Stability results for a class of neutral functional differential equations. A Liapunov like analysis.

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STABILITY RESULTS FOR A CLASS OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS. A LIAPUNOV LIKE ANALYSIS

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This paper addresses the stability problem of a special class of dynamical systems of coupled differential and difference equations arising from the mathematical description of various engineering systems that contain lossless propagation media (pipes or electrical lines).

More explicitely, there are obtained sufficient stability conditions including delay information using a suitably chosen quadratic Liapunov functional of Liapunov-Krasovskii type under appropriate system transformation; at its turn this system transformation induces additional dynamics that will be also analyzed.
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1. INTRODUCTION

It is pointed out in the book of Hale and Verduyn Lunel [14] that neutral functional differential equations (NFDE) are met when dealing with oscillatory systems with some interconnections between them. The time for interaction is important: it is a straightforward way to speak about *propagation phenomena*. *Lossless propagation* is associated to transmission lines without losses; such lines correspond in engineering to LC electrical lines, or to lossless steam, water or gas pipes. Some examples with respect to this topics are to find in Hale and Lunel [14] as well as in the paper of Halanay and Răsvan [13].

In general, by *lossless propagation* it is understood the phenomenon associated with long transmission lines for (some) physical signals. In engineering, this problem is strongly related to *electric* and *electronic applications*, e.g. circuit structures consisting of multipoles connected through LC transmission lines (a long list of references may be provided, starting with a pioneering paper of Brayton [3] and going up to a quite recent book of Marinov and Neittaanmäki [18]). Some propagation phenomena may be also met in *power distribution* systems if the distribution area is quite large (see, e.g. Karaev [15]). We shall note that the lossless propagation occurs also for *non-electric* 'signals' as water, steam or gas flows and pressures.

The mathematical model is described in all these cases by a *mixed initial* and *boundary value problem* for hyperbolic partial differential equations modelling the lossless propagation. The boundary conditions are of special type being in "feedback connection" with some system described by ordinary differential equations.

This sends to the so-called "derivative boundary conditions" considered by Cooke [4] (see also Cooke and Krumme [5]) but also to the even more general boundary conditions of Abolinia and Myshkis [1], described by Volterra operators. Integration along characteristics of the hyperbolic partial differential equations (which is in fact the method of d'Alembert) mentioned in the cited references allows the association of a certain system of functional equations to the mixed problem; more precisely, a one-to-one correspondence may be established and proved between the solutions of the mixed problem for hyperbolic partial differential equations and the initial value (Cauchy) problem for the associated system of functional equations.

In certain cases, some of them considered in the above references, this system of functional differential equations reads as follows:

$$\text{(1)} \qquad \begin{cases} \dot{x}_1(t) &= Ax_1(t) + Bx_2(t-\tau) \\ & \qquad + f(x_1(t), x_2(t), x_2(t-\tau)) \\ x_2(t) &= Cx_1(t) + Dx_2(t-\tau) \\ & \qquad + g(x_1(t), x_2(t), x_2(t-\tau)), \end{cases}$$

which is a a differential equation coupled with a difference equation.

An earlier approach [22] suggested the treatment of (1) as a special case of neutral systems by letting $x_2(t) = \dot{z}(t)$. This last approach was used in the construction of a Popov-like theory in the input-output approach for absolute stability [22], forced nonlinear oscillations [11] and approximation by ordinary differential equations [12] (which "projected back" on the partial differential equations gave the method of lines).

All these considerations show that (1) represents a type of system that display a self-contained interest. Its linearized version is:

$$\text{(2)} \qquad \begin{cases} \dot{x}_1(t) = Ax_1(t) + Bx_2(t-\tau) \\ x_2(t) = Cx_1(t) + Dx_2(t-\tau), \end{cases}$$

where x_1 and x_2 describe the *differential* and *difference equations*, $\tau > 0$ is the delay, A, B, C and D are real matrices of appropriate dimensions and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ represents the vector of the state variables, $x \in \mathbb{R}^n$. Note that $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ $(n_1 + n_2 = n)$.

Furthermore, in the sequel we assume that the difference operator $\mathscr{D}(\phi) = \phi(0) - D\phi(-\tau)$ is stable, which is equivalent to the location of the eigenvalues of the matrix D inside the unit disk. This property guarantees the stability of the difference operator $\mathscr{D}$ for all positive delay values. At its turn stability of $\mathscr{D}$ for all delays ensures a "smooth" dependence of the qualitative properties of (2) with respect to the delays.

The paper extends the time-domain approach proposed in [21] to a more general framework (including various model transformations of the original system), and is organized as follows: system transformations are presented in section 2, and their corresponding additional eigenvalues characterization in Section 3. Sections 4 and 5 are devoted to the stability results and proof ideas. Various control interpretations are also included. Some concluding remarks end the paper. The notations are standard.

2. SYSTEM TRANSFORMATIONS

One of the methods largely used for delay-differential equations in deriving the so called *delay-dependent* stability (including information on the delay size) results is based on the Leibniz rule:

$$x_2(t-\tau) = x_2(t) - \int_{-\tau}^{0} \dot{x}_2(t+\theta)d\theta,$$

to transform the original system (2) to a distributed delay system of the form:

$$\left\{\begin{array}{l} \dot{x}_1(t) = Ax_1(t) + Bx_2(t) - B\int_{-\tau}^{0} \dot{x}_2(t+\theta)d\theta \\ x_2(t) = Cx_1(t) + Dx_2(t-\tau). \end{array}\right. \tag{3}$$

Since:

$$\int_{-\tau}^{0} \dot{x}_2(t+\theta)d\theta = C\int_{-\tau}^{0}[Ax_1(t+\theta) + Bx_2(t+\theta-\tau)]d\theta - D[x(t-\tau) - x(t-2\tau)],$$

the system (3) can be rewritten as follows:

$$\left\{\begin{array}{lcl} \dot{x}_1(t) & = & Ax_1(t) + Bx_2(t) - BD[x_2(t-\tau) - x_2(t-2\tau)] \\ & & -BCA\int_{-\tau}^{0} x_1(t+\theta)d\theta - BCB\int_{-2\tau}^{-\tau} x_2(t+\theta)d\theta \\ x_2(t) & = & Cx_1(t) + Dx_2(t-\tau). \end{array}\right. \tag{4}$$

Such a process is generically called *model(or system) transformation*.Note that the model transformations are largely used in the retarded case (i.e. for FDE of delayed type) [8], [9] (and the references therein) for deriving *delay-dependent* stability results. For the neutral case (and let us remember that (2) *is* of neutral type, as previously specified) the results are not so numerous, the method being not sufficiently exploited; a quite recent reference is [20] where the above transformation is called *fixed first order*.

Consider now a slightly different version of the above procedure, where the method above will by applied *not* for the *whole* delayed state $x_2(t-\tau)$, but only for some "part" of it. Let $M \in \mathbb{R}^{n_1 \times n_2}$ be a real matrix, and apply the same procedure as above, but *only* for $Mx_2(t-\tau)$. Then, the system (3) rewrites as follows:

$$\left\{\begin{array}{lcl} \dot{x}_1(t) & = & Ax_1(t) + Mx_2(t) + (B-M)x_2(t-\tau) \\ & & -MD[x_2(t-\tau) - x_2(t-2\tau)] \\ & & -MCA\int_{-\tau}^{0} x_1(t+\theta)d\theta - MCB\int_{-2\tau}^{-\tau} x_2(t+\theta)d\theta \\ x_2(t) & = & Cx_1(t) + Dx_2(t-\tau). \end{array}\right. \tag{5}$$

It is clear that if one takes $M = B$ in (5), we shall recover the previous model transformation (4), and if $M = 0$, (5) reduces to the original system.

This second transformation is called a *parameterized (first-order) model transformation*. The advantage in using (5) will be presented in the sequel and consists in inducing a further degree of freedom in the model (the matrix M), that may lead to an appropriate *control problem*, as seen below.

3. RELATIONS BETWEEN THE TWO SYSTEMS. THE ADDITIONAL EIGENVALUES

A Let us focus on the two systems (2) and (5). If they are considered as such, then, as already known from other reference concerning systems of this kind (i.e. defined by coupled delay differential and difference equations or, as called sometimes, by functional differential and algebraic equations), system (2) may define a dynamical system on e.g. $\mathbb{R}^{n_1} \times \mathbf{L}^p(-\tau, 0; \mathbb{R}^{n_2})$ while (5) may define a dynamical system e.g. on $\mathbb{R}^{n_1} \times \mathbf{L}^p(-2\tau, 0; \mathbb{R}^{n_2})$ since, generally speaking x_2 may display discontinuities that propagate. On the other hand one cannot leave aside the fact that (5) is merely an instrument for obtaining stability results concerning (2) hence making a connection between the two systems is necessary; at its turn this connection strongly relies on the Leibniz rule. Consequently x_2 has to be at least absolutely continuous and especially continuous at 0 : this would require the initial condition for (2) to be in $\mathbb{R}^{n_1} \times \mathbf{W}_p^{(1)}(-\tau, 0; \mathbb{R}^{n_2})$ and satisfy the condition of the continuity at 0

$$\phi_2(0) = Cx_1^0 + D\phi_2(-\tau) \tag{6}$$

where (x_1^0, ϕ_2) is the initial condition for (2).

If we start from the solutions of (5) then other restrictions on system's initial conditions may be necessary in order to establish a connection between the two systems. We may actually state and prove the following result

THEOREM 3.1 *Consider system (2) and let $(x_1^\phi(t), x_2^\phi(t))$ be a solution defined by the initial condition $(x_1^0, \phi_2) \in \mathbb{R}^{n_1} \times \mathbf{W}_2^{(1)}(-\tau, 0; \mathbb{R}^{n_2})$ satisfying (6). Then the functions $(z_1^\psi(t), z_2^\psi(t))$ defined on $t > 0$ by*

$$z_i^\psi(t) \equiv x_i^\phi(t+\tau)\ ,\ i = 1,2 \tag{7}$$

are a solution of (5) with the initial condition defined by

$$\left\{ \begin{array}{rcl} \psi_1(\theta) & = & e^{A(\tau+\theta)}x_1^0 + \int_{-\tau}^{\theta} e^{A(\theta-\lambda)}B\phi_2(\lambda)d\lambda\ ,\ -\tau \le \theta \le 0 \\ \psi_2(\theta) & = & \left\{ \begin{array}{c} \phi_2(\tau+\theta)\ ,\ -2\tau \le \theta \le -\tau \\ C\psi_1(\theta) + D\phi_2(\theta)\ ,\ -\tau \le \theta \le 0 \end{array} \right. \end{array} \right. \tag{8}$$

Conversely, consider system (5) and let $(z_1^\psi(t), z_2^\psi(t))$ be a solution of it defined by the initial condition $(\psi_1, \psi_2) \in \mathbf{W}_2^{(1)}(-\tau, 0; \mathbb{R}^{n_1}) \times \mathbf{W}_2^{(1)}(-2\tau, 0; \mathbb{R}^{n_2})$. If these initial conditions are subject, additionally, to the following conditions on $[-\tau, 0]$

$$
(9) \qquad \begin{cases} \dot{\psi}_1 & \equiv & A\psi_1(\theta) + B\psi_2(\theta - \tau) \\ \psi_2(\theta) & \equiv & C\psi_1(\theta) + D\psi_2(\theta - \tau) \end{cases}
$$

i.e. they satisfy (2) on $[-\tau, 0]$ *then* $(z_1^{\psi}(t), z_2^{\psi}(t))$ *is also a solution of (2) on* $t > 0$.

The proof of this theorem is based on straightforward computation and is reported to the APPENDIX. Here we shall only give some comments on its significance. It is not difficult to see that for each initial condition of (2) the solution is uniquely defined, being constructed by the method of the steps; therefore (7) and (8) define in a unique way a solution of (5) together with its initial condition hence we could say that *any solution of (2) generates a solution of (5)*. The converse is no longer true: if we consider a solution of (5) defined by some initial condition, on $[-\tau, 0]$ for the first component and on $[-2\tau, 0]$ for the second one, this will not be, generally speaking, a solution of (2). Only if we require this property *for the initial conditions* on $[-\tau, 0]$ then the solution of (5) will be also a solution of (2).

But this fact is rather easy to explain: the initial conditions of (5) are arbitrary on the intervals $[-\tau, 0]$ and $[-2\tau, 0]$ and we would like the solution defined by them to coincide with a solution of (2) which is defined only by some initial condition on $[-\tau, 0]$ (moreover its first component is defined by a pointwise initial condition); we have consequently to restrict from the beginning the class of the initial conditions by making them to verify (2) on $[-\tau, 0]$. Therefore system (2) will be verified on $t > -\tau$ what also looks quite natural if we think about an "inversion" of (7) and (8). We shall comment more on this feature after discussing some spectral issues of the two systems.

B As pointed out in [20] (both for retarded and neutral cases), the "difference" between the dynamical behaviors of the transformed systems with respect to the original system can be explained by the corresponding *additional eigenvalues* induced by the (fixed or parameterized) transformation under consideration.

In order to analyze these additional eigenvalues, let us focus on the roots of the characteristic equations associated to (2) and (5).

Thus, we have:

$$
(10) \qquad \Delta_o(s) = \mathbf{det} \begin{bmatrix} sI_{n_1} - A & -Be^{-s\tau} \\ -C & I_{n_2} - De^{-s\tau} \end{bmatrix}
$$

for the basic system (2), and

$$
(11) \qquad \Delta_t(s) = \mathbf{det} \begin{bmatrix} sI_{n_1} - A + MCA\frac{1-e^{-s\tau}}{s} & -Q_t(s) \\ -C & I_{n_2} - De^{-s\tau} \end{bmatrix},
$$

with:

$$
Q_t(s) = M\left(I_{n_2} - De^{-s\tau} + De^{-2s\tau} - CBe^{-s\tau}\frac{1-e^{-s\tau}}{s}\right) + (B - M)e^{-s\tau},
$$

for the parameterized transformed model (5). (The case (4) is recovered by taking $M = B$)

After some simple manipulation we obtain that:

$$\Delta_t(s) = \mathbf{det}\left[I_{n_1} - MC\frac{1-e^{-s\tau}}{s}\right]\cdot\Delta_o(s). \tag{12}$$

Since the second model transformation includes the first one as a particular case, the results below are directly derived for the parameterized model transformation case. Based on (12), we have:

PROPOSITION 3.1 [Additional eigenvalues] *Let $s = s_{ik}$, $k = 1,2,3,...$ be all the solutions of the equation*

$$1 - \lambda_i(MC)\frac{1-e^{-\tau s}}{s} = 0, \tag{13}$$

where $\lambda_i(MC)$, is the ith eigenvalue of matrix MC. Then s_{ik}, $i = 1,2,...,n_1$; $k = 1,2,3,...$ are all the additional eigenvalues of system (5).

The complete set of eigenvalues of (5) consists of the solutions of (13), and the eigenvalues of the original system (2), which are the solutions of $\Delta_0(s) = 0$.

If $M = B$, one recovers the fixed first-order model transformation (4).

From here we may obtain an additional explanation of the restrictions put on the initial conditions of (5) in order that the corresponding solutions be also solutions of (2). Any Euler (exponential) solution of (5) or any linear combination of Euler solutions of (5) is a solution of (2) if it corresponds to eigenvalues of (2); since such solutions are analytic they have to satisfy (2) on some non-zero interval e.g. the interval $[-\tau, 0]$.

C The relationship between the solutions of the two systems as well as those between their characteristic equations give some insight about stability problems: clearly stability of (5) implies stability of (2) and every zero of $\Delta_0(s)$ is also a zero of $\Delta_t(s)$. The entire function $\Delta_t(s)$ may have additional zeros which do not necessarily have negative real parts. When the basic system (2) is (exponentially) stable these additional zeros, if any, are responsible for the instability of the transformed system (5) (these remarks are in the spirit of [16]).

Consider now the problem of the additional eigenvalues defined by (13). Since MC has a finite number of eigenvalue, there is a finite number of additional eigenvalues chains for (5). Our discussion will now follow the line of [16]. Let $\lambda_i(MC)$ be some eigenvalue of MC. If we consider equation (13) then it is easily seen that its roots $s_{ik} = \sigma_{ik} + \imath\omega_{ik}$ are determined from the two equations below

$$\begin{cases} \sigma_{ik} & - \quad \mu_i(1-e^{-\sigma_{ik}\tau}\cos\omega_{ik}\tau) - \nu_i e^{-\sigma_{ik}\tau}\sin\omega_{ik}\tau = 0 \\ \omega_{ik} & + \quad \mu_i e^{-\sigma_{ik}\tau}\sin\omega_{ik}\tau - \nu_i(1-e^{-\sigma_{ik}\tau}\cos\omega_{ik}\tau) = 0 \end{cases} \tag{14}$$

Our main concern is, as mentioned previously, the sign of σ_{ik} according to the properties of λ_i. Since $\tau > 0$ we may multiply the above equations by τ to obtain the equations

$$\begin{cases} x & - \quad \mu_i\tau(1-e^{-x}\cos y) - \nu_i\tau e^{-x}\sin y = 0 \\ y & + \quad \mu_i\tau e^{-x}\sin y - \nu_i\tau(1-e^{-x}\cos y) = 0 \end{cases} \tag{15}$$

The problem now reduces to that of discussing the sign of x_{ik}, where (x_{ik}, y_{ik}) are solutions of the system (15), according to the properties of the complex/real parameter $z_i = \lambda_i \tau$. The result reads as follows (according to [16], but in a slightly different form)

PROPOSITION 3.2 *If $z_i \in \mathbb{C}$ belongs to the open convex domain delimited by the curve*

$$\Re(z) = (\Im(z))\tan(\Im(z)) \ , \ -\pi < \Im(z) < \pi \tag{16}$$

and containing the origin of the corresponding complex plane, then all *the roots (x_{ik}, y_{ik}) of (15) are such that $x_{ik} < 0$*

Outline of proof The proof relies on what is called in Control *D-Decomposition* with respect to a (complex) parameter. For quasi-polynomials, as well as for polynomials, the technique is due to Iu. I. Neimark [19]. It strongly relies on the continuous dependence of the roots of a (quasi-)polynomial of its coefficients, as well as on the elementary fact that, when "moving" from the LHP (Left Half-plane) of $\mathbb{C}$ to the RHP (Right Half-plane) of $\mathbb{C}$, any root has to cross the imaginary axis $\imath\mathbb{R}$. For these reasons we need two pre-requisites: a) to find conditions for the crossing of $\imath\mathbb{R}$ i.e. for the existence of the roots $(0, y_{ik})$ of (15) and b) to find at least one case when the condition of the proposition is satisfied. The first question is solved by considering the version of (15) for $x = 0$ namely

$$\begin{cases} \mu_i\tau(1-\cos y) + \nu_i\tau\sin y = 0 \\ y + \mu_i\tau\sin y - \nu_i\tau(1-\cos y) = 0 \end{cases} \tag{17}$$

These equations define a compatible system having real y as solutions provided $\mu_i\tau = \nu_i\tau\cot\nu_i\tau$. In the plane of the complex variables $z_i = \mu_i\tau + \imath\nu_i\tau$ this is a family of curves Γ_k defined for $\Im z \in (-k\pi, -(k-1)\pi) \cup ((k-1)\pi, k\pi)$, $k = 1, 2, 3, \ldots$. If D_k is the domain delimited by Γ_k and containing the origin, then $D_1 \subset D_2 \subset \ldots \subset D_k \subset \ldots$.

Inside of Γ_1 there are points corresponding to the conditions of the Proposition. Indeed, if $\nu_i = 0$ in (15) then a direct computation shows that there are no solutions with $x > 0$ provided $\mu_i\tau < 1$ (i.e. for all $\mu_i < 0$ as well as for $0 < \mu_i\tau < 1$. The proposition is thus proved since it may be shown by direct check that if $z_i \in D_k$, $k > 1$ there are always some roots of (15) with $x > 0$. It appears that each time when z crosses a curve Γ_k from "inside" to "outside" (leaving D_k and entering D_{k+1}), at least one root of (13) crosses $\imath\mathbb{R}$ (this direct check may be performed e.g. for real z_i with $\Re(z_i) > 1$).

Some remarks are necessary. First of all, the result of this proposition completes (but not competes with) Theorem 1 of [9]. Second, the straightforward "small delay" condition for (13) to have its roots in $\mathbb{C}$ namely $\|MC\|\tau < 1$ where $\|MC\|$ is some matrix norm, is *rather conservative* i.e. far from the necessary and sufficient conditions. Regardless the simple fact that this estimate is norm-dependent, assume for a while that we have taken the spectral norm i.e. the modulus of the largest eigenvalue. Proposition 3.1 shows that in this case $\|MC\|\tau < 1$ is non-conservative if all eigenvalues are real but if there are complex eigenvalues then D_1 contains points of modulus larger than 1.

In fact the results above give the limitations of the model transformation method for deriving *delay-dependent* stability results.

It is clear that if the basic and transformed systems are such that no additional eigenvalues in $\mathbb{C}^+$ appear, the delay bound derived using the Liapunov-Krasovskii approach only will give the *conservatism* of the method.

Further remarks in the retarded case can be found in [8], [9]. Note also that the same ideas (model transformation construction, additional eigenvalues characterization) hold in the ('standard') neutral case (C invertible) as it has been proved in [20].

4. STABILITY FOR SMALL DELAYS

It is a well established fact that many applied research dealing with stability of time delay systems is concerned with what is contained in a happily coined contribution title due to J. Kurzweil [17]. In our case this means that it is supposed that the system without delays is exponentially stable and, viewing the time delays terms as some kind of perturbation, it is checked the first strictly positive delay value for which stability is lost. With respect to parameter uncertainty the sharpest approach would probably be application of bifurcation theory. Nevertheless in many cases there is considered another approach: instead of seeking for the first delay corresponding to stability loss, it is considered the finding of the delay that still corresponds to an exponentially stable system, with the additional condition of computational feasibility *via* a commercially available software.

The most popular approach is that of the construction of simple quadratic functionals (called Liapunov-Krasovskii functionals) whose sign conditions (usually for their derivatives along system's solutions) are expressed by *Linear Matrix Inequalities (LMI)*. Dozens (if not hundreds) of papers on this subject are published each year in the scientific literature; the reader is sent to [20] for the state of the art.

The same approach is taken here but for system (2) which is, as already pointed out, somehow different and is based on a sound motivation. It is necessary also to mention that the above discussed system transformation is used mainly in order to obtain the so-called *delay-dependent stability conditions*, actually such conditions allowing to obtain the best possible estimate of the delay for which exponential stability still holds. The Liapunov functional is thus constructed looking at the transformed system; its derivative is also computed along the solutions of *this system* which is shown to be exponentially stable; consequently the basic system results exponentially stable; as a by-product, exponential stability of the additional dynamics system is obtained; as pointed out in [16], this system is in fact described by

$$z(t) = MC\int_{-\tau}^{0} z(t+\theta)d\theta \tag{18}$$

A We shall state first a general stability result based on Liapunov functionals, for the transformed system (5)

THEOREM 4.1 *If there exist positive definite matrices $P > 0$, $S_i > 0$, $i = 1,2,3$, of appropriate dimensions such that the following Linear Matrix Inequality holds*

$$
(19) \qquad \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{pmatrix} \quad < \quad 0,
$$

where

$$
H_{11} = \begin{pmatrix} \begin{array}{c}(A+MC)^T P + P(A+MC) \\ +C^T(S_1+\tau S_4)C + \tau S_3\end{array} & -PMCA & -PMCB \\ -A^T C^T M^T P & -\tau^{-1} S_3 & 0 \\ -B^T C^T M^T P & 0 & -\tau^{-1} S_4 \end{pmatrix},
$$

$$
H_{12} = \begin{pmatrix} C^T(\tau S_4 + S_1)D + P(B-M) & PMD \\ 0 & 0 \\ 0 & 0 \end{pmatrix},
$$

$$
H_{22} = \begin{pmatrix} D^T(\tau S_4 + S_1)D + S_2 - S_1 & 0 \\ 0 & -S_2 \end{pmatrix},
$$

then system (5) is exponentially stable for all delays $\tau > 0$ *and all matrices* M *such that (19) holds.*

Proof The theorem is a standard Liapunov-like result and the proof is such. Using the matrices P, S_i from the statement, we define the following Liapunov-Krasovskii functional on $\mathscr{C}(-\tau,0;\mathbb{R}^{n_1}) \times \mathscr{C}(-2\tau,0;\mathbb{R}^{n_2})$

$$
\begin{aligned}
V(\psi_1,\psi_2) &= \psi_1(0)^T P \psi_1(0) + \int_{-\tau}^{0} \psi_2(\theta)^T S_1 \psi_2(\theta) d\theta \\
&+ \int_{-2\tau}^{-\tau} \psi_2(\theta)^T S_2 \psi_2(\theta) d\theta + \int_{-\tau}^{0} \left(\int_{\theta}^{0} \psi_1(\lambda)^T S_3 \psi_1(\lambda) d\lambda \right) d\theta \\
&+ \int_{-2\tau}^{-\tau} \left(\int_{\theta}^{0} \psi_2(\lambda)^T S_4 \psi_2(\lambda) d\lambda \right) d\theta. \qquad (20)
\end{aligned}
$$

We may differentiate $V(x_{1t},x_{2t})$ along the solutions of (5) and find, after standard manipulation that is not reproduced here, the following derivative functional on the above mentioned state space

$$
\begin{aligned}
W(\psi_1,\psi_2) = \psi_1(0)^T &(P(A+MC) + (A+MC)^T P + C^T(S_1+\tau S_4)C + \tau S_3)\psi_1(0) \\
&- \psi_1(0)^T PMCA \int_{-\tau}^{0} \psi_1(\theta) d\theta - \psi_1(0)^T PMCB \int_{-2\tau}^{-\tau} \psi_2(\theta) d\theta \\
&- \left(\int_{-\tau}^{0} \psi_1^T(\theta) d\theta \right) A^T C^T M^T P \psi_1(0) - \left(\int_{-2\tau}^{-\tau} \psi_2^T(\theta) d\theta \right) B^T C^T M^T P \psi_1(0)
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{-\tau}^{0}\psi_1^T(\theta)S_3\psi_1(\theta)d\theta-\int_{-2\tau}^{-\tau}\psi_2^T(\theta)S_4\psi_2(\theta)d\theta\\
&+\psi_1^T(0)(P(B-M)+C^T(S_1+\tau S_4)D)\psi_2(-\tau)+\psi_1^T(0)PMD\psi_2(-2\tau)\\
&+\psi_2^T(-\tau)((B-M)^TP+D^T(S_1+\tau S_4)C)\psi_1(0)+\psi_2^T(-2\tau)D^TM^TP\psi_1(0)\\
&+\psi_2^T(-\tau)(S_2-S_1+D^T(S_1+\tau S_4)D)\psi_2(-\tau)-\psi_2^T(-2\tau)S_2\psi_2(-2\tau)
\end{aligned}
\tag{21}
$$

We may recognize here a finite dimensional quadratic form with respect to the following five vector arguments

$$
\psi_1(0)\ ,\ \int_{-\tau}^{0}\psi_1(\theta)d\theta\ ,\ \int_{-2\tau}^{-\tau}\psi_2(\theta)d\theta\ ,\ \psi_2(-\tau)\ ,\ \psi_2(-2\tau)
$$

except two quadratic integrals. These quadratic integrals are estimated using a suitable extension of the inequality of Jensen [7] to obtain

$$
\begin{aligned}
\tau\int_{-\tau}^{0}\psi_1^T(\theta)S_3\psi_1(\theta)d\theta &\geq \left(\int_{-\tau}^{0}\psi_1^T(\theta)d\theta\right)S_3\left(\int_{-\tau}^{0}\psi_1(\theta)d\theta\right)\\
\tau\int_{-2\tau}^{-\tau}\psi_2^T(\theta)S_4\psi_2(\theta)d\theta &\geq \left(\int_{-2\tau}^{-\tau}\psi_2^T(\theta)d\theta\right)S_4\left(\int_{-2\tau}^{-\tau}\psi_2(\theta)d\theta\right)
\end{aligned}
\tag{22}
$$

In this way $W(\psi_1,\psi_2)$ is evaluated by a quadratic form with respect to the above described arguments

$$
W(\psi_1,\psi_2)\leq\mathscr{H}\left(\psi_1(0)\ ,\ \int_{-\tau}^{0}\psi_1(\theta)d\theta\ ,\ \int_{-2\tau}^{-\tau}\psi_2(\theta)d\theta\ ,\ \psi_2(-\tau)\ ,\ \psi_2(-2\tau)\right)
$$

the matrix of this quadratic form being the negative definite matrix of (19).

Any standard Liapunov-like theorem will give now asymptotic stability. This stability is even exponential and this follows from various arguments, all of them being quite standard in Liapunov theory for ordinary or functional differential equations [10]. For instance, if we use the properties of the quadratic forms, we may obtain the inequality

$$
\frac{d}{dt}V(x_{1t},x_{2t})\leq\gamma V(x_{1t},x_{2t})\leq 0 \tag{23}
$$

for some $\gamma>0$ to get exponential decrease to 0 of the solution. But we may just use the simple fact that our system is linear, with constant coefficients and its asymptotic behavior is a consequence of the solutions' structure imposed by the roots of some characteristic equation - the exponential (Euler) solutions being the eigenfunctions of the corresponding linear operator associated to the equation. A third argument might be given by the Persidskii type result stating that for general linear systems uniform asymptotic stability is but exponential. The proof is thus completed.

B We turn now back to the case of the small delays: let $\tau>0$ approach 0. For $\tau=0$ the transformation that associates (5) to (2) becomes trivial hence the two systems should coincide: this is obvious from visual inspection. We obtain the system

$$(24) \qquad \begin{cases} \dot{x}_1 = Ax_1 + Bx_2 \\ (I-D)x_2 = Cx_1, \end{cases}$$

and, assuming $\mathbf{det}(I-D) \neq 0$ (which will turn later to be automatically fulfilled due to a stronger but nonetheless necessary assumption), the linear system of ordinary differential equations is obtained

$$(25) \qquad \dot{x}_1 = (A + B(I-D)^{-1}C)x_1$$

Let now $\tau \to 0$ in the expression of the Liapunov functional: as expected, only the pointwise part matters hence, with a slight notation abuse, we associate to (25) the quadratic Liapunov function $V(x_1) = x_1^T P x_1$ with $P > 0$. Consider now the derivative functional $W(\psi_1, \psi_2)$ from (21) and let $\tau \to 0$; clearly the integrals are $O(\tau)$ hence we obtain

$$\begin{aligned} W(\psi_1,\psi_2) = {} & \psi_1(0)^T (P(A+MC) + (A+MC)^T P + C^T S_1 C)\psi_1(0) \\ & + \psi_1^T(0)(P(B-M) + C^T S_1 D)\psi_2(-\tau) + \psi_1^T(0) PMD\psi_2(-2\tau) \\ & + \psi_2^T(-\tau)((B-M)^T P + D^T S_1 C)\psi_1(0) + \psi_2^T(-2\tau) D^T M^T P \psi_1(0) \\ & + \psi_2^T(-\tau)(S_2 - S_1 + D^T S_1 D)\psi_2(-\tau) - \psi_2^T(-2\tau) S_2 \psi_2(-2\tau) \end{aligned}$$

This expression may be still simplified since for $\tau = 0$ we take $\psi_2(-\tau) = \psi_2(-2\tau) = \psi_2(0)$ as shown by the degenerate transformation and by the degenerate system (24). Using new notation we obtain, after some manipulation

$$W(x_1,x_2) = \widehat{W}(x_1) = x_1^T (P(A + B(I-D)^{-1}C) + (A + B(I-D)^{-1}C)^T P) x_1$$

that is exactly the derivative function of $V(x_1) = x_1^T P x_1$ with respect to system (25). We still have to consider the quadratic form that estimates W i.e. $\mathscr{H}(\cdot)$. But here the things are quite clear: the only terms with problems arising from the estimate are

$$\tau^{-1}\left(\int_{-\tau}^{0} \psi_1^T(\theta)d\theta\right) S_3 \left(\int_{-\tau}^{0} \psi_1(\theta)d\theta\right)$$

and

$$\tau^{-1}\left(\int_{-2\tau}^{-\tau} \psi_2^T(\theta)d\theta\right) S_4 \left(\int_{-2\tau}^{-\tau} \psi_2(\theta)d\theta\right)$$

which are both $O(\tau)$. Therefore using the previous arguments we obtain that $\lim_{\tau\to 0} \mathscr{H}(\cdot) = W(\cdot)$. *It follows that validity of the theorem implies its validity for* $\tau \to 0$. Therefore exponential stability of the delay-less system is a necessary condition for exponential stability in the small delay case.

Let us discuss now the sufficiency of this condition: assuming exponential stability for the system without delays, this property will still be valid, from continuity reasons, for $\tau > 0$ sufficiently small. Moreover, due to such terms in W as those given by the above integrals, a sufficiently small increase of $\tau > 0$ will add sufficiently large negative terms in W thus ensuring exponential stability.

At the same time it is now the place to point out another necessary condition. For $\tau > 0$, even arbitrarily small, we have to take into account the complete inequality (19) of Theorem 4.1. A necessary condition is $H_{22} < 0$ which is equivalent to

$$S_2 - S_1 + D^T(S_1 + \tau S_4)D < 0$$

since $S_2 > 0$. This is nothing more but

$$D^T(S_1 + \tau S_4)D - (S_1 + \tau S_4) + S_2 + \tau S_4 < 0$$

This discrete time Liapunov inequality holds for $S_i > 0$, $i = 1, 2, 4$ and any $\tau > 0$ provided D is a discrete time stable matrix i.e. its eigenvalues are located inside the unit disk. The necessity of this condition is consistent with the stability of the difference operator for system (2); at its term this condition is necessary for the robustness of stability with respect to small delay variations. Or, passing from $\tau = 0$ to $\tau > 0$ even arbitrarily small is such a variation that could destroy stability of the delay-less system. Therefore we may state the following stability result for small delays thus proved

PROPOSITION 4.1 *Let system (25) be exponentially stable and the matrix D have its eigenvalues inside the unit disk (with moduli strictly less than 1). Then system (5) (and, therefore, system (2)) is exponentially stable for sufficiently small* $\tau > 0$.

5. ESTIMATES OF THE TIME DELAY. CONTROL INTERPRETATIONS

Proposition 4.1 just ensures stability preservation for small delays without giving any estimate of this small delay. On the other hand we have at our disposal Theorem 4.1 with its LMI (Linear Matrix Inequality) (19). As follows from (19) finding the estimates for the delay bound τ is a standard LMI-based (quasi-convex) optimization problem much alike to a state feedback construction (see [20] for standard delay equations).Furthermore, one can interpret the delay-dependent stability of the above lossless propagation models as a *multi-objective control problem*, since one needs to find some model transformation to guarantee simultaneously the following constraints:

a) the stability equivalence between the original and the transformed systems (see Section 3),

b) the stability of the system free of delay (the basic assumption in Proposition 4.1), and

c) the largest value for the delay bound τ^*.

In fact fulfillment of these conditions is dependent on the choice of $P > 0$, $S_i > 0$ subject to (19) but also on the choice of M which, while accounting especially for the fulfillment of a) allows some freedom in the choice of P and S_i in order to maximize the estimates for the admissible value of τ. Throughout C the product MC accounts also for fulfillment of b), these two requirements being competitive (contradictory); this will put an additional limit on the admissible value of τ.

6. CONCLUDING REMARKS

This paper has focused on the *delay-dependent* stability of some linear lossless propagation models. In order to use some simple quadratic Liapunov-Krasovskii functionals for the stability analysis, some *model transformations* of the original system have been proposed.

As known, the difference between sufficient and necessary and sufficient conditions is called *"conservatism"*. The method proposed here also has its conservatism expressed in the upper bound of the delay defining "small delay" domain of exponential stability. It is felt that the advantage of the LMI based method lies in its finite dimensional character (what means numerical tractability using available commercial software) and (hopefully) numerical efficiency (see the basic monograph [2] or the comments and hints in [6]).

APPENDIX

We shall present here the proof of Theorem 3.1 which will be performed in two steps

1^o Let $(x_1^\phi(t), x_2^\phi(t)), t > 0$ be a solution of (2) defined by the initial condition(x_1^0, ϕ_2) satisfying the conditions of the Theorem. On $[0,\tau]$ we shall have the following identities representing the fact that $(x_1^\phi(t), x_2^\phi(t))$ is a solution of (2) with the corresponding initial condition

$$\left\{ \begin{array}{l} \dot{x}_1^\phi(t) \equiv A x_1^\phi(t) + B\phi_2(t-\tau) \\ x_2^\phi(t) = C x_1^\phi(t) + D\phi_2(t-\tau), \end{array} \right. \tag{26}$$

Let $t > \tau$: $x_1^\phi(t)$ and $x_2^\phi(t)$ verify (2) with the initial condition $x_1^\phi(\tau)$ and $x_2^\phi(t)$ constructed on $[0,\tau]$ from (26). We may write

$$\begin{aligned} x_2^\phi(t-\tau) &= x_2^\phi(t) - \int_{-\tau}^0 \dot{x}_2^\phi(t+\theta)d\theta = x_2^\phi(t) - C\int_{-\tau}^0 \dot{x}_1^\phi(t+\theta)d\theta \\ &-D\int_{-\tau}^0 \dot{x}_2^\phi(t-\tau+\theta)d\theta = x_2^\phi(t) - D(x_2^\phi(t-\tau) - x_2^\phi(t-2\tau)) \\ &-CA\int_{-\tau}^0 x_1^\phi(t+\theta)d\theta - CB\int_{-2\tau}^{-\tau} x_2^\phi(t+\theta)d\theta \end{aligned} \tag{27}$$

and if we re-write the first equation of (2) using an arbitrary matrix M

$$\dot{x}_1^{\phi}(t) \equiv Ax_1^{\phi}(t) + (B-M)x_2^{\phi}(t-\tau) + Mx_2^{\phi}(t-\tau)$$

then substituting $Mx_2^{\phi}(t-\tau)$ by (27) multiplied by M we obtain the first equation of (5). Since the second one is common to (2) and (5) we deduce (5) to be verified for $t > \tau$. Now if we define

$$z_i^{\psi}(t) = x_i^{\phi}(t+\tau)\ ,\ i = 1,2\ ,\ t > 0\ , \tag{28}$$

$$\psi_1(\theta) = x_1^{\phi}(\theta+\tau)\ ,\ -\tau \le \theta \le 0\ , \tag{29}$$

that is

$$\psi_1(\theta) = e^{A(\theta+\tau)}x_1^0 + \int_{-\tau}^{\theta} e^{A(\theta-\lambda)}B\phi_2(\lambda)d\lambda\ , \tag{30}$$

and

$$\psi_2(\theta) = \begin{cases} x_2^{\phi}(\theta+\tau) & ,\quad -\tau \le \theta \le 0 \\ \phi_2(\theta+\tau) & ,\quad -2\tau \le \theta \le -\tau \end{cases}\ , \tag{31}$$

i.e.

$$\psi_2(\theta) = \begin{cases} C\psi_1(\theta) + D\phi_2(\theta) & ,\quad -\tau \le \theta \le 0 \\ \phi_2(\theta+\tau) & ,\quad -2\tau \le \theta \le -\tau \end{cases}\ , \tag{32}$$

the first part of Theorem 3.1 is proved.

2^o Conversely, let $(z_1^{\psi}(t), z_2^{\psi}(t))$ be a solution of (5) with the initial conditions (z_1^0, ψ_1, ψ_2) with $\psi_1 \in \mathbf{W}_2^{(1)}(-\tau, 0; \mathbb{R}^{n_1})$, $\psi_2 \in \mathbf{W}_2^{(1)}(-2\tau, 0; \mathbb{R}^{n_2})$ and also subject to the continuity condition at the origin

$$\psi_2(0) = Cz_1^0 + D\psi_2(-\tau) \tag{33}$$

Consider the interval $0 \le t \le \tau$ and write down system (5) for this interval, using the initial conditions

$$\begin{cases} \dot{z}_1^{\psi}(t) & \equiv & (A+MC)z_1^{\psi}(t) + (B-M)\psi_2(t-\tau) \\ & - & MCA\int_{-\tau}^{-t}\psi_1(t+\theta)d\theta - MCA\int_{-t}^{0} z_1^{\psi}(t+\theta)d\theta \\ & - & MCB\int_{-2\tau}^{-\tau}\psi_2(t+\theta)d\theta \\ z_2^{\psi}(t) & \equiv & Cz_1^{\psi}(t) + D\psi_2(t-\tau) \end{cases} \tag{34}$$

Denoting

$$\Omega(t) := \dot{z}_1^{\psi}(t) - Az_1^{\psi}(t) - B\psi_2(t-\tau)$$

we may write

$$\Omega(t) \equiv M\left[Cz_1^{\psi}(t) - \psi_2(t-\tau) + D\psi_2(t-2\tau) - C\int_{-\tau}^{-t}(A\psi_1(t+\theta) + B\psi_2(t-\tau+\theta))d\theta \right.$$
$$\left. -C\int_{-t}^{0}(Az_1^{\psi}(t+\theta) + B\psi_2(t-\tau+\theta))d\theta\right]$$

The continuity conditions subject to which are the initial conditions allow application of the Leibniz formula to obtain

$$\Omega(t) \equiv M[Cz_1^0 - C\psi_1(0) - \psi_2(t-\tau) + C\psi_1(t-\tau)]$$
$$-MC\int_{-\tau}^{-t}(A\psi_1(t+\theta) + B\psi_2(t-\tau+\theta) - \psi_1(t+\theta))d\theta + MC\int_{-t}^{0}\Omega(t+\theta)d\theta$$

This is nothing else but the Volterra integral equation

(35) $$\Omega(t) = \Gamma(t) + MC\int_0^t \Omega(\lambda)d\lambda$$

Under the assumptions on the initial conditions - see (8) and (9) - it follows that $\Gamma(t) \equiv 0$ hence $\Omega(t) \equiv 0$. Therefore first equation of (2) is fulfilled on $[0,\tau]$ as well as the second one which is common to both (2) and (5). Since (8) and (9) signify that (z_1^0, ψ_1, ψ_2) have to be such that they define a continuous solution of (2) we deduce that $(z_1^{\psi}(t), z_2^{\psi}(t))$ extended on $[-\tau,0]$ by the initial conditions are a solution of (2) on $[-\tau,\tau]$. Let now $t > \tau$, more precisely $\tau \le t \le 2\tau$. It can be shown as above that the solution of (5) on $[\tau,2\tau]$, extended with the solution on $[0,\tau]$ which, as shown, satisfies (2) and, therefore, may be viewed as an initial condition for (5) on $[0,\tau]$, satisfies (2) on $[0,2\tau]$ etc. The proof is completed by induction.

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