}^N \times \mathbb{Z}^m \times \mathbb{Z}^l$  and every  $(x, u, \xi, v) \in \Omega \times \mathbb{R}^m \times \mathbb{M} \times \mathbb{R}^l$

$$W(x + z, u + z', \xi, v + \tau) = W(x, u, \xi, v)$$

For each  $\varepsilon > 0$ , we consider  $I_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$  defined by

$$I_\varepsilon(u, \nu; O) := \begin{cases} \int_O W\left(\frac{x}{\varepsilon}, \frac{u}{\varepsilon}, \nabla u(x), \frac{1}{\varepsilon} \frac{d\nu}{d\mathcal{L}_N}(x)\right) d\mathcal{L}_N(x) & \text{if } \nu \ll \mathcal{L}_N \\ \infty & \text{otherwise.} \end{cases}$$

Consider the following conditions:

$(\mathcal{H}'_1)$  there exist  $C, c > 0$  such that for every  $(x, u, \xi, v) \in \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{M} \times \mathbb{R}^l$

$$c(|\xi|^p + |v|) \leq W(x, u, \xi, v) \leq C(1 + |\xi|^p + |v|).$$

*Remark 3.5.* Assume that  $(\mathcal{H}'_0)$  and  $(\mathcal{H}'_1)$  hold. For each  $v \in \mathbb{R}^l$  and each  $\varepsilon \in ]0, 1[$  we set  $\lfloor \frac{v}{\varepsilon} \rfloor_l := (\lfloor \frac{v_1}{\varepsilon} \rfloor, \dots, \lfloor \frac{v_l}{\varepsilon} \rfloor) \in \mathbb{Z}^l$  where  $\lfloor \frac{v_i}{\varepsilon} \rfloor = \max\{z \in \mathbb{Z} : z \leq \frac{v_i}{\varepsilon}\}$  is the integer part of  $\frac{v_i}{\varepsilon}$  for all  $i \in \{1, \dots, l\}$ . Then we can write  $\frac{v}{\varepsilon} = \lfloor \frac{v}{\varepsilon} \rfloor_l + (\frac{v}{\varepsilon} - \lfloor \frac{v}{\varepsilon} \rfloor_l)$ , and we remark that  $\frac{v}{\varepsilon} - \lfloor \frac{v}{\varepsilon} \rfloor_l \in [0, 1]^l$ , so

$$W\left(\frac{x}{\varepsilon}, \frac{u}{\varepsilon}, \xi, \frac{v}{\varepsilon}\right) = W\left(\frac{x}{\varepsilon}, \frac{u}{\varepsilon}, \xi, \frac{v}{\varepsilon} - \lfloor \frac{v}{\varepsilon} \rfloor_l\right) \leq C\left(1 + |\xi|^p + \left|\frac{v}{\varepsilon} - \lfloor \frac{v}{\varepsilon} \rfloor_l\right|\right) \leq 2C(1 + |\xi|^p)$$

for all  $(x, u, \xi) \in \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{M}$  and all  $\varepsilon > 0$ .

The  $\Gamma$ -convergence of  $\{I_\varepsilon\}_{\varepsilon > 0}$  (not depending on additional measure variable) was studied by [BDM78] and [AB84] in the scalar case, for problems related to the homogenization of Riemannian metrics. Later, a generalisation to the vectorial case was obtained by [E91] in the periodic setting. The case of almost periodic setting was studied by Braides [Bra92]. The following homogenization result is an extension, in the periodic setting, of Braides-E results for integrands depending on periodic vector measures.

**Theorem 3.4.** *Assume that  $(\mathcal{H}'_0)$  and  $(\mathcal{H}'_1)$  hold. For every  $O \in \mathcal{O}(\Omega)$  the family  $\{I_\varepsilon(\cdot, \cdot; O)\}_\varepsilon$   $\Gamma$ -converges to  $I_0(\cdot, \cdot; O)$  at each  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  given by*

$$I_0(u, \nu; O) = \int_O \mathcal{H}_0 W(\nabla u(x)) d\mathcal{L}_N(x)$$

where for every  $\xi \in \mathbb{M}$

$$\mathcal{H}_0 W(\xi) := \liminf_{k \rightarrow \infty} \left\{ \int_{k\mathbb{Y}} W(y, \xi y + \varphi(y), \xi + \nabla \varphi(y), \psi(y)) d\mathcal{L}_N(y) : \varphi \in W_0^{1,p}(k\mathbb{Y}; \mathbb{R}^m) \text{ and } \psi \in L^1(k\mathbb{Y}; \mathbb{R}^l), \int_{k\mathbb{Y}} \psi(y) d\mathcal{L}_N(y) = 0 \right\}.$$

#### 4. PROOF OF PROPOSITIONS 2.1 AND 2.2

**Proof of Proposition 2.1 (i).** Let  $O \in \mathcal{O}(\Omega)$ . Let  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  be such that  $I_-(u, \nu; O) < \infty$ . There exists a sequence  $\{(u_\varepsilon, \nu_\varepsilon)\}_{\varepsilon > 0} \subset W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  such that  $(u_\varepsilon, \nu_\varepsilon) \rightarrow (u, \nu)$  in  $L^p(\Omega; \mathbb{R}^m) \times \mathcal{M}(O; \mathbb{R}^l)$  as  $\varepsilon \rightarrow 0$ , and

$$I_-(u, \nu; O) = \lim_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon, \nu_\varepsilon; O) < \infty \quad \text{and} \quad \sup_{\varepsilon > 0} I_\varepsilon(u_\varepsilon, \nu_\varepsilon; O) < \infty. \quad (4.1)$$

Using the coercivity condition  $(H_1)$  and (4.1), we obtain the existence of a subsequence (not relabelled) such that  $|\nu_\varepsilon| \xrightarrow{*} \mu$  in  $\mathcal{M}_+(O)$ . For each  $\varepsilon > 0$  we set

$$\Theta_\varepsilon := f_\varepsilon\left(\cdot, u_\varepsilon(\cdot), \nabla u_\varepsilon(\cdot), \frac{d\nu_\varepsilon}{d\mathcal{L}_N}(\cdot)\right) \mathcal{L}_N \llcorner_O \in \mathcal{M}^+(O).$$

Using (4.1) there exists a subsequence (not relabelled)  $\{\Theta_\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}^+(O)$  and  $\Theta \in \mathcal{M}^+(O)$  such that  $\Theta_\varepsilon \xrightarrow{*} \Theta$  in  $\mathcal{M}^+(O)$ . The Lebesgue decomposition theorem gives that

$$\Theta = \frac{d\Theta}{d\mathcal{L}_N}(\cdot)\mathcal{L}_N + \frac{d\Theta}{d|\nu^s|}(\cdot)|\nu^s| + \Theta^s$$

The measure  $\Theta^s$  being the singular part of the Lebesgue decomposition of  $\Theta$  with respect to  $\mathcal{L}_N + |\nu^s|$ . Therefore we have

$$I_-(u, \nu; O) = \lim_{\varepsilon \rightarrow 0} I_\varepsilon(u_\varepsilon, \nu_\varepsilon; O) = \overline{\lim}_{\varepsilon \rightarrow 0} \Theta_\varepsilon(O) \geq \Theta(O) \geq \sum_{\sigma \in \{\mathcal{L}_N, |\nu^s|\}} \int_O \frac{d\Theta}{d\sigma}(x) d\sigma(x).$$

Moreover, we have for every  $t \in ]0, 1[$

$$\frac{d\Theta}{d\mathcal{L}_N}(x) = \lim_{\rho \rightarrow 0} \frac{\Theta(B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \geq \overline{\lim}_{\rho \rightarrow 0} \frac{\Theta(\overline{B_{t\rho}(x)})}{\mathcal{L}_N(B_\rho(x))} \geq \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{\Theta_\varepsilon(B_{t\rho}(x))}{\mathcal{L}_N(B_\rho(x))} \quad \mathcal{L}_N\text{-a.e. in } O,$$

hence for  $\mathcal{L}_N$ -a.e.  $x \in O$

$$\frac{d\Theta}{d\mathcal{L}_N}(x) \geq \overline{\lim}_{t \rightarrow 1^-} \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{\mathcal{L}_N(B_\rho(x))} \int_{B_{t\rho}(x)} f_\varepsilon\left(z, u_\varepsilon(z), \nabla u_\varepsilon(z), \frac{d\nu_\varepsilon}{d\mathcal{L}_N}(z)\right) d\mathcal{L}_N.$$

Similarly, we have

$$\frac{d\Theta}{d|\nu^s|}(x) = \lim_{\rho \rightarrow 0} \frac{\Theta(B_\rho(x))}{|\nu^s|(B_\rho(x))} \geq \overline{\lim}_{\rho \rightarrow 0} \frac{\Theta(\overline{B_{t\rho}(x)})}{|\nu^s|(B_\rho(x))} \geq \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{\Theta_\varepsilon(B_{t\rho}(x))}{|\nu^s|(B_\rho(x))} \quad |\nu^s|\text{-a.e. in } O,$$

and then

$$\frac{d\Theta}{d|\nu^s|}(x) \geq \overline{\lim}_{t \rightarrow 1^-} \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{|\nu^s|(B_\rho(x))} \int_{B_{t\rho}(x)} f_\varepsilon\left(z, u_\varepsilon, \nabla u_\varepsilon, \frac{d\nu_\varepsilon}{d\mathcal{L}_N}\right) d\mathcal{L}_N \quad |\nu^s|\text{-a.e. in } O. \blacksquare$$

**Proof of Proposition 2.1 (ii).** Taking Lemma 7.6 into account, which gives an integral representation of the lower Vitali envelope with density the derivative of the set function  $m_+(u, \nu; \cdot) = \overline{\lim}_{\varepsilon \rightarrow 0} m_\varepsilon(u, \nu; \cdot)$ , we see that it is sufficient to show that for every  $(u, \nu, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l) \times \mathcal{O}(\Omega)$

$$I_+(u, \nu; O) \leq V_{m_+(u, \nu; \cdot)}^-(O).$$

Let  $(u, \nu, O) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l) \times \mathcal{O}(\Omega)$  be such that  $V_{m_+(u, \nu; \cdot)}^-(O) < \infty$  where

$$V_{m_+(u, \nu; \cdot)}^-(O) = \liminf_{\delta \rightarrow 0} \left\{ \sum_{i \in I} m_+(u, \nu; B_i) : \{B_i\}_{i \in I} \in \mathcal{V}^\delta(O) \right\}.$$

where for every  $\delta > 0$

$$\mathcal{V}^\delta(O) := \left\{ \{B_i\}_{i \in I} : B_i \text{ is an open ball, } \mu(\partial B_i) = 0, \overline{B}_i \subset O, \text{diam}(B_i) \in ]0, \delta[ \right. \\ \left. I \text{ is countable, } \mu\left(O \setminus \bigcup_{i \in I} B_i\right) = 0, \text{ and } \overline{B}_i \cap \overline{B}_j = \emptyset \text{ for all } i \neq j \right\}.$$

Fix  $\delta \in ]0, 1[$ . There exists  $\{B_i^\delta\}_{i \in I_\delta} \in \mathcal{V}^\delta(O)$  such that

$$V_{m_+(u, \nu; \cdot)}^-(O) + \delta \geq \sum_{i \in I_\delta} m_+(u, \nu; B_i^\delta). \quad (4.2)$$



We claim that

$$\overline{\lim}_{\varepsilon \rightarrow 0} \sum_{i \in I_\delta} m_\varepsilon(u, \nu; B_i^\delta) \leq \delta + \sum_{i \in I_\delta} m_+(u, \nu; B_i^\delta). \quad (4.3)$$

Indeed, consider  $\alpha_{u, \nu}$  given by (H<sub>2</sub>') and set  $\mu := \mathcal{L}_N + |\nu^s|$ . Since  $\mu(O \setminus \cup_{i \in I_\delta} B_i^\delta) = 0$  and  $\alpha_{u, \nu} \ll \mu$  we have  $\alpha_{u, \nu}(O \setminus \cup_{i \in I_\delta} B_i^\delta) = 0$ . It follows that there exists a finite subset  $J_\delta \subset I_\delta$  satisfying  $\sum_{i \in I_\delta \setminus J_\delta} \alpha_{u, \nu}(B_i^\delta) < \delta$ . Now, for every  $\varepsilon > 0$  we have

$$\begin{aligned} \sum_{i \in I_\delta} m_\varepsilon(u, \nu; B_i^\delta) &= \sum_{i \in J_\delta} m_\varepsilon(u, \nu; B_i^\delta) + \sum_{i \in I_\delta \setminus J_\delta} m_\varepsilon(u, \nu; B_i^\delta) \\ &\leq \sum_{i \in J_\delta} m_\varepsilon(u, \nu; B_i^\delta) + \sum_{i \in I_\delta \setminus J_\delta} \alpha_{u, \nu}(B_i^\delta) \\ &\leq \sum_{i \in J_\delta} m_\varepsilon(u, \nu; B_i^\delta) + \delta. \end{aligned}$$

Since  $J_\delta$  is finite, we obtain, by passing to the limit  $\varepsilon \rightarrow 0$ , that

$$\begin{aligned} \overline{\lim}_{\varepsilon \rightarrow 0} \sum_{i \in I_\delta} m_\varepsilon(u, \nu; B_i^\delta) &\leq \overline{\lim}_{\varepsilon \rightarrow 0} \sum_{i \in J_\delta} m_\varepsilon(u, \nu; B_i^\delta) + \delta \\ &\leq \sum_{i \in J_\delta} \overline{\lim}_{\varepsilon \rightarrow 0} m_\varepsilon(u, \nu; B_i^\delta) + \delta \\ &\leq \sum_{i \in I_\delta} m_+(u, \nu; B_i^\delta) + \delta. \end{aligned}$$

which proves (4.3).

Next, fix  $\varepsilon \in ]0, 1[$ . For each  $i \in I_\delta$  there exists  $(u_{i, \varepsilon}^\delta, \nu_{i, \varepsilon}^\delta) \in u + W_0^{1,p}(B_i^\delta; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  such that  $\nu_{i, \varepsilon}^\delta(B_i^\delta) = \nu(B_i^\delta)$ ,  $\nu_{i, \varepsilon}^\delta \ll \mathcal{L}_N$  and

$$m_\varepsilon(u, \nu; B_i^\delta) + \varepsilon \frac{\mathcal{L}_N(B_i^\delta)}{\mathcal{L}_N(O)} \geq I_\varepsilon(u_{i, \varepsilon}^\delta, \nu_{i, \varepsilon}^\delta; B_i^\delta). \quad (4.4)$$

Set

$$\nu_\varepsilon^\delta := \sum_{i \in I_\delta} \nu_{i, \varepsilon}^\delta \mathbf{1}_{B_i^\delta} \quad \text{and} \quad u_\varepsilon^\delta := \sum_{i \in I_\delta} u_{i, \varepsilon}^\delta \mathbf{1}_{B_i^\delta} + u \mathbf{1}_{\Omega \setminus \cup_{i \in I_\delta} B_i^\delta}.$$

Using (H<sub>1</sub>') and (H<sub>2</sub>') we have  $\nu_\varepsilon^\delta \in \mathcal{M}(O; \mathbb{R}^l)$ , indeed

$$\begin{aligned} |\nu_\varepsilon^\delta|(O) &= \left| \sum_{i \in I_\delta} \nu_{i, \varepsilon}^\delta \mathbf{1}_{B_i^\delta} \right|(O) \leq \sum_{i \in I_\delta} \left( |\nu_{i, \varepsilon}^\delta \mathbf{1}_{B_i^\delta}|(O) \right) = \sum_{i \in I_\delta} \int_{B_i^\delta} \left| \frac{d\nu_{i, \varepsilon}^\delta}{d\mathcal{L}_N} \right| d\mathcal{L}_N \\ &\leq \frac{1}{c} \sum_{i \in I_\delta} m_\varepsilon(u, \nu; B_i^\delta) + \varepsilon \frac{\mathcal{L}_N(B_i^\delta)}{\mathcal{L}_N(O)} \\ &\leq \frac{1}{c} (\alpha_{u, \nu}(O) + 1). \end{aligned} \quad (4.5)$$

In the same way, using (H<sub>1</sub>'), (4.4) and (H<sub>2</sub>'), we have

$$\begin{aligned} \int_O |\nabla u_\varepsilon^\delta|^p d\mathcal{L}_N &= \sum_{i \in I_\delta} \int_{B_i^\delta} |\nabla u_i^\delta|^p d\mathcal{L}_N \leq \frac{1}{c} \sum_{i \in I_\delta} I_\varepsilon(u_{i, \varepsilon}^\delta, \nu_{i, \varepsilon}^\delta; B_i^\delta) \\ &\leq \frac{1}{c} \left( \sum_{i \in I_\delta} m_\varepsilon(u, \nu; B_i^\delta) + \varepsilon \right) \leq \frac{1}{c} (\alpha_{u, \nu}(O) + 1). \end{aligned}$$

Thus

$$\sup_{\varepsilon \in ]0,1[, \delta \in ]0,1[} \left( |\nu_\varepsilon^\delta| (O) + \int_O |\nabla u_\varepsilon^\delta|^p d\mathcal{L}_N \right) < \infty. \quad (4.6)$$

Let  $U \subset O$  be an open set. We consider the sets

$$U^\delta := \bigcup \{B_i^\delta : B_i^\delta \cap U \neq \emptyset\} \quad \text{and} \quad U_\delta := \bigcup \{B_i^\delta : B_i^\delta \subset U\}.$$

Setting  $K_\delta := \{i \in I_\delta : B_i^\delta \cap U \neq \emptyset \text{ and } B_i^\delta \cap O \setminus U \neq \emptyset\}$  and arguing as in (4.5), we have

$$\begin{aligned} |\nu_\varepsilon^\delta| (U^\delta \setminus U_\delta) &\leq \sum_{i \in K_\delta} |\nu_{i,\varepsilon}^\delta| (B_i^\delta) \leq \frac{1}{c} \sum_{i \in K_\delta} I_\varepsilon (u_{i,\varepsilon}^\delta, \nu_{i,\varepsilon}^\delta; B_i^\delta) \\ &\leq \frac{1}{c} \left( \sum_{i \in K_\delta} m_\varepsilon (u, \nu; B_i^\delta) + \varepsilon \frac{\mathcal{L}_N (B_i^\delta)}{\mathcal{L}_N (O)} \right) \\ &\leq \max \left\{ \frac{1}{c}, \frac{1}{c \mathcal{L}_N (O)} \right\} (\alpha_{u,\nu} + \mathcal{L}_N) (U^\delta \setminus U_\delta). \end{aligned} \quad (4.7)$$

For every  $\varepsilon \in ]0,1[$  we have  $\lim_{\delta \rightarrow 0} |u_\varepsilon^\delta - u|_{p,\Omega} = 0$ , indeed, there exists  $C > 0$  such that for every  $(\varepsilon, \delta) \in ]0,1[^2$

$$\begin{aligned} \int_\Omega |u_\varepsilon^\delta - u|^p d\mathcal{L}_N &= \sum_{i \in I_\delta} \int_{B_i^\delta} |u_{i,\varepsilon}^\delta - u|^p d\mathcal{L}_N \\ &\leq C \delta^p \sum_{i \in I_\delta} \int_{B_i^\delta} |\nabla u_{i,\varepsilon}^\delta - \nabla u|^p d\mathcal{L}_N \\ &\leq C 2^{p-1} \delta^p \left( \int_O |\nabla u_\varepsilon^\delta|^p d\mathcal{L}_N + \int_O |\nabla u|^p d\mathcal{L}_N \right) \\ &\leq C 2^{p-1} \delta^p \left( \sup_{\varepsilon, \delta} \int_O |\nabla u_\varepsilon^\delta|^p d\mathcal{L}_N + \int_O |\nabla u|^p d\mathcal{L}_N \right), \end{aligned}$$

which proves the claim by letting  $\delta \rightarrow 0$ .

Using a diagonalization argument, there exists  $\{\delta(\varepsilon)\}_{\varepsilon > 0}$  decreasing with  $\lim_{\varepsilon \rightarrow 0} \delta(\varepsilon) = 0$  such that if we set  $u_\varepsilon := u_\varepsilon^{\delta(\varepsilon)}$  and  $\nu_\varepsilon := \nu_\varepsilon^{\delta(\varepsilon)}$ , we have  $u_\varepsilon \rightarrow u$  in  $L^p(\Omega; \mathbb{R}^l)$  and by using (4.4), (4.3) and (4.2)

$$\begin{aligned} \overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon (u_\varepsilon, \nu_\varepsilon; O) &\leq \overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon (u_\varepsilon^\delta, \nu_\varepsilon^\delta; O) = \overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \sum_{i \in I_\delta} I_\varepsilon (u_{i,\varepsilon}^\delta, \nu_{i,\varepsilon}^\delta; B_i^\delta) \\ &\leq \overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \sum_{i \in I_\delta} m_\varepsilon (u, \nu; B_i^\delta) \\ &\leq \overline{\lim}_{\delta \rightarrow 0} \sum_{i \in I_\delta} m_+ (u, \nu; B_i^\delta) \\ &\leq V_{m_+(u,\nu;\cdot)}^- (O). \end{aligned}$$

Finally, taking (4.5) and (4.7) into account and applying Lemma 7.5 with  $\{\nu_\varepsilon\}_{\varepsilon \in ]0,1[}$  we have  $\nu_\varepsilon \xrightarrow{*} \nu$  in  $\mathcal{M}(O; \mathbb{R}^l)$ , and thus

$$I_+(u, \nu; O) \leq \overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon (u_\varepsilon, \nu_\varepsilon; O) \leq V_{m_+(u,\nu;\cdot)}^- (O). \quad \blacksquare$$

**Proof of Proposition 2.2 (i).** Let  $O \in \mathcal{O}(\Omega)$ . Let  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$ , and let  $\{(u_\varepsilon, \nu_\varepsilon)\}_{\varepsilon>0} \subset W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  be a sequence satisfying

$$(u_\varepsilon, \nu_\varepsilon) \rightarrow (u, \nu) \text{ in } L^p(\Omega; \mathbb{R}^m) \times \mathcal{M}(O; \mathbb{R}^l) \quad \text{and} \quad \sup_{\varepsilon>0} I_\varepsilon(u_\varepsilon, \nu_\varepsilon; O) < \infty.$$

Up to a subsequence, we can assume that

$$\Theta_\varepsilon := f_\varepsilon \left( \cdot, u_\varepsilon(\cdot), \nabla u_\varepsilon(\cdot), \frac{d\nu_\varepsilon}{d\mathcal{L}_N}(\cdot) \right) \mathcal{L}_N \xrightarrow{*} \Theta \text{ in } \mathcal{M}_+(O).$$

Fix  $\sigma \in \{\mathcal{L}_N, |\nu^s|\}$  and set  $\beta := (1 + |u|^p) \mathcal{L}_N$ . Fix  $x \in O$  satisfying

$$\frac{d\Theta}{d\sigma}(x) = \lim_{\rho \rightarrow 0} \frac{\Theta(B_\rho(x))}{\sigma(B_\rho(x))} < \infty; \quad (4.8)$$

$$\frac{d\beta}{d\sigma}(x) = \lim_{\rho \rightarrow 0} \frac{\beta(B_\rho(x))}{\sigma(B_\rho(x))} < \infty; \quad (4.9)$$

$$\frac{d|\nu|}{d\sigma}(x) = \lim_{\rho \rightarrow 0} \frac{|\nu|(B_\rho(x))}{\sigma(B_\rho(x))} < \infty; \quad (4.10)$$

$$|u(x)|^p + |\nabla u(x)|^p < \infty; \quad (4.11)$$

$$\lim_{\rho \rightarrow 0} \frac{1}{\rho^p \sigma(B_\rho(x))} \int_{B_\rho(x)} |u - u_{x,\rho}^\sigma|^p d\mathcal{L}_N = 0; \quad (4.12)$$

$$\lim_{\rho \rightarrow 0} \frac{\sigma(B_{t\rho}(x))}{\sigma(B_\rho(x))} \geq t^N \quad \text{for all } t \in ]0, 1[. \quad (4.13)$$

Note that (4.12) is satisfied  $\sigma$ -a.e. in  $O$  by the  $L^p$ -differentiation theorem when  $\sigma = \mathcal{L}_N$ , and by Lemma 7.4 when  $\sigma = |\nu^s|$ . Moreover, (4.13) is satisfied  $\sigma$ -a.e. in  $O$  since Lemma 7.3.

Set the limit operator

$$\lim_{\rho \rightarrow 0}^\sigma := \begin{cases} \overline{\lim}_{\rho \rightarrow 0} & \text{if } \sigma = \mathcal{L}_N \\ \underline{\lim}_{\rho \rightarrow 0} & \text{if } \sigma = |\nu^s|. \end{cases}$$

By using Corollary 7.1 we have

$$\lim_{\rho \rightarrow 0}^\sigma \lim_{\varepsilon \rightarrow 0} \frac{m_\varepsilon \left( u_{x,\rho}^\sigma, \frac{d\nu}{d\sigma}(x) \sigma; B_\rho(x) \right)}{\sigma(B_\rho(x))} = \lim_{\rho \rightarrow 0}^\sigma \lim_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}^\sigma, \nu; B_\rho(x))}{\sigma(B_\rho(x))} \quad \sigma\text{-a.e. in } O. \quad (4.14)$$

It follows that it is enough to show that

$$\lim_{\rho \rightarrow 0}^\sigma \lim_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}^\sigma, \nu; B_\rho(x))}{\sigma(B_\rho(x))} \leq \overline{\lim}_{t \rightarrow 1^-} \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{I_\varepsilon(u_\varepsilon, \nu_\varepsilon; B_{t\rho}(x))}{\sigma(B_\rho(x))} \quad \sigma\text{-a.e. in } O.$$

**Step 1: cut-off method.** Let  $t \in ]0, 1[$  and  $\rho \in ]0, 1[$  with  $B_\rho(x) \subset O$ . Consider a cut-off function  $\varphi \in W_0^{1,\infty}(B_\rho(x); [0, 1])$  between  $\overline{B_{t^2\rho}(x)}$  and  $O \setminus B_{t\rho}(x)$ , which means that  $\varphi \equiv 1$  on  $\overline{B_{t^2\rho}(x)}$  and  $\varphi \equiv 0$  on  $O \setminus B_{t\rho}(x)$  and such that there exists  $C_0 > 0$  verifying  $|\nabla \varphi|_\infty \leq \frac{C_0}{t(1-t)\rho}$ . Set  $\nu_\varepsilon^{t^2\rho} := \varphi \nu_\varepsilon = (\varphi \nu_{\varepsilon,1}, \dots, \varphi \nu_{\varepsilon,j}, \dots, \varphi \nu_{\varepsilon,l}) \in \mathcal{M}(\Omega; \mathbb{R}^l)$  defined by  $\nu_{\varepsilon,j}(B) := \int_B \varphi(z) d\nu_{\varepsilon,j}(z)$  for all Borel set  $B \subset O$ , and  $u^\sigma := \varphi u_\varepsilon + (1 - \varphi) u_{x,\rho}^\sigma \in$

$u_{x,\rho}^\sigma + W_0^{1,p}(B_\rho(x); \mathbb{R}^m)$ . We have

$$\begin{aligned} & \frac{m_\varepsilon \left( u_{x,\rho}^\sigma, \nu_\varepsilon^{t^2\rho}; B_\rho(x) \right)}{\sigma(B_\rho(x))} \\ & \leq \frac{1}{\sigma(B_\rho(x))} I_\varepsilon \left( u^\sigma, \nu_\varepsilon^{t^2\rho}; B_\rho(x) \right) \\ & \leq \frac{I_\varepsilon(u_\varepsilon, \nu_\varepsilon; B_{t\rho}(x))}{\sigma(B_\rho(x))} + \frac{I_\varepsilon \left( u^\sigma, \nu_\varepsilon^{t^2\rho}; B_{t\rho}(x) \setminus B_{t^2\rho}(x) \right)}{\sigma(B_\rho(x))} + \frac{I_\varepsilon(u_{x,\rho}^\sigma, 0; B_\rho(x) \setminus B_{t^2\rho}(x))}{\sigma(B_\rho(x))}. \end{aligned}$$

Using the growth condition (H<sub>1</sub>), we have for some  $C', C_1 > 0$  depending on  $C, C_0$  and  $p$  only, that

$$\begin{aligned} & I_\varepsilon \left( u^\sigma, \nu_\varepsilon^{t^2\rho}; B_{t\rho}(x) \setminus B_{t^2\rho}(x) \right) + I_\varepsilon \left( u_{x,\rho}^\sigma, 0; B_\rho(x) \setminus B_{t^2\rho}(x) \right) \\ & \leq C' \left\{ \int_{B_\rho(x) \setminus B_{t^2\rho}(x)} 1 + |u_{x,\rho}^\sigma|^p + |\nabla u_{x,\rho}^\sigma|^p d\mathcal{L}_N \right. \\ & \quad + \int_{B_{t\rho}(x) \setminus B_{t^2\rho}(x)} |u_\varepsilon|^p d\mathcal{L}_N + \int_{B_{t\rho}(x) \setminus B_{t^2\rho}(x)} |\nabla u_\varepsilon|^p + \left| \frac{dv_\varepsilon}{d\mathcal{L}_N} \right| d\mathcal{L}_N \\ & \quad \left. + \left( \frac{1}{t(1-t)\rho} \right)^p \int_{B_{t\rho}(x) \setminus B_{t^2\rho}(x)} |u_\varepsilon - u_{x,\rho}^\sigma|^p d\mathcal{L}_N \right\} \\ & \leq C_1 \left\{ \int_{B_\rho(x) \setminus B_{t^2\rho}(x)} 1 + |u_{x,\rho}^\sigma|^p + |\nabla u_{x,\rho}^\sigma|^p + |u|^p d\mathcal{L}_N + \frac{1}{(t(1-t)\rho)^p} \int_{B_\rho(x)} |u - u_{x,\rho}^\sigma|^p d\mathcal{L}_N \right. \\ & \quad \left. + \int_{B_{t\rho}(x) \setminus B_{t^2\rho}(x)} |\nabla u_\varepsilon|^p + \left| \frac{dv_\varepsilon}{d\mathcal{L}_N} \right| d\mathcal{L}_N + \left( 1 + \frac{1}{(t(1-t)\rho)^p} \right) \int_{B_\rho(x)} |u_\varepsilon - u|^p d\mathcal{L}_N \right\}. \end{aligned}$$

By setting

$$\begin{aligned} \Delta_{x,1}^\sigma(t, \rho) &:= \frac{C_1}{\sigma(B_\rho(x))} \int_{B_\rho(x) \setminus B_{t^2\rho}(x)} 1 + |u_{x,\rho}^\sigma|^p + |\nabla u_{x,\rho}^\sigma|^p + |u|^p d\mathcal{L}_N, \\ \Delta_{x,2}^\sigma(t, \rho) &:= \frac{C_1}{(t(1-t))^p} \frac{1}{\rho^p \sigma(B_\rho(x))} \int_{B_\rho(x)} |u - u_{x,\rho}^\sigma|^p d\mathcal{L}_N, \\ \Delta_{x,3}^\sigma(t, \rho, \varepsilon) &:= \frac{C_1}{\sigma(B_\rho(x))} \left( 1 + \frac{1}{(t(1-t)\rho)^p} \right) \int_{B_\rho(x)} |u_\varepsilon - u|^p d\mathcal{L}_N, \\ \text{and } \Delta_{x,4}^\sigma(t, \rho, \varepsilon) &:= \frac{C_1}{\sigma(B_\rho(x))} \int_{B_{t\rho}(x) \setminus B_{t^2\rho}(x)} |\nabla u_\varepsilon|^p + \left| \frac{dv_\varepsilon}{d\mathcal{L}_N} \right| d\mathcal{L}_N, \end{aligned}$$

we can write

$$\begin{aligned} \frac{m_\varepsilon \left( u_{x,\rho}^\sigma, \nu_\varepsilon^{t^2\rho}; B_\rho(x) \right)}{\sigma(B_\rho(x))} & \leq \frac{I_\varepsilon(u_\varepsilon, \nu_\varepsilon; B_{t\rho}(x))}{\sigma(B_\rho(x))} \\ & \quad + \Delta_{x,1}^\sigma(t, \rho) + \Delta_{x,2}^\sigma(t, \rho) + \Delta_{x,3}^\sigma(t, \rho, \varepsilon) + \Delta_{x,4}^\sigma(t, \rho, \varepsilon). \end{aligned}$$

Note that  $\overline{\lim}_{\varepsilon \rightarrow 0} \Delta_{x,3}^\sigma(t, \rho, \varepsilon) = 0$ , and  $\overline{\lim}_{\rho \rightarrow 0} \Delta_{x,2}^\sigma(t, \rho) = 0$  since (4.12). Next, using the coercivity condition  $(H_1)$ , we have

$$\begin{aligned} \overline{\lim}_{\varepsilon \rightarrow 0} \Delta_{x,4}^\sigma(t, \rho, \varepsilon) &\leq \overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{\sigma(B_\rho(x))} \int_{B_{t\rho}(x) \setminus B_{t^2\rho}(x)} |\nabla u_\varepsilon|^p + \left| \frac{d\nu_\varepsilon}{d\mathcal{L}_N} \right| d\mathcal{L}_N \\ &\leq \frac{C_1}{c} \left( \overline{\lim}_{\varepsilon \rightarrow 0} \frac{\Theta_\varepsilon(B_{t\rho}(x) \setminus B_{t^2\rho}(x))}{\sigma(B_\rho(x))} \right) \\ &\leq \frac{C_1}{c} \left( \overline{\lim}_{\varepsilon \rightarrow 0} \frac{\Theta_\varepsilon(\overline{B}_{t\rho}(x))}{\sigma(B_\rho(x))} - \overline{\lim}_{\varepsilon \rightarrow 0} \frac{\Theta_\varepsilon(B_{t^2\rho}(x))}{\sigma(B_\rho(x))} \right) \\ &\leq \frac{C_1}{c} \left( \frac{\Theta(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\Theta(B_{t^2\rho}(x)) \sigma(B_{t^2\rho}(x))}{\sigma(B_{t^2\rho}(x)) \sigma(B_\rho(x))} \right). \end{aligned}$$

It follows that

$$\begin{aligned} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}^\sigma, \nu_\varepsilon^{t^2\rho}; B_\rho(x))}{\sigma(B_\rho(x))} &\leq \overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{\sigma(B_\rho(x))} I_\varepsilon(u_\varepsilon, \nu_\varepsilon; B_{t\rho}(x)) + \Delta_{x,1}^\sigma(t, \rho) + \Delta_{x,2}^\sigma(t, \rho) \\ &\quad + \frac{C_1}{c} \left( \frac{\Theta(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\Theta(B_{t^2\rho}(x)) \sigma(B_{t^2\rho}(x))}{\sigma(B_{t^2\rho}(x)) \sigma(B_\rho(x))} \right). \end{aligned}$$

By using Lemma 7.2 with  $r = t^2\rho$ , we have

$$\begin{aligned} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}^\sigma, \nu; B_\rho(x))}{\sigma(B_\rho(x))} &\leq \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}^\sigma, \nu_\varepsilon^{t^2\rho}; B_\rho(x))}{\sigma(B_\rho(x))} \\ &\quad + 2C \left( \left( 1 - \frac{\sigma(B_{t^2\rho}(x))}{\sigma(B_\rho(x))} \right) \chi_{t^2}^\sigma(\rho) + \chi_1^\sigma(\rho) - \chi_{t^2}^\sigma(\rho) \right) \end{aligned}$$

where  $\chi_s^\sigma(\rho) := \frac{|\nu|(B_{s\rho}(x))}{\sigma(B_{s\rho}(x))}$  for all  $s \in ]0, 1]$ , and where  $C > 0$  is the constant growth appearing in  $(H_1)$ . Therefore we can write

$$\begin{aligned} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}^\sigma, \nu; B_\rho(x))}{\sigma(B_\rho(x))} &\leq \overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{\sigma(B_\rho(x))} I_\varepsilon(u_\varepsilon, \nu_\varepsilon; B_{t\rho}(x)) + \Delta_{x,1}^\sigma(t, \rho) + \Delta_{x,2}^\sigma(t, \rho) \\ &\quad + \frac{C_1}{c} \left( \frac{\Theta(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\Theta(B_{t^2\rho}(x)) \sigma(B_{t^2\rho}(x))}{\sigma(B_{t^2\rho}(x)) \sigma(B_\rho(x))} \right) \\ &\quad + 2C \left( \left( 1 - \frac{\sigma(B_{t^2\rho}(x))}{\sigma(B_\rho(x))} \right) \chi_{t^2}^\sigma(\rho) + \chi_1^\sigma(\rho) - \chi_{t^2}^\sigma(\rho) \right). \end{aligned} \tag{4.15}$$

**Step 2: upper bound estimate for  $\Delta_{x,1}^\sigma(t, \rho)$ .** We specify  $\sigma$  in order to obtain an upper bound for

$$\frac{1}{\sigma(B_\rho(x))} \int_{B_\rho(x) \setminus B_{t^2\rho}(x)} |u_{x,\rho}^\sigma|^p + |\nabla u_{x,\rho}^\sigma|^p d\mathcal{L}_N.$$

In case  $\sigma = \mathcal{L}_N$ , there exists  $C_2 > 1$ , depending only on  $p$ , such that

$$\begin{aligned} & \frac{1}{\mathcal{L}_N(B_\rho(x))} \int_{B_\rho(x) \setminus B_{t^2\rho}(x)} |u_x|^p + |\nabla u_x|^p d\mathcal{L}_N \\ & \leq C_2 \frac{1}{\mathcal{L}_N(B_\rho(x))} \int_{B_\rho(x) \setminus B_{t^2\rho}(x)} |u(x)|^p + |\nabla u(x)|^p d\mathcal{L}_N \\ & = C_2 (|u(x)|^p + |\nabla u(x)|^p) (1 - t^{2N}). \end{aligned}$$

In case  $\sigma = |\nu^s|$ , noticing that  $\nabla u_{x,\rho} \equiv 0$  and using Jensen inequality, we have

$$\begin{aligned} \frac{1}{\sigma(B_\rho(x))} \int_{B_\rho(x) \setminus B_{t^2\rho}(x)} |u_{x,\rho}|^p d\mathcal{L}_N & \leq \frac{\mathcal{L}_N(B_\rho(x) \setminus B_{t^2\rho}(x))}{\sigma(B_\rho(x))} \frac{1}{\mathcal{L}_N(B_\rho(x))} \int_{B_\rho(x)} |u|^p d\mathcal{L}_N \\ & = (1 - t^{2N}) \frac{1}{\sigma(B_\rho(x))} \int_{B_\rho(x)} |u|^p d\mathcal{L}_N. \end{aligned}$$

It follows that

$$\begin{aligned} & \frac{1}{\sigma(B_\rho(x))} \int_{B_\rho(x) \setminus B_{t^2\rho}(x)} |u_{x,\rho}^\sigma|^p + |\nabla u_{x,\rho}^\sigma|^p d\mathcal{L}_N \\ & \leq C_2 (1 - t^{2N}) \left( |u(x)|^p + |\nabla u(x)|^p + \frac{1}{\sigma(B_\rho(x))} \int_{B_\rho(x)} |u|^p d\mathcal{L}_N \right). \end{aligned}$$

Next, recalling that  $\beta = (1 + |u|^p)\mathcal{L}_N$  and setting  $\kappa_s^\sigma(\rho) := \frac{\beta(B_{s\rho}(x))}{\sigma(B_{s\rho}(x))}$  for all  $s \in ]0, 1]$ , we have

$$\frac{C_1}{\sigma(B_\rho(x))} \int_{B_\rho(x) \setminus B_{t^2\rho}(x)} 1 + |u|^p d\mathcal{L}_N = C_1 \left( \left( 1 - \frac{\sigma(B_{t^2\rho}(x))}{\sigma(B_\rho(x))} \right) \kappa_{t^2}^\sigma(\rho) + \kappa_1^\sigma(\rho) - \kappa_{t^2}^\sigma(\rho) \right).$$

Hence

$$\begin{aligned} \Delta_{x,1}^\sigma(t, \rho) & \leq C_2 C_1 (1 - t^{2N}) \left( |u(x)|^p + |\nabla u(x)|^p + \frac{\beta(B_\rho(x))}{\sigma(B_\rho(x))} \right) \\ & \quad + C_1 \left( \left( 1 - \frac{\sigma(B_{t^2\rho}(x))}{\sigma(B_\rho(x))} \right) \kappa_{t^2}^\sigma(\rho) + \kappa_1^\sigma(\rho) - \kappa_{t^2}^\sigma(\rho) \right). \quad (4.16) \end{aligned}$$

**Step 3: end of the proof of Proposition 2.2 (i).** Using (4.16) in (4.15), we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}^\sigma, \nu; B_\rho(x))}{\sigma(B_\rho(x))} & \leq \lim_{\varepsilon \rightarrow 0} \frac{1}{\sigma(B_\rho(x))} I_\varepsilon(u_\varepsilon, \nu_\varepsilon; B_{t\rho}(x)) + \Delta_{x,2}^\sigma(t, \rho) \\ & \quad + \frac{C_1}{c} \left( \frac{\Theta(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\Theta(B_{t^2\rho}(x)) \sigma(B_{t^2\rho}(x))}{\sigma(B_{t^2\rho}(x)) \sigma(B_\rho(x))} \right) \\ & \quad + 2C \left( \left( 1 - \frac{\sigma(B_{t^2\rho}(x))}{\sigma(B_\rho(x))} \right) \chi_{t^2}^\sigma(\rho) + \chi_1^\sigma(\rho) - \chi_{t^2}^\sigma(\rho) \right) \\ & \quad + C_2 C_1 (1 - t^{2N}) \left( |u(x)|^p + |\nabla u(x)|^p + \frac{\beta(B_\rho(x))}{\sigma(B_\rho(x))} \right) \\ & \quad + C_1 \left( \left( 1 - \frac{\sigma(B_{t^2\rho}(x))}{\sigma(B_\rho(x))} \right) \kappa_{t^2}^\sigma(\rho) + \kappa_1^\sigma(\rho) - \kappa_{t^2}^\sigma(\rho) \right). \end{aligned}$$

First, (4.13) gives that  $\lim_{\rho \rightarrow 0} \left(1 - \frac{\sigma(B_{t^2\rho}(x))}{\sigma(B_\rho(x))}\right) \leq 1 - t^{2N}$ , thus using (4.8) we obtain

$$\lim_{\rho \rightarrow 0} \frac{\Theta(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\Theta(B_{t^2\rho}(x))}{\sigma(B_{t^2\rho}(x))} \frac{\sigma(B_{t^2\rho}(x))}{\sigma(B_\rho(x))} \leq \frac{d\Theta}{d\sigma}(x) (1 - t^{2N}). \quad (4.17)$$

Second, using (4.9) and (4.10), we have

$$\lim_{\rho \rightarrow 0} \kappa_1^\sigma(\rho) = \lim_{\rho \rightarrow 0} \kappa_{t^2}^\sigma(\rho) = \frac{d\beta}{d\sigma}(x) < \infty, \text{ and } \lim_{\rho \rightarrow 0} \chi_1^\sigma(\rho) = \lim_{\rho \rightarrow 0} \chi_{t^2}^\sigma(\rho) = \frac{d|\nu|}{d\sigma}(x) < \infty.$$

Since (4.11), (4.12), and (4.17), we obtain for some  $C_3 > 0$  depending only on  $C, c, C_1$  and  $C_2$

$$\begin{aligned} \lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}^\sigma, \nu; B_\rho(x))}{\sigma(B_\rho(x))} &\leq \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{I_\varepsilon(u_\varepsilon, \nu_\varepsilon; B_{t\rho}(x))}{\sigma(B_\rho(x))} + C_3 (1 - t^{2N}) \left( |u(x)|^p \right. \\ &\quad \left. + |\nabla u(x)|^p + \frac{d\beta}{d\sigma}(x) + \frac{d\Theta}{d\sigma}(x) + \frac{d|\nu|}{d\sigma}(x) \right). \end{aligned}$$

Thus, letting  $t \rightarrow 1$  we conclude

$$\lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}^\sigma, \nu; B_\rho(x))}{\sigma(B_\rho(x))} \leq \overline{\lim}_{t \rightarrow 1} \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{\sigma(B_\rho(x))} I_\varepsilon(u_\varepsilon, \nu_\varepsilon; B_{t\rho}(x)). \blacksquare$$

**Proof of Proposition 2.2 (ii).** Fix  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  and  $O \in \mathcal{O}(\Omega)$ . Let  $\{\rho_k\}_{k \in \mathbb{N}} \subset ]0, 1[$  be a sequence such that  $\lim_{k \rightarrow \infty} \rho_k = 0$ . We define the following limit operator

$$\lim_{\sigma} := \begin{cases} \overline{\lim}_{k \rightarrow \infty} & \text{if } \sigma = \mathcal{L}_N \\ \overline{\lim}_{k \rightarrow \infty} & \text{if } \sigma = |\nu^s|. \end{cases}$$

We have to prove that for every  $\sigma \in \{\mathcal{L}_N, |\nu^s|\}$

$$\lim_{k \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon\left(u_{x,\rho_k}^\sigma, \frac{d\nu}{d\sigma}(x) \sigma; B_{\rho_k}(x)\right)}{\sigma(B_{\rho_k}(x))} \geq \lim_{R_\mu^x \ni \rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u, \nu; B_\rho(x))}{\sigma(B_\rho(x))} \quad \sigma\text{-a.e. in } O.$$

By Corollary 7.1 (i) and (ii), we have

$$\lim_{k \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon\left(u_{x,\rho_k}^\sigma, \frac{d\nu}{d\sigma}(x) \sigma; B_{\rho_k}(x)\right)}{\sigma(B_{\rho_k}(x))} = \lim_{k \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho_k}^\sigma, \nu; B_{\rho_k}(x))}{\sigma(B_{\rho_k}(x))} \quad \sigma\text{-a.e. in } O.$$

Set  $\mu := \mathcal{L}_N + |\nu^s|$ . For each  $k \in \mathbb{N}$  and  $x \in O$ , we set  $R_k^x := \{t \in ]1, 2[: \mu(\partial B_{t\rho_k}(x)) > 0\}$ . We see that  $R_k^x$  is a countable subset of  $]1, 2[$  since  $\partial B_{t\rho_k}(x) \cap \partial B_{\tau\rho_k}(x) = \emptyset$  for all  $t \neq \tau$  and  $\mu(O) < \infty$ . Setting  $R^x := \cup_{k \in \mathbb{N}} R_k^x$ , we then have  $\mu(\partial B_{t\rho_k}(x)) = 0$  for all  $t \in ]1, 2[ \setminus R^x$  and all  $k \in \mathbb{N}$ . Hence, for  $\sigma$ -a.e.  $x \in O$  and for every  $t \in ]1, 2[ \setminus R^x$

$$\overline{\lim}_{k \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u, \nu; B_{t\rho_k}(x))}{\sigma(B_{t\rho_k}(x))} = \lim_{R_\mu^x \ni \rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u, \nu; B_\rho(x))}{\sigma(B_\rho(x))}.$$

So, we are reduced to prove that

$$\lim_{k \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho_k}^\sigma, \nu; B_{\rho_k}(x))}{\sigma(B_{\rho_k}(x))} \geq \overline{\lim}_{]1, 2[ \setminus R^x \ni t \rightarrow 1} \overline{\lim}_{k \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u, \nu; B_{t\rho_k}(x))}{\sigma(B_{t\rho_k}(x))} \quad \sigma\text{-a.e. in } O. \quad (4.18)$$

We set  $\beta := (1 + |u|^p + |\nabla u|^p)\mathcal{L}_N + |\nu|$ . Fix  $x \in O$  satisfying

$$|u(x)|^p + |\nabla u(x)|^p < \infty; \quad (4.19)$$

$$\lim_{k \rightarrow \infty} \frac{1}{\sigma(B_{\rho_k}(x))} \int_{B_{\rho_k}(x)} |u|^p d\mathcal{L}_N < \infty; \quad (4.20)$$

$$\lim_{k \rightarrow \infty} \frac{\beta(B_{\rho_k}(x))}{\sigma(B_{\rho_k}(x))} = \frac{d\beta}{d\sigma}(x) < \infty; \quad (4.21)$$

$$\lim_{k \rightarrow \infty} \frac{1}{\rho_k^p \sigma(B_{\rho_k}(x))} \int_{B_{\rho_k}(x)} |u_{x, \rho_k}^\sigma - u|^p d\mathcal{L}_N = 0 \quad (4.22)$$

$$\overline{\lim}_{k \rightarrow \infty} \frac{\beta(B_{\rho_k}(x))}{\beta(B_{\tau \rho_k}(x))} \geq \frac{1}{\tau^N} \quad \text{for all } \tau > 0. \quad (4.23)$$

Fix  $\varepsilon > 0$ ,  $t \in ]1, 2[ \setminus \mathbb{R}^x$  and  $k \in \mathbb{N}$ . There exists  $(v, \lambda) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  with  $v \in u_{x, \rho_k}^\sigma + W_0^{1,p}(B_{\rho_k}(x); \mathbb{R}^m)$ ,  $\lambda \ll \mathcal{L}_N$ ,  $\lambda(B_{\rho_k}(x)) = \nu(B_{\rho_k}(x))$  satisfying

$$\rho_k \sigma(B_{\rho_k}(x)) + m_\varepsilon(u_{x, \rho_k}^\sigma, \nu; B_{\rho_k}(x)) \geq I_\varepsilon(v, \lambda; B_{\rho_k}(x)).$$

Consider a cut-off function  $\varphi_k \in W_0^{1,\infty}(B_{t\rho_k}(x); [0, 1])$  between  $\overline{B_{\rho_k}(x)}$  and  $\overline{O} \setminus B_{t\rho_k}(x)$ , i.e.  $\varphi_k \equiv 1$  on  $\overline{B_{\rho_k}(x)}$ ,  $\varphi_k \equiv 0$  on  $\overline{O} \setminus B_{t\rho_k}(x)$ , and for some  $C_0 > 0$  (independent of  $t$  and  $k$ )

$$|\nabla \varphi_k|_\infty \leq \frac{C_0}{\rho_k(t-1)}. \quad (4.24)$$

Set  $u_k^\sigma := \varphi_k v + (1 - \varphi_k) u \in u + W_0^{1,p}(B_{t\rho_k}(x); \mathbb{R}^m)$ , and  $\lambda_k := \frac{d\lambda_k}{d\mathcal{L}_N} \mathcal{L}_N$  where

$$\frac{d\lambda_k}{d\mathcal{L}_N} := \frac{d\lambda}{d\mathcal{L}_N} \mathbb{1}_{B_{\rho_k}(x)} + \frac{\nu(B_{t\rho_k}(x) \setminus B_{\rho_k}(x))}{\mathcal{L}_N(B_{t\rho_k}(x) \setminus B_{\rho_k}(x))} \mathbb{1}_{B_{t\rho_k}(x) \setminus B_{\rho_k}(x)}.$$

It is clear that  $\lambda_k(B_{t\rho_k}(x)) = \nu(B_{t\rho_k}(x))$ .

Now, using growth condition (H<sub>1</sub>) we have

$$\begin{aligned} & \frac{m_\varepsilon(u, \nu; B_{t\rho_k}(x))}{\sigma(B_{t\rho_k}(x))} \\ & \leq \frac{I_\varepsilon(u_k^\sigma, \lambda_k; B_{t\rho_k}(x))}{\sigma(B_{t\rho_k}(x))} \\ & = \frac{I_\varepsilon(v, \lambda; B_{\rho_k}(x))}{\sigma(B_{t\rho_k}(x))} + \frac{I_\varepsilon(u_k^\sigma, \lambda_k; B_{t\rho_k}(x) \setminus B_{\rho_k}(x))}{\sigma(B_{t\rho_k}(x))} \\ & \leq \frac{I_\varepsilon(v, \lambda; B_{\rho_k}(x))}{\sigma(B_{\rho_k}(x))} + \frac{I_\varepsilon(u_k^\sigma, \lambda_k; B_{t\rho_k}(x) \setminus B_{\rho_k}(x))}{\sigma(B_{t\rho_k}(x))} \\ & \leq \rho_k + \frac{m_\varepsilon(u_{x, \rho_k}^\sigma, \nu; B_{\rho_k}(x))}{\sigma(B_{\rho_k}(x))} + \frac{I_\varepsilon(u_k^\sigma, \lambda_k; B_{t\rho_k}(x) \setminus B_{\rho_k}(x))}{\sigma(B_{t\rho_k}(x))} \\ & \leq \rho_k + \frac{m_\varepsilon(u_{x, \rho_k}^\sigma, \nu; B_{\rho_k}(x))}{\sigma(B_{\rho_k}(x))} \\ & \quad + \frac{C}{\sigma(B_{t\rho_k}(x))} \int_{B_{t\rho_k}(x) \setminus B_{\rho_k}(x)} \left( 1 + |u_k^\sigma|^p + |\nabla u_k^\sigma|^p + \left| \frac{\nu(B_{t\rho_k}(x) \setminus B_{\rho_k}(x))}{\mathcal{L}_N(B_{t\rho_k}(x) \setminus B_{\rho_k}(x))} \right| \right) d\mathcal{L}_N \\ & \leq \rho_k + \frac{m_\varepsilon(u_{x, \rho_k}^\sigma, \nu; B_{\rho_k}(x))}{\sigma(B_{\rho_k}(x))} + 4^p C \Delta_x^\sigma(t, k), \end{aligned}$$



where

$$\begin{aligned} \Delta_x^\sigma(t, k) := & \frac{1}{\sigma(B_{t\rho_k}(x))} \int_{B_{t\rho_k}(x) \setminus B_{\rho_k}(x)} 1 + |u_{x, \rho_k}^\sigma|^p + |\nabla u_{x, \rho_k}^\sigma|^p + |u|^p + |\nabla u|^p \\ & + \frac{C_0^p}{\rho_k^p (t-1)^p} |u_{x, \rho_k}^\sigma - u|^p + \left| \frac{\nu(B_{t\rho_k}(x) \setminus B_{\rho_k}(x))}{\mathcal{L}_N(B_{t\rho_k}(x) \setminus B_{\rho_k}(x))} \right| d\mathcal{L}_N. \end{aligned}$$

To prove (4.18) we need to show that

$$\overline{\lim}_{t \rightarrow 1^+} \lim_{k \rightarrow \infty}^\sigma \Delta_x^\sigma(t, k) = 0 \quad \sigma\text{-a.e. in } O.$$

In the same way as for (4.16), we prove that there exists  $C_1 > 0$  depending only on  $p$  such that

$$\begin{aligned} & \frac{1}{\sigma(B_{t\rho_k}(x))} \int_{B_{t\rho_k}(x) \setminus B_{\rho_k}(x)} |u_{x, \rho_k}^\sigma|^p + |\nabla u_{x, \rho_k}^\sigma|^p d\mathcal{L}_N \\ & \leq C_1 \left( (t^N - 1) \frac{1}{\sigma(B_{\rho_k}(x))} \int_{B_{\rho_k}(x)} |u|^p d\mathcal{L}_N + \left(1 - \frac{1}{t^N}\right) (|u(x)|^p + |\nabla u(x)|^p) \right). \end{aligned} \quad (4.25)$$

Since  $\beta = (1 + |u|^p + |\nabla u|^p)\mathcal{L}_N + |\nu|$ , we have

$$\begin{aligned} & \frac{((1 + |u|^p + |\nabla u|^p)\mathcal{L}_N)(B_{t\rho_k}(x) \setminus B_{\rho_k}(x))}{\sigma(B_{t\rho_k}(x))} + \frac{|\nu(B_{t\rho_k}(x) \setminus B_{\rho_k}(x))|}{\sigma(B_{t\rho_k}(x))} \\ & \leq \frac{\beta(B_{t\rho_k}(x) \setminus B_{\rho_k}(x))}{\sigma(B_{t\rho_k}(x))}. \end{aligned} \quad (4.26)$$

We specify  $\sigma$  in order to pass to the limit  $k \rightarrow \infty$  for the last term of (4.26), which can be written as

$$\frac{\beta(B_{t\rho_k}(x) \setminus B_{\rho_k}(x))}{\sigma(B_{t\rho_k}(x))} = \frac{\beta(B_{t\rho_k}(x))}{\sigma(B_{t\rho_k}(x))} - \frac{\sigma(B_{\rho_k}(x))}{\sigma(B_{t\rho_k}(x))} \frac{\beta(B_{\rho_k}(x))}{\sigma(B_{\rho_k}(x))}.$$

In case  $\sigma = \mathcal{L}_N$ , we use (4.21) to obtain

$$\lim_{k \rightarrow \infty} \frac{\beta(B_{t\rho_k}(x) \setminus B_{\rho_k}(x))}{\mathcal{L}_N(B_{t\rho_k}(x))} = \frac{d\beta}{d\mathcal{L}_N}(x) \left(1 - \frac{1}{t^N}\right). \quad (4.27)$$

When  $\sigma = |\nu^s|$ , using (4.23) and (4.21) we obtain

$$\varliminf_{k \rightarrow \infty} \frac{\beta(B_{t\rho_k}(x) \setminus B_{\rho_k}(x))}{|\nu^s|(B_{t\rho_k}(x))} \leq \frac{d\beta}{d|\nu^s|}(x) \left(1 - \frac{1}{t^N}\right). \quad (4.28)$$

Thus (4.25) and (4.26) imply that

$$\begin{aligned} \Delta_x^\sigma(t, k) \leq & C_1 \left( (t^N - 1) \frac{1}{\sigma(B_{\rho_k}(x))} \int_{B_{\rho_k}(x)} |u|^p d\mathcal{L}_N + \left(1 - \frac{1}{t^N}\right) (|u(x)|^p + |\nabla u(x)|^p) \right) \\ & + \frac{\beta(B_{t\rho_k}(x) \setminus B_{\rho_k}(x))}{\sigma(B_{t\rho_k}(x))} + \frac{C_0^p}{\rho_k^p (t-1)^p \sigma(B_{\rho_k}(x))} \int_{B_{t\rho_k}(x)} |u_{x, \rho_k}^\sigma - u|^p d\mathcal{L}_N. \end{aligned}$$

Taking (4.19), (4.20), (4.22), (4.27) and (4.28) into account, by passing to the limit  $k \rightarrow \infty$  we have

$$\lim_{k \rightarrow \infty}^\sigma \Delta_x^\sigma(t, k) \leq \lim_{k \rightarrow \infty}^\sigma \frac{\beta(B_{t\rho_k}(x) \setminus B_{\rho_k}(x))}{\sigma(B_{t\rho_k}(x))} \leq \left(1 - \frac{1}{t^N}\right) \frac{d\beta}{d\sigma}(x).$$

Letting  $t \rightarrow 1$  we obtain  $\overline{\lim}_{]1,2[ \setminus \mathbb{R}^x \ni t \rightarrow 1} \lim_{k \rightarrow \infty} \sigma \Delta_x^\sigma(t, k) = 0$   $\sigma$ -a.e. in  $O$  and the proof is finished. ■

## 5. PROOF OF RELAXATION THEOREMS

**5.1. Proof of Theorem 3.1.** In order to apply Theorem 2.1, we have to show that the condition  $(H_3)$  is satisfied, then we establish the formula for  $\mathcal{Q}_0 f$ . We set

$$t_{x,\rho} := \frac{|\nu^s|(B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \text{ for all } \rho > 0 \text{ and all } x \in \Omega.$$

**Step1: Proof of  $(H_3)$  and formula for  $\mathcal{Q}_0^s f^\infty$ .** Let  $O \in \mathcal{O}(\Omega)$ ,  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  and  $v \in L^1_{|\nu^s|}(\Omega; \mathbb{S}^l)$ . Let  $x \in O \cap \mathbb{L}_u$  be such that

$$\lim_{\rho \rightarrow 0} \frac{((1 + |u|^p)\mathcal{L}_N)(B_\rho(x))}{|\nu^s|(B_\rho(x))} = 0 \quad \text{and} \quad |v(x)| = 1. \quad (5.1)$$

Fix  $\rho \in ]0, 1[$ . Using the growth condition  $(\mathcal{R}_1)$  and Jensen inequality, we have

$$\begin{aligned} \frac{m(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} &\leq \frac{C}{|\nu^s|(B_\rho(x))} \int_{B_\rho(x)} 1 + |u_{x,\rho}|^p + |v(x) t_{x,\rho}| d\mathcal{L}_N \\ &\leq C \left( \frac{\mathcal{L}_N(B_\rho(x))}{|\nu^s|(B_\rho(x))} + \frac{1}{|\nu^s|(B_\rho(x))} \int_{B_\rho(x)} |u|^p d\mathcal{L}_N + 1 \right) \\ &= \frac{\alpha_1(B_\rho(x))}{|\nu^s|(B_\rho(x))} + C, \end{aligned} \quad (5.2)$$

where  $\alpha_1 := C(1 + |u|^p)\mathcal{L}_N$ . We introduce the following notation:

$$\begin{aligned} m_x(\bar{u}(x), v(x) |\nu^s|; B_\rho(x)) \\ := \inf_{(\varphi, \psi) \in \mathcal{A}_0(B_\rho(x))} \int_{B_\rho(x)} f(x, \bar{u}(x) + \varphi, \nabla \varphi, t_{x,\rho}(\psi + v(x))) d\mathcal{L}_N. \end{aligned}$$

We have

$$\begin{aligned} \frac{m_x(\bar{u}(x), v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} &\leq \frac{C}{|\nu^s|(B_\rho(x))} \int_{B_\rho(x)} 1 + |\bar{u}(x)|^p + |v(x) t_{x,\rho}| d\mathcal{L}_N \\ &\leq C \left( \frac{\mathcal{L}_N(B_\rho(x))}{|\nu^s|(B_\rho(x))} (1 + |\bar{u}(x)|^p) + 1 \right) = \frac{\alpha_2(B_\rho(x))}{|\nu^s|(B_\rho(x))} + C, \end{aligned} \quad (5.3)$$

where  $\alpha_2 := C(1 + |\bar{u}(x)|^p)\mathcal{L}_N$ .

There exists  $(\varphi_1, \psi_1) \in \mathcal{A}_0(B_\rho(x))$  such that

$$\begin{aligned} &\frac{m(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} + \rho \\ &\geq \frac{1}{|\nu^s|(B_\rho(x))} \int_{B_\rho(x)} f(z, u_{x,\rho} + \varphi_1(z), \nabla \varphi_1(z), \psi_1(z) + v(x) t_{x,\rho}) d\mathcal{L}_N(z) \\ &\geq c \frac{1}{|\nu^s|(B_\rho(x))} \int_{B_\rho(x)} |\nabla \varphi_1(z)|^p + |\psi_1(z) + v(x) t_{x,\rho}| d\mathcal{L}_N(z), \end{aligned} \quad (5.4)$$

where we have used the coercivity condition in  $(\mathcal{R}_1)$ . Similarly, there exists  $(\varphi_2, \psi_2) \in \mathcal{A}_0(B_\rho(x))$  such that

$$\begin{aligned} & \frac{m_x(\bar{u}(x), v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} + \rho \\ & \geq \frac{1}{|\nu^s|(B_\rho(x))} \int_{B_\rho(x)} f(x, \bar{u}(x) + \varphi_2, \nabla \varphi_2, \psi_2 + v(x) t_{x,\rho}) d\mathcal{L}_N \\ & \geq c \frac{1}{|\nu^s|(B_\rho(x))} \int_{B_\rho(x)} |\nabla \varphi_2|^p + |\psi_2 + v(x) t_{x,\rho}| d\mathcal{L}_N. \end{aligned} \quad (5.5)$$

Using (5.2) we have

$$\sum_{i=1}^2 \frac{1}{|\nu^s|(B_\rho(x))} \int_{B_\rho(x)} |\nabla \varphi_i|^p + |\psi_i + v(x) t_{x,\rho}| d\mathcal{L}_N \leq \frac{1}{c} \frac{(\alpha_1 + \alpha_2)(B_\rho(x))}{|\nu^s|(B_\rho(x))} + \frac{2(\rho + C)}{c}. \quad (5.6)$$

Using  $(\mathcal{R}_2)$  and (5.6), we have

$$\begin{aligned} & \frac{m(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} - \frac{m_x(\bar{u}(x), v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} \\ & \geq \frac{1}{|\nu^s|(B_\rho(x))} \int_{B_\rho(x)} f(z, u_{x,\rho} + \varphi_1, \nabla \varphi_1, \psi_1 + v(x) t_{x,\rho}) \\ & \quad - f(x, \bar{u}(x) + \varphi_1, \nabla \varphi_1, \psi_1 + v(x) t_{x,\rho}) d\mathcal{L}_N - \rho \\ & \geq -C_1 \frac{|u_{x,\rho} - \bar{u}(x)|^q + \rho^{q'}}{|\nu^s|(B_\rho(x))} \int_{B_\rho(x)} 1 + |\nabla \varphi_1|^p + |\psi_1 + t_{x,\rho} v(x)| d\mathcal{L}_N - \rho \\ & \geq -C_1 \left( |u_{x,\rho} - \bar{u}(x)|^q + \rho^{q'} \right) \left( \frac{\mathcal{L}_N(B_\rho(x))}{|\nu^s|(B_\rho(x))} + \frac{1}{c} \frac{(\alpha_1 + \alpha_2)(B_\rho(x))}{|\nu^s|(B_\rho(x))} + \frac{2(\rho + C)}{c} \right) - \rho. \end{aligned}$$

In the same way, we have

$$\begin{aligned} & \frac{m(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} - \frac{m_x(\bar{u}(x), v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} \\ & \leq C_1 \frac{|u_{x,\rho} - \bar{u}(x)|^q + \rho^{q'}}{|\nu^s|(B_\rho(x))} \int_{B_\rho(x)} 1 + |\nabla \varphi_2|^p + |\psi_2 + t_{x,\rho} v(x)| d\mathcal{L}_N + \rho \\ & \leq C_1 \left( |u_{x,\rho} - \bar{u}(x)|^q + \rho^{q'} \right) \left( \frac{\mathcal{L}_N(B_\rho(x))}{|\nu^s|(B_\rho(x))} + \frac{1}{c} \frac{(\alpha_1 + \alpha_2)(B_\rho(x))}{|\nu^s|(B_\rho(x))} + \frac{2(\rho + C)}{c} \right) + \rho. \end{aligned}$$

Therefore, setting  $\alpha_0 := \mathcal{L}_N + \frac{1}{c}(\alpha_1 + \alpha_2)$  we can write

$$\begin{aligned} & \left| \frac{m(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} - \frac{m_x(\bar{u}(x), v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} \right| \\ & \leq C_1 \left( |u_{x,\rho} - \bar{u}(x)|^q + \rho^{q'} \right) \left( \frac{\alpha_0(B_\rho(x))}{|\nu^s|(B_\rho(x))} + \frac{2(\rho + C)}{c} \right) + \rho. \end{aligned} \quad (5.7)$$

Considering balls of the form  $B_\rho(x) = x + \rho Y$  with  $Y = ]-1, 1[^N$ , and changing variables, we have

$$\frac{m_x(\bar{u}(x), v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} = \inf_{(\varphi, \psi) \in \mathcal{A}_0(Y)} \int_Y \frac{1}{t_{x,\rho}} f(x, \bar{u}(x) + \rho \varphi, \nabla \varphi, t_{x,\rho}(\psi + v(x))) d\mathcal{L}_N.$$

Set

$$\frac{m_x^0(\bar{u}(x), v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} := \inf_{(\varphi, \psi) \in \mathcal{A}_0(Y)} \int_Y \frac{1}{t_{x,\rho}} f(x, \bar{u}(x), \nabla \varphi, t_{x,\rho}(\psi + v(x))) d\mathcal{L}_N.$$

We need the following lemma to obtain an estimate of the difference between  $m_x$  and  $m_x^0$ .

**Lemma 5.1.** *Let  $x \in \Omega$  and  $\rho \in ]0, 1[$  be such that  $t_{x,\rho} \geq 1$ . For every  $\delta > 0$  there exists  $M(x, \delta) \geq 3$  such that for every  $M \geq M(x, \delta)$*

$$\begin{aligned} \inf_{\substack{(\varphi, \psi) \in \mathcal{A}_0(Y) \\ \|\varphi\|_{L^\infty(Y; \mathbb{R}^m)} \leq M}} \int_Y \frac{1}{t_{x,\rho}} f(x, \bar{u}(x) + \rho\varphi, \nabla \varphi, t_{x,\rho}(\psi + v(x))) d\mathcal{L}_N \\ \leq \delta + \frac{m_x(\bar{u}(x), v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))}. \end{aligned} \quad (5.8)$$

and

$$\begin{aligned} \inf_{\substack{(\varphi, \psi) \in \mathcal{A}_0(Y) \\ \|\varphi\|_{L^\infty(Y; \mathbb{R}^m)} \leq M}} \int_Y \frac{1}{t_{x,\rho}} f(x, \bar{u}(x), \nabla \varphi, t_{x,\rho}(\psi + v(x))) d\mathcal{L}_N \\ \leq \delta + \frac{m_x^0(\bar{u}(x), v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))}. \end{aligned} \quad (5.9)$$

*Proof.* We show (5.8) only, the proof of (5.9) being similar. Let  $M \geq 3$  and  $\delta \in ]0, 1[$ . Let  $h_M : \mathbb{R}^m \rightarrow \mathbb{R}^m$  be a Lipschitz function such that

$$h_M(a) = \begin{cases} a & \text{if } |a| \leq M \\ 0 & \text{if } |a| > M + \frac{1}{\delta} \end{cases} \quad \text{and } \|\nabla h_M\|_{L^\infty(\mathbb{R}^m; \mathbb{R}^m)} \leq 3\delta.$$

For instance, we can take  $h_M(a) := \frac{\text{dist}(a; \mathbb{R}^m \setminus B_{M+1/\delta})}{\text{dist}(a; \mathbb{R}^m \setminus B_{M+1/\delta}) + \text{dist}(a; B_M)} a$  for all  $a \in \mathbb{R}^m$ , where  $B_{M'} := \{a \in \mathbb{R}^m : |a| < M'\}$  for all  $M' > 0$ . There exists  $(\varphi_\delta, \psi_\delta) \in \mathcal{A}_0(Y)$  such that

$$\begin{aligned} \frac{\delta}{2} + \inf_{(\varphi, \psi) \in \mathcal{A}_0(Y)} \int_Y \frac{1}{t_{x,\rho}} f(x, \bar{u}(x) + \rho\varphi, \nabla \varphi, t_{x,\rho}(\psi + v(x))) d\mathcal{L}_N \\ \geq \int_Y \frac{1}{t_{x,\rho}} f(x, \bar{u}(x) + \rho\varphi_\delta, \nabla \varphi_\delta, t_{x,\rho}(\psi_\delta + v(x))) d\mathcal{L}_N. \end{aligned}$$

Set  $\tilde{\varphi}_{\delta, M} := h_M \circ \varphi_\delta$ . By [ADM90, Corollary 3.2, pp. 701], and since  $h_M(0) = 0$  we have  $\tilde{\varphi}_{\delta, M} \in W_0^{1,p}(Y; \mathbb{R}^m)$ . Moreover, for  $\mathcal{L}_N$ -a.e.  $y \in Y$

$$|\tilde{\varphi}_{\delta, M}(y)| \leq \begin{cases} |\varphi_\delta(y)| & \text{if } |\varphi_\delta(y)| \leq M \\ 3 & \text{if } M < |\varphi_\delta(y)| \leq M + \frac{1}{\delta}, \\ 0 & \text{if } |\varphi_\delta(y)| > M + \frac{1}{\delta} \end{cases}$$

$$|\tilde{\varphi}_{\delta, M}(y)| \leq M \text{ and } |\nabla \tilde{\varphi}_{\delta, M}(y)| \leq 3\delta |\nabla \varphi_\delta(y)|,$$

and we can write

$$\begin{aligned}
& \inf_{\substack{(\varphi, \psi) \in \mathcal{A}_0(Y) \\ \|\varphi\|_{L^\infty(Y; \mathbb{R}^m)} \leq M}} \int_Y \frac{1}{t_{x, \rho}} f(x, \bar{u}(x) + \rho\varphi, \nabla\varphi, t_{x, \rho}(\psi + v(x))) d\mathcal{L}_N \\
& \leq \int_Y \frac{1}{t_{x, \rho}} f(x, \bar{u}(x) + \rho\tilde{\varphi}_{\delta, M}, \nabla\tilde{\varphi}_{\delta, M}, t_{x, \rho}(\psi_\delta + v(x))) d\mathcal{L}_N \\
& \leq \int_{\{|\varphi_\delta| \leq M\}} \frac{1}{t_{x, \rho}} f(x, \bar{u}(x) + \rho\varphi_\delta, \nabla\varphi_\delta, t_{x, \rho}(\psi_\delta + v(x))) d\mathcal{L}_N \\
& \quad + \int_{\{|\varphi_\delta| > M + \frac{1}{\delta}\}} \frac{1}{t_{x, \rho}} f(x, \bar{u}(x), 0, t_{x, \rho}(\psi_\delta + v(x))) d\mathcal{L}_N \\
& \quad + \int_{\{M < |\varphi_\delta| \leq M + \frac{1}{\delta}\}} \frac{1}{t_{x, \rho}} f(x, \bar{u}(x) + \rho\tilde{\varphi}_{\delta, M}, \nabla\tilde{\varphi}_{\delta, M}, t_{x, \rho}(\psi_\delta + v(x))) d\mathcal{L}_N \\
& \leq \frac{\delta}{2} + \frac{m_x(\bar{u}(x), v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} + \Delta(x, \rho, M, \delta),
\end{aligned}$$

where

$$\begin{aligned}
\Delta(x, \rho, M, \delta) & := C \frac{1}{t_{x, \rho}} \int_{\{|\varphi_\delta| > M + \frac{1}{\delta}\}} 1 + |\bar{u}(x)|^p + |t_{x, \rho}(\psi_\delta + v(x))| d\mathcal{L}_N \\
& \quad + C2^p \frac{1}{t_{x, \rho}} \int_{\{M < |\varphi_\delta| \leq M + \frac{1}{\delta}\}} 1 + |\bar{u}(x)|^p + \rho^p |\tilde{\varphi}_{\delta, M}|^p + |\nabla\tilde{\varphi}_{\delta, M}|^p + |t_{x, \rho}(\psi_\delta + v(x))| d\mathcal{L}_N.
\end{aligned}$$

Using the estimates on  $|\tilde{\varphi}_{\delta, M}(\cdot)|$  and  $|\nabla\tilde{\varphi}_{\delta, M}(\cdot)|$ , we obtain

$$\Delta(x, \rho, M, \delta) \leq C2^p \int_{\{|\varphi_\delta| > M\}} 1 + |\bar{u}(x)|^p + (3 + |\varphi_\delta|)^p + 3^p |\nabla\varphi_\delta|^p + |\psi_\delta + v(x)| d\mathcal{L}_N.$$

It follows that  $\lim_{M \rightarrow \infty} \sup_{\rho \in ]0, 1[} \Delta(x, \rho, M, \delta) = 0$ , thus there exists  $M(x, \delta) \geq 3$  such that  $\sup_{\rho \in ]0, 1[} \Delta(x, \rho, M, \delta) \leq \frac{\delta}{2}$  for all  $M \geq M(x, \delta)$ , which gives (5.8).  $\blacksquare$

Let  $\rho \in ]0, 1[$  be such that  $t_{x, \rho} \geq 1$ . Fix  $\delta > 0$  and  $M \geq M(x, \delta)$  where  $M(x, \delta)$  is given by Lemma 5.1. Using estimates (5.8), (5.9) and proceeding as for the estimate (5.7), we have

$$\begin{aligned}
& \left| \frac{m_x(\bar{u}(x), v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} - \frac{m_x^0(\bar{u}(x), v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} \right| \\
& \leq 2\delta + 2C_1 M^q \rho^q \left( \frac{\alpha'_0(B_\rho(x))}{|\nu^s|(B_\rho(x))} + \frac{C + \rho}{c} \right) + \rho, \quad (5.10)
\end{aligned}$$

where  $\alpha'_0 = \mathcal{L}_N + \frac{1}{c}\alpha_2$ .

Since  $\lim_{\rho \rightarrow 0} t_{x, \rho} = \infty$ , there exists  $\rho_x \in ]0, 1[$  such that for every  $\rho \in ]0, \rho_x[$  we have  $t_{x, \rho} > \max(1, T)$ , where  $T$  is given by  $(\mathcal{R}_3)$ . Then, for every  $(\varphi, \psi) \in \mathcal{A}_0(Y)$  and every  $\rho \in ]0, \rho_x[$

$$\left| \int_Y \frac{1}{t_{x, \rho}} f(x, \bar{u}(x), \nabla\varphi, t_{x, \rho}(\psi + v(x))) d\mathcal{L}_N - \int_Y f^\infty(x, \bar{u}(x), \nabla\varphi, \psi + v(x)) d\mathcal{L}_N \right| \leq \frac{\beta}{t_{x, \rho}^r}.$$

It follows that for every  $\rho \in ]0, \rho_x[$

$$\left| \frac{m_x^0(\bar{u}(x), v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} - m_x^\infty(\bar{u}(x), v(x) |\nu^s|; Y) \right| \leq \frac{\beta}{t_{x, \rho}^r}, \quad (5.11)$$

where

$$m_x^\infty(\bar{u}(x), v(x) |\nu^s|; Y) := \inf_{(\varphi, \psi) \in \mathcal{A}_0(Y)} \int_Y f^\infty(x, \bar{u}(x), \nabla\varphi, \psi + v(x)) d\mathcal{L}_N. \quad (5.12)$$

Combining (5.7), (5.10) and (5.11) we obtain for every  $\rho \in ]0, \rho_x[$

$$\begin{aligned} & \left| \frac{m(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} - m_x^\infty(\bar{u}(x), v(x) |\nu^s|; Y) \right| \\ & \leq 2\delta + 2C_1 \left( |u_{x,\rho} - \bar{u}(x)|^q + \rho^{q'} + M^q \rho^q \right) \left( \frac{\alpha_0(B_\rho(x))}{|\nu^s|(B_\rho(x))} + \frac{\rho + C}{c} \right) + 2\rho + \frac{\beta}{t_{x,\rho}^r}. \end{aligned} \quad (5.13)$$

Passing to the limits  $\rho \rightarrow 0$  and then  $\delta \rightarrow 0$  in (5.13) by taking (5.1) into account, the condition (H<sub>3</sub>) is satisfied, and we obtain the formula for  $\mathcal{Q}_0^s f^\infty$

$$\lim_{\rho \rightarrow 0} \frac{m(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} = m_x^\infty(\bar{u}(x), v(x) |\nu^s|; Y) = \mathcal{Q}_0^s f^\infty(x, \bar{u}(x), v(x)).$$

**Step 2: formula for  $\mathcal{Q}_0 f$ .** Let  $O \in \mathcal{O}(\Omega)$ ,  $u \in \mathbb{R}^m$ ,  $\xi \in \mathbb{M}$  and  $v \in \mathbb{R}^l$ . We have to show that

$$\mathcal{Q}_0 f(x, u, \xi, v) = \overline{\lim}_{\rho \rightarrow 0} \frac{m(u + \xi(\cdot - x), v \mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \quad \mathcal{L}_N\text{-a.e. in } O.$$

Fix  $\rho \in ]0, 1[$  and  $x \in O$ . We see, by taking the open balls  $B_\rho(x)$  of the form  $x + \rho Y$ , that

$$\begin{aligned} & \frac{m(u + \xi(\cdot - x), v \mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \\ & = \inf_{(\varphi, \psi) \in \mathcal{A}_0(Y)} \int_Y f(x + \rho y, u + \rho \xi y + \rho \varphi, \xi + \nabla\varphi, v + \psi) d\mathcal{L}_N. \end{aligned}$$

We set

$$\mathcal{Q}_1 f(x, u, \xi, v) := \inf_{(\varphi, \psi) \in \mathcal{A}_0(Y)} \int_Y f(x, u + \rho \varphi, \xi + \nabla\varphi, v + \psi) d\mathcal{L}_N.$$

Next, we estimate the difference between  $m$  and  $\mathcal{Q}_1 f$ . Using the growth condition ( $\mathcal{R}_1$ ), we have

$$\begin{aligned} \frac{m(u + \xi(\cdot - x), v \mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} & \leq C \int_Y 1 + |u + \rho \xi y|^p + |\xi|^p + |v| d\mathcal{L}_N \\ & \leq 2^p C (1 + |u|^p + |\xi|^p + |v|). \end{aligned} \quad (5.14)$$

In the same way, we have

$$\mathcal{Q}_1 f(x, u, \xi, v) \leq 2C (1 + |u|^p + |\xi|^p + |v|). \quad (5.15)$$

Furthermore, there exists  $(\varphi_1, \psi_1) \in \mathcal{A}_0(Y)$  such that

$$\begin{aligned} \frac{m(u + \xi(\cdot - x), v \mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} + \rho & \geq \int_Y f(x + \rho y, u + \rho \xi y + \rho \varphi_1, \xi + \nabla\varphi_1, v + \psi_1) d\mathcal{L}_N \\ & \geq c \int_Y |\xi + \nabla\varphi_1|^p + |v + \psi_1| d\mathcal{L}_N. \end{aligned} \quad (5.16)$$

Similarly, there exists  $(\varphi_2, \psi_2) \in \mathcal{A}_0(Y)$  such that

$$\begin{aligned} \mathcal{Q}_1 f(x, u, \xi, v) + \rho & \geq \int_Y f(x, u + \rho \varphi_2, \xi + \nabla\varphi_2, v + \psi_2) d\mathcal{L}_N \\ & \geq c \int_Y |\xi + \nabla\varphi_2|^p + |v + \psi_2| d\mathcal{L}_N. \end{aligned} \quad (5.17)$$

Combining (5.14), (5.15), (5.16) and (5.17), we deduce that for every  $i \in \{1, 2\}$

$$\int_Y |\xi + \nabla \varphi_i|^p + |v + \psi_i| d\mathcal{L}_N \leq \frac{2^{p+1}C}{c} (1 + |u|^p + |\xi|^p + |v|) + \frac{\rho}{c}. \quad (5.18)$$

Using  $(\mathcal{R}_2)$  and (5.18), we have

$$\begin{aligned} & \frac{m(u + \xi(\cdot - x), v\mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} - \mathcal{Q}_1 f(x, u, \xi, v) \\ & \geq \int_Y f(x + \rho y, u + \rho \xi y + \rho \varphi_1, \xi + \nabla \varphi_1, v + \psi_1) d\mathcal{L}_N \\ & \quad - \int_Y f(x, u + \rho \varphi_1, \xi + \nabla \varphi_1, v + \psi_1) d\mathcal{L}_N - \rho \\ & \geq -C_1 \int_Y (|\rho \xi y|^q + |\rho y|^{q'}) (1 + |\nabla \varphi_1 + \xi|^p + |v + \psi_1|) d\mathcal{L}_N - \rho \\ & \geq -2^{p+1}C_1 (\rho^q + \rho^{q'}) (1 + |\xi|^q) \left(1 + \frac{C}{c} (1 + |u|^p + |\xi|^p + |v|) + \frac{\rho}{c}\right) - \rho, \end{aligned}$$

and

$$\begin{aligned} & \frac{m(u + \xi(\cdot - x), v\mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} - \mathcal{Q}_1 f(x, u, \xi, v) \\ & \leq \int_Y f(x + \rho y, u + \rho \xi y + \varphi_2, \xi + \nabla \varphi_2, v + \psi_1) \\ & \quad - \int_Y f(x, u + \varphi_1(y), \xi + \nabla \varphi_2, v + \psi_2) d\mathcal{L}_N + \rho \\ & \leq 2^{p+1}C_1 (\rho^q + \rho^{q'}) (1 + |\xi|^q) \left(1 + \frac{C}{c} (1 + |u|^p + |\xi|^p + |v|) + \frac{\rho}{c}\right) + \rho, \end{aligned}$$

which gives

$$\begin{aligned} & \left| \frac{m(u + \xi(\cdot - x), v\mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} - \mathcal{Q}_1 f(x, u, \xi, v) \right| \\ & \leq 2^{p+1}C_1 (\rho^q + \rho^{q'}) (1 + |\xi|^q) \left(1 + \frac{C}{c} (1 + |u|^p + |\xi|^p + |v|) + \frac{\rho}{c}\right) + \rho. \quad (5.19) \end{aligned}$$

Now, we proceed as in **Step 1** by setting for every  $M \geq 1$

$$\begin{aligned} \mathcal{Q}_0^M f(x, u, \xi, v) & := \inf_{\substack{(\varphi, \psi) \in \mathcal{A}_0(Y) \\ \|\varphi\|_{L^\infty(Y; \mathbb{R}^m)} \leq M}} \int_Y f(x, u, \xi + \nabla \varphi, v + \psi) d\mathcal{L}_N \\ \mathcal{Q}_1^M f(x, u, \xi, v) & := \inf_{\substack{(\varphi, \psi) \in \mathcal{A}_0(Y) \\ \|\varphi\|_{L^\infty(Y; \mathbb{R}^m)} \leq M}} \int_Y f(x, u + \varphi, \xi + \nabla \varphi, v + \psi) d\mathcal{L}_N. \end{aligned}$$

Similarly to the Lemma 5.1 we can show that for every  $\delta > 0$  there exists  $M(x, u, v, \delta) \geq 3$  such that for every  $M \geq M(x, u, v, \delta)$

$$\mathcal{Q}_0^M f(x, u, \xi, v) \leq \delta + \mathcal{Q}_0 f(x, u, \xi, v) \quad \text{and} \quad \mathcal{Q}_1^M f(x, u, \xi, v) \leq \delta + \mathcal{Q}_1 f(x, u, \xi, v).$$

We also have for every  $i \in \{0, 1\}$

$$\mathcal{Q}_i^M f(x, u, \xi, v) \leq 2C (1 + |u|^p + |\xi|^p + |v|)$$

and there exists  $(\varphi'_i, \psi'_i) \in \mathcal{A}_0(Y)$  with  $\|\varphi'_i\|_{L^\infty(Y; \mathbb{R}^m)} \leq M$  such that

$$\mathcal{Q}_i^M f(x, u, \xi, v) \geq c \int_Y |\xi + \nabla \varphi'_i|^p + |v + \psi'_i| d\mathcal{L}_N.$$

We deduce, using  $(\mathcal{R}_2)$ , that

$$|\mathcal{Q}_0^M f(x, u, \xi, v) - \mathcal{Q}_1^M f(x, u, \xi, v)| \leq 2C_1 \rho^q M^q \left( 1 + \frac{1}{c} (\rho + C(1 + |u|^p + |\xi|^p + |v|)) \right) + \rho.$$

It follows that

$$\begin{aligned} & |\mathcal{Q}_0 f(x, u, \xi, v) - \mathcal{Q}_1 f(x, u, \xi, v)| \\ & \leq |\mathcal{Q}_0 f(x, u, \xi, v) - \mathcal{Q}_0^M f(x, u, \xi, v)| + |\mathcal{Q}_0^M f(x, u, \xi, v) - \mathcal{Q}_1^M f(x, u, \xi, v)| \\ & \quad + |\mathcal{Q}_1^M f(x, u, \xi, v) - \mathcal{Q}_1 f(x, u, \xi, v)| \\ & \leq 2C_1 \rho^q M^q \left( 1 + \frac{1}{c} (\rho + C(1 + |u|^p + |\xi|^p + |v|)) \right) + 2\delta + \rho. \end{aligned} \quad (5.20)$$

Using the estimates (5.19) and (5.20), we obtain

$$\begin{aligned} & \left| \frac{m(u + \xi(\cdot - x), v \mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} - \mathcal{Q}_0 f(x, u, \xi, v) \right| \\ & \leq \left| \frac{m(u + \xi(\cdot - x), v \mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} - \mathcal{Q}_1 f(x, u, \xi, v) \right| + |\mathcal{Q}_1 f(x, u, \xi, v) - \mathcal{Q}_0 f(x, u, \xi, v)| \\ & \leq 2\delta + \rho + 2^{p+1} C_1 \left( \rho^q (1 + M^q + |\xi|^q) + \rho^{q'} \right) \left( 1 + \frac{C}{c} (1 + |u|^p + |\xi|^p + |v|) + \frac{\rho}{c} \right). \end{aligned}$$

Passing to the limits  $\rho \rightarrow 0$  and then  $\delta \rightarrow 0$ , we find that

$$\mathcal{Q}_0 f(x, u, \xi, v) = \lim_{\rho \rightarrow 0} \frac{m(u + \xi(\cdot - x), v \mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))},$$

which completes the proof.  $\blacksquare$

**5.2. Proof of Theorem 3.2.** We proceed as in the proof of Theorem 3.1, replacing  $\bar{u}(x)$  by  $0 \in \mathbb{R}^m$  for the singular part. For the formula of  $\mathcal{Q}_0 f$ , we refer to the **Step 2** of the proof §5.1.

Let  $O \in \mathcal{O}(\Omega)$ ,  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ ,  $\nu \in \mathcal{M}(\Omega; \mathbb{R}^l)$  and  $v \in L^1_{|\nu^s|}(\Omega; \mathbb{S}^l)$ . Let  $x \in O$  be such that

$$\lim_{\rho \rightarrow 0} \frac{((1 + |u|^p) \mathcal{L}_N)(B_\rho(x))}{|\nu^s|(B_\rho(x))} = 0 \quad \text{and} \quad |v(x)| = 1. \quad (5.21)$$



Fix  $\rho \in ]0, 1[$ . Using  $(\mathcal{R}_3)$ , Jensen inequality and proceeding as for the estimate (5.7), replacing  $\bar{u}(x)$  by  $0 \in \mathbb{R}^m$ , we have

$$\begin{aligned}
& \left| \frac{m(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} - \frac{m_x(0, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} \right| \\
& \leq C_2 \left( |u_{x,\rho}|^p + \rho^{q'} \right) \frac{\mathcal{L}_N(B_\rho(x))}{|\nu^s|(B_\rho(x))} + \rho \\
& \leq C_2 \left( \frac{1}{|\nu^s|(B_\rho(x))} \int_{B_\rho(x)} |u|^p d\mathcal{L}_N + \rho^{q'} \frac{\mathcal{L}_N(B_\rho(x))}{|\nu^s|(B_\rho(x))} \right) + \rho \\
& \leq C_2 \frac{((1 + |u|^p)\mathcal{L}_N)(B_\rho(x))}{|\nu^s|(B_\rho(x))} + \rho. \tag{5.22}
\end{aligned}$$

We follow the same path as in the proof **Step 1** §5.1. Thus, similarly to the Lemma 5.1, for every  $\delta > 0$  we can find  $M'(x, \delta) \geq 3$  such that for every  $M \geq M'(x, \delta)$  we have the analogue of (5.10)

$$\left| \frac{m_x(0, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} - \frac{m_x^0(0, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} \right| \leq 2\delta + 2C_2\rho^q M^q + \rho.$$

Since  $\lim_{\rho \rightarrow 0} t_{x,\rho} = \infty$ , there exists  $\rho_x \in ]0, 1[$  such that for every  $\rho \in ]0, \rho_x[$  we have  $t_{x,\rho} > T$ , where  $T$  is given by  $(\mathcal{R}_3)$ . So, for every  $\rho \in ]0, \rho_x[$

$$\begin{aligned}
& \left| \frac{m(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} - m_x^\infty(0, v(x) |\nu^s|; Y) \right| \\
& \leq C_2 \frac{((1 + |u|^p)\mathcal{L}_N)(B_\rho(x))}{|\nu^s|(B_\rho(x))} + 2C_2\rho^q M^q + 2\rho + \frac{\beta}{t_{x,\rho}^r} + 2\delta, \tag{5.23}
\end{aligned}$$

where  $m_x^\infty$  is given by (5.12) (replacing  $\bar{u}(x)$  by 0). Passing to the limit  $\rho \rightarrow 0$  by taking (5.21) into account, and then  $\delta \rightarrow 0$  in (5.23), we get at the same time that  $(H_3)$  is satisfied and

$$\lim_{\rho \rightarrow 0} \frac{m(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} = m_x^\infty(0, v(x) |\nu^s|; Y) = \mathcal{Q}_0^s f^\infty(x, v(x)). \blacksquare$$

## 6. PROOF OF HOMOGENIZATION THEOREMS

We use a subadditive theorem (due to [AK81]) to prove  $(H_2)$ ,  $(H_3)$  and the formulas for the limit integrands (we can find a development of subadditive theorems and their applications to the characterization of  $\Gamma$ -limits in [LM02]). Let  $\mathcal{S} : \mathcal{O}_b(\mathbb{R}^N) \rightarrow \mathbb{R}_+$  be a set function defined on  $\mathcal{O}_b(\mathbb{R}^N)$  the set of bounded open subsets of  $\mathbb{R}^N$ . Consider the following conditions:

(i) there exists  $C > 0$  such that for all  $O \in \mathcal{O}_b(\mathbb{R}^N)$

$$\mathcal{S}(O) \leq C\mathcal{L}_N(O);$$

(ii)  $\mathcal{S}$  is subadditive, i.e. that for every  $U \in \mathcal{O}_b(\mathbb{R}^N)$ ,  $V \in \mathcal{O}_b(\mathbb{R}^N)$  and  $O \in \mathcal{O}_b(\mathbb{R}^N)$  with  $U \cap V = \emptyset$ ,  $U \subset O$ ,  $V \subset O$  and  $\mathcal{L}_N(O \setminus (U \cup V)) = 0$  it holds

$$\mathcal{S}(O) \leq \mathcal{S}(U) + \mathcal{S}(V);$$

(iii)  $\mathcal{S}$  is invariant, i.e. for every  $z \in \mathbb{Z}^N$  and every  $O \in \mathcal{O}_b(\mathbb{R}^N)$  we have

$$\mathcal{S}(z + O) = \mathcal{S}(O);$$

(iv)  $\mathcal{S}$  is almost-invariant, i.e. for every  $z \in \mathbb{Z}^N$  and every  $O \in \mathcal{O}_b(\mathbb{R}^N)$  the set  $T := \{z \in \mathbb{Z}^N : \mathcal{S}(z + O) = \mathcal{S}(O)\}$  is relatively dense in  $\mathbb{R}^N$ , i.e. there exists  $L > 0$  such that

$$T + [0, L]^N = \mathbb{R}^N.$$

The following result is a particular case of [LM02, Theorem 3.1, pp. 30].

**Theorem 6.1.** *If (i), (ii) and (iii) then for every open cube  $B \in \mathcal{O}_b(\mathbb{R}^N)$  we have*

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{S}\left(\frac{1}{\varepsilon}B\right)}{\mathcal{L}_N\left(\frac{1}{\varepsilon}B\right)} = \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}(k\mathbb{Y})}{k^N}.$$

*If (i), (ii) and (iv) then for every open cube  $B \in \mathcal{O}_b(\mathbb{R}^N)$  of the form  $\prod_{i=1}^N]a_i, b_i[$  with  $(a_i, b_i) \in \mathbb{R}^2$ , we have*

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{S}\left(\frac{1}{\varepsilon}B\right)}{\mathcal{L}_N\left(\frac{1}{\varepsilon}B\right)} = \lim_{k \rightarrow \infty} \frac{\mathcal{S}(k\mathbb{Y})}{k^N}.$$

**6.1. Proof of Theorem 3.3.** In order to apply Theorem 2.1 we have to verify (H<sub>2</sub>) and (H<sub>3</sub>) with  $f_\varepsilon(x, u, \xi, v) := W\left(x, \frac{x}{\varepsilon}, \xi, v\right)$  for all  $(x, \xi, v) \in \Omega \times \mathbb{M} \times \mathbb{R}^l$  and all  $\varepsilon > 0$ ; (H<sub>1</sub>) being a consequence of  $(\mathcal{H}_1)$ .

**Proof of (H<sub>2</sub>).** For each  $(x, \xi, v) \in \Omega \times \mathbb{M} \times \mathbb{R}^l$  we define  $\mathcal{S}_{x, \xi, v} : \mathcal{O}_b(\mathbb{R}^N) \rightarrow \mathbb{R}_+$  by

$$\mathcal{S}_{x, \xi, v}(O) := \inf_{(\varphi, \psi) \in \mathcal{A}_0(O)} \int_O W\left(x, y, \xi + \nabla\varphi(y), v + \psi(y)\right) d\mathcal{L}_N(y).$$

We notice that (i), (ii) and (iii) of Theorem 6.1 are satisfied. Thus, for every  $\rho > 0$  and every  $x \in \Omega$  we have

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{S}_{x, \xi, v}\left(\frac{1}{\varepsilon}B_\rho(x)\right)}{\mathcal{L}_N\left(\frac{1}{\varepsilon}B_\rho(x)\right)} = \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}_{x, \xi, v}(k\mathbb{Y})}{k^N}.$$

Moreover, for every  $\varepsilon > 0$ ,  $\rho > 0$ ,  $u \in \mathbb{R}^m$  and every  $x \in \Omega$ , we have

$$\frac{\mathcal{S}_{x, \xi, v}\left(\frac{1}{\varepsilon}B_\rho(x)\right)}{\mathcal{L}_N\left(\frac{1}{\varepsilon}B_\rho(x)\right)} = \frac{m_\varepsilon^x(u + \xi, v; \mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))},$$

where

$$\frac{m_\varepsilon^x(u + \xi, v; \mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} := \inf_{(\varphi, \psi) \in \mathcal{A}_0(B_\rho(x))} \int_{B_\rho(x)} W\left(x, \frac{y}{\varepsilon}, \xi + \nabla\varphi(y), v + \psi(y)\right) d\mathcal{L}_N(y).$$

It follows that

$$\lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{m_\varepsilon^x(u + \xi, v; \mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} = \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}_{x, \xi, v}(k\mathbb{Y})}{k^N} = \mathcal{H}_0 W(x, \xi, v).$$

Next, we prove (H<sub>2</sub>) by using the condition  $(\mathcal{H}_2)$ . Let  $v \in L^1(\Omega; \mathbb{R}^l)$  and  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ . Fix  $x \in \Omega$  such that  $|v(x)| < \infty$  and  $|\xi| := |\nabla u(x)| < \infty$ . There exist  $C_1 > 0$  and  $q' > 0$  such that for every  $(\varphi, \psi) \in \mathcal{A}_0(B_\rho(x))$  we have

$$\begin{aligned} & \left| \int_{B_\rho(x)} W\left(x, \frac{y}{\varepsilon}, \xi + \nabla\varphi, v(x) + \psi\right) - W\left(y, \frac{y}{\varepsilon}, \xi + \nabla\varphi, v(x) + \psi\right) d\mathcal{L}_N(y) \right| \\ & \leq C_1 \rho^{q'} \int_{B_\rho(x)} 1 + |\xi + \nabla\varphi|^p + |v(x) + \psi| d\mathcal{L}_N. \quad (6.1) \end{aligned}$$

Using the growth conditions, we have

$$\max\left(\frac{m_\varepsilon^x(u_x, v(x); \mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))}, \frac{m_\varepsilon(u_x, v(x); \mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))}\right) \leq C(1 + |\xi|^p + |v(x)|).$$

Moreover, for every  $i \in \{1, 2\}$  there exists  $(\varphi_i, \psi_i) \in \mathcal{A}_0(B_\rho(x))$  such that

$$\begin{aligned} \frac{m_\varepsilon^x(u_x, v(x)\mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} + \rho &\geq \int_{B_\rho(x)} W\left(x, \frac{y}{\varepsilon}, \xi + \nabla\varphi_1(y), v(x) + \psi_1(y)\right) d\mathcal{L}_N(y), \\ \frac{m_\varepsilon(u_x, v(x)\mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} + \rho &\geq \int_{B_\rho(x)} W\left(y, \frac{y}{\varepsilon}, \xi + \nabla\varphi_2(y), v(x) + \psi_2(y)\right) d\mathcal{L}_N(y). \end{aligned}$$

Taking (6.1) into account, we have

$$\begin{aligned} &\left| \frac{m_\varepsilon^x(u_x, v(x)\mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} - \frac{m_\varepsilon(u_x, v(x)\mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \right| \\ &\leq C_1 \rho^{q'} \sum_{i=1}^2 \int_{B_\rho(x)} 1 + |\xi + \nabla\varphi_i|^p + |v(x) + \psi_i| d\mathcal{L}_N \\ &\leq 2C_1 \rho^{q'} (\rho + C(1 + |\xi|^p + |v(x)|)). \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x, v(x)\mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} &= \lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x, v(x)\mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \\ &= \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}_{x, \xi, v(x)}(k\mathbb{Y})}{k^N}. \end{aligned}$$

**Proof of (H<sub>3</sub>).** Let  $O \in \mathcal{O}(\Omega)$ ,  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  and  $v \in L^1_{|\nu^s|}(\Omega; \mathbb{S}^l)$ . Let  $x \in O$  be such that

$$\lim_{\rho \rightarrow 0} \frac{\mathcal{L}_N(B_\rho(x))}{|\nu^s|(B_\rho(x))} = 0 \quad \text{and} \quad |v(x)| = 1. \quad (6.2)$$

Fix  $\rho > 0$  and  $\varepsilon > 0$ . Set  $t_{x,\rho} := \frac{|\nu^s|(B_\rho(x))}{\mathcal{L}_N(B_\rho(x))}$ . We have

$$\begin{aligned} &\frac{m_\varepsilon(u_{x,\rho}, v(x)|\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} \\ &= \inf_{(\phi, \psi) \in \mathcal{A}_0(B_\rho(x))} \int_{B_\rho(x)} \frac{1}{t_{x,\rho}} W\left(y, \frac{y}{\varepsilon}, \nabla\varphi(y), t_{x,\rho}(v(x) + \psi(y))\right) d\mathcal{L}_N(y). \end{aligned}$$

Since  $\lim_{\rho \rightarrow 0} t_{x,\rho} = \infty$ , there exists  $\rho_x > 0$  such that for every  $\rho \in ]0, \rho_x[$  we have  $t_{x,\rho} > T$ , where  $T$  is given by ( $\mathcal{H}_3$ ). Fix  $\rho \in ]0, \rho_x[$ . Then, for every  $(\varphi, \psi) \in \mathcal{A}_0(B_\rho(x))$  we have

$$\begin{aligned} &\left| \int_{B_\rho(x)} \frac{1}{t_{x,\rho}} W\left(y, \frac{y}{\varepsilon}, \nabla\varphi(y), t_{x,\rho}(v(x) + \psi(y))\right) d\mathcal{L}_N(y) \right. \\ &\quad \left. - \int_{B_\rho(x)} W^\infty\left(y, \frac{y}{\varepsilon}, \nabla\varphi(y), v(x) + \psi(y)\right) d\mathcal{L}_N(y) \right| \leq \frac{\beta}{t_{x,\rho}^r}. \end{aligned}$$

Therefore

$$\left| \frac{m_\varepsilon(u_{x,\rho}, v(x)|\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} - \frac{m_\varepsilon^\infty(0, v(x)|\nu^s|; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \right| \leq \frac{\beta}{t_{x,\rho}^r}, \quad (6.3)$$

where

$$m_\varepsilon^\infty(0, v(x)|\nu^s|; B_\rho(x)) := \inf_{(\phi, \psi) \in \mathcal{A}_0(B_\rho(x))} \int_{B_\rho(x)} W^\infty\left(y, \frac{y}{\varepsilon}, \nabla\varphi, v(x) + \psi\right) d\mathcal{L}_N.$$

We observe that  $W^\infty$  has linear growth with respect to  $v$ , i.e. for every  $(x, y, \xi, v) \in \Omega \times \mathbb{R}^N \times \mathbb{M} \times \mathbb{S}^l$  we have

$$c|v| \leq W^\infty(x, y, \xi, v) \leq C|v|.$$

Next, arguing as in the *Proof of (H<sub>2</sub>)*, we have

$$\left| \frac{m_\varepsilon^\infty(0, v(x) |\nu^s|; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} - \frac{m_\varepsilon^{\infty, x}(0, v(x) |\nu^s|; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \right| \leq 2C_1 \rho^{q'} (\rho + C) \quad (6.4)$$

where

$$\frac{m_\varepsilon^{\infty, x}(0, v(x) |\nu^s|; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} = \inf_{(\phi, \psi) \in \mathcal{A}_0(B_\rho(x))} \int_{B_\rho(x)} W^\infty\left(x, \frac{y}{\varepsilon}, \nabla \varphi, v(x) + \psi\right) d\mathcal{L}_N.$$

For each  $x \in \Omega$  and each  $v \in \mathbb{S}^l$  we define  $\mathcal{S}_{x,v}^\infty : \mathcal{O}_b(\mathbb{R}^N) \rightarrow \mathbb{R}_+$  by

$$\mathcal{S}_{x,v}^\infty(O) := \inf_{(\phi, \psi) \in \mathcal{A}_0(O)} \int_O W^\infty(x, y, \nabla \varphi(y), v + \psi(y)) d\mathcal{L}_N(y).$$

It is direct to see that  $\mathcal{S}_{x,v}^\infty$  satisfies (i), (ii) and (iii) of Theorem 6.1, hence

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{S}_{x,v}^\infty\left(\frac{1}{\varepsilon} B_\rho(x)\right)}{\mathcal{L}_N\left(\frac{1}{\varepsilon} B_\rho(x)\right)} = \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}_{x,v}^\infty(k\mathbb{Y})}{k^N} = \mathcal{H}_0^s W^\infty(x, v).$$

But, we have

$$\frac{\mathcal{S}_{x,v(x)}^\infty\left(\frac{1}{\varepsilon} B_\rho(x)\right)}{\mathcal{L}_N\left(\frac{1}{\varepsilon} B_\rho(x)\right)} = \frac{m_\varepsilon^{\infty, x}(0, v(x) |\nu^s|; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))},$$

thus, gathering (6.3) and (6.4) we have

$$\begin{aligned} -\frac{\beta}{t_{x,\rho}^r} - C_1 \rho^{q'} (\rho + C) + \frac{\mathcal{S}_{x,v(x)}^\infty\left(\frac{1}{\varepsilon} B_\rho(x)\right)}{\mathcal{L}_N\left(\frac{1}{\varepsilon} B_\rho(x)\right)} &\leq \frac{m_\varepsilon(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} \\ &\leq \frac{\beta}{t_{x,\rho}^r} + C_1 \rho^{q'} (\rho + C) + \frac{\mathcal{S}_{x,v(x)}^\infty\left(\frac{1}{\varepsilon} B_\rho(x)\right)}{\mathcal{L}_N\left(\frac{1}{\varepsilon} B_\rho(x)\right)}. \end{aligned}$$

Passing to the limits  $\varepsilon \rightarrow 0$  and  $\rho \rightarrow 0$ , we obtain

$$\begin{aligned} \lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} &= \lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} \\ &= \mathcal{H}_0^s W^\infty(x, v(x)). \blacksquare \end{aligned}$$

**6.2. Proof of Theorem 3.4.** For every  $v = (v_1, \dots, v_l) \in \mathbb{R}^l$  we set

$$[v]_l := ([v_1], \dots, [v_l]) \in \mathbb{Z}^l.$$

Similarly, we define  $[R]_m \in \mathbb{Z}^m$  for  $R \in \mathbb{R}^m$ .

The following lemmas will be used to show that (H<sub>2</sub>) and (H<sub>3</sub>) are fulfilled. The first one shows that the limit integrands are independent of the measure variable. The second lemma deals with estimates on  $m_\varepsilon$  with shift on the variable  $u$ .

**Lemma 6.1.** *For every  $\varepsilon > 0$ , every  $v \in \mathbb{R}^l$ , every  $B \in \mathcal{O}(\Omega)$  and every  $\sigma \in \mathcal{M}^+(\Omega)$*

$$\sup \left\{ \frac{|m_\varepsilon(u, v\sigma; B) - m_\varepsilon(u, 0; B)|}{\sigma(B)} : u \in W^{1,p}(\Omega; \mathbb{R}^m) \right\} \leq C \left| v - \frac{\varepsilon}{t_B} \left[ \frac{vt_B}{\varepsilon} \right]_l \right|$$

where  $t_B := \frac{\sigma(B)}{\mathcal{L}_N(B)}$  and  $C$  is the constant growth condition appearing in ( $\mathcal{H}_1^l$ ).

*Proof.* Let  $\varepsilon > 0$  and  $v \in \mathbb{R}^l$ . By using Lemma 7.1 and periodicity, we see that for every  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ ,  $\sigma \in \mathcal{M}^+(\Omega)$  and  $B \in \mathcal{O}(\Omega)$  we have  $m_\varepsilon(u, v\sigma; B) = m_\varepsilon(u, vt_B \mathcal{L}_N; B) = m_\varepsilon(u, (vt_B - \varepsilon \lfloor \frac{vt_B}{\varepsilon} \rfloor) \mathcal{L}_N; B)$ . The estimate follows using again Lemma 7.1.  $\blacksquare$

**Lemma 6.2.** *There exists  $C > 0$  such that for every  $t \in ]1, 2[$ , every  $\rho \in ]0, 1[$ , every  $x \in \Omega$  with  $B_{t\rho}(x) \subset \Omega$ , every  $\varepsilon > 0$ , every  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  and every  $v \in W_{\text{loc}}^{1,p}(\mathbb{R}^N; \mathbb{R}^m)$  we have*

- (i)  $m_\varepsilon(u + v, 0; B_{t\rho}(x)) \leq \varepsilon + m_\varepsilon(u, 0; B_\rho(x)) + C\Delta_0(t\rho, \rho, |v|_{W^{1,p}(B_{2\rho}(x); \mathbb{R}^m)})$ ;
- (ii)  $m_\varepsilon(u, 0; B_\rho(x)) \leq \varepsilon + m_\varepsilon(u + v, 0; B_{\frac{1}{t}\rho}(x)) + C\Delta_0(\rho, \frac{1}{t}\rho, |v|_{W^{1,p}(B_{2\rho}(x); \mathbb{R}^m)})$

where the function  $\Delta_0 : ]0, 2[ \times ]0, 1[ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is defined by

$$\Delta_0(r_1, r_2, s) := \int_{B_{r_1}(x) \setminus B_{r_2}(x)} 1 + |\nabla u|^p d\mathcal{L}_N + \left(1 + \frac{1}{(r_1 - r_2)^p}\right) s$$

and satisfies for every  $\rho \in ]0, 1[$

$$\lim_{\tau \rightarrow 1^+} \overline{\lim}_{s \rightarrow 0} \Delta_0(\tau\rho, \rho, s) = \lim_{\tau \rightarrow 1^-} \overline{\lim}_{s \rightarrow 0} \Delta_0(\rho, \tau\rho, s) = 0.$$

*Proof.* Let  $t \in ]1, 2[$ ,  $\rho \in ]0, 1[$ ,  $x \in \Omega$  with  $B_{t\rho}(x) \subset \Omega$ . Let  $\varepsilon > 0$ ,  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  and  $v \in W_{\text{loc}}^{1,p}(\mathbb{R}^N; \mathbb{R}^m)$ . There exists  $(w, \psi) \in \mathcal{A}_0(B_\rho(x))$  such that

$$\varepsilon + m_\varepsilon(u, 0; B_\rho(x)) \geq I_\varepsilon(u + w, \psi; B_\rho(x)).$$

We consider a cut-off function  $\varphi_t \in W_0^{1,\infty}(B_{t\rho}(x); [0, 1])$  between  $\overline{B_\rho(x)}$  and  $\overline{O} \setminus B_{t\rho}(x)$ , satisfying  $\varphi_t \equiv 1$  on  $\overline{B_\rho(x)}$ ,  $\varphi_t \equiv 0$  on  $\overline{O} \setminus B_{t\rho}(x)$ , and  $|\nabla \varphi_t|_\infty \leq \frac{C_0}{\rho(t-1)}$  for some  $C_0 > 0$  (independent of  $\rho$  and  $t$ ). We set  $w_t := \varphi_t w + (1 - \varphi_t)v \in v + W_0^{1,p}(B_{t\rho}(x); \mathbb{R}^m)$ . We denote by  $\tilde{\psi}$  the extension of  $\psi$  to  $B_{t\rho}(x)$  with  $\tilde{\psi} \equiv 0$  in  $B_{t\rho}(x) \setminus B_\rho(x)$ . We have

$$\begin{aligned} m_\varepsilon(u + v, 0; B_{t\rho}(x)) &\leq I_\varepsilon(u + w_t, \tilde{\psi}; B_{t\rho}(x)) \\ &= I_\varepsilon(u + w, \psi; B_\rho(x)) + I_\varepsilon(u + w_t, 0; B_{t\rho}(x) \setminus B_\rho(x)) \\ &\leq \varepsilon + m_\varepsilon(u, 0; B_\rho(x)) + I_\varepsilon(u + (1 - \varphi_t)v, 0; B_{t\rho}(x) \setminus B_\rho(x)). \end{aligned}$$

Using the growth condition  $(\mathcal{H}'_1)$ , we have for some  $C > 0$  depending only on  $p$

$$\begin{aligned} &I_\varepsilon(u + (1 - \varphi_t)v, 0; B_{t\rho}(x) \setminus B_\rho(x)) \\ &\leq C \int_{B_{t\rho}(x) \setminus B_\rho(x)} 1 + |\nabla u|^p + |\nabla v|^p + |\nabla \varphi_t|_\infty^p |v|^p d\mathcal{L}_N \\ &\leq C \left\{ \int_{B_{t\rho}(x) \setminus B_\rho(x)} 1 + |\nabla u|^p d\mathcal{L}_N + \left(1 + \frac{C_0^p}{\rho^p(t-1)^p}\right) |v|_{W^{1,p}(B_{2\rho}(x); \mathbb{R}^m)}^p \right\} \\ &\leq \max(C, C \max(1, C_0^p)) \Delta_0\left(t\rho, \rho, |v|_{W^{1,p}(B_{2\rho}(x); \mathbb{R}^m)}^p\right) \end{aligned}$$

which proves (i). Now, to show (ii), we observe that

$$m_\varepsilon(u, 0; B_\rho(x)) = m_\varepsilon((u + v) - v, 0; B_\rho(x)),$$

thus, by proceeding similarly as for the proof of (i) above, for some  $C' > 0$  depending only on  $p$

$$\begin{aligned}
& m_\varepsilon(u, 0; B_\rho(x)) \\
& \leq \varepsilon + m_\varepsilon\left(u + v, 0; B_{\frac{1}{t}\rho}(x)\right) \\
& + C' \left\{ \int_{B_\rho(x) \setminus B_{\frac{1}{t}\rho}(x)} 1 + |\nabla u|^p d\mathcal{L}_N + \left(1 + \frac{C_0^p}{\rho^p \left(1 - \frac{1}{t}\right)^p}\right) |v|_{W^{1,p}(B_2(x); \mathbb{R}^m)}^p \right\} \\
& \leq \varepsilon + m_\varepsilon\left(u + v, 0; B_{\frac{1}{t}\rho}(x)\right) + \max(C', C' \max(1, C_0^p)) \Delta_0\left(\rho, \frac{1}{t}\rho, |v|_{W^{1,p}(B_2(x); \mathbb{R}^m)}^p\right). \blacksquare
\end{aligned}$$

6.2.1. *Proofs of (H<sub>2</sub>) and formula of  $\mathcal{H}_0 W(\cdot)$ .* Let  $O \in \mathcal{O}(\Omega)$ ,  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  and  $v \in L^1(\Omega; \mathbb{R}^l)$ , we have to show

$$\overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x, v(x)\mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} = \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x, v(x)\mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \quad \mathcal{L}_N\text{-a.e. in } O.$$

For each  $x \in \Omega$  we set  $A := \nabla u(x) \in \mathbb{M}$  and  $R := u(x) - Ax \in \mathbb{R}^m$ , thus  $u_x(\cdot) = A \cdot + R$ . By Lemma 6.1 we see that for every  $x \in \Omega$

$$\begin{aligned}
\overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x, v(x)\mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} &= \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x, 0; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \\
\overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x, v(x)\mathcal{L}_N; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} &= \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x, 0; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))},
\end{aligned}$$

so, we are reduced to show

$$\overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(A \cdot + R, 0; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} = \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(A \cdot + R, 0; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))}. \quad (6.5)$$

For each  $\xi \in \mathbb{M}$  we define  $\mathcal{S}_\xi : \mathcal{O}_b(\mathbb{R}^N) \rightarrow \mathbb{R}_+$  by

$$\mathcal{S}_\xi(O) := \inf_{(\varphi, \psi) \in \mathcal{A}_0(O)} \int_O W(y, \xi y + \varphi(y), \xi + \nabla \varphi(y), \psi(y)) d\mathcal{L}_N(y).$$

**Step 1: formula  $\mathcal{H}_0 W(\xi)$  for  $\xi \in \mathbb{Q}^{m \times N}$ .** In this first step, we show that for every  $\xi \in \mathbb{Q}^{m \times N}$  and  $\rho > 0$  we have

$$\lim_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(\xi \cdot, 0; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} = \lim_{k \rightarrow \infty} \frac{\mathcal{S}_\xi(kY)}{k^N} = \mathcal{H}_0 W(\xi). \quad (6.6)$$

Let  $\xi = (\xi_{ij})_{1 \leq i \leq m, 1 \leq j \leq N} \in \mathbb{Q}^{m \times N}$ , be a matrix with rational numbers entries, with  $\xi_{ij} = \frac{p_{ij}}{q_{ij}}$  where  $p_{ij} \in \mathbb{Z}$  and  $q_{ij} \in \mathbb{Z}^*$  for all  $(i, j) \in \{1, \dots, m\} \times \{1, \dots, N\}$ . Observe that  $\mathcal{S}_\xi$  satisfies (i) and (ii) of Theorem 6.1. Moreover  $\mathcal{S}_\xi$  satisfies (iv). Indeed, using periodicity, we need to show that the  $T_\xi := \{z \in \mathbb{Z}^N : \xi z \in \mathbb{Z}^m\}$  is relatively dense in  $\mathbb{R}^N$ . Define  $Z_\xi = (Z_\xi^1, \dots, Z_\xi^N) \in \mathbb{Z}^N$  by setting  $Z_\xi^j := \prod_{i=1}^m q_{ij}$  for all  $j \in \{1, \dots, N\}$ . Set  $L_\xi := \max_{1 \leq j \leq N} \prod_{i=1}^m |q_{ij}|$  and let  $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ . Then  $x \in [0, L_\xi]^N + T_\xi$ , indeed, we have  $\xi Z_\xi^x \in \mathbb{Z}^m$  where  $Z_\xi^x := \left(Z_\xi^{x,1}, \dots, Z_\xi^{x,N}\right)$  with  $Z_\xi^{x,j} := Z_\xi^j \left\lfloor \frac{x_j}{Z_\xi^j} \right\rfloor$  for all  $j \in \{1, \dots, N\}$ , and

$$|x - Z_\xi^x| \leq \max_{1 \leq j \leq N} |Z_\xi^j| \leq L_\xi.$$

Thus, by Theorem 6.1 we have

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{S}_\xi \left( \frac{1}{\varepsilon} B_\rho(x) \right)}{\mathcal{L}_N \left( \frac{1}{\varepsilon} B_\rho(x) \right)} = \lim_{k \rightarrow \infty} \frac{\mathcal{S}_\xi(k\mathbb{Y})}{k^N} \quad \text{for all } \rho > 0 \text{ and all } x \in \Omega.$$

Moreover, we see that for every  $\varepsilon > 0$ ,  $\rho > 0$ ,  $\xi \in \mathbb{M}$  and every  $x \in \Omega$

$$\frac{\mathcal{S}_\xi \left( \frac{1}{\varepsilon} B_\rho(x) \right)}{\mathcal{L}_N \left( \frac{1}{\varepsilon} B_\rho(x) \right)} = \frac{m_\varepsilon(\xi \cdot, 0; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))},$$

which, by passing to the limit  $\varepsilon \rightarrow 0$ , gives (6.6).

**Step 2: end of the proof of (H<sub>2</sub>).** In this second step, we show (6.5). Fix  $x \in O$  and  $\rho > 0$ . We set  $R_\varepsilon := R - \varepsilon \lfloor \frac{R}{\varepsilon} \rfloor_m$  and choose a sequence  $\{A_\delta\}_{\delta > 0} \subset \mathbb{Q}^{m \times N}$  such that  $\lim_{\delta \rightarrow 0} A_\delta = A$ . Using the periodicity with respect to the second variable, we see that  $m_\varepsilon(A \cdot + R, 0; B_\rho(x)) = m_\varepsilon(A \cdot + R_\varepsilon, 0; B_\rho(x))$  for all  $\varepsilon > 0$ . Let  $t \in ]1, 2[$  and  $\delta > 0$ . By Lemma 6.2, we have

$$\begin{aligned} \frac{m_\varepsilon(A \cdot + R, 0; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} &= \frac{m_\varepsilon(A \cdot + R_\varepsilon, 0; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \\ &\leq \frac{\varepsilon}{\mathcal{L}_N(B_\rho(x))} + \frac{1}{t^N} \frac{m_\varepsilon(A_\delta \cdot, 0; B_{\frac{1}{t}\rho}(x))}{\mathcal{L}_N(B_{\frac{1}{t}\rho}(x))} + \frac{C\Delta_0(\rho, \frac{1}{t}\rho, |(A_\delta - A) \cdot - R_\varepsilon|_{W^{1,p}(B_2(x); \mathbb{R}^m)})}{\mathcal{L}_N(B_\rho(x))} \\ &= \frac{\varepsilon}{\mathcal{L}_N(B_\rho(x))} + \frac{1}{t^N} \frac{\mathcal{S}_{A_\delta} \left( \frac{1}{\varepsilon} B_{\frac{1}{t}\rho}(x) \right)}{\mathcal{L}_N \left( \frac{1}{\varepsilon} B_{\frac{1}{t}\rho}(x) \right)} + \frac{C\Delta_0(\rho, \frac{1}{t}\rho, |(A_\delta - A) \cdot - R_\varepsilon|_{W^{1,p}(B_2(x); \mathbb{R}^m)})}{\mathcal{L}_N(B_\rho(x))}, \end{aligned}$$

and

$$\begin{aligned} t^N \frac{\mathcal{S}_{A_\delta} \left( \frac{1}{\varepsilon} B_{t\rho}(x) \right)}{\mathcal{L}_N \left( \frac{1}{\varepsilon} B_{t\rho}(x) \right)} &= t^N \frac{m_\varepsilon(A_\delta \cdot, 0; B_{t\rho}(x))}{\mathcal{L}_N(B_{t\rho}(x))} \leq \frac{\varepsilon}{\mathcal{L}_N(B_\rho(x))} + \frac{m_\varepsilon(A \cdot + R, 0; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \\ &\quad + \frac{C\Delta_0(t\rho, \rho, |(A - A_\delta) \cdot + R_\varepsilon|_{W^{1,p}(B_2(x); \mathbb{R}^m)})}{\mathcal{L}_N(B_\rho(x))}. \end{aligned}$$

Letting  $\varepsilon \rightarrow 0$  and using (6.6), then passing to the limits  $\delta \rightarrow 0$  and then  $t \rightarrow 1^+$  we obtain

$$\lim_{\delta \rightarrow 0} \lim_{k \rightarrow \infty} \frac{\mathcal{S}_{A_\delta}(k\mathbb{Y})}{k^N} = \lim_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(A \cdot + R, 0; B_\rho(x))}{\mathcal{L}_N(B_\rho(x))}$$

which proves (H<sub>2</sub>).

**Step 3: formula  $\mathcal{H}_0\mathcal{W}(\xi)$  for  $\xi \in \mathbb{M}$ .** For every  $\xi \in \mathbb{M}$  and every  $\{\xi_\delta\}_{\delta > 0} \subset \mathbb{Q}^{m \times N}$  satisfying  $\lim_{\delta \rightarrow 0} \xi_\delta = \xi$ , we have

$$\lim_{\delta \rightarrow 0} \lim_{k \rightarrow \infty} \frac{\mathcal{S}_{\xi_\delta}(k\mathbb{Y})}{k^N} = \lim_{k \rightarrow \infty} \frac{\mathcal{S}_\xi(k\mathbb{Y})}{k^N}.$$

Indeed, let  $t \in ]1, 2[$  and  $\delta > 0$ . Using Lemma 6.2 and noticing that  $\mathbb{Y} = B_{\frac{1}{2}}(\frac{1}{2}e) = \frac{1}{2}e + \frac{1}{2}[-1, 1]^N$  with  $e := (1, 1, \dots, 1) \in \mathbb{R}^N$ , we have

$$\begin{aligned}
t^N \frac{\mathcal{S}_{\xi_\delta} \left( k B_{\frac{t}{2}} \left( \frac{e}{2} \right) \right)}{\mathcal{L}_N \left( k \left( B_{\frac{t}{2}} \left( \frac{e}{2} \right) \right) \right)} &= m_{\frac{1}{k}} \left( \xi_\delta, 0; B_{\frac{t}{2}} \left( \frac{e}{2} \right) \right) \tag{6.7} \\
&\leq \frac{1}{k} + m_{\frac{1}{k}} \left( \xi_\cdot, 0; B_{\frac{1}{2}} \left( \frac{e}{2} \right) \right) + C \Delta_0 \left( \frac{t}{2}, \frac{1}{2}, |(\xi - \xi_\delta) \cdot |_{W^{1,p}(B_1(\frac{e}{2}); \mathbb{R}^m)} \right) \\
&\leq \frac{2}{k} + m_{\frac{1}{k}} \left( \xi_\delta, 0; B_{\frac{1}{2t}} \left( \frac{e}{2} \right) \right) + C \Delta_0 \left( \frac{t}{2}, \frac{1}{2}, |(\xi - \xi_\delta) \cdot |_{W^{1,p}(B_1(\frac{e}{2}); \mathbb{R}^m)} \right) \\
&\quad + C \Delta_0 \left( \frac{1}{2}, \frac{1}{2t}, |(\xi_\delta - \xi) \cdot |_{W^{1,p}(B_1(\frac{e}{2}); \mathbb{R}^m)} \right) \\
&= \frac{2}{k} + \frac{1}{t^N} \frac{\mathcal{S}_{\xi_\delta} \left( k B_{\frac{1}{2t}} \left( \frac{e}{2} \right) \right)}{\mathcal{L}_N \left( k \left( B_{\frac{1}{2t}} \left( \frac{e}{2} \right) \right) \right)} + C \Delta_0 \left( \frac{t}{2}, \frac{1}{2}, |(\xi - \xi_\delta) \cdot |_{W^{1,p}(B_1(\frac{e}{2}); \mathbb{R}^m)} \right) \\
&\quad + C \Delta_0 \left( \frac{1}{2}, \frac{1}{2t}, |(\xi_\delta - \xi) \cdot |_{W^{1,p}(B_1(\frac{e}{2}); \mathbb{R}^m)} \right).
\end{aligned}$$

Applying **Step 1**, we have for every  $\delta > 0$  and  $t \in ]1, 2[$

$$\lim_{k \rightarrow \infty} \frac{\mathcal{S}_{\xi_\delta} \left( \frac{1}{t} k B_{\frac{t}{2}} \left( \frac{e}{2} \right) \right)}{\left( \frac{1}{t} k \right)^N} = \lim_{k \rightarrow \infty} \frac{\mathcal{S}_{\xi_\delta} \left( \frac{1}{t} k B_{\frac{1}{2t}} \left( \frac{e}{2} \right) \right)}{\left( \frac{1}{t} k \right)^N} = \lim_{k \rightarrow \infty} \frac{\mathcal{S}_{\xi_\delta}(k\mathbb{Y})}{k^N}.$$

Passing to the limits  $\delta \rightarrow 0$  and  $t \rightarrow 1$  in (6.7) we obtain

$$\lim_{\delta \rightarrow 0} \lim_{k \rightarrow \infty} \frac{\mathcal{S}_{\xi_\delta}(k\mathbb{Y})}{k^N} = \lim_{k \rightarrow \infty} m_{\frac{1}{k}}(\xi_\cdot, 0; Y) = \lim_{k \rightarrow \infty} \frac{\mathcal{S}_\xi(k\mathbb{Y})}{k^N} = \mathcal{H}_0 W(\xi).$$

6.2.2. *Proof of (H<sub>3</sub>).* Let  $O \in \mathcal{O}(\Omega)$ . Let  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ ,  $\nu \in \mathcal{M}(\Omega; \mathbb{R}^l)$  and  $v \in L^1_{|\nu^s|}(\Omega; \mathbb{S}^l)$ . We have to show that

$$\lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} = \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}, v(x) |\nu^s|; B_\rho(x))}{|\nu^s|(B_\rho(x))} \quad |\nu^s|\text{-a.e. in } O.$$

By Lemma 6.1, it is enough to prove that

$$\lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}, 0; B_\rho(x))}{|\nu^s|(B_\rho(x))} = \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}, 0; B_\rho(x))}{|\nu^s|(B_\rho(x))} \quad |\nu^s|\text{-a.e. in } O. \tag{6.8}$$

Let  $x \in O$  be such that  $\lim_{\rho \rightarrow 0} \frac{\mathcal{L}_N(B_\rho(x))}{|\nu^s|(B_\rho(x))} = 0$ . Fix  $\rho \in ]0, 1[$  and  $\varepsilon > 0$ . Using Remark 3.5, we have

$$\begin{aligned}
\frac{m_\varepsilon(u_{x,\rho}, 0; B_\rho(x))}{|\nu^s|(B_\rho(x))} &= \inf_{(\phi, \psi) \in \mathcal{A}_0(B_\rho(x))} \int_{B_\rho(x)} \frac{1}{t_{x,\rho}} W \left( \frac{y}{\varepsilon}, \frac{u_{x,\rho}}{\varepsilon} + \varphi, \nabla \varphi, t_{x,\rho} \left( \frac{1}{\varepsilon} \psi \right) \right) d\mathcal{L}_N \\
&\leq \int_{B_\rho(x)} \frac{1}{t_{x,\rho}} W \left( \frac{y}{\varepsilon}, \frac{u_{x,\rho}}{\varepsilon}, 0, 0 \right) d\mathcal{L}_N(y) \leq 2C \frac{1}{t_{x,\rho}}.
\end{aligned}$$

Since  $\lim_{\rho \rightarrow 0} t_{x,\rho} = \infty$ , it follows that

$$\lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_{x,\rho}, 0; B_\rho(x))}{|\nu^s|(B_\rho(x))} = 0,$$



which establishes (6.8) and completes the proof. ■

## 7. AUXILIARY RESULTS

**7.1. Some properties of  $m_\varepsilon$ .** In this part, we assume that  $(H_1)$  holds.

The following result can be seen as a “measure version” of [BFM98, Lemma 3.1]. The constant  $C > 0$ , in Lemma 7.1 below, is the one appearing in the growth condition  $(H_1)$ .

**Lemma 7.1.** *Let  $B \in \mathcal{O}(\Omega)$ .*

(i) *For every  $\varepsilon > 0$ , every  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  and every  $(\lambda, \nu) \in \mathcal{M}(\Omega; \mathbb{R}^l) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  we have*

$$|m_\varepsilon(u, \lambda; B) - m_\varepsilon(u, \nu; B)| \leq C |\lambda(B) - \nu(B)|.$$

*In particular, we also have*

$$m_\varepsilon\left(u, \frac{\nu(B)}{\mathcal{L}_N(B)} \mathcal{L}_N; B\right) = m_\varepsilon(u, \nu; B).$$

(ii) *For every  $\varepsilon > 0$ , every  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  and every  $(\lambda, \nu) \in \mathcal{M}(\Omega; \mathbb{R}^l) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  we have*

$$|m_\varepsilon(u_B, \lambda; B) - m_\varepsilon(u_B, \nu; B)| \leq C |\lambda(B) - \nu(B)|,$$

*where  $u_B := \int_B u d\mathcal{L}_N$ . In particular, we also have*

$$m_\varepsilon\left(u_B, \frac{\nu(B)}{\mathcal{L}_N(B)} \mathcal{L}_N; B\right) = m_\varepsilon(u_B, \nu; B).$$

(iii) *For every  $\varepsilon > 0$ ,  $\rho > 0$ ,  $x \in \Omega$  with  $B_\rho(x) \subset \Omega$ ,  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  and every  $(\lambda, \nu) \in \mathcal{M}(\Omega; \mathbb{R}^l) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  we have*

$$|m_\varepsilon(u_x, \lambda; B_\rho(x)) - m_\varepsilon(u_x, \nu; B_\rho(x))| \leq C |\lambda(B_\rho(x)) - \nu(B_\rho(x))|.$$

*In particular, we also have*

$$m_\varepsilon\left(u_x, \frac{\nu(B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \mathcal{L}_N; B_\rho(x)\right) = m_\varepsilon(u_x, \nu; B_\rho(x)).$$

*Proof.* Fix  $B \in \mathcal{O}(\Omega)$ ,  $\varepsilon > 0$ ,  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  and  $\lambda, \nu \in \mathcal{M}(\Omega; \mathbb{R}^l)$ . Let  $\delta > 0$ . There exist  $u_\delta \in u + W_0^{1,p}(B; \mathbb{R}^m)$  and  $\nu_\delta \in \mathcal{M}(\Omega; \mathbb{R}^l)$  with  $\nu_\delta \ll \mathcal{L}_N$  and  $\nu_\delta(B) = \nu(B)$  such that

$$m_\varepsilon(u, \nu; B) + \delta \geq I_\varepsilon(u_\delta, \nu_\delta; B).$$

Let  $k \in \mathbb{N}^*$  and  $B_k := \{x \in B : \text{dist}(x, \partial B) > \frac{1}{k}\}$ . We set

$$\lambda_\delta := \nu_\delta \mathbf{1}_{B_k} + \left( \frac{1}{\mathcal{L}_N(B \setminus B_k)} (\lambda(B) - \nu_\delta(B_k)) \right) \mathcal{L}_N \mathbf{1}_{B \setminus B_k}.$$

We observe that  $\lambda_\delta \ll \mathcal{L}_N$  and  $\lambda_\delta(B) = \lambda(B)$ , so we can write

$$\begin{aligned} m_\varepsilon(u, \lambda; B) &\leq I_\varepsilon(u_\delta, \lambda_\delta; B) \\ &= I_\varepsilon(u_\delta, \nu_\delta; B_k) + I_\varepsilon(u_\delta, \lambda_\delta; B \setminus B_k) \\ &\leq m_\varepsilon(u, \nu; B) + \delta + I_\varepsilon(u_\delta, \lambda_\delta; B \setminus B_k). \end{aligned} \tag{7.1}$$

Using the growth condition  $(H_1)$ , we have

$$I_\varepsilon(u_\delta, \lambda_\delta; B \setminus B_k) \leq C \int_{B \setminus B_k} 1 + |u_\delta|^p + |\nabla u_\delta|^p + \left| \frac{1}{\mathcal{L}_N(B \setminus B_k)} (\lambda(B) - \nu_\delta(B_k)) \right| d\mathcal{L}_N.$$

Since  $\lim_{k \rightarrow \infty} \nu_\delta(B_k) = \nu_\delta(B) = \nu(B)$  and  $\lim_{k \rightarrow \infty} \mathcal{L}_N(B \setminus B_k) = 0$ , we have

$$\overline{\lim}_{k \rightarrow \infty} I_\varepsilon(u_\delta, \lambda_\delta; B \setminus B_k) \leq C |\lambda(B) - \nu(B)|.$$

So, inequality (7.1) becomes

$$m_\varepsilon(u, \lambda; B) \leq m_\varepsilon(u, \nu; B) + \delta + C |\lambda(B) - \nu(B)|$$

and the proof of (i) is complete by letting  $\delta \rightarrow 0$ . The proofs of (ii) and (iii) are similar. ■

Using Lemma 7.1 we obtain the following result which is used in the proof of Proposition 2.2.

**Corollary 7.1.** *Let  $\sigma \in \mathcal{M}_+(\Omega)$ ,  $\nu \in \mathcal{M}(\Omega; \mathbb{R}^l)$  and  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ . We have*

$$\begin{aligned} (i) \quad & \limsup_{\rho \rightarrow 0} \sup_{\varepsilon > 0} \left| \frac{m_\varepsilon\left(u, \frac{d\nu}{d\sigma}(x)\sigma; B_\rho(x)\right)}{\sigma(B_\rho(x))} - \frac{m_\varepsilon(u, \nu; B_\rho(x))}{\sigma(B_\rho(x))} \right| = 0 \quad \sigma\text{-a.e. in } \Omega; \\ (ii) \quad & \limsup_{\rho \rightarrow 0} \sup_{\varepsilon > 0} \left| \frac{m_\varepsilon\left(u_{x,\rho}, \frac{d\nu}{d\sigma}(x)\sigma; B_\rho(x)\right)}{\sigma(B_\rho(x))} - \frac{m_\varepsilon(u_{x,\rho}, \nu; B_\rho(x))}{\sigma(B_\rho(x))} \right| = 0 \quad \sigma\text{-a.e. in } \Omega; \\ (iii) \quad & \limsup_{\rho \rightarrow 0} \sup_{\varepsilon > 0} \left| \frac{m_\varepsilon\left(u_x, \frac{d\nu}{d\sigma}(x)\sigma; B_\rho(x)\right)}{\sigma(B_\rho(x))} - \frac{m_\varepsilon(u_x, \nu; B_\rho(x))}{\sigma(B_\rho(x))} \right| = 0 \quad \sigma\text{-a.e. in } \Omega. \end{aligned}$$

*Proof.* We only give the proof of (i), those of (ii) and (iii) being similar. Fix  $x \in \Omega$  satisfying

$$\lim_{\rho \rightarrow 0} \left| \frac{d\nu}{d\sigma}(x) - \frac{\nu(B_\rho(x))}{\sigma(B_\rho(x))} \right| = 0.$$

Using Lemma 7.1 (i) we have for every  $\rho > 0$

$$\sup_{\varepsilon > 0} \left| \frac{m_\varepsilon\left(u, \frac{d\nu}{d\sigma}(x)\sigma; B_\rho(x)\right)}{\sigma(B_\rho(x))} - \frac{m_\varepsilon(u, \nu; B_\rho(x))}{\sigma(B_\rho(x))} \right| \leq C \left| \frac{d\nu}{d\sigma}(x) - \frac{\nu(B_\rho(x))}{\sigma(B_\rho(x))} \right|.$$

Passing to the limit  $\rho \rightarrow 0$ , the proof of (i) is complete. ■

The following lemma is used in the proof of Proposition 2.2 (i).

**Lemma 7.2.** *Let  $\sigma \in \mathcal{M}_+(\Omega)$ ,  $\nu \in \mathcal{M}(\Omega; \mathbb{R}^l)$ ,  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  and  $O \in \mathcal{O}(\Omega)$ . Let  $\{\nu_\varepsilon\}_\varepsilon \subset \mathcal{M}(\Omega; \mathbb{R}^l)$  be such that  $\nu_\varepsilon \xrightarrow{*} \nu$  in  $\mathcal{M}(O; \mathbb{R}^l)$  and  $|\nu_\varepsilon| \xrightarrow{*} \mu$  in  $\mathcal{M}_+(O)$  as  $\varepsilon \rightarrow 0$ . Let  $x \in O$  and  $\rho > 0$  with  $B_\rho(x) \subset O$ . Let  $t \in ]0, 1[$  and  $r \in [t^2\rho, t\rho[$ . Let  $\varphi_r \in \mathcal{C}_c(B_\rho(x); [0, 1])$  be such that  $\varphi_r \equiv 1$  in  $\overline{B_r(x)}$  and  $\varphi_r \equiv 0$  in  $\overline{B_\rho(x)} \setminus B_{t\rho}(x)$ . For each  $\varepsilon > 0$  we set  $\nu_\varepsilon^r = (\nu_{\varepsilon,1}^r, \dots, \nu_{\varepsilon,l}^r) \in \mathcal{M}(O; \mathbb{R}^l)$  defined by*

$$\nu_{\varepsilon,i}^r(B) := \int_B \varphi_r d\nu_{\varepsilon,i} \text{ for all Borel set } B \subset O \text{ where } \nu_\varepsilon = (\nu_{\varepsilon,1}, \dots, \nu_{\varepsilon,i}, \dots, \nu_{\varepsilon,l}).$$

Then

$$\begin{aligned} \overline{\lim}_{\varepsilon \rightarrow 0} \left| \frac{m_\varepsilon(u_x, \nu; B_\rho(x))}{\sigma(B_\rho(x))} - \frac{m_\varepsilon(u_x, \nu_\varepsilon^r; B_\rho(x))}{\sigma(B_\rho(x))} \right| \\ \leq 2C \left( \left( 1 - \frac{\sigma(B_{t^2\rho}(x))}{\sigma(B_\rho(x))} \right) \chi_{t^2}^\sigma(\rho) + \chi_1^\sigma(\rho) - \chi_{t^2}^\sigma(\rho) \right), \end{aligned}$$

where  $C > 0$  is the constant growth appearing in  $(H_1)$ , and where

$$\chi_s^\sigma(\rho) := \frac{|\nu|(B_{s\rho}(x))}{\sigma(B_{s\rho}(x))} \quad \text{for all } s \in ]0, 1].$$

The same conclusion holds replacing  $u_x$  by  $u_{x,\rho}$ .

*Proof.* We see that  $\nu_\varepsilon^r \xrightarrow{*} \nu^r$  in  $\mathcal{M}(O; \mathbb{R}^l)$  and  $|\nu_\varepsilon^r| \xrightarrow{*} \mu^r$  in  $\mathcal{M}_+(O)$  as  $\varepsilon \rightarrow 0$ , where  $\mu^r := \varphi_r \mu$  verifies  $\mu^r(\partial B_\rho(x)) = 0$  since the support of  $\varphi_r$  is include in  $\overline{B_{t\rho}}(x)$ . So, by [FL07, Corollary 1.204, pp. 131] we have  $\lim_{\varepsilon \rightarrow 0} \nu_\varepsilon^r(B_\rho(x)) = \nu^r(B_\rho(x))$ . Using Lemma 7.1, we have for every  $\varepsilon > 0$

$$\begin{aligned} & \left| \frac{m_\varepsilon(u_x, \nu; B_\rho(x))}{\sigma(B_\rho(x))} - \frac{m_\varepsilon(u_x, \nu_\varepsilon^r; B_\rho(x))}{\sigma(B_\rho(x))} \right| \\ & \leq C \left| \frac{\nu(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\nu_\varepsilon^r(B_\rho(x))}{\sigma(B_\rho(x))} \right| \\ & \leq C \left( \left| \frac{\nu(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\nu^r(B_\rho(x))}{\sigma(B_\rho(x))} \right| + \left| \frac{\nu^r(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\nu_\varepsilon^r(B_\rho(x))}{\sigma(B_\rho(x))} \right| \right). \end{aligned}$$

Passing to the limit  $\varepsilon \rightarrow 0$  we obtain

$$\overline{\lim}_{\varepsilon \rightarrow 0} \left| \frac{m_\varepsilon(u_x, \nu; B_\rho(x))}{\sigma(B_\rho(x))} - \frac{m_\varepsilon(u_x, \nu_\varepsilon^r; B_\rho(x))}{\sigma(B_\rho(x))} \right| \leq C \left| \frac{\nu(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\nu^r(B_\rho(x))}{\sigma(B_\rho(x))} \right|. \quad (7.2)$$

Since  $\nu(B_r) = \nu^r(B_r)$ ,  $|\nu^r| \leq |\nu|$  and  $B_r(x) \supset B_{t^2\rho}(x)$  it follows that

$$\begin{aligned} \left| \frac{\nu(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\nu^r(B_\rho(x))}{\sigma(B_\rho(x))} \right| &= \left| \frac{\nu(B_\rho(x) \setminus B_r)}{\sigma(B_\rho(x))} - \frac{\nu^r(B_\rho(x) \setminus B_r)}{\sigma(B_\rho(x))} \right| \\ &\leq 2 \frac{|\nu|(B_\rho(x) \setminus B_r)}{\sigma(B_\rho(x))} \\ &\leq 2 \frac{|\nu|(B_\rho(x) \setminus B_{t^2\rho}(x))}{\sigma(B_\rho(x))}. \end{aligned} \quad (7.3)$$

Moreover, we can write

$$\begin{aligned} \frac{|\nu|(B_\rho(x) \setminus B_{t^2\rho}(x))}{\sigma(B_\rho(x))} &= \frac{|\nu|(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{\sigma(B_{t^2\rho}(x))}{\sigma(B_\rho(x))} \frac{|\nu|(B_{t^2\rho}(x))}{\sigma(B_{t^2\rho}(x))} \\ &= \left( 1 - \frac{\sigma(B_{t^2\rho}(x))}{\sigma(B_\rho(x))} \right) \frac{|\nu|(B_{t^2\rho}(x))}{\sigma(B_{t^2\rho}(x))} + \frac{|\nu|(B_\rho(x))}{\sigma(B_\rho(x))} - \frac{|\nu|(B_{t^2\rho}(x))}{\sigma(B_{t^2\rho}(x))}. \end{aligned} \quad (7.4)$$

The assertion of the lemma follows by combining (7.2), (7.3) and (7.4).  $\blacksquare$

**7.2. Some results on measures.** The following result (see [FM93, Lemma 2.13, pp. 46] and [Alb93, Theorem 5.8, pp. 33]) is needed in the proof of Proposition 2.2.

**Lemma 7.3.** *Let  $\sigma$  be a nonnegative Radon measure on  $\Omega$ . For  $\sigma$ -a.e.  $x \in \Omega$  and for every  $t \in ]0, 1[$ , we have*

$$\overline{\lim}_{\rho \rightarrow 0} \frac{\sigma(B_{t\rho}(x))}{\sigma(B_\rho(x))} \geq t^N.$$

In the proof of Theorem 2.1, we need the following lemma which plays the same role for the singular part, as the  $L^p$ -differentiation theorem, with the mean value  $u_{x,\rho}$  replacing the affine tangent map  $u_x$  of  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ .

**Lemma 7.4.** *Let  $u \in W^{1,p}(\Omega; \mathbb{R}^m)$  and let  $\sigma \in \mathcal{M}_+(\Omega)$  be a positive Radon measure singular with respect to  $\mathcal{L}_N$ . Then*

$$\lim_{\rho \rightarrow 0} \frac{1}{\sigma(B_\rho(x))} \int_{B_\rho(x)} \left| \frac{u_{x,\rho} - u(z)}{\rho} \right|^p d\mathcal{L}_N(z) = 0 \quad \sigma\text{-a.e. in } \Omega.$$

*Proof.* Since  $\sigma \perp |\nabla u|^p \mathcal{L}_N$ , we have

$$\lim_{\rho \rightarrow 0} \frac{1}{\sigma(B_\rho(x))} \int_{B_\rho(x)} |\nabla u(z)|^p d\mathcal{L}_N(z) = 0 \quad \sigma\text{-a.e. in } \Omega. \quad (7.5)$$

Fix  $x \in \Omega$  satisfying (7.5). By Poincaré inequality there exists  $C > 0$  such that for every  $\rho > 0$  we have

$$\int_{B_\rho(x)} |u_{x,\rho} - u(z)|^p d\mathcal{L}_N(z) \leq C\rho^p \int_{B_\rho(x)} |\nabla u(z)|^p d\mathcal{L}_N(z).$$

Dividing by  $\rho^p \sigma(B_\rho(x))$  and letting  $\rho \rightarrow 0$ , we obtain the result. ■

The following result is used in the proof of Proposition 2.1 (ii) (see §4).

**Lemma 7.5.** *Let  $O \in \mathcal{O}(\Omega)$  and  $\nu \in \mathcal{M}(O; \mathbb{R}^l)$ . Let  $\{\delta(n)\}_{n \in \mathbb{N}} \subset ]0, 1[$  be a decreasing sequence satisfying  $\lim_{n \rightarrow \infty} \delta(n) = 0$ . For each  $n \in \mathbb{N}$  let  $\{B_i^n\}_{i \in I_n} \subset \mathcal{O}(O)$  be a countable family of pairwise disjoint open subsets with  $\text{diam}(B_i^n) \in ]0, \delta(n)[$  for all  $i \in I_n$ . Let  $\{\nu_i^n\}_{i \in I_n} \subset \mathcal{M}(O; \mathbb{R}^l)$  be such that  $\nu_i^n(B_i^n) = \nu(B_i^n)$  for all  $i \in I_n$ . Set  $\nu^n := \sum_{i \in I_n} \nu_i^n \mathbf{1}_{B_i^n}$ , i.e.*

$$\nu^n(B) = \sum_{i \in I_n} \nu_i^n(B_i^n \cap B) \quad \text{for all Borel set } B \subset O \text{ and all } n \in \mathbb{N}.$$

Assume that

- (i)  $\sum_{i \in I_n} (|\nu_i^n \mathbf{1}_{B_i^n}|(O)) < \infty$  for all  $n \in \mathbb{N}$ ;
- (ii)  $(\mathcal{L}_N + |\nu^s|) \left( O \setminus \bigcup_{i \in I_n} B_i^n \right) = 0$  for all  $n \in \mathbb{N}$ ;
- (iii) there exists  $\alpha \in \mathcal{M}_+(O)$  such that for every  $n \in \mathbb{N}$  and every open set  $U \subset O$

$$\alpha \ll \mathcal{L}_N + |\nu^s| \quad \text{and} \quad |\nu^n|(\mathcal{S}_n(U)) \leq \alpha(\mathcal{S}_n(U)),$$

$$\text{where } \mathcal{S}_n(U) := \bigcup \left\{ B_j^n : B_j^n \cap U \neq \emptyset \text{ and } B_j^n \cap O \setminus U \neq \emptyset \right\}.$$

Then  $\nu^n \xrightarrow{*} \nu$  in  $\mathcal{M}(O; \mathbb{R}^l)$ .

*Proof.* The assumption (i) insures that  $\nu^n \in \mathcal{M}(O; \mathbb{R}^l)$  for all  $n \in \mathbb{N}$ .

**Step 1.** In this first step we show that for every  $n \in \mathbb{N}$  and every open subset  $U \subset O$

$$\nu^n(U) - \nu(U) = \nu(U^n \setminus U) - \nu^n(U^n \setminus U). \quad (7.6)$$

where

$$U^n := \bigcup_{j \in J_n} B_j^n \quad \text{where} \quad J_n := \{i \in I_n : B_i^n \cap U \neq \emptyset\}.$$

Fix  $n \in \mathbb{N}$ . Let  $U \subset O$  be an open set. We have  $\nu^n(U \setminus U^n) = 0$  since  $\nu^n(U \cap U^n) = \nu^n(U)$ , indeed, we have

$$\nu^n(U^n \cap U) = \sum_{i \in I_n} \nu_i^n \left( B_i^n \cap \left( \bigcup_{j \in J_n} B_j^n \right) \cap U \right) = \sum_{j \in J_n} \nu_j^n (B_j^n \cap U) = \nu^n(U).$$

Moreover, we have

$$\nu^n(U^n) = \sum_{j \in J_n} \nu_j^n (B_j^n) = \sum_{j \in J_n} \nu (B_j^n) = \nu(U^n).$$

Thus, we can write, since  $U^n \cup (U \setminus U^n) = U \cup (U^n \setminus U)$ , that

$$\nu^n(U) = \nu(U) - \nu(U \setminus U^n) + \nu(U^n \setminus U) - \nu^n(U^n \setminus U). \quad (7.7)$$

But  $\nu(U \setminus U^n) = 0$ , indeed, we have  $U \setminus U^n = U \setminus \bigcup_{j \in I_n} B_j^n$  since  $B_j^n \cap U = \emptyset$  when  $j \notin J_n$ , and by using the fact that  $|\nu| \ll \mathcal{L}_N + |\nu^s|$ , we obtain

$$|\nu(U \setminus U^n)| \leq |\nu| \left( U \setminus \bigcup_{j \in I_n} B_j^n \right) \leq |\nu| \left( O \setminus \bigcup_{j \in I_n} B_j^n \right) = 0.$$

Thus, (7.7) becomes (7.6).

**Step 2.** Let  $f \in \mathcal{C}_c(O; \mathbb{R}^l)$ . We have to show that

$$\lim_{n \rightarrow \infty} \int_O f d\nu^n = \int_O f d\nu \quad \text{where} \quad \int_O f d\nu^n = \sum_{s=1}^l \int_O f_s d\nu_s^n \quad \text{for all } n \in \mathbb{N},$$

where  $f = (f_1, \dots, f_s, \dots, f_l)$ ,  $\nu^n = (\nu_1^n, \dots, \nu_s^n, \dots, \nu_l^n)$ ,  $f_s \in \mathcal{C}_c(O)$  and  $\nu_s^n \in \mathcal{M}(O; \mathbb{R})$  is a signed measure. Reasoning component by component we can assume that  $\nu^n$  is a signed measure for all  $n \in \mathbb{N}$  and  $f \in \mathcal{C}_c(O)$ . We have for every  $n \in \mathbb{N}$

$$\int_O f d\nu^n = \int_O f^+ d\nu^n - \int_O f^- d\nu^n$$

where  $f^+ = \max\{f, 0\}$  and  $f^- = \max\{-f, 0\}$ .

Set  $M := \max\{\sup_O f^+, \sup_O f^-\} < \infty$ . We have

$$\int_O f d\nu^n = \int_0^M \nu^n([f^+ > t]) dt - \int_0^M \nu^n([f^- > t]) dt. \quad (7.8)$$

Set  $U_t := [g > t] := \{x \in O : g(x) > t\}$  for all  $t \geq 0$  and all  $g \in \{f^+, f^-\}$ . To finish the proof we need to prove that

$$\lim_{n \rightarrow \infty} \int_0^M \nu^n(U_t) dt = \int_0^M \nu(U_t) dt.$$

By (7.6) we see that

$$\left| \int_0^M \nu^n(U_t) - \nu(U_t) dt \right| = \left| \int_0^M \nu(U_t^n \setminus U_t) dt - \int_0^M \nu^n(U_t^n \setminus U_t) dt \right|,$$

thus, it is sufficient to show that

$$\overline{\lim}_{n \rightarrow \infty} \left( \left| \int_0^M \nu(U_t^n \setminus U_t) dt \right| + \left| \int_0^M \nu^n(U_t^n \setminus U_t) dt \right| \right) = 0.$$

But, by (iii) we have for every  $n \in \mathbb{N}$

$$\left| \int_0^M \nu^n(U_t^n \setminus U_t) dt \right| \leq \int_0^M |\nu^n(\mathcal{S}_n(U_t))| dt \leq \int_0^M \alpha(\mathcal{S}_n(U_t)) dt.$$

Therefore we are reduced to prove that  $\overline{\lim}_{n \rightarrow \infty} \int_0^M \alpha(\mathcal{S}_n(U_t)) dt = 0$ .

**Step 3.** Fix  $n \in \mathbb{N}$ . Set  $K_n := \{j \in J_n : B_j^n \subset U_t\}$  and  $U_{t,n} := \cup_{j \in K_n} B_j^n$ . We have  $\mathcal{S}_n(U_t) = U_t^n \setminus U_{t,n}$ . Since  $\text{diam}(B_i^n) < \delta(n)$ , we have  $U_t^n \subset \mathcal{N}_{\delta(n)}[U_t] := \{x \in O : \text{dist}(x, U_t) < \delta(n)\}$ , thus  $\alpha(\mathcal{S}_n(U_t)) \leq \alpha(\mathcal{N}_{\delta(n)}[U_t] \setminus U_{t,n})$ . Now, we give an estimate from above of  $\alpha(\mathcal{N}_{\delta(n)}[U_t] \setminus U_{t,n})$  as  $n \rightarrow \infty$ . Since  $\alpha \ll \mathcal{L}_N + |\nu^s|$ , (ii) and  $U_t \subset O$ , we have  $\alpha(U_t \setminus \cup_{i \in I_n} B_i^n) = 0$ . But we can write

$$\begin{aligned} 0 &= \alpha \left( U_t \setminus \left( \bigcup_{i \in J_n} B_i^n \cup \bigcup_{i \in I_n \setminus J_n} B_i^n \right) \right) = \alpha \left( U_t \setminus \left( \left( \bigcup_{i \in J_n \setminus K_n} B_i^n \right) \cup U_{t,n} \right) \right) \\ &= \alpha((U_t \setminus U_{t,n}) \setminus F_n) \end{aligned}$$

where  $F_n := \cup_{i \in J_n \setminus K_n} B_i^n$ . Hence  $\alpha(U_t \setminus U_{t,n}) = \alpha((U_t \setminus U_{t,n}) \cap F_n)$ . Observe that  $F_n \subset \{x \in O : \text{dist}(x, \partial U_t) \leq 2\delta(n)\}$  which gives  $\overline{\lim}_{n \rightarrow \infty} \alpha(F_n) \leq \alpha(\partial U_t)$ . The sequence  $\{U_t \setminus U_{t,n}\}_{n \in \mathbb{N}}$  is decreasing thus  $\lim_{n \rightarrow \infty} \alpha(U_t \setminus U_{t,n}) \leq \alpha(\partial U_t)$ . In the same way, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \alpha(\mathcal{S}_n(U_t)) &\leq \lim_{n \rightarrow \infty} \alpha(\mathcal{N}_{\delta(n)}[U_t] \setminus U_{t,n}) \\ &= \overline{\lim}_{n \rightarrow \infty} \left\{ \alpha((\mathcal{N}_{\delta(n)}[U_t] \setminus U_t) \setminus U_{t,n}) + \alpha(U_t \setminus U_{t,n}) \right\} \leq 2\alpha(\partial U_t) \end{aligned}$$

since  $\{\mathcal{S}_n(U_t)\}_{n \in \mathbb{N}}$  is decreasing and  $\cap_{n \in \mathbb{N}} \mathcal{N}_{\delta(n)}[U_t] = \overline{U_t}$ . Now, by noticing that  $\partial U_t \subset [g = t]$  and  $\int_0^M \alpha([g = t]) dt = 0$  (since Fubini-Tonelli theorem), we apply the Lebesgue dominated convergence theorem to find

$$\lim_{n \rightarrow \infty} \int_0^M \alpha(\mathcal{S}_n(U_t)) dt = 0.$$

Repeating the same arguments with  $|\nu|$  in place of  $\alpha$ , we also have

$$\overline{\lim}_{n \rightarrow \infty} \left| \int_0^M \nu(U_t^n \setminus U_t) dt \right| = 0. \blacksquare$$

**7.3. Integral representation of the Vitali envelope of a set function.** This part is devoted to the integral representation of the Vitali envelope of a set function defined on open subsets of  $\Omega$ , it is partly inspired by [BB00, BFM98, DMM86]. Then we apply it to the set function  $m_+(u, \nu; \cdot) := \overline{\lim}_{\varepsilon \rightarrow 0} m_\varepsilon(u, \nu; \cdot)$ .

Let  $\mu \in \mathcal{M}_+(\Omega)$ . For each open set  $O \subset \Omega$ , we denote by  $\mathcal{Q}_o(O) \subset \mathcal{O}(O)$  the set of all open balls  $B$  of  $O$  such that their boundaries have zero measure, i.e.  $\mu(\partial B) = 0$ .

7.3.1. *Vitali envelopes of set functions.* Let  $G : \mathcal{Q}_o(\Omega) \rightarrow \mathbb{R}$  be a set function. We define the lower Vitali envelope of  $G$  with respect to  $\mu$

$$\mathcal{O}(\Omega) \ni O \longmapsto V_G^-(O) := \sup_{\varepsilon > 0} \inf \left\{ \sum_{i \in I} G(B_i) : \{B_i\}_{i \in I} \in \mathcal{V}^\varepsilon(O) \right\}$$

and the upper Vitali envelope with respect to  $\mu$

$$\mathcal{O}(\Omega) \ni O \longmapsto V_G^+(O) := \inf_{\varepsilon > 0} \sup \left\{ \sum_{i \in I} G(B_i) : \{B_i\}_{i \in I} \in \mathcal{V}^\varepsilon(O) \right\},$$

where for every  $\varepsilon > 0$

$$\mathcal{V}^\varepsilon(O) := \left\{ \{B_i\}_{i \in I} \subset \mathcal{Q}_o(\Omega) : I \text{ is countable, } \mu \left( O \setminus \bigcup_{i \in I} B_i \right) = 0, \bar{B}_i \subset O, \right. \\ \left. \text{diam}(B_i) \in ]0, \varepsilon[ \text{ and } \bar{B}_i \cap \bar{B}_j = \emptyset \text{ for all } i \neq j \right\}.$$

*Remarks 7.1.*

(i) We have  $-V_{-G}^- = V_G^+$ .

(ii) If  $G$  is the trace on  $\mathcal{Q}_o(\Omega)$  of a positive Borel measure  $\lambda$  on  $\Omega$  which is absolutely continuous with respect to  $\mu$  then  $V_G^\pm(O) = \lambda(O)$  for all  $O \in \mathcal{O}(\Omega)$ .

7.3.2. *Derivatives of set function.* Let  $G : \mathcal{Q}_o(\Omega) \rightarrow \mathbb{R}$  be a set function. Define the lower and the upper derivatives at  $x \in \Omega$  of  $G$  with respect to  $\mu$  as follows

$$D_\mu^- G(x) := \liminf_{\rho \rightarrow 0} \left\{ \frac{G(B)}{\mu(B)} : x \in B \in \mathcal{Q}_o(\Omega), \text{diam}(B) \leq \rho \right\}; \\ D_\mu^+ G(x) := \limsup_{\rho \rightarrow 0} \left\{ \frac{G(B)}{\mu(B)} : x \in B \in \mathcal{Q}_o(\Omega), \text{diam}(B) \leq \rho \right\}.$$

We say that  $G$  is  $\mu$ -differentiable in  $O \in \mathcal{O}(\Omega)$  if for  $\mu$ -a.e.  $x \in O$  it holds

$$-\infty < D_\mu^- G(x) = D_\mu^+ G(x) < \infty.$$

In this case we denote the common value by  $D_\mu G(x)$  and

$$D_\mu G(x) = \lim_{R_\mu^x \ni \rho \rightarrow 0} \frac{G(B_\rho(x))}{\mu(B_\rho(x))},$$

where we recall that  $R_\mu^x \subset ]0, \infty[$  is a countable set and  $\rho \notin R_\mu^x$  if and only if  $\mu(\partial B_\rho(x)) = 0$ .

*Remarks 7.2.*

(i) We have  $-D_\mu^-(-G) = D_\mu^+ G$ .

(ii) For every  $O \in \mathcal{O}(\Omega)$  and every  $x \in O$  we have

$$D_\mu^- G(x) := \liminf_{\rho \rightarrow 0} \left\{ \frac{G(B)}{\mu(B)} : x \in B \in \mathcal{Q}_o(O), \text{diam}(B) \leq \rho \right\}; \\ D_\mu^+ G(x) := \limsup_{\rho \rightarrow 0} \left\{ \frac{G(B)}{\mu(B)} : x \in B \in \mathcal{Q}_o(O), \text{diam}(B) \leq \rho \right\}.$$

(iii) If for each  $\rho > 0$  we set  $D_{\mu, \rho}^- G(x) := \inf \left\{ \frac{G(B)}{\mu(B)} : x \in B \in \mathcal{Q}_o(O), \text{diam}(B) \leq \rho \right\}$  for every  $x \in O \in \mathcal{O}(\Omega)$ , then it is not difficult to see that  $\{x \in O : D_{\mu, \rho}^- G(x) < c\}$  is open for all  $c \in \mathbb{R}$ . It follows that  $D_{\mu, \rho}^- G$  is measurable for all  $\rho > 0$ . Note that the

function  $\rho \mapsto D_{\mu,\rho}^- G(x)$  is nondecreasing for all  $x \in O$ . Thus  $D_\mu^- G$  is measurable since for every  $x \in O$

$$D_\mu^- G(x) = \lim_{\rho \rightarrow 0} D_{\mu,\rho}^- G(x).$$

The same conclusion holds for  $D_\mu^+ G$ .

(iv) If  $G = \lambda$  is a Borel finite measure absolutely continuous with respect to  $\mu$  then  $\lambda$  is  $\mu$ -differentiable in  $O$  and

$$D_\mu \lambda(x) = \frac{d\lambda}{d\mu}(x) \quad \mu\text{-a.e. in } O \quad \text{where} \quad \frac{d\lambda}{d\mu}(x) = \lim_{\rho \rightarrow 0} \frac{\lambda(B_\rho(x))}{\mu(B_\rho(x))} \quad \mu\text{-a.e. in } O.$$

The relation between lower and upper Vitali envelopes and lower and upper derivatives of a set function respectively, is given by the following result.

**Proposition 7.1.** *Let  $\mu \in \mathcal{M}_+(\Omega)$ . Let  $G : \mathcal{Q}_o(\Omega) \rightarrow \mathbb{R}$  be a set function satisfying*

( $\mathcal{E}_0$ ) *there exists  $\alpha \in \mathcal{M}_+(\Omega)$  with  $\alpha \ll \mu$  satisfying*

$$|G(B)| \leq \alpha(B) \quad \text{for all } B \in \mathcal{Q}_o(\Omega).$$

*Then for every  $O \in \mathcal{O}(\Omega)$  we have*

$$V_G^-(O) = \int_O D_\mu^- G(x) d\mu(x) \quad \text{and} \quad V_G^+(O) = \int_O D_\mu^+ G(x) d\mu(x).$$

7.3.3. *Integral representation of Vitali envelopes.* We consider the following two conditions on a set function  $G : \mathcal{O}(\Omega) \rightarrow \mathbb{R}$ :

( $\mathcal{E}_1$ ) *the set function  $G$  is dominated by a positive measure absolutely continuous with respect to  $\mu$ , i.e. there exists  $\alpha \in \mathcal{M}_+(\Omega)$  with  $\alpha \ll \mu$  satisfying*

$$|G(O)| \leq \alpha(O) \quad \text{for all } O \in \mathcal{O}(\Omega) \text{ with } \mu(\partial O) = 0;$$

( $\mathcal{E}_2$ ) *the set function  $G$  is subadditive, i.e. for every  $U, V, O \in \mathcal{O}(\Omega)$  with  $U \cap V = \emptyset$ ,  $U \subset O$ ,  $V \subset O$  and  $\mu(O \setminus (U \cup V)) = 0$  it holds*

$$G(O) \leq G(U) + G(V).$$

*Remark 7.1.* Under ( $\mathcal{E}_1$ ), since  $\alpha \ll \mu$ , we note that

$$\begin{aligned} -\frac{d\alpha}{d\mu}(x) &\leq D_\mu^- G(x) \leq D_\mu^+ G(x) \leq \frac{d\alpha}{d\mu}(x) \quad \mu\text{-a.e. in } \Omega \\ -\alpha(O) &\leq V_G^-(O) \leq V_G^+(O) \leq \alpha(O) \quad \text{for all } O \in \mathcal{O}(\Omega). \end{aligned}$$

It follows, taking Remark 7.2 (iii) into account, that  $D_\mu^- G$  and  $D_\mu^+ G$  belong to  $L_\mu^1(\Omega)$ .

Under the domination and subadditivity conditions, a set function defined on open sets is  $\mu$ -differentiable, and the lower and upper Vitali envelopes are equal and admit an integral representation with density its derivative.

**Theorem 7.1.** *Let  $\mu \in \mathcal{M}_+(\Omega)$ . Let  $G : \mathcal{O}(\Omega) \rightarrow \mathbb{R}$  be a set function satisfying ( $\mathcal{E}_1$ ) and ( $\mathcal{E}_2$ ). Then  $V_G^+(O) = V_G^-(O)$  for all  $O \in \mathcal{O}(\Omega)$ ,  $G$  is  $\mu$ -differentiable and for every  $O \in \mathcal{O}(\Omega)$*

$$V_G^+(O) = V_G^-(O) = \int_O \lim_{R_\mu \ni \rho \rightarrow 0} \frac{G(B_\rho(x))}{\mu(B_\rho(x))} d\mu(x). \quad (7.9)$$



**Application to  $G(\cdot) = m_+(u, \nu; \cdot)$  and  $\mu = \mathcal{L}_N + |\nu^s|$ .** For each pair  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  we consider  $V_{m_+(u, \nu; \cdot)}^-$  the lower Vitali envelope of

$$m_+(u, \nu; \cdot) := \overline{\lim}_{\varepsilon \rightarrow 0} m_\varepsilon(u, \nu; \cdot)$$

with respect to the measure  $\mu = \mathcal{L}_N + |\nu^s|$ .

Let us show that the set function  $m_+(u, \nu; \cdot) : \mathcal{O}(\Omega) \rightarrow \mathbb{R}_+$  is subadditive. Let  $(U_1, U_2, W) \in \mathcal{O}(\Omega)^3$  be such that  $U_1 \subset W$ ,  $U_2 \subset W$  with  $\mu(W \setminus (U_1 \cup U_2)) = 0$  and  $U_1 \cap U_2 = \emptyset$ . We can assume that  $m_+(u, \nu; U_1) < \infty$  and  $m_+(u, \nu; U_2) < \infty$ . So, there exists  $\varepsilon_0 > 0$  such that  $m_\varepsilon(u, \nu; U_i) < \infty$  for all  $\varepsilon \in ]0, \varepsilon_0[$  and all  $i \in \{1, 2\}$ . Let  $\varepsilon \in ]0, \varepsilon_0[$ . There exists  $(v_i, \lambda_i) \in (u + W_0^{1,p}(U_i; \mathbb{R}^m)) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  such that  $\lambda_i \ll \mathcal{L}_N$ ,  $\lambda_i(U_i) = \nu(U_i)$  and

$$\varepsilon + \sum_{i=1}^2 m_\varepsilon(u, \nu; U_i) \geq \sum_{i=1}^2 I_\varepsilon(v_i, \nu_i; U_i).$$

Set  $v_0 := \sum_{i=1}^2 v_i \mathbf{1}_{U_i} + u \mathbf{1}_{\Omega \setminus (U_1 \cup U_2)} \in u + W_0^{1,p}(U_1 \cup U_2; \mathbb{R}^m)$  and  $\lambda_0 := \sum_{i=1}^2 \lambda_i \mathbf{1}_{U_i}$ . We see that  $\lambda_0 \ll \mathcal{L}_N$  and  $\lambda_0(W) = \lambda_0(U_1 \cup U_2) = \lambda_1(U_1) + \lambda_2(U_2) = \nu(U_1 \cup U_2) = \nu(W)$  since  $\nu \ll \mu$ . Therefore we obtain

$$\varepsilon + \sum_{i=1}^2 m_\varepsilon(u, \nu; U_i) \geq I_\varepsilon(v_0, \lambda_0; W) \geq m_\varepsilon(u, \nu; W).$$

Passing to the limit  $\varepsilon \rightarrow 0$ , we obtain our claim.

Now, we assume that  $(H'_2)$  holds, i.e. for every  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  there exists  $\alpha_{u, \nu} \in \mathcal{M}_+(\Omega)$  with  $\alpha_{u, \nu} \ll \mu$  such that

$$\sup_{\varepsilon > 0} m_\varepsilon(u, \nu; U) \leq \alpha_{u, \nu}(U) \quad \text{for all } U \in \mathcal{O}(\Omega).$$

Whence we proved the following result.

**Lemma 7.6.** *Assume that  $(H'_2)$  holds. Then for every  $(u, \nu) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$  the set function  $m_+(u, \nu; \cdot)$  is  $\mu$ -differentiable and for every  $O \in \mathcal{O}(\Omega)$  we have*

$$V_{m_+(u, \nu; \cdot)}^-(O) = \sum_{\sigma \in \{\mathcal{L}_N, |\nu^s|\}} \int_O \lim_{R_\mu^x \ni \rho \rightarrow 0} \frac{m_+(u, \nu; B_\rho(x))}{\sigma(B_\rho(x))} d\sigma(x).$$

**7.4. Proof of Theorem 7.1.** We divide the proof into two steps.

**Step 1: A sufficient condition for the equality of the lower and upper Vitali envelopes.** In this first step, we establish a sufficient condition for the equality of the lower and upper Vitali envelopes. We claim that if

$$G(B) \leq V_G^-(B) \quad \text{for all } B \in \mathcal{Q}_o(\Omega),$$

then  $V_G^-(O) = V_G^+(O)$  for all  $O \in \mathcal{O}(\Omega)$ . Indeed, let  $O \in \mathcal{O}(\Omega)$  be an open set, and let  $\varepsilon > 0$ . There exists  $\{B_i\}_{i \in I} \subset \mathcal{V}^\varepsilon(O)$  such that

$$\sup \left\{ \sum_{i \in I} G(B_i) : \{B_i\}_{i \in I} \in \mathcal{V}^\varepsilon(O) \right\} \leq \sum_{i \in I} G(B_i) + \varepsilon.$$

Since the integral representation for  $V_G^-$  given by Proposition 7.1, we have

$$\sup \left\{ \sum_{i \in I} G(B_i) : \{B_i\}_{i \in I} \in \mathcal{V}^\varepsilon(O) \right\} \leq \sum_{i \in I} G(B_i) + \varepsilon \leq \sum_{i \in I} V_G^-(B_i) + \varepsilon = V_G^-(O) + \varepsilon,$$

which, by passing to the limit  $\varepsilon \rightarrow 0$ , gives  $V_G^+(O) \leq V_G^-(O)$ , and so the equality holds.

**Step 2: end of the proof.** Taking *Step 1* into account, we only have to show that  $G(B) \leq V_G^-(B)$  for all  $B \in \mathcal{Q}_o(\Omega)$ . Indeed, if we have  $V_G^-(O) = V_G^+(O)$  for all  $O \in \mathcal{O}(\Omega)$  then, by Proposition 7.1 and Remark 7.1,  $G$  is  $\mu$ -differentiable and the integral representation formula (7.9) holds.

Let  $\delta > 0$ . By the absolute continuity of  $\alpha$  with respect to  $\mu$ , there exists  $\eta_0$  such that for every Borel set  $A \subset B$  satisfying  $\mu(A) < \eta_0$  we have  $\alpha(A) < \frac{\delta}{2}$ .

Let  $B \in \mathcal{Q}_o(\Omega)$ . Let  $\varepsilon \in ]0, \eta_0[$ . There exists  $\{B_i\}_{i \in I} \in \mathcal{V}^\varepsilon(O)$  such that

$$\varepsilon + \inf \left\{ \sum_{i \in I} G(B_i) : \{B_i\}_{i \in I} \in \mathcal{V}^\varepsilon(B) \right\} \geq \sum_{i \in I} G(B_i).$$

There exists a finite subset  $I_\varepsilon \subset I$  such that  $\mu(B \setminus \cup_{i \in I_\varepsilon} B_i) < \varepsilon$ . We set  $W_\varepsilon := \cup_{i \in I_\varepsilon} B_i$ . Note that  $\mu(\partial W_\varepsilon) = 0$  and that  $\mu(\partial(B \setminus \overline{W_\varepsilon})) = 0$ . Using  $(\mathcal{E}_1)$  and  $(\mathcal{E}_2)$  we have

$$\begin{aligned} \varepsilon + \inf \left\{ \sum_{i \in I} G(B_i) : \{B_i\}_{i \in I} \in \mathcal{V}^\varepsilon(O) \right\} &\geq \sum_{i \in I} G(B_i) \\ &= \sum_{i \in I_\varepsilon} G(B_i) + \sum_{i \in I \setminus I_\varepsilon} G(B_i) \\ &\geq G(W_\varepsilon) - \alpha \left( \bigcup_{i \in I \setminus I_\varepsilon} B_i \right) \\ &\geq G(B) - G(B \setminus \overline{W_\varepsilon}) - \alpha \left( \bigcup_{i \in I \setminus I_\varepsilon} B_i \right) \\ &\geq G(B) - \alpha(B \setminus \overline{W_\varepsilon}) - \alpha \left( \bigcup_{i \in I \setminus I_\varepsilon} B_i \right). \end{aligned}$$

But,  $\alpha(\cup_{i \in I \setminus I_\varepsilon} B_i) = \alpha(B \setminus W_\varepsilon)$  and  $\alpha(B \setminus \overline{W_\varepsilon}) = \alpha(B \setminus W_\varepsilon)$  since  $\alpha \ll \mu$ . It follows that

$$\varepsilon + \inf \left\{ \sum_{i \in I} G(B_i) : \{B_i\}_{i \in I} \in \mathcal{V}^\varepsilon(B) \right\} \geq G(B) - 2\alpha(B \setminus W_\varepsilon) \geq G(B) - \delta.$$

Passing to the limits  $\varepsilon \rightarrow 0$  and  $\delta \rightarrow 0$ , we obtain  $V_G^-(B) \geq G(B)$  which completes the proof. ■

**7.5. Proof of Proposition 7.1.** The proof is based on the following lemma.

**Lemma 7.7.** *Let  $G : \mathcal{Q}_o(\Omega) \rightarrow \mathbb{R}$  be a set function satisfying  $(\mathcal{E}_0)$ . Let  $O \in \mathcal{O}(\Omega)$ . If  $D_\mu^- G(x) = 0$  (resp.  $D_\mu^+ G(x) = 0$ )  $\mu$ -a.e. in  $O$  then  $V_G^-(O) = 0$  (resp.  $V_G^+(O) = 0$ ).*

By using Remark 7.1 we see that for every  $\star \in \{+, -\}$

$$\int_\Omega |D_\mu^\star G(x)| d\mu(x) \leq \int_\Omega \left| \frac{d\alpha}{d\mu}(x) \right| d\mu(x) = \alpha(\Omega) < \infty.$$

Let  $\star \in \{+, -\}$  and  $O \in \mathcal{O}(\Omega)$ . Set  $H^\star := G - D_\mu^\star G(\cdot) \mu$ . It is sufficient to prove that  $D^\star H^\star(x) = 0$   $\mu$ -a.e. in  $O$ . Indeed, we observe that by  $(\mathcal{E}_0)$

$$|H^\star(B)| \leq |G(B)| + \left| \int_B D_\mu^\star G d\mu \right| \leq \alpha(B) + \int_B |D_\mu^\star G| d\mu \quad \text{for all } B \in \mathcal{Q}_o(\Omega),$$

that is  $H^\star$  satisfies  $(\mathcal{E}_0)$ . Applying Lemma 7.7, we get  $V_H^\star(O) = 0$ , but it is easy to verify that  $V_H^\star(O) = V_G^\star(O) - \int_O D_\mu^\star G d\mu$ .

Let  $x \in O$  be such that  $D_\mu^* G(x) = \lim_{\rho \rightarrow 0} \int_{B_\rho(x)} D_\mu^* G d\mu < \infty$ . For each  $\rho > 0$ , set

$$\mathcal{B}_{x,\rho}(O) := \{B \in \mathcal{Q}_o(O) : x \in B \in O, \text{diam}(B) \leq \rho\}.$$

We have for every  $\rho > 0$

$$\begin{aligned} \inf_{B \in \mathcal{B}_{x,\rho}(O)} \frac{G(B)}{\mu(B)} - \sup_{B \in \mathcal{B}_{x,\rho}(O)} \int_B D_\mu^- G d\mu &\leq \inf_{B \in \mathcal{B}_{x,\rho}(O)} \frac{H^-(B)}{\mu(B)} \\ &\leq \inf_{B \in \mathcal{B}_{x,\rho}(O)} \frac{G(B)}{\mu(B)} - \inf_{B \in \mathcal{B}_{x,\rho}(O)} \int_B D_\mu^- G d\mu, \end{aligned}$$

and

$$\begin{aligned} \sup_{B \in \mathcal{B}_{x,\rho}(O)} \frac{G(B)}{\mu(B)} - \sup_{B \in \mathcal{B}_{x,\rho}(O)} \int_B D_\mu^+ G d\mu &\leq \sup_{B \in \mathcal{B}_{x,\rho}(O)} \frac{H^+(B)}{\mu(B)} \\ &\leq \sup_{B \in \mathcal{B}_{x,\rho}(O)} \frac{G(B)}{\mu(B)} - \inf_{B \in \mathcal{B}_{x,\rho}(O)} \int_B D_\mu^+ G d\mu. \end{aligned}$$

It follows, by taking Remark 7.2 (iv) into account and by passing to the limit  $\rho \rightarrow 0$ , that  $D_\mu^* H(x) = 0$ . ■

7.5.1. *Proof of Lemma 7.7.* Since Remarks 7.1 (i) and 7.2 (i), we are reduced to prove the result only for the lower Vitali envelope.

**Step 1: we show that if  $D_\mu^- G(x) \leq 0$   $\mu$ -a.e. in  $O$  then  $V_G^-(O) \leq 0$ .** It is enough to show that for every  $\varepsilon > 0$  if

$$D_\mu^- G(x) < \varepsilon \quad \mu\text{-a.e. in } O \quad \text{then} \quad \inf_{\{B_i\}_{i \in I} \in \mathcal{V}^\varepsilon(O)} \sum_{i \in I} G(B_i) \leq \varepsilon \mu(O).$$

Fix  $\varepsilon > 0$ . Let  $N \subset O$  with  $\mu(N) = 0$  be such that  $O \setminus N = \{x \in O : D_\mu^- G(x) < \varepsilon\}$ . If  $x \in O \setminus N$  then for some  $\delta > 0$

$$\inf \left\{ \frac{G(B)}{\mu(B)} : B \in \mathcal{B}_{x,\rho}(O) \right\} < \varepsilon - \delta \quad \text{for all } \rho \in ]0, \varepsilon[.$$

For each  $\rho \in ]0, \varepsilon[$  there exists  $B_{x,\rho} \in \mathcal{B}_{x,\rho}(O)$  such that

$$\frac{G(B_{x,\rho})}{\mu(B_{x,\rho})} - \delta \leq \inf \left\{ \frac{G(B)}{\mu(B)} : B \in \mathcal{B}_{x,\rho}(O) \right\} < \varepsilon - \delta. \quad (7.10)$$

Consider the family  $\mathcal{K}_\varepsilon := \{\overline{B_{x,\rho}}\}_{x \in O \setminus N, \rho \in ]0, \varepsilon[}$  of closed balls such that (7.10) holds. The family  $\mathcal{K}_\varepsilon$  is a fine cover of  $O \setminus N$ , i.e.

$$O \setminus N \subset \bigcup_{B \in \mathcal{K}_\varepsilon} B \quad \text{and} \quad \inf \{\text{diam}(B) : B \in \mathcal{K}_{\varepsilon,x}\} = 0 \quad \text{for all } x \in O \setminus N$$

where  $\mathcal{K}_{\varepsilon,x} := \{\overline{B_{x,\rho}}\}_{\rho \in ]0, \varepsilon[} \subset \mathcal{K}_\varepsilon$ . By Vitali covering theorem, there exists a countable pairwise disjoint family  $\{\overline{B_i}\}_{i \in I} \subset \mathcal{K}_\varepsilon$  such that

$$\mu \left( (O \setminus N) \setminus \bigcup_{i \in I} \overline{B_i} \right) = 0, \quad G(B_i) < \varepsilon \mu(B_i) \quad \text{and} \quad \text{diam}(B_i) < \varepsilon \quad \text{for all } i \in I. \quad (7.11)$$

From (7.11) we have  $\mu(O \setminus \bigcup_{i \in I} \overline{B_i}) = 0$  since  $\mu(N) = 0$ . So, we have  $\sum_{i \in I} \mu(\overline{B_i}) = \sum_{i \in I} \mu(B_i) = \mu(O)$ . Summing over  $i \in I$  the first inequality in (7.11) we obtain

$$\inf \left\{ \sum_{i \in I} G(B_i) : \{B_i\}_{i \in I} \in \mathcal{V}^\varepsilon(O) \right\} \leq \varepsilon \sum_{i \in I} \mu(B_i) = \varepsilon \mu(O).$$

**Step 2: we show that if  $D_\mu^- G(x) \geq 0$   $\mu$ -a.e. in  $O$  then  $V_G^-(O) \geq 0$ .** Assume that there exists a measurable set  $N \subset O$  with  $\mu(N) = 0$  such that  $D_\mu^- G(x) \geq 0$  for all  $x \in O \setminus N$ .

Let  $\delta > 0$ . By the absolute continuity of  $\alpha$  with respect to  $\mu$  there exists  $\eta_0$  such that for every measurable set  $A \subset B$  satisfying  $\mu(A) < \eta_0$  we have  $\alpha(A) < \delta$ .

Let  $\eta \in ]0, \eta_0[$ . By Egoroff theorem, there exists a measurable  $K_\eta \subset O \setminus N$  such that  $\mu(O \setminus K_\eta) < \eta$ , and there exists  $\rho_0 > 0$  such that for every  $\rho \in ]0, \rho_0[$  we have

$$\inf \{ D_{\mu, \rho}^- G(x) : x \in K_\eta \} > -\eta \quad \text{where} \quad D_{\mu, \rho}^- G(x) = \inf_{B \in \mathcal{B}_{x, \rho}(O)} \frac{G(B)}{\mu(B)} \quad \text{for all } x \in O.$$

Let  $\rho \in ]0, \rho_0[$ . There exists  $\{B_i\}_{i \in I} \in \mathcal{V}^\rho(O)$  such that

$$V_G^-(O) + \rho \geq V_G^\rho(O) + \rho \geq \sum_{i \in I} G(B_i) \quad \text{where} \quad V_G^\rho(O) := \inf_{\{B_i\}_{i \in I} \in \mathcal{V}^\rho(O)} \sum_{i \in I} G(B_i).$$

Let  $I_\eta := \{i \in I : B_i \cap K_\eta \neq \emptyset\}$ . Choose  $x_i \in B_i \cap K_\eta$  for all  $i \in I_\eta$ . It follows that

$$\begin{aligned} V_G^-(O) + \rho &\geq \sum_{i \in I_\eta} G(B_i) + \sum_{i \in I \setminus I_\eta} G(B_i) = \sum_{i \in I_\eta} \frac{G(B_i)}{\mu(B_i)} \mu(B_i) - \alpha \left( \bigcup_{i \in I \setminus I_\eta} B_i \right) \\ &\geq \sum_{i \in I_\eta} D_{\mu, \rho}^- G(x_i) \mu(B_i) - \alpha(O \setminus K_\eta) \\ &\geq -\eta \mu(O) - \alpha(O \setminus K_\eta) \geq -\eta \mu(O) - \delta. \end{aligned}$$

By passing to the limits  $\rho \rightarrow 0$ ,  $\eta \rightarrow 0$  and  $\delta \rightarrow 0$  we obtain the result. ■

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