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Omar Anza Hafsa, Jean-Philippe Mandallena

Γ-CONVERGENCE OF NONCONVEX UNBOUNDED INTEGRALS IN CHEEGER-SOBOLEV SPACES

OMAR ANZA HAFSA AND JEAN-PHILIPPE MANDALLENA

Abstract. We study Γ-convergence of nonconvex integrals of the calculus of variations in the setting of Cheeger-Sobolev spaces when the integrands have not polynomial growth and can take infinite values. Applications to relaxation and homogenization are also developed.

Contents

Key words and phrases. Γ-convergence, Nonconvex unbounded integral, Ru-usc, General growth conditions, Metric measure space, Cheeger-Sobolev space.

1. Introduction

Let (X,d,μ) be a metric measure space, where (X,d) is separable and complete and μ is a doubling positive Radon measure on X which satisfies the annular decay property, supporting a weak $(1,p)$-Poincaré inequality with $p>1$. Let $m,N\geqslant 1$ be two integers, let $O\subset X$ be a bounded open set such that $\mu(\overline{O}\backslash O)=0$ and let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space. In this paper we consider a family of stochastic integrals $E_t : H^{1,p}_\mu(O;\mathbb{R}^m)\times\Omega\to[0,\infty]$ defined by

$$E_t(u,\omega):=\int_O L_t\big(x,\nabla_\mu u(x),\omega\big)d\mu(x), \tag{1.1}$$

where $L_t : O\times\mathbb{M}\times\Omega\to[0,\infty]$ is a Borel measurable stochastic integrand[1] depending on a parameter $t>0$, not necessarily convex with respect to $\xi\in\mathbb{M}$, where $\mathbb{M}$ denotes the space of real $m\times N$ matrices, and possibly taking infinite values. The space $H^{1,p}_\mu(O;\mathbb{R}^m)$ denotes the class of p-Cheeger-Sobolev functions from Ω to $\mathbb{R}^m$ and $\nabla_\mu u$ is the μ-gradient of u.

The object of the present paper is to deal with the problem of computing the almost sure Γ-convergence (see Definitions 2.1) of the stochastic family $\{E_t\}_{t>0}$, as $t\to\infty$, to a stochastic integral $E_{\rm lim} : H^{1,p}_\mu(O;\mathbb{R}^m)\times\Omega\to[0,\infty]$ of the type

$$E_{\rm lim}(u,\omega)=\int_O L_{\rm lim}\big(x,\nabla_\mu u(x),\omega\big)d\mu(x) \tag{1.2}$$

with $L_{\rm lim} : O\times\mathbb{M}\times\Omega\to[0,\infty]$ not depending on the parameter t. When $L_{\rm lim}$ is independent of the variable x, the procedure of passing from (1.1) to (1.2) is referred as stochastic homogenization. If furthermore $L_{\rm lim}$ is independent of the variable ω then $E_{\rm lim}$ is said to be deterministic, otherwise $E_{\rm lim}$ is said to be stochastic. When $\{L_t\}_{t>0}$ is deterministic, i.e. L_t is independent of the variable ω for all $t>0$, the procedure of passing from (1.1) to (1.2) is referred as deterministic homogenization.

This Γ-convergence problem was already studied in [AHM17] in the case where L_t has p-growth. Here we treat the case where L_t has not necessarily p-growth and can take infinite values (see Sect. 2 for more details).

For related works in the Euclidean case, i.e. when $(X,d,\mu)=(\mathbb{R}^N,|\cdot-\cdot|,\mathcal{L}_N)$ where $\mathcal{L}_N$ is the Lebesgue measure on $\mathbb{R}^N$, we refer the reader to [Mar78, Bra85, DMM86, Mül87, JKO94, MM94, BG95, BD98, AM02, AM04, AHM11, AHLM11, AHMZ15, DG16, AHCM17] and the references therein.

One motivation for developing Γ-convergence, and more generally calculus of variations, in the setting of metric measure spaces comes from applications to hyperelasticity. In fact, the interest of considering a general measure is that its support can be interpretated as a hyperelastic structure together with its singularities like for example thin dimensions, corners, junctions, etc. Such mechanical "singular" objects naturally lead to develop calculus of variations in the setting of metric measure spaces. Indeed, for example, a low multi-dimensional structures can be described by a finite number of smooth compact manifolds S_i

[1]Throughout the paper, by a Borel measurable stochastic integrand $L : O\times\mathbb{M}\times\Omega\to[0,\infty]$ we mean that L is $(\mathscr{B}(X)\otimes\mathscr{B}(\mathbb{M})\otimes\mathscr{F},\mathscr{B}(\overline{\mathbb{R}}))$-measurable, where $\mathscr{B}(X)$, $\mathscr{B}(\mathbb{M})$ and $\mathscr{B}(\overline{\mathbb{R}})$ denote the Borel σ-algebra on X, $\mathbb{M}$ and $\overline{\mathbb{R}}$ respectively.

of dimension k_i on which a superficial measure $\mu_i = \mathcal{H}^{k_i}|_{S_i}$ is attached. Such a situation leads to deal with the finite union of manifolds S_i, i.e. $X = \cup_i S_i$, together with the finite sum of measures μ_i, i.e. $\mu = \sum_i \mu_i$, whose mathematical framework is that of metric measure spaces (for more examples, we refer the reader to [BBS97, Zhi02, CJLP02] and [CPS07, Chapter 2, §10] and the references therein).

Another motivation is the development of the calculus of variations on "singular" spaces, which are of interest for geometers and physicists, like Carnot groups, glued spaces, Laakso spaces, Bourdon-Pajot spaces, Gromov-Hausdorff limit spaces, spaces statisfying generalized Ricci bounds (see [KM16] for more details). Indeed, all these spaces are examples of doubling metric measure spaces satisfying a Poincaré inequality on which the theory of Γ-convergence on Cheeger-Sobolev spaces could be applied.

The plan of the paper is as follows. In Sect. 2 we state the main result of the paper, see Theorem 2.5 (and also Proposition 2.8 whose proof is given at the end of Sect. 2). Theorem 2.5 is a Γ-convergence result of $\{E_t\}_{t>0}$ as $t \to \infty$ to $E_{\lim}$ in the setting of metric measure spaces and in a unbounded framework. Classically, its proof is a consequence of Proposition 2.6 (the lower bound) and Proposition 2.7 (the upper bound). Sect. 3 is devoted to several auxiliary definitions and results needed for understanding and proving our Γ-convergence result: in §3.1 we provide materials about Cheeger-Sobolev spaces; in §3.2 we recall the concept of (family of) ru-usc[2] integrand(s) and its main properties that will be used in the proof of Propositions 2.6, 2.7 and 2.8; the proof of Proposition 2.7 also needs the use of the Viatli envelope of a set function which is recalled in §3.3. Sect. 4 is devoted to the proofs of Propositions 2.6 and 2.7 and Theorem 2.5. Finally, applications to relaxation and homogenization are developed in Sect. 5.

Notation. The open and closed balls centered at $x \in X$ with radius $\rho > 0$ are denoted by:

$$B_\rho(x) := \Big\{ y \in X : d(x,y) < \rho \Big\};$$
$$\overline{B}_\rho(x) := \Big\{ y \in X : d(x,y) \leqslant \rho \Big\}.$$

For $x \in X$ and $\rho > 0$ we set

$$\partial B_\rho(x) := \overline{B}_\rho(x) \backslash B_\rho(x) = \Big\{ y \in X : d(x,y) = \rho \Big\}.$$

For $A \subset X$, the diameter of A (resp. the distance from a point $x \in X$ to the subset A) is defined by $\mathrm{diam}(A) := \sup_{x,y \in A} d(x,y)$.
The symbol $⨍$ stands for the mean-value integral

$$⨍_B f d\mu = \frac{1}{\mu(B)} \int_B f d\mu.$$

For $\mathbb{F} \subset \mathbb{M}$, where $\mathbb{M}$ denotes the space of real $m \times N$ matrices, the interior and the closure of $\mathbb{F}$ are respectively denoted by $\mathrm{int}(\mathbb{F})$ and $\overline{\mathbb{F}}$.

[2]The abbreviation ru-usc means radially uniformly upper semicontinuous.

2. The Γ-convergence result

We begin by recalling the definition of the almost sure Γ-convergence. (For more details on the theory of Γ-convergence we refer to [DM93].)

Definition 2.1. We say that $\{E_t\}_{t>0}$ almost surely $\Gamma(L^p_\mu)$-converges as $t\to\infty$ to the functional $E_{\rm lim}: H^{1,p}_\mu(O;\mathbb{R}^m)\times\Omega\to[0,\infty]$ if there exists $\Omega'\in\mathscr{F}$ with $\mathbb{P}(\Omega')=1$ such that for every $\omega\in\Omega'$, one has:

Γ-$\underline{\lim}$: for every $u\in H^{1,p}_\mu(O;\mathbb{R}^m)$, $\Gamma(L^p_\mu)$-$\underline{\lim}_{t\to\infty}E_t(u,\omega)\geqslant E_{\rm lim}(u,\omega)$ with

$$\Gamma(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}}\,E_t(u,\omega):=\inf\left\{\underline{\lim_{t\to\infty}}\,E_t(u_t,\omega):u_t\to u\text{ in }L^p_\mu(O;\mathbb{R}^m)\right\},$$

or equivalently, for every $u\in H^{1,p}_\mu(O;\mathbb{R}^m)$ and every $\{u_t\}_{t>0}\subset H^{1,p}_\mu(O;\mathbb{R}^m)$ such that $u_t\to u$ in $L^p_\mu(O;\mathbb{R}^m)$,

$$\underline{\lim_{t\to\infty}}\,E_t(u_\varepsilon,\omega)\geqslant E_{\rm lim}(u,\omega);$$

Γ-$\overline{\lim}$: for every $u\in H^{1,p}_\mu(O;\mathbb{R}^m)$, $\Gamma(L^p_\mu)$-$\overline{\lim}_{t\to\infty}E_t(u,\omega)\leqslant E_{\rm lim}(u,\omega)$ with

$$\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}}\,E_t(u,\omega):=\inf\left\{\overline{\lim_{t\to\infty}}\,E_t(u_t,\omega):u_t\to u\text{ in }L^p_\mu(O;\mathbb{R}^m)\right\},$$

or equivalently, for every $u\in H^{1,p}_\mu(O;\mathbb{R}^m)$ there exists $\{u_t\}_{t>0}\subset H^{1,p}_\mu(O;\mathbb{R}^m)$ such that $u_t\to u$ in $L^p_\mu(O;\mathbb{R}^m)$ and

$$\overline{\lim_{t\to\infty}}\,E_t(u_t,\omega)\leqslant E_{\rm lim}(u,\omega).$$

Referring to the next section for any unfamiliar notation or definition, in what follows we state the main results of the paper. Let $m,N\geqslant 1$ be two integers and let $\mathbb{M}$ be the space of real $m\times N$ matrices. Let $G:\mathbb{M}\to[0,\infty]$ be a Borel measurable integrand satisfying the following conditions:

(C_1) there exists $\gamma>0$ such that for every $\xi,\zeta\in\mathbb{M}$ and every $\tau\in]0,1[$,

$$G(\tau\xi+(1-\tau)\zeta)\leqslant\gamma(1+G(\xi)+G(\zeta));$$

(C_2) $0\in\mathrm{int}(\mathbb{G})$, where $\mathbb{G}$ denotes the effective domain of G, i.e. $\mathbb{G}:=\{\xi\in\mathbb{M}:G(\xi)<\infty\}$.

Remark 2.2. If (C_1) is satisfied then $\mathbb{G}$ is convex. If moreover (C_2) holds, then there exists $r>0$ such that $\sup_{|\xi|\leqslant r}G(\xi)<\infty$, see [AHM12b, Lemma 4.1].

Let $\mathscr{Q}_\mu G:O\times\mathbb{M}\to[0,\infty]$ be defined by

$$\mathscr{Q}_\mu G(x,\xi):=\overline{\lim_{\rho\to0}}\inf\left\{\fint_{B_\rho(x)}G(\xi+\nabla_\mu w(y))d\mu(y):w\in H^{1,p}_{\mu,0}(B_\rho(x);\mathbb{R}^m)\right\},$$

where the space $H^{1,p}_{\mu,0}(B_\rho(x);\mathbb{R}^m)$ is defined as the closure of

$$\mathrm{Lip}_0(B_\rho(x);\mathbb{R}^m):=\Big\{u\in\mathrm{Lip}(O;\mathbb{R}^m):u=0\text{ on }O\backslash B_\rho(x)\Big\}$$

with respect to the $H^{1,p}_\mu$-norm, where $\mathrm{Lip}(O;\mathbb{R}^m) := [\mathrm{Lip}(O)]^m$ with $\mathrm{Lip}(O)$ denoting the algebra of Lipschitz functions from O to $\mathbb{R}$. (The integrand $\mathscr{Q}_\mu G$ is called the $H^{1,p}_\mu$-quasiconvexification of G. For more details on the notion of $H^{1,p}_\mu$-quasiconvexity, we refer to [AHM20a].) Denote the the effective domain of $\mathscr{Q}_\mu G(x,\cdot)$ by $\mathscr{Q}_\mu\mathbb{G}_x$. We further suppose that:

- (C_3) for every $x \in O$, $\tau\overline{\mathscr{Q}_\mu\mathbb{G}_x} \subset \mathrm{int}(\mathscr{Q}_\mu\mathbb{G}_x)$ for all $\tau \in]0,1[$;
- (C_4) for every $u \in H^{1,p}_\mu(O;\mathbb{R}^m)$, if $\int_O \mathscr{Q}_\mu G(x,\nabla_\mu u(x))d\mu < \infty$ and if $\nabla_\mu u(x) \in \mathrm{int}(\mathscr{Q}_\mu\mathbb{G}_x)$ for μ-a.a. $x \in O$ then $\int_O G(\nabla_\mu u(x))d\mu < \infty$;
- (C_5) for every $x \in O$, $\mathscr{Q}_\mu G(x,\cdot)$ is lsc[3] on $\mathrm{int}(\mathscr{Q}_\mu\mathbb{G}_x)$.

Remark 2.3. Under (C_1) and (C_2) if $G = \mathscr{Q}_\mu G$, i.e. G is $H^{1,p}_\mu$-quasiconvex, then (C_3) and (C_4) hold. In particular, since convexity implies $H^{1,p}_\mu$-quasiconvexity (see [AHM20a]), if G is convex then (C_3) and (C_4) hold.

Let (X,d,μ) be a metric measure space, where (X,d) is separable and complete and μ is a doubling positive Radon measure on X which satisfies the annular decay property, supporting a weak $(1,p)$-Poincaré inequality with

$$p > \kappa := \frac{\ln(C_d)}{\ln(2)} \text{ where } C_d \geqslant 1 \text{ is the doubling constant.}$$

Let $O \subset X$ be a bounded open set such that $\mu(\overline{O}\backslash O) = 0$ and let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space. Throughout the paper, we consider a family $\{L_t : O \times \mathbb{M} \times \Omega \to [0,\infty]\}_{t>0}$ of Borel measurable stochastic integrands depending on a parameter $t > 0$ (destined to tend to infinity) and satisfying the following conditions:

- (C_6) $\{L_t\}_{t>0}$ is p-coercive, i.e. there exists $c > 0$ such that for every $t > 0$, every $x \in O$, every $\xi \in \mathbb{M}$ and every $\omega \in \Omega$,
$$L_t(x,\xi,\omega) \geqslant c|\xi|^p;$$
- (C_7) $\{L_t\}_{t>0}$ has G-growth, i.e. there exist $\alpha,\beta > 0$ such that for every $x \in O$, every $\xi \in \mathbb{M}$ and every $\omega \in \Omega$,
$$\alpha G(\xi) \leqslant L_t(x,\xi,\omega) \leqslant \beta(1+G(\xi)).$$

Remark 2.4. If (C_1) and (C_7) hold then the effective domain $\mathbb{L}_{t,x,\omega}$ of $L_t(x,\cdot,\omega)$ is equal to $\mathbb{G}$ and so is convex and does not depend on x and ω.

The p-growth case, i.e. when $G(\xi) = |\xi|^p$, was already studied in [AHM17]. The object of this paper is to deal with the G-growth case. For this, in addition, we need to suppose that

- (C_8) for every $\omega \in \Omega$, $\{L_t\}_{t>0}$ is ru-usc at ω, i.e. for every $\omega \in \Omega$, there exists $\{a_t(\cdot,\omega)\}_{t>0} \subset L^1_\mu(O;]0,\infty])$ with
$$\overline{\lim_{t\to\infty}} \int_O a_t(x,\omega)d\mu(x) < \infty \tag{2.1}$$

[3]The abbreviation lsc means lower semicontinuous.

and

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \fint_{B_\rho(\cdot)} a_t(y,\omega)d\mu(y) =: a_\infty(\cdot,\omega) \in L^1_\mu(O) \tag{2.2}$$

such that

$$\overline{\lim_{\tau\to 1^-}} \sup_{t>0} \Delta^{a_t}_{L_t}(\tau,\omega) \leqslant 0,$$

where $\Delta^{a_t}_{L_t} : [0,1]\times\Omega \to]-\infty,\infty]$ is given by

$$\Delta^{a_t}_{L_t}(\tau,\omega) := \sup_{x\in O}\sup_{\xi\in\mathbb{L}_{t,x,\omega}} \frac{L_t(x,\tau\xi,\omega)-L_t(x,\xi,\omega)}{a_t(x,\omega)+L_t(x,\xi,\omega)} \tag{2.3}$$

with $\mathbb{L}_{t,x,\omega}$ denoting the effective domain of $L_t(x,\cdot,\omega)$.

For each $t>0$ and each $\rho>0$, let $\mathscr{H}^\rho_\mu L_t : O\times\mathbb{M}\times\Omega\to[0,\infty]$ be defined by

$$\mathscr{H}^\rho_\mu L_t(x,\xi,\omega) := \inf\left\{\fint_{B_\rho(x)} L_t(y,\xi+\nabla_\mu w(y),\omega)d\mu(y) : w\in H^{1,p}_{\mu,0}(B_\rho(x);\mathbb{R}^m)\right\}.$$

For each $t>0$, let $E_t : H^{1,p}_\mu(O;\mathbb{R}^m)\times\Omega\to[0,\infty]$ be defined by (1.1). The main result of the paper is the following Γ-convergence result.

Theorem 2.5 (Γ-$\lim$)**.** *Assume that $p>\kappa$. If* (C_1)–(C_8) *hold and if*

(C_9) *there exists $\Omega'\in\mathscr{F}$ with $\mathbb{P}(\Omega')=1$ such that for every $\omega\in\Omega'$, one has*

$$\overline{\lim_{\rho\to 0}}\,\underline{\lim_{t\to\infty}} \mathscr{H}^\rho_\mu L_t(x,\xi,\omega) \geqslant \overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \mathscr{H}^\rho_\mu L_t(x,\xi,\omega)$$

for all $x\in O$ and all $\xi\in\mathbb{Q}_\mu\mathbb{G}_x$,

then $\{E_t\}_{t>0}$ almost surely $\Gamma(L^p_\mu)$-converges as $t\to\infty$ to the functional $E_{\lim} : H^{1,p}_\mu(O;\mathbb{R}^m)\times\Omega\to[0,\infty]$ defined by (1.2) *with $L_{\lim} : O\times\mathbb{M}\times\Omega\to[0,\infty]$ given by*

$$L_{\lim}(x,\xi,\omega) = \underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \mathscr{H}^\rho_\mu L_t(x,\tau\xi,\omega)$$

Theorem 2.5 is a consequence of the following two propositions.

Proposition 2.6 (Γ-$\underline{\lim}$)**.** *Assume that $p>\kappa$. If* (C_1), (C_2) *and* (C_6)–(C_8) *hold then, for every $\omega\in\Omega$, one has*

$$\Gamma(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}} E_t(u,\omega) \geqslant \int_O \underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\underline{\lim_{t\to\infty}} \mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x)$$

for all $u\in H^{1,p}_\mu(O;\mathbb{R}^m)$.

Proposition 2.7 (Γ-$\overline{\lim}$)**.** *Assume that $p>\kappa$. If* (C_1)–(C_8) *hold then, for every $\omega\in\Omega$, one has*

$$\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}} E_t(u,\omega) \leqslant \int_O \underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x)$$

for all $u\in H^{1,p}_\mu\big(O;\mathbb{R}^m\big)$.

Let $L_\infty : O \times \mathbb{M} \times \Omega \to [0,\infty]$ be defined by

$$L_\infty(x,\xi,\omega) := \overline{\lim_{\rho\to 0}}\, \overline{\lim_{t\to\infty}}\, \mathscr{H}^\rho_\mu L_t(x,\xi,\omega).$$

(Note that if (C_9) is satisfied then $L_\infty(\cdot,\cdot,\omega) = \overline{\lim}_{\rho\to 0} \lim_{t\to\infty} \mathscr{H}^\rho_\mu L_t(\cdot,\cdot,\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.) Let $\widehat{L}_\infty : O \times \mathbb{M} \times \Omega \to [0,\infty]$ be given by

$$\widehat{L}_\infty(x,\xi,\omega) := \underline{\lim_{\tau\to 1^-}}\, L_\infty(x,\tau\xi,\omega)$$

and, for each $x \in O$ and each $\omega \in \Omega$, let $\overline{L}_\infty(x,\cdot,\omega)$ denotes the lsc envelope of $L_\infty(x,\cdot,\omega)$. The following proposition makes more precise the formula of the limit integrand $L_{\rm lim}$ in Theorem 2.5.

Proposition 2.8. *Assume that* (C_3), (C_7) *and* (C_8) *hold.*

(i) *For every* $\omega \in \Omega$,

$$\widehat{L}_\infty(x,\xi,\omega) = \lim_{\tau\to 1^-} L_\infty(x,\tau\xi,\omega) = \begin{cases} \lim\limits_{\tau\to 1^-} L_\infty(x,\tau\xi,\omega) & \text{if } \xi \in \overline{\mathscr{Q}_\mu \mathbb{G}_x} \\ \infty & \text{otherwise.} \end{cases}$$

So, in Theorem 2.5 *we have* $L_{\rm lim} = \widehat{L}_\infty$.

(ii) *Suppose furthermore that for every* $\omega \in \Omega$ *and every* $x \in O$, $L_\infty(x,\cdot,\omega)$ *is lsc on* $\mathrm{int}(\mathscr{Q}_\mu \mathbb{G}_x)$. *Then*

$$\widehat{L}_\infty(x,\xi,\omega) = \overline{L}_\infty(x,\xi,\omega) = \begin{cases} L_\infty(x,\xi,\omega) & \text{if } \xi \in \mathrm{int}(\mathscr{Q}_\mu \mathbb{G}_x) \\ \lim\limits_{\tau\to 1^-} L_\infty(x,\tau\xi,\omega) & \text{if } \xi \in \partial\mathscr{Q}_\mu \mathbb{G}_x \\ \infty & \text{otherwise.} \end{cases} \tag{2.4}$$

In such a case, in Theorem 2.5, $L_{\rm lim}$ *is given by* (2.4).

Proof of Proposition 2.8. From (C_8) and proposition 3.14, we can assert that for every $\omega \in \Omega$, L_∞ is ru-usc at ω. Moreover, by (C_7) it is easily seen that for every $x \in O$ and every $\omega \in \Omega$, the effective domain of $L_\infty(x,\cdot,\omega)$ is equal to $\mathscr{Q}_\mu \mathbb{G}_x$. So, taking (C_3) into account, Proposition 2.8 follows from Theorem 3.12. ■

3. Auxiliary results

3.1. **Cheeger-Sobolev spaces.** Let (X,d,μ) be a separable and complete metric measure space. Here and subsequently, we assume that μ is doubling on X, i.e. there exists a constant $C_d \geqslant 1$ such that

$$\mu\left(B_\rho(x)\right) \leqslant C_d \mu\left(B_{\frac{\rho}{2}}(x)\right) \tag{3.1}$$

for μ-a.a. $x \in X$ and all $\rho > 0$, and X supports a weak $(1,p)$-Poincaré inequality with $p > 1$, i.e. there exist $C_P > 0$ and $\sigma \geqslant 1$ such that for μ-a.e. $x \in X$ and every $\rho > 0$,

$$\fint_{B_\rho(x)} \left| u - \fint_{B_\rho(x)} u d\mu \right| d\mu \leqslant \rho C_P \left(\fint_{B_{\sigma\rho}(x)} v^p d\mu \right)^{\frac{1}{p}} \tag{3.2}$$

for every $u \in L^p_\mu(O)$, every p-weak upper gradient[4] $v \in L^p_\mu(O)$ for u and every open set $O \subset X$ such that $B_{\sigma\rho}(x) \subset O$.

Remark 3.1. As μ is doubling, for μ-a.e. $\bar{x} \in X$ and every $r > 0$, we have $\mu(B_\rho(x))/\mu(B_r(\bar{x})) \geqslant 4^{-\kappa}(\rho/r)^\kappa$ for all $x \in B_r(\bar{x})$ and all $0 < \rho \leqslant r$, where $\kappa := \frac{\ln(C_d)}{\ln(2)}$ (see [Haj03, Lemma 4.7]).

We further assume that (X, d, μ) satisfies the annular decay property, i.e. there exist $\delta > 0$ and $C_A \geqslant 1$ such that

$$\mu\left(B_{\sigma r}(x) \backslash B_r(x)\right) \leqslant C_A \left(1 - \frac{1}{\sigma}\right)^\delta \mu(B_{\sigma r}(x)) \tag{3.3}$$

for all $x \in X$, all $r > 0$ and all $\sigma \in]1, \infty[$.

Remark 3.2. From [Buc99, Corollary 2.2] and [CM98, Lemma 3.3] (see also [Che99, Proposition 6.12] and [HKST15, Proposition 11.5.3 pp. 328]), under (3.1) and (3.2), if moreover (X, d) is a length space, i.e. the distance between any two points equals infimum of lengths of curves connecting the points, then (3.3) holds.

Remark 3.3. If (3.3) holds then $\mu\big(\overline{B}_r(x) \backslash B_r(x)\big) = 0$ for all $x \in X$ and all $r > 0$, i.e. the boundary of any ball is of zero measure. Indeed, given $x \in X$ and $r > 0$, we have $1 \geqslant \frac{\mu(B_r(x))}{\mu(\overline{B}_r(x))} \geqslant \frac{\mu(B_r(x))}{\mu(B_{\sigma r}(x))} \geqslant 1 - C_A(1 - \frac{1}{\sigma})^\delta$ for all $\sigma \in]1, \infty[$. Hence, by letting $\sigma \to 1$, we obtain $\frac{\mu(B_r(x))}{\mu(\overline{B}_r(x))} = 1$, i.e. $\mu(B_r(x)) = \mu(\overline{B}_r(x))$.

Let $O \subset X$ be a bounded open set. Denote the algebra of Lipschitz functions from O to $\mathbb{R}$ by $\mathrm{Lip}(O)$. (Note that, by Hopf-Rinow's theorem (see [BH99, Proposition 3.7, pp. 35]), the closure of O is compact, and so every Lipschitz function from O to $\mathbb{R}$ is bounded.) Let $\mathrm{Lip}(O; \mathbb{R}^m) := [\mathrm{Lip(O)}]^m$ and let $\nabla_\mu : \mathrm{Lip}(O; \mathbb{R}^m) \to L^\infty_\mu(O; \mathbb{M})$ be given by

$$\nabla_\mu u := \begin{pmatrix} D_\mu u_1 \\ \vdots \\ D_\mu u_m \end{pmatrix} \text{ with } u = (u_1, \cdots, u_m),$$

where $D_\mu : \mathrm{Lip}(O) \to L^\infty_\mu(O; \mathbb{R}^N)$ is the differential of Cheeger (see [Che99, Theorem 4.38] and [Kei04, Definition 2.1.1 and Theorem 2.3.1] for more details). The p-Cheeger-Sobolev space $H^{1,p}_\mu(O; \mathbb{R}^m)$ is defined as the completion of $\mathrm{Lip}(O; \mathbb{R}^m)$ with respect to the norm

$$\|u\|_{H^{1,p}_\mu(O;\mathbb{R}^m)} := \|u\|_{L^p_\mu(O;\mathbb{R}^m)} + \|\nabla_\mu u\|_{L^p_\mu(O;\mathbb{M})}. \tag{3.4}$$

As $\|\nabla_\mu u\|_{L^p_\mu(O;\mathbb{M})} \leqslant \|u\|_{W^{1,p}_\mu(O;\mathbb{R}^m)}$ for all $u \in \mathrm{Lip}(O; \mathbb{R}^m)$, the linear map ∇_μ from $\mathrm{Lip}(O; \mathbb{R}^m)$ to $L^p_\mu(O; \mathbb{M})$ has a unique extension to $H^{1,p}_\mu(O; \mathbb{R}^m)$ which will still be denoted by ∇_μ and will be called the μ-gradient. For more details on the various possible extensions of the

[4]A Borel function $v : O \to [0, \infty]$ is said to be an upper gradient for $u : O \to \mathbb{R}$ if $|u(c(1)) - u(c(0))| \leqslant \int_0^1 v(c(s))ds$ for all continuous rectifiable curves $c : [0, 1] \to O$. A function $v \in L^p_\mu(O)$ is said to be a p-weak upper gradient for $u \in L^p_\mu(O)$ if there exist $\{u_n\}_n \subset L^p_\mu(O)$ and $\{v_n\}_n \subset L^p_\mu(O)$ such that for each $n \geqslant 1$, v_n is an upper gradient for u_n, $u_n \to u$ in $L^p_\mu(O)$ and $v_n \to v$ in $L^p_\mu(O)$. For more details we refer to [HK98, Che99].

classical theory of the Sobolev spaces to the setting of metric measure spaces, we refer to [Hei07, §10-14] (see also [Che99, Sha00, GT01, Haj03]).

The following proposition brings together useful known properties for dealing with calculus of variations in the metric measure setting. (For a proof we refer to [HKST15] and [AHM20a, §7].)

Proposition 3.4. *Under* (3.1), (3.2) *and* (3.3) *the following properties hold*:

(i) *O satisfies the Vitali covering theorem, i.e. for every $A \subset O$ and every family $\mathscr{B}$ of closed balls in O, if $\inf\{\rho > 0 : \overline{B}_\rho(x) \in \mathscr{B}\} = 0$ for all $x \in A$ (we say that $\mathscr{B}$ is a fine cover of A) then there exists a countable disjoint subfamily $\mathscr{B}'$ of $\mathscr{B}$ such that $\mu(A \setminus \cup_{B\in\mathscr{B}'} B) = 0$; in other words, $A \subset \big(\cup_{B\in\mathscr{B}'} B\big) \cup N$ with $\mu(N) = 0$;*

(ii) *the μ-gradient is closable in $H^{1,p}_\mu(O;\mathbb{R}^m)$, i.e. for every $u \in H^{1,p}_\mu(O;\mathbb{R}^m)$ and every open set $A \subset O$, if $u(x) = 0$ for μ-a.a. $x \in A$ then $\nabla_\mu u(x) = 0$ for μ-a.a. $x \in A$;*

(iii) *O supports a p-Sobolev inequality, i.e. there exists $C_S > 0$ such that*

$$\left(\int_{B_\rho(x)} |v|^p d\mu\right)^{\frac{1}{p}} \leqslant \rho C_S \left(\int_{B_\rho(x)} |\nabla_\mu v|^p d\mu\right)^{\frac{1}{p}}$$

for all $0 < \rho \leqslant \rho_0$, with $\rho_0 > 0$, and all $v \in H^{1,p}_{\mu,0}(B_\rho(x);\mathbb{R}^m)$, where, for each open set $A \subset O$, $H^{1,p}_{\mu,0}(A;\mathbb{R}^m)$ is the closure of $\mathrm{Lip}_0(A;\mathbb{R}^m)$ with respect to $H^{1,p}_\mu$-norm defined in (3.4) *with*

$$\mathrm{Lip}_0(A;\mathbb{R}^m) := \big\{u \in \mathrm{Lip}(O;\mathbb{R}^m) : u = 0 \text{ on } O\setminus A\big\};$$

(iv) *for every $u \in H^{1,p}_\mu(O;\mathbb{R}^m)$ and μ-a.e. $x \in O$ there exists $u_x \in H^{1,p}_\mu(O;\mathbb{R}^m)$ such that*:

$$\nabla_\mu u_x(y) = \nabla_\mu u(x) \text{ for } \mu\text{-a.a. } y \in O;$$
$$\lim_{\rho\to 0} \frac{1}{\rho} \|u - u_x\|_{L^\infty_\mu(B_\rho(x);\mathbb{R}^m)} = 0 \text{ if } p > \kappa,$$

where $\kappa := \frac{\ln(C_d)}{\ln(2)}$ with $C_d \geqslant 1$ given by the inequality (3.1);

(v) *for every $x \in O$, every $\rho > 0$ and every $\lambda \in]0,1[$ there exists a Urysohn function $\varphi \in \mathrm{Lip}(O)$ for the pair $(O\setminus B_\rho(x), \overline{B}_{\lambda\rho}(x))$*[5] *such that*

$$\|D_\mu \varphi\|_{L^\infty_\mu(O;\mathbb{R}^N)} \leqslant \frac{\theta}{\rho(1-\lambda)}$$

for some $\theta > 0$;

(vi) *for μ-a.e. $x \in O$,*

$$\lim_{\lambda\to 1^-} \underline{\lim_{\rho\to 0}} \frac{\mu(B_{\lambda\rho}(x))}{\mu(B_\rho(x))} = \lim_{\lambda\to 1^-} \overline{\lim_{\rho\to 0}} \frac{\mu(B_{\lambda\rho}(x))}{\mu(B_\rho(x))} = 1;$$

[5]Given a metric space (O,d), by a Urysohn function from O to $\mathbb{R}$ for the pair $(O\setminus V, K)$, where $K \subset V \subset O$ with K compact and V open, we mean a continuous function $\varphi : O \to \mathbb{R}$ such that $\varphi(x) \in [0,1]$ for all $x \in O$, $\varphi(x) = 0$ for all $x \in O\setminus V$ and $\varphi(x) = 1$ for all $x \in K$.

(vii) *for every* $u \in H^{1,p}_\mu(O;\mathbb{R}^m)$ *and every* $\varphi \in \mathrm{Lip}(O)$,

$$\nabla_\mu(\varphi u) = \varphi \nabla_\mu u + D_\mu \varphi \otimes u.$$

Remark 3.5. As μ is a Radon measure and O satisfies the Vitali covering theorem, for every open set $A \subset O$ and every $\varepsilon > 0$ there exists a countable family $\{B_{\rho_i}(x_i)\}_{i\in I}$ of disjoint open balls of A with $x_i \in A$, $\rho_i \in]0,\varepsilon[$ such that $\mu\big(A \setminus \cup_{i\in I} B_{\rho_i}(x_i)\big) = 0$. By the annular decay property, see (3.3), we also have $\mu(\partial B_{\rho_i}(x_i)) = 0$ for all $i \in I$ (see Remark 3.3).

In the framework of the p-Cheeger-Sobolev spaces with $p > \kappa := \ln(C_d)/\ln(2)$, where $C_d \geqslant 1$ is the doubling constant, we also have the following L^∞_μ-compactness result.

Theorem 3.6. *Assume that* $p > \kappa$ *and* $\mu\big(\overline{O}\setminus O\big) = 0$. *If* $u \in H^{1,p}_\mu(O;\mathbb{R}^m)$ *and* $\{u_n\}_n \subset H^{1,p}_\mu(O;\mathbb{R}^m)$ *are such that*

$$\lim_{n\to\infty} \|u_n - u\|_{L^p_\mu(O;\mathbb{R}^m)} = 0 \text{ and } \sup_{n\geqslant 1} \|\nabla_\mu u_n\|_{L^p_\mu(O;\mathbb{M})} < \infty, \tag{3.5}$$

then, up to a subsequence,

$$\lim_{n\to\infty} \|u_n - u\|_{L^\infty_\mu(O;\mathbb{R}^m)} = 0. \tag{3.6}$$

Proof of Theorem 3.6. Since (X,d,μ) is a complete doubling metric space, (X,d,μ) is proper, i.e. every closed ball is compact (see [HKST15, Lemma 4.1.14]), and so $(\overline{O}, d|_{\overline{O}\times\overline{O}})$ is compact. Thus, as $\mu\big(\overline{O}\setminus O\big) = 0$ we can assert that $(\overline{O}, d|_{\overline{O}\times\overline{O}}, \mu|_{\overline{O}})$ is a compact doubling metric measure space supporting a weak $(1,p)$-Poincaré inequality. In what follows, to simplify the notation we set $(Y,\delta,\nu) := (\overline{O}, d|_{\overline{O}\times\overline{O}}, \mu|_{\overline{O}})$.

Step 1: two auxiliary lemmas. We need the following two lemmas (cf. Lemmas 3.7 and 3.8).

Lemma 3.7. *If* $p > \kappa$ *then for every* $r > 0$ *and* ν*-a.e.* $\bar{x} \in Y$ *there exists* $C(r,\bar{x}) > 0$ *such that*

$$|u(y) - u(z)| \leqslant C(r,\bar{x})\delta(y,z)^{1-\frac{\kappa}{p}} \left(\int_{B_{6\sigma r}(\bar{x})} |\nabla_\nu u|^p d\nu \right)^{\frac{1}{p}}$$

for all $u \in H^{1,p}_\nu(Y;\mathbb{R}^m)$ *and all* $y,z \in B_r(\bar{x})$, *where* $\sigma \geqslant 1$ *is given by* (3.2).

Proof of Lemma 3.7. From [Haj03, Theorem 9.7] we can assert that there exists $c > 0$ such that

$$|w(y) - w(z)| \leqslant c r^{\frac{\kappa}{p}} \delta(y,z)^{1-\frac{\kappa}{p}} \left(\fint_{B_{6\sigma r}(\bar{x})} g_w^p d\nu \right)^{\frac{1}{p}} \tag{3.7}$$

for all $w \in H^{1,p}_\nu(Y)$, all $\bar{x} \in Y$, all $r > 0$ and all $y,z \in B_r(\bar{x})$, where $\sigma \geqslant 1$ is given by (3.2) and $g_w \in L^p_\nu(Y)$ denotes the minimal p-weak upper gradient for w. On the other hand, from Remark 3.1 it is easy to see that for every $r > 0$ and ν-a.e. $\bar{x} \in Y$ there exists $\theta(r,\bar{x}) > 0$ such that

$$\nu(B_r(\bar{x})) \geqslant \theta(r,\bar{x}) r^\kappa.$$

But $g_w \leqslant \alpha|D_\nu w|$ with $\alpha \geqslant 1$ (see [Che99, §4]) and so $⨍_{B_{6\sigma r}(\bar{x})} g_w^p d\nu \leqslant \alpha^p ⨍_{B_{6\sigma r}(\bar{x})} |D_\nu w|^p d\nu$. Thus, for every $r > 0$, ν-a.e. $\bar{x} \in Y$ and every $y, z \in B_r(\bar{x})$, (3.7) can be rewritten as follows

$$|w(y) - w(z)| \leqslant C(r, \bar{x})\delta(y, z)^{1-\frac{\kappa}{p}} \left(\int_{B_{6\sigma r}(\bar{x})} |D_\nu w|^p d\nu \right)^{\frac{1}{p}}$$

with $C(r, \bar{x}) = c\alpha/\theta(r, \bar{x}) > 0$. It follows that for every $r > 0$ and ν-a.e. $\bar{x} \in Y$, we have

$$\begin{aligned} |u(y) - u(z)| &\leqslant C(r, \bar{x})\delta(y, z)^{1-\frac{\kappa}{p}} \sum_{i=1}^m \left(\int_{B_{6\sigma r}(\bar{x})} |D_\nu u_i|^p d\nu \right)^{\frac{1}{p}} \\ &\leqslant C(r, \bar{x})\delta(y, z)^{1-\frac{\kappa}{p}} \left(\int_{B_{6\sigma r}(\bar{x})} \sum_{i=1}^m |D_\nu u_i|^p d\nu \right)^{\frac{1}{p}} \\ &= C(r, \bar{x})\delta(y, z)^{1-\frac{\kappa}{p}} \left(\int_{B_{6\sigma r}(\bar{x})} |\nabla_\nu u|^p d\nu \right)^{\frac{1}{p}} \end{aligned}$$

for all $u \in H^{1,p}_\nu(Y; \mathbb{R}^m)$ and all $y, z \in B_r(\bar{x})$, and the proof of Lemma 3.7 is complete. ■

Denote the space of continuous functions from Y to $\mathbb{R}^m$ by $C(Y; \mathbb{R}^m)$. As a consequence of Lemma 3.7 we have the following result.

Lemma 3.8. *If $p > \kappa$ then $H^{1,p}_\nu(Y; \mathbb{R}^m)$ continuously embeds into $C(Y; \mathbb{R}^m)$, i.e.*

$$H^{1,p}_\nu(Y; \mathbb{R}^m) \subset C(Y; \mathbb{R}^m)$$

and there exists $K_0 > 0$ such that

$$\|u\|_{C(Y;\mathbb{R}^m)} \leqslant K_0 \|u\|_{H^{1,p}_\nu(Y;\mathbb{R}^m)} \tag{3.8}$$

for all $u \in H^{1,p}_\nu(X; \mathbb{R}^m)$. Moreover, there exists $K_1 > 0$ such that

$$|u(y) - u(z)| \leqslant K_1 \delta(y, z)^{1-\frac{\kappa}{p}} \|\nabla_\nu u\|_{L^p_\nu(Y;\mathbb{M})} \tag{3.9}$$

for all $u \in H^{1,p}_\nu(Y; \mathbb{R}^m)$ and all $y, z \in Y$.

Proof of Lemma 3.8. Applying Lemma 3.7 with $r = \operatorname{diam}(Y)$ and for a fixed $\bar{x} = x_0 \in Y$, where $\operatorname{diam}(Y) = \sup\{\delta(y, z) : y, z \in Y\} < \infty$ because (Y, δ) is compact, we see that

$$\begin{aligned} |u(y) - u(z)| &\leqslant C\left(\operatorname{diam}(Y), x_0\right) \delta(y, z)^{1-\frac{\kappa}{p}} \|\nabla_\nu u\|_{L^p_\nu(Y;\mathbb{M})} \\ &\leqslant C\left(\operatorname{diam}(Y), x_0\right) \operatorname{diam}(Y)^{1-\frac{\kappa}{p}} \|\nabla_\nu u\|_{L^p_\nu(Y;\mathbb{M})} \end{aligned} \tag{3.10}$$

for all $u \in H^{1,p}_\nu(Y; \mathbb{R}^m)$ and all $y, z \in Y$. Hence (3.9) holds with $K_1 = C\left(\operatorname{diam}(Y), x_0\right)$ and every $u \in H^{1,p}_\nu(Y; \mathbb{R}^m)$ is $(1 - \frac{\kappa}{p})$-Hölder continuous. In particular, it follows that $H^{1,p}_\nu(Y; \mathbb{R}^m) \subset C(Y; \mathbb{R}^m)$. On the other hand, given any $u \in H^{1,p}_\nu(Y; \mathbb{R}^m)$ and any $y \in Y$, we have $|u(y)|^p \leqslant 2^p \left(|u(y) - u(z)|^p + |u(z)|^p\right)$ for all $z \in Y$, and consequently

$$\nu(Y)^{\frac{1}{p}} |u(y)| \leqslant 2^{1+\frac{1}{p}} \left(\int_Y |u(y) - u(z)|^p d\nu(z) \right)^{\frac{1}{p}} + 2^{1+\frac{1}{p}} \|u\|_{L^p_\nu(Y;\mathbb{R}^m)}. \tag{3.11}$$

But, by (3.10) we have

$$\left(\int_Y |u(y)-u(z)|^p d\nu(z)\right)^{\frac{1}{p}} \leqslant \nu(Y)^{\frac{1}{p}} C\left(\operatorname{diam}(Y), x_0\right) \operatorname{diam}(Y)^{1-\frac{\kappa}{p}} \|\nabla_\nu u\|_{L^p_\nu(Y;\mathbb{M})}. \tag{3.12}$$

Hence, combining (3.11) and (3.12) we deduce that for every $y \in Y$,

$$\begin{aligned} |u(y)| &\leqslant 2^{1+\frac{1}{p}} C\left(\operatorname{diam}(Y), x_0\right) \operatorname{diam}(Y)^{1-\frac{\kappa}{p}} \|\nabla_\nu u\|_{L^p_\nu(Y;\mathbb{M})} + \frac{2^{1+\frac{1}{p}}}{\nu(Y)^{\frac{1}{p}}} \|u\|_{L^p_\nu(Y;\mathbb{R}^m)} \\ &\leqslant K_0 \|u\|_{H^{1,p}_\nu(Y;\mathbb{R}^m)} \end{aligned}$$

with $K_0 = \sup\left\{2^{1+\frac{1}{p}} C\left(\operatorname{diam}(Y), x_0\right) \operatorname{diam}(Y)^{1-\frac{\kappa}{p}}, \frac{2^{1+\frac{1}{p}}}{\nu(Y)^{\frac{1}{p}}}\right\}$, and (3.8) follows. ■

Step 2: end of the proof of Theorem 3.6. As $\mu\big(\overline{O}\backslash O\big) = 0$, from (3.5) we deduce that

$$\lim_{n\to\infty} \|u_n - u\|_{L^p_\nu(Y;\mathbb{R}^m)} = 0 \text{ and } \sup_{n\geqslant 1} \|\nabla_\nu u_n\|_{L^p_\nu(Y;\mathbb{M})} < \infty,$$

and so $\sup_{n\geqslant 1} \|u_n\|_{H^{1,p}_\nu(Y;\mathbb{R}^m)} < \infty$. By Lemma 3.8 we can assert that $\sup_{n\geqslant 1} \|u_n\|_{C(Y;\mathbb{R}^m)} < \infty$, i.e. $\{u_n\}_n$ is bounded in $C(Y;\mathbb{R}^m)$ with (Y,δ) a compact metric space. Moreover, using (3.9) we see that $\{u_n\}_n$ is equicontinuous. Consequently, up to a subsequence,

$$\lim_{n\to\infty} \|u_n - u\|_{L^\infty_\nu(Y;\mathbb{R}^m)} = 0$$

by Arzelà-Ascoli's theorem, and (3.6) follows because $\mu\big(\overline{O}\backslash O\big) = 0$. ■

3.2. **Ru-usc integrands.** Let (X,d,μ) be a metric measure space, let $O \subset X$ be an open set, let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space and let $L : O\times\mathbb{M}\times\Omega \to [0,\infty]$ be a Borel measurable stochastic integrand. For each $\{a(\cdot,\omega)\}_\omega \subset L^1_\mu(O;]0,\infty])$ we define $\Delta^a_L : [0,1]\times\Omega \to]-\infty,\infty]$ by

$$\Delta^a_L(\tau,\omega) := \sup_{x\in O} \sup_{\xi\in\mathbb{L}_{x,\omega}} \frac{L(x,\tau\xi,\omega) - L(x,\xi,\omega)}{a(x,\omega) + L(x,\xi,\omega)},$$

where $\mathbb{L}_{x,\omega}$ denotes the effective domain of $L(x,\cdot,\omega)$.

Definition 3.9. Let $\omega \in \Omega$. We say that L is radially uniformly upper semicontinuous (ru-usc) at ω if there exists $a(\cdot,\omega) \in L^1_\mu(O;]0,\infty])$ such that

$$\overline{\lim_{\tau\to 1^-}}\ \Delta^a_L(\tau,\omega) \leqslant 0.$$

The concept of ru-usc integrand was introduced in [AH10] and then developed in [AHM11, AHM12a, AHM12b, Man13, AHM14, AHMZ15, AHM18].

Remark 3.10. If L is ru-usc at $\omega\in\Omega$ then $\overline{\lim}_{\tau\to 1^-} L(x,\tau\xi,\omega) \leqslant L(x,\xi,\omega)$ for all $x \in O$ and all $\xi \in \mathbb{L}_{x,\omega}$. On the other hand, given $\omega\in\Omega$, if there exist $x\in O$ and $\xi\in\mathbb{L}_{x,\omega}$ such that $L(x,\cdot,\omega)$ is lsc at ξ then, for each $a(\cdot,\omega)\in L^1_\mu(O;]0,\infty])$, $\underline{\lim}_{\tau\to 1^-}\Delta^a_L(\tau,\omega) \geqslant 0$, and so if in addition L is ru-usc at ω then $\lim_{\tau\to 1^-}\Delta^a_L(\tau,\omega) = 0$ for some $a(\cdot,\omega)\in L^1_\mu(O;]0,\infty])$.

Remark 3.11. Given $\omega\in\Omega$, if, for every $x\in O$, $L(x,\cdot,\omega)$ is convex and $0\in\mathbb{L}_{x,\omega}$, then L is ru-usc at ω.

The interest of Definition 3.9 comes from the following theorem. (For a proof we refer to [AHM11, Theorem 3.5] and also [AHM12b, §4.2].) Let $\widehat{L} : O \times \mathbb{M} \times \Omega \to [0,\infty]$ be defined by

$$\widehat{L}(x,\xi,\omega) := \varliminf_{\tau\to 1^-} L(x,\tau\xi,\omega).$$

Theorem 3.12. *Let $\omega \in \Omega$. If L is ru-usc at ω with $a(\cdot,\omega)$ and if for every $x \in O$,*

$$\tau\overline{\mathbb{L}}_{x,\omega} \subset \operatorname{int}(\mathbb{L}_{x,\omega}) \text{ for all } \tau \in]0,1[,$$

where $\mathbb{L}_{x,\omega}$ denotes the effective domain of $L(x,\cdot,\omega)$, then:

(i) $\widehat{L}(x,\xi,\omega) := \varliminf_{\tau\to 1^-} L(x,\tau\xi,\omega) = \begin{cases} \varliminf_{\tau\to 1^-} L(x,\tau\xi,\omega) & \text{if } \xi \in \overline{\mathbb{L}}_{x,\omega} \\ \infty & \text{otherwise;} \end{cases}$

(ii) *$\widehat{L}$ is ru-usc at ω with $a(\cdot,\omega)$.*

If moreover $L(x,\cdot,\omega)$ is lsc on $\operatorname{int}(\mathbb{L}_{x,\omega})$ then:

(iii) $\widehat{L}(x,\xi,\omega) = \begin{cases} L(x,\xi,\omega) & \text{if } \xi \in \operatorname{int}(\mathbb{L}_{x,\omega}) \\ \varliminf_{\tau\to 1^-} L(x,\tau\xi,\omega) & \text{if } \xi \in \partial\mathbb{L}_{x,\omega} \\ \infty & \text{otherwise;} \end{cases}$

(iv) *for every $x \in O$, $\widehat{L}(x,\cdot,\omega)$ is the lsc envelope of $L(x,\cdot,\omega)$.*

The following definition extends Definition 3.9 to a family $\{L_t\}_{t>0}$ of Borel measurable stochastic integrands $L_t : O\times\mathbb{M}\times\Omega \to [0,\infty]$. (When $L_t = L$ for all $t > 0$ we retrieve Definition 3.9.)

Definition 3.13. Let $\omega \in \Omega$. We say that $\{L_t\}_{t>0}$ is ru-usc at ω if there exists $\{a_t(\cdot,\omega)\}_{t>0} \subset L^1_\mu(O;]0,\infty])$, satisfying (2.1) and (2.2), such that

$$\overline{\varlimsup_{\tau\to 1^-}} \sup_{t>0} \Delta^{a_t}_{L_t}(\tau,\omega) \leqslant 0.$$

For each $t > 0$ and each $\rho > 0$, let $\mathscr{H}^\rho_\mu L_t : O \times \mathbb{M} \times \Omega \to [0,\infty]$ be defined by

$$\mathscr{H}^\rho_\mu L_t(x,\xi,\omega) := \inf\left\{ \fint_{B_\rho(x)} L_t(y,\xi+\nabla_\mu w(y),\omega)d\mu(y) : w \in H^{1,p}_{\mu,0}(B_\rho(x);\mathbb{R}^m) \right\},$$

where the space $H^{1,p}_{\mu,0}(B_\rho(x);\mathbb{R}^m)$ is defined as the closure of

$$\operatorname{Lip}_0(B_\rho(x);\mathbb{R}^m) := \Big\{ u \in \operatorname{Lip}(O;\mathbb{R}^m) : u = 0 \text{ on } O\backslash B_\rho(x) \Big\}$$

with respect to the $H^{1,p}_\mu$-norm, where $\operatorname{Lip}(O;\mathbb{R}^m) := [\operatorname{Lip}(O)]^m$ with $\operatorname{Lip}(O)$ denoting the algebra of Lipschitz functions from O to $\mathbb{R}$. Let $L_\infty : O \times \mathbb{M} \times \Omega \to [0,\infty]$ be given by

$$L_\infty(x,\xi,\omega) := \overline{\varlimsup_{\rho\to 0}}\, \overline{\varlimsup_{t\to\infty}}\, \mathscr{H}^\rho_\mu L_t(x,\xi,\omega). \tag{3.13}$$

The following proposition shows that ru-usc is conserved under the operation characterized by (3.13).

Proposition 3.14. *Let $\omega \in \Omega$. If $\{L_t\}_{t>0}$ is ru-usc at ω with $\{a_t(\cdot,\omega)\}_{t>0}$ then L_∞ is ru-usc at ω with $a_\infty(\cdot,\omega)$ given by* (2.2).

Proof of Proposition 3.14. Fix any $\tau \in [0,1]$, any $x \in O$ and any $\xi \in \mathbb{L}_{\infty,x,\omega}$, where $\mathbb{L}_{\infty,x,\omega}$ is the effective domain of $L_\infty(x,\cdot,\omega)$. Then $L_\infty(x,\xi,\omega) = \overline{\lim}_{\rho\to 0}\,\overline{\lim}_{t\to\infty} \mathscr{H}^\rho_\mu L_t(x,\xi,\omega) < \infty$ and without loss of generality we can suppose that $\mathscr{H}^\rho_\mu L_t(x,\xi,\omega) < \infty$ for all $\rho > 0$ and all $t > 0$. Fix any $\rho > 0$ and any $t > 0$. By definition, there exists $\{w_n\}_n \subset H^{1,p}_{\mu,0}(B_\rho(x);\mathbb{R}^m)$ such that:

$$\mathscr{H}^\rho_\mu L_t(x,\xi,\omega) = \lim_{n\to\infty} \fint_{B_\rho(x)} L_t(y,\xi+\nabla_\mu w_n(y),\omega)d\mu(y); \tag{3.14}$$

$$\xi + \nabla_\mu w_n(y) \in \mathbb{L}_{t,y,\omega} \text{ for all } n \geqslant 1 \text{ and } \mu\text{-a.a. } y \in B_\rho(x), \tag{3.15}$$

where $\mathbb{L}_{t,y,\omega}$ denotes the effective domain of $L_t(y,\cdot,\omega)$. Moreover, for every $n \geqslant 1$,

$$\mathscr{H}^\rho_\mu L_t(x,\tau\xi,\omega) \leqslant \fint_{B_\rho(x)} L_t\big(y,\tau(\xi+\nabla_\mu w_n(y)),\omega\big)d\mu(y)$$

since $\tau w_n \in H^{1,p}_{\mu,0}(B_\rho(x);\mathbb{R}^m)$, and so

$$\delta^\tau_{\rho,t}(x,\xi,\omega) \leqslant \underline{\lim}_{n\to\infty} \fint_{B_\rho(x)} \big(L_t(y,\tau(\xi+\nabla_\mu w_n(y)),\omega) - L_t(y,\xi+\nabla_\mu w_n(y),\omega)\big)d\mu(y) \tag{3.16}$$

with $\delta^\tau_{\rho,t}(x,\xi,\omega) := \mathscr{H}^\rho_\mu L_t(x,\tau\xi,\omega) - \mathscr{H}^\rho_\mu L_t(x,\xi,\omega)$. Taking (3.15) into account, for every $n \geqslant 1$ and μ-a.e. $y \in B_\rho(x)$, one has

$$\lambda^\tau_{t,n}(y,\xi,\omega) \leqslant \Delta^{a_t}_{L_t}(\tau,\omega)\big(a_t(y,\omega) + L_t(y,\xi+\nabla_\mu w_n(y),\omega)\big),$$

with $\lambda^\tau_{t,n}(y,\xi,\omega) := L_t\big(y,\tau(\xi+\nabla_\mu w_n(y)),\omega\big) - L_t\big(y,\xi+\nabla_\mu w_n(y),\omega\big)$, hence

$$\fint_{B_\rho(x)} \lambda^\tau_{t,n}(y,\xi,\omega)d\mu \leqslant \Delta^{a_t}_{L_t}(\tau,\omega)\left(\fint_{B_\rho(x)} a_t(y,\omega)d\mu + \fint_{B_\rho(x)} L_t(y,\xi+\nabla_\mu w_n(y),\omega)d\mu\right)$$

for all $n \geqslant 1$. Letting $n \to \infty$ and using (3.14) and (3.16), it follows that

$$\begin{aligned}\delta^\tau_{\rho,t}(x,\xi,\omega) &\leqslant \Delta^{a_t}_{L_t}(\tau,\omega)\left(\fint_{B_\rho(x)} a_t(y,\omega)d\mu(y) + \mathscr{H}^\rho_\mu L_t(x,\xi,\omega)\right)\\ &\leqslant \Delta(\tau,\omega)\left(\fint_{B_\rho(x)} a_t(y,\omega)d\mu(y) + \mathscr{H}^\rho_\mu L_t(x,\xi,\omega)\right)\end{aligned} \tag{3.17}$$

for all $\rho > 0$ and all $t > 0$, where $\Delta(\tau,\omega) := \sup_{s>0} \Delta^{a_s}_{L_s}(\tau,\omega)$. By letting $t \to \infty$ and $\rho \to 0$ in (3.17), we get

$$L_\infty(x,\tau\xi,\omega) - L_\infty(x,\xi,\omega) \leqslant \Delta(\tau,\omega)\big(a_\infty(x,\omega) + L_\infty(x,\xi,\omega)\big)$$

with $a_\infty(\cdot,\omega) \in L^1_\mu(O;]0,\infty])$ given by (2.2), which implies that $\Delta^{a_\infty}_{L_\infty}(\tau,\omega) \leqslant \Delta(\tau,\omega)$ for all $\tau \in [0,1]$. As $\{L_t\}_{t>0}$ is ru-usc at ω with $\{a_t(\cdot,\omega)\}_{t>0}$, i.e. $\overline{\lim}_{\tau\to 1^-} \Delta(\tau,\omega) \leqslant 0$, we conclude that $\overline{\lim}_{\tau\to 1^-} \Delta^{a_\infty}_{L_\infty}(\tau,\omega) \leqslant 0$ which means that L_∞ is ru-usc at ω with $a_\infty(\cdot,\omega)$. ■

Remark 3.15. In the proof of Proposition 3.14 we do not need (2.1) . In fact, (2.1) will be used in the proof of the Γ-convergence result (see Sect. 4).

3.3. **Integral representation of the Vitali envelope of a set function.** What follows was first developed in [BFM98, BB00] (see also [AHM16, AHM17, AHCM17, AHM18]). Let (O,d) be a metric space, let $\mathcal{O}(O)$ be the class of open subsets of O and let μ be a positive finite Radon measure on O. We begin with the concept of the Vitali envelope of a set function.

For each $\varepsilon>0$ and each $A\in\mathcal{O}(O)$, we denote the class of countable families $\{B_i:=B_{\rho_i}(x_i)\}_{i\in I}$ of disjoint open balls of A with $x_i\in A$, $\rho_i\in]0,\varepsilon[$ and $\mu(\partial B_i)=0$ such that $\mu(A\setminus\cup_{i\in I}B_i)=0$ by $\mathscr{V}_\varepsilon(A)$.

Definition 3.16. Given $\mathcal{S}:\mathcal{O}(O)\to[0,\infty]$, for each $\varepsilon>0$ we define $\mathcal{S}^\varepsilon:\mathcal{O}(O)\to[0,\infty]$ by

$$\mathcal{S}^\varepsilon(A):=\inf\left\{\sum_{i\in I}\mathcal{S}(B_i):\{B_i\}_{i\in I}\in\mathscr{V}_\varepsilon(A)\right\}.$$

By the Vitali envelope of $\mathcal{S}$ we call the set function $\mathcal{S}^*:\mathcal{O}(O)\to[-\infty,\infty]$ defined by

$$\mathcal{S}^*(A):=\sup_{\varepsilon>0}\mathcal{S}^\varepsilon(A)=\lim_{\varepsilon\to0}\mathcal{S}^\varepsilon(A).$$

The interest of Definition 3.16 comes from the following integral representation result. (For a proof we refer to [AHM18, §3.3] or [AHCM17, §A.4].)

Theorem 3.17. *Let $\mathcal{S}:\mathcal{O}(O)\to[0,\infty]$ be a set function satisfying the following two conditions*:

(i) *there exists a finite Radon measure ν on Ω which is absolutely continuous with respect to μ such that $\mathcal{S}(A)\leqslant\nu(A)$ for all $A\in\mathcal{O}(O)$;*
(ii) *$\mathcal{S}$ is subadditive, i.e. $\mathcal{S}(A)\leqslant\mathcal{S}(B)+\mathcal{S}(C)$ for all $A,B,C\in\mathcal{O}(O)$ with $B,C\subset A$, $B\cap C=\varnothing$ and $\mu(A\setminus B\cup C)=0$.*

Then $\lim_{\rho\to0}\frac{\mathcal{S}(B_\rho(\cdot))}{\mu(B_\rho(\cdot))}\in L^1_\mu(O)$ and for every $A\in\mathcal{O}(O)$, one has

$$\mathcal{S}^*(A)=\int_A\lim_{\rho\to0}\frac{\mathcal{S}(B_\rho(x))}{\mu(B_\rho(x))}d\mu(x).$$

4. Proofs

4.1. **Proof of the lower bound.** Here we prove Proposition 2.6.

Proof of Proposition 2.6. Fix $\omega\in\Omega$. Let $u\in H^{1,p}_\mu(O;\mathbb{R}^m)$ and let $\{u_t\}_{t>0}\subset H^{1,p}_\mu(O;\mathbb{R}^m)$ be such that $\|u_t-u\|_{L^p_\mu(O;\mathbb{R}^m)}\to0$. We have to prove that

$$\underline{\lim_{t\to\infty}}\,E_t(u_t,\omega)\geqslant\int_O\underline{\lim_{\tau\to1^-}}\,\overline{\lim_{\rho\to0}}\,\underline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x).\tag{4.1}$$

Without loss of generality we can assume that $\underline{\lim}_{t\to\infty}E_t(u_t,\omega)=\lim_{t\to\infty}E_t(u_t,\omega)<\infty$, and so

$$\sup_{t>0}E_t(u_t,\omega)<\infty.\tag{4.2}$$

In particular, $\sup_{t>0}\|\nabla_\mu u_t\|_{L^p_\mu(O;\mathbb{M})}<\infty$ because $\{L_t\}_{t>0}$ is p-coercive, see (C_6). Then

$$\nabla_\mu u_t(x)\in\mathbb{G}\text{ for all }t>0\text{ and }\mu\text{-a.a. }x\in O\tag{4.3}$$

and, up to a subsequence,

$$u_t \rightharpoonup u \text{ in } H^{1,p}_\mu(O;\mathbb{R}^m). \tag{4.4}$$

As $\mathbb{G}$ is convex, see (C_1) and Remark 2.2, from (4.3) and (4.4) it follows that

$$\nabla_\mu u(x) \in \overline{\mathbb{G}} \text{ for } \mu\text{-a.a. } x \in O. \tag{4.5}$$

As $\|u_t - u\|_{H^{1,p}_\mu(O;\mathbb{R}^m)} \to 0$, $\sup_{t>0} \|\nabla_\mu u_t\|_{L^p_\mu(O;\mathbb{M})} < \infty$ and $p > \kappa$, from Theorem 3.6 we deduce that, up to a subsequence,

$$\|u_t - u\|_{L^\infty_\mu(O;\mathbb{R}^m)} \to 0. \tag{4.6}$$

Step 1: localization. For each $t > 0$, we define the (positive) Radon measure ν_t on O by

$$\nu_t := L_t(\cdot, \nabla_\mu u_t(\cdot), \omega)d\mu.$$

From (4.2) we see that $\sup_{t>0} \nu_t(O) < \infty$, and so there exists a (positive) Radon measure ν on O such that, up to a subsequence, $\nu_t \rightharpoonup \nu$ weakly. By Lebesgue's decomposition theorem, we have $\nu = \nu^a + \nu^s$ where ν^a and ν^s are (positive) Radon measures on O such that $\nu^a \ll \mu$ and $\nu^s \perp \mu$. Thus, to prove (4.1) it suffices to show that

$$\nu^a \geqslant \underline{\lim}_{\tau\to 1^-} \overline{\lim_{\rho\to 0}} \underline{\lim}_{t\to\infty} \mathscr{H}^\rho_\mu L_t(\cdot, \tau\nabla_\mu u(\cdot), \omega)d\mu. \tag{4.7}$$

From Radon-Nikodym's theorem we have $\nu^a = f(\cdot)d\mu$ with

$$f(\cdot) := \lim_{\rho\to 0} \frac{\nu(B_\rho(\cdot))}{\mu(B_\rho(\cdot))} \in L^1_\mu(O;[0,\infty[),$$

and so to prove (4.7) it is sufficient to establish that for μ-a.e. $x_0 \in O$,

$$f(x_0) = \lim_{\rho\to 0} \frac{\nu(B_\rho(x_0))}{\mu(B_\rho(x_0))} \geqslant \underline{\lim}_{\tau\to 1^-} \overline{\lim_{\rho\to 0}} \underline{\lim}_{t\to\infty} \mathscr{H}^\rho_\mu L_t(x_0, \tau\nabla_\mu u(x_0), \omega). \tag{4.8}$$

Fix $x_0 \in O\backslash N$ where $N \subset O$ is a suitable set such that $\mu(N) = 0$. As $\nu(O) < \infty$, without loss of generality we can assume that $\nu(\partial B_\rho(x_0)) = 0$ for all $\rho > 0$, which implies, by Alexandrov's theorem, that $\nu(B_\rho(x_0)) = \lim_{t\to\infty} \nu_t(B_\rho(x_0))$. Consequently, to prove (4.8) it suffices to show that

$$\lim_{\rho\to 0} \lim_{t\to\infty} \fint_{B_\rho(x_0)} L_t(x, \nabla_\mu u_t(x), \omega)d\mu \geqslant \underline{\lim}_{\tau\to 1^-} \overline{\lim_{\rho\to 0}} \underline{\lim}_{t\to\infty} \mathscr{H}^\rho_\mu L_t(x_0, \tau\nabla_\mu u(x_0), \omega). \tag{4.9}$$

On the other hand, as $\mathbb{G}$ is convex, see (C_1) and Remark 2.2, and $0 \in \mathrm{int}(\mathbb{G})$, see (C_2), from (4.3) we can assert for every $\sigma \in]0,1[$,

$$\sigma\nabla_\mu u_t(x) \in \mathbb{G} \text{ for all } t > 0 \text{ and for } \mu\text{-a.a. } x \in O.$$

Hence, given any $\tau \in]0,1[$, we see that for every $t > 0$ and every $\rho > 0$,

$$\begin{aligned} \fint_{B_\rho(x_0)} L_t(x, \tau\nabla_\mu u_t(x), \omega)d\mu \;\leqslant\; & \big(1 + \Delta(\tau,\omega)\big) \fint_{B_\rho(x_0)} L_t(x, \nabla_\mu u_t(x), \omega)d\mu \\ & + \Delta(\tau,\omega) \fint_{B_\rho(x_0)} a_t(x,\omega)d\mu \end{aligned}$$

with $\Delta(\tau,\omega) := \sup_{t>0} \Delta_{L_t}^{a_t}(\tau,\omega)$, where $\Delta_{L_t}^{a_t}(\tau,\omega)$ is given by (2.3). Letting $t \to \infty$ and $\rho \to 0$ we obtain

$$\begin{aligned}\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \fint_{B_\rho(x_0)} L_t(x,\tau\nabla_\mu u_t(x),\omega)d\mu \;\leqslant\; \big(1+\Delta(\tau,\omega)\big)\lim_{\rho\to 0}\lim_{t\to\infty}\fint_{B_\rho(x_0)} L_t(x,\nabla_\mu u_t(x),\omega)d\mu \\ + \overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\Delta(\tau,\omega)\fint_{B_\rho(x_0)} a_t(x,\omega)d\mu.\end{aligned}$$

But, from (C_8) we have

$$\overline{\lim_{\tau\to 1^-}}\,\Delta(\tau,\omega) \leqslant 0$$

and, by (2.2),

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \fint_{B_\rho(x_0)} a_t(x,\omega)d\mu < \infty$$

with $\fint_{B_\rho(x_0)} a_t(x,\omega)d\mu \geqslant 0$, hence

$$\overline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\Delta(\tau,\omega)\fint_{B_\rho(x_0)} a_t(x,\omega)d\mu \leqslant 0,$$

and consequently

$$\overline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \fint_{B_\rho(x_0)} L_t(x,\tau\nabla_\mu u_t(x),\omega)d\mu \leqslant \lim_{\rho\to 0}\lim_{t\to\infty}\fint_{B_\rho(x_0)} L_t(x,\nabla_\mu u_t(x),\omega)d\mu.$$

Thus, to prove (4.9) it is sufficient to show that

$$\overline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \fint_{B_\rho(x_0)} L_t(x,\tau\nabla_\mu u_t(x),\omega)d\mu \geqslant \underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\underline{\lim_{t\to\infty}}\, \mathscr{H}_\mu^\rho L_t(x_0,\tau\nabla_\mu u(x_0),\omega). \quad (4.10)$$

Step 2: cut-off method. Fix any $t > 0$, any $\tau \in]0,1[$, any $\sigma \in]\tau,1[$, any $\lambda \in]0,1[$ and any $\rho > 0$. By Proposition 3.4(v) there is a Uryshon function $\varphi \in \mathrm{Lip}(O)$ for the pair $(O\backslash B_\rho(x_0), \overline{B}_{\lambda\rho}(x_0))$ such that $\|D_\mu\varphi\|_{L^\infty_\mu(O;\mathbb{R}^N)} \leqslant \frac{\theta}{\rho(1-\lambda)}$ for some $\theta > 0$ (which does not depend on ρ). Define $v_t \in H^{1,p}_\mu(O;\mathbb{R}^m)$ by

$$v_t := \varphi u_t + (1-\varphi)u_{x_0} = \varphi(u_t - u_{x_0}) + u_{x_0}$$

with $u_{x_0} \in H^{1,p}_\mu(O;\mathbb{R}^m)$ given by Proposition 3.4(iv). (Note that $\nabla_\mu u_{x_0}(x) = \nabla_\mu u(x_0)$ for μ-a.a. $x \in O$.) Then

$$\tau v_t - \tau u_{x_0} \in H^{1,p}_{\mu,0}(B_\rho(x_0);\mathbb{R}^m) \quad (4.11)$$

and, using Proposition 3.4(vii),

$$\tau\nabla_\mu v_t = \begin{cases} \tau\nabla_\mu u_t & \text{in } \overline{B}_{\lambda\rho}(x_0) \\ \frac{\tau}{\sigma}\big(\varphi\sigma\nabla_\mu u_t + (1-\varphi)\sigma\nabla_\mu u(x_0)\big) + \big(1-\frac{\tau}{\sigma}\big)\,\Psi_{t,\rho} & \text{in } B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)\end{cases} \quad (4.12)$$

with $\Psi_{t,\rho} := \frac{\tau}{1-\frac{\tau}{\sigma}} D_\mu\varphi \otimes (u_t - u_{x_0})$. Using the right inequality in (C_7) it follows that

$$\begin{aligned}
\fint_{B_\rho(x_0)} L_t(x,\tau\nabla_\mu v_t,\omega)d\mu &= \frac{1}{\mu(B_\rho(x_0))}\int_{\overline{B}_{\lambda\rho}(x_0)} L_t(x,\tau\nabla_\mu u_t,\omega)d\mu \\
&\quad + \frac{1}{\mu(B_\rho(x_0))}\int_{B_\rho(x_0)\setminus\overline{B}_{\lambda\rho}(x_0)} L_t(x,\tau\nabla_\mu v_t,\omega)d\mu \\
&\leqslant \fint_{B_\rho(x_0)} L_t(x,\tau\nabla_\mu u_t,\omega)d\mu + \beta\frac{\mu(B_\rho(x_0)\setminus\overline{B}_{\lambda\rho}(x_0))}{\mu(B_\rho(x_0))} \\
&\quad + \frac{\beta}{\mu(B_\rho(x_0))}\int_{B_\rho(x_0)\setminus\overline{B}_{\lambda\rho}(x_0)} G(\tau\nabla_\mu v_t)d\mu. \qquad (4.13)
\end{aligned}$$

On the other hand, taking (4.12) into account and using (C_1) and the left inequality in (C_7), we have

$$\begin{aligned}
G(\tau\nabla_\mu v_t) &\leqslant c_1\left(1 + G(\sigma\nabla_\mu u_t) + G(\sigma\nabla_\mu u(x_0)) + G(\Psi_{t,\rho})\right) \\
&\leqslant c_1\left(1 + \frac{1}{\alpha}L_t(x,\sigma\nabla_\mu u_t,\omega) + G(\sigma\nabla_\mu u(x_0)) + G(\Psi_{t,\rho})\right) \qquad (4.14)
\end{aligned}$$

with $c_1 := 2(\gamma+\gamma^2) > 0$. Note that from (C_2) and (4.5) we can assert that $\sigma\nabla_\mu u(x_0) \in \mathbb{G}$, and so

$$G(\sigma\nabla_\mu u(x_0)) < \infty.$$

Moreover, it is easy to see that

$$\begin{aligned}
\|\Psi_{t,\rho}\|_{L^\infty_\mu(B_\rho(x_0);\mathbb{M})} &\leqslant \frac{\theta\tau}{(1-\frac{\tau}{\sigma})(1-\lambda)}\frac{1}{\rho}\|u-u_{x_0}\|_{L^\infty_\mu(B_\rho(x_0);\mathbb{R}^m)} \\
&\quad + \frac{\theta\tau}{\rho(1-\frac{\tau}{\sigma})(1-\lambda)}\|u_t-u\|_{L^\infty_\mu(O;\mathbb{R}^m)},
\end{aligned}$$

where

$$\lim_{\rho\to 0}\frac{\theta\tau}{(1-\frac{\tau}{\sigma})(1-\lambda)}\frac{1}{\rho}\|u-u_{x_0}\|_{L^\infty_\mu(B_\rho(x_0);\mathbb{R}^m)} = 0 \qquad (4.15)$$

by Proposition 3.4(iv), i.e., $\lim_{\rho\to 0}\frac{1}{\rho}\|u-u_{x_0}\|_{L^\infty_\mu(B_\rho(x_0);\mathbb{R}^m)} = 0$, and

$$\lim_{t\to\infty}\frac{\theta\tau}{\rho(1-\frac{\tau}{\sigma})(1-\lambda)}\|u_t-u\|_{L^\infty_\mu(O;\mathbb{R}^m)} = 0 \qquad (4.16)$$

by (4.6), i.e., $\lim_{t\to\infty}\|u_t-u\|_{L^\infty_\mu(O;\mathbb{R}^m)} = 0$. From (C_1) and (C_2) there exists $r>0$ such that

$$c_2 := \sup_{|\xi|\leqslant r} G(\xi) < \infty$$

(see Remark 2.2). By (4.15) there exists $\bar\rho > 0$ such that $\frac{\theta\tau}{(1-\frac{\tau}{\sigma})(1-\lambda)}\frac{1}{\rho}\|u-u_{x_0}\|_{L^\infty_\mu(B_\rho(x_0);\mathbb{R}^m)} < \frac{r}{2}$ for all $\rho\in]0,\bar\rho[$. Fix any $\rho\in]0,\bar\rho[$. Taking (4.16) into account we can assert that there exists $t_\rho > 0$ such that

$$G(\Psi_{t,\rho}) \leqslant c_2 \text{ for all } t\in]0,t_\rho[. \qquad (4.17)$$

Thus, from (4.13), (4.14) and (4.17) we deduce that

$$\fint_{B_\rho(x_0)} L_t(x,\tau\nabla_\mu v_t,\omega)d\mu \leqslant \fint_{B_\rho(x_0)} L_t(x,\tau\nabla_\mu u_t,\omega)d\mu + c_3(\sigma)\gamma_{\rho,\lambda} + \frac{\beta c_1}{\alpha}\Gamma_{t,\rho,\lambda,\sigma}$$

for all $t \in]0,t_\rho[$ with:

$$\begin{aligned} c_3(\sigma) &:= \beta c_1\left(1+\frac{1}{c_1}+G(\sigma\nabla_\mu u(x_0))+c_2\right) \in]0,\infty[\ ;\\ \gamma_{\rho,\lambda} &:= \frac{\mu(B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0))}{\mu(B_\rho(x_0))}\ ;\\ \Gamma_{t,\rho,\lambda,\sigma} &:= \frac{1}{\mu(B_\rho(x_0))}\int_{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0))} L_t(x,\sigma\nabla_\mu u_t,\omega)d\mu. \end{aligned}$$

But, taking (4.11) into account, we see that

$$\mathscr{H}^\rho_\mu L_t(x_0,\tau\nabla_\mu u(x_0),\omega) \leqslant \fint_{B_\rho(x_0)} L_t(x,\tau\nabla_\mu v_t,\omega)d\mu,$$

hence, for every $\rho>0$, every $t\in]0,t_\rho[$, every $\lambda\in]0,1[$, every $\tau\in]0,1[$ and every $\sigma\in]\tau,1[$, we have

$$\mathscr{H}^\rho_\mu L_t(x_0,\tau\nabla_\mu u(x_0),\omega) \leqslant \fint_{B_\rho(x_0)} L_t(x,\tau\nabla_\mu u_t,\omega)d\mu + c_3(\sigma)\gamma_{\rho,\lambda} + \frac{\beta c_1}{\alpha}\Gamma_{t,\rho,\lambda,\sigma}. \tag{4.18}$$

Step 3: passing to the limit. Letting $t\to\infty$, $\rho\to 0$, $\lambda\to 1^-$, $\sigma\to 1^-$ and $\tau\to 1^-$ in (4.18), we obtain

$$\begin{aligned} \underline{\lim}_{\tau\to1^-}\overline{\lim}_{\rho\to0}\underline{\lim}_{t\to\infty}\mathscr{H}^\rho_\mu L_t(x_0,\tau\nabla_\mu u(x_0),\omega) \ \leqslant\ & \overline{\lim}_{\tau\to1^-}\overline{\lim}_{\rho\to0}\overline{\lim}_{t\to\infty}\fint_{B_\rho(x_0)} L_t(x,\tau\nabla_\mu u_t,\omega)d\mu\\ &+\overline{\lim}_{\sigma\to1^-}c_3(\sigma)\,\overline{\lim}_{\lambda\to1^-}\overline{\lim}_{\rho\to0}\gamma_{\rho,\lambda}\\ &+\frac{\beta c_1}{\alpha}\,\overline{\lim}_{\sigma\to1^-}\overline{\lim}_{\lambda\to1^-}\overline{\lim}_{\rho\to0}\overline{\lim}_{t\to\infty}\Gamma_{t,\rho,\lambda,\sigma}. \end{aligned} \tag{4.19}$$

Substep 3-1: proving that $\overline{\lim}_{\lambda\to1^-}\overline{\lim}_{\rho\to0}\gamma_{\rho,\lambda}=0$. As the boundary of any ball is of zero measure (see Remark 3.3), we have

$$\gamma_{\rho,\lambda} = 1-\frac{\mu(B_{\lambda\rho}(x_0))}{\mu(B_\rho(x_0))},$$

hence

$$\overline{\lim}_{\rho\to0}\gamma_{\rho,\lambda} = 1-\underline{\lim}_{\rho\to0}\frac{\mu(B_{\lambda\rho}(x_0))}{\mu(B_\rho(x_0))},$$

and so, by using Proposition (3.4)(vi),

$$\lim_{\lambda\to1^-}\overline{\lim}_{\rho\to0}\gamma_{\rho,\lambda}=0. \tag{4.20}$$

Substep 3-2: proving that $\overline{\lim}_{\sigma\to 1^-}\overline{\lim}_{\lambda\to 1^-}\overline{\lim}_{\rho\to 0}\overline{\lim}_{t\to\infty}\Gamma_{t,\rho,\lambda,\sigma}=0$. For every $t\in]0,t_\rho[$, we have

$$\begin{aligned}\Gamma_{t,\rho,\lambda,\sigma} &\leqslant \big(1+\Delta(\sigma,\omega)\big)\frac{\nu_t\left(B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0))\right)}{\mu(B_\rho(x_0))}\\ &\quad+\Delta(\sigma,\omega)\frac{1}{\mu(B_\rho(x_0))}\int_{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0))}a_t(x,\omega)d\mu. \end{aligned}\tag{4.21}$$

But $\overline{\lim}_{\sigma\to 1^-}\Delta(\sigma,\omega)\leqslant 0$ by (C_8), and by (2.2) we have

$$\overline{\lim_{\lambda\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\frac{1}{\mu(B_\rho(x_0))}\int_{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0))}a_t(x,\omega)d\mu\leqslant\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\fint_{B_\rho(x_0)}a_t(x,\omega)d\mu<\infty$$

with $\frac{1}{\mu(B_\rho(x_0))}\int_{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0))}a_t(x,\omega)d\mu\geqslant 0$, hence

$$\overline{\lim_{\sigma\to 1^-}}\,\overline{\lim_{\lambda\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\Delta(\sigma,\omega)\frac{1}{\mu(B_\rho(x_0))}\int_{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0))}a_t(x,\omega)d\mu\leqslant 0.\tag{4.22}$$

Since (X,d,μ) is a complete doubling metric space, (X,d,μ) is proper, i.e. every closed ball is compact. Hence $\overline{B}_\rho(x_0)$ is compact, and so $\overline{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)}$ is compact because $\overline{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)}\subset\overline{B}_\rho(x_0)$. As $\nu_t\to\nu$ weakly, by Alexandrov's theorem, we have

$$\overline{\lim_{t\to\infty}}\,\nu_t\left(\overline{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)}\right)\leqslant\nu\left(\overline{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)}\right),$$

and consequently

$$\overline{\lim_{t\to\infty}}\,\nu_t\left(B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)\right)\leqslant\nu\left(\overline{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)}\right).\tag{4.23}$$

As $\overline{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)}\subset\overline{B}_\rho(x_0)\subset O\backslash\mathrm{supp}(\nu^s)$, where $\mathrm{supp}(\nu^s)$ denotes the support of ν^s, we have

$$\nu\left(\overline{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)}\right)=\nu^a\left(\overline{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)}\right)=\int_{\overline{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)}}f(x)d\mu.\tag{4.24}$$

But $\overline{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)}\backslash\left(B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)\right)=\left(\overline{B}_\rho(x_0)\backslash B_\rho(x_0)\right)\cup\left(\overline{B}_{\lambda\rho}(x_0)\backslash B_{\lambda\rho}(x_0)\right)$, hence

$$\mu\left(\overline{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)}\backslash\left(B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)\right)\right)\leqslant\mu\left(\overline{B}_\rho(x_0)\backslash B_\rho(x_0)\right)+\mu\left(\overline{B}_{\lambda\rho}(x_0)\backslash B_{\lambda\rho}(x_0)\right),$$

and so, since the boundary of any ball is of zero measure,

$$\mu\left(\overline{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)}\backslash\left(B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)\right)\right)=0.\tag{4.25}$$

From (4.23), (4.24) and (4.25) (and by using again the fact that the boundary of any ball is of zero measure) it follows that

$$\begin{aligned}\overline{\lim_{t\to\infty}}\,\frac{\nu_t\left(B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)\right)}{\mu(B_\rho(x_0))} &\leqslant \frac{1}{\mu(B_\rho(x_0))}\int_{B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)} f(x)d\mu\\ &= \fint_{B_\rho(x_0)} f(x)d\mu - \frac{1}{\mu(B_\rho(x_0))}\int_{B_{\lambda\rho}(x_0)} f(x)d\mu\\ &= \fint_{B_\rho(x_0)} f(x)d\mu - \frac{\mu(B_{\lambda\rho}(x_0))}{\mu(B_\rho(x_0))}\fint_{B_{\lambda\rho}(x_0)} f(x)d\mu.\end{aligned}$$

Thus, recalling that $f\in L^1_\mu(O;[0,\infty[)$,

$$\begin{aligned}\overline{\lim_{\rho\to0}}\,\overline{\lim_{t\to\infty}}\,\frac{\nu_t\left(B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)\right)}{\mu(B_\rho(x_0))} &\leqslant \lim_{\rho\to0}\fint_{B_\rho(x_0)} f(x)d\mu - \underline{\lim_{\rho\to0}}\,\frac{\mu(B_{\lambda\rho}(x_0))}{\mu(B_\rho(x_0))}\lim_{\rho\to0}\fint_{B_{\lambda\rho}(x_0)} f(x)d\mu\\ &= \left(1-\underline{\lim_{\rho\to0}}\,\frac{\mu(B_{\lambda\rho}(x_0))}{\mu(B_\rho(x_0))}\right)f(x_0),\end{aligned}$$

and so, by using Proposition (3.4)(vi),

$$\lim_{\lambda\to1^-}\overline{\lim_{\rho\to0}}\,\overline{\lim_{t\to\infty}}\,\frac{\nu_t\left(B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)\right)}{\mu(B_\rho(x_0))}=0.$$

with $\frac{\nu_t\left(B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)\right)}{\mu(B_\rho(x_0))}\geqslant 0$. Consequently, by using (C_8),

$$\overline{\lim_{\sigma\to1^-}}\,\lim_{\lambda\to1^-}\,\overline{\lim_{\rho\to0}}\,\overline{\lim_{t\to\infty}}\left(1+\Delta(\sigma,\omega)\right)\frac{\nu_t\left(B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0)\right)}{\mu(B_\rho(x_0))}\leqslant 0. \tag{4.26}$$

From (4.21), (4.22) and (4.26) we deduce that

$$\overline{\lim_{\sigma\to1^-}}\,\overline{\lim_{\lambda\to1^-}}\,\overline{\lim_{\rho\to0}}\,\overline{\lim_{t\to\infty}}\,\Gamma_{t,\rho,\lambda,\sigma}=0. \tag{4.27}$$

Substep 3-3: end of the proof. Combining (4.20) and (4.27) with (4.19) we obtain (4.10), and the proof of the lower bound is complete. ■

4.2. **Proof of the upper bound.** Here we prove Proposition 2.7.

Proof of Proposition 2.7. Fix $\omega\in\Omega$. For each $u\in H^{1,p}_\mu(O;\mathbb{R}^m)$, let $\overline{\mathrm{m}}_{u,\omega}:\mathcal{O}(O)\to[0,\infty]$ be defined by

$$\overline{\mathrm{m}}_{u,\omega}(A):=\overline{\lim_{t\to\infty}}\,\mathrm{m}^t_{u,\omega}(A).$$

with, for each $t>0$, $\mathrm{m}^t_{u,\omega}:\mathcal{O}(O)\to[0,\infty]$ given by

$$\mathrm{m}^t_{u,\omega}(A)=\inf\left\{\int_A L_t(x,\nabla_\mu v(x),\omega)d\mu(x) : v-u\in H^{1,p}_{\mu,0}(A;\mathbb{R}^m)\right\}. \tag{4.28}$$

For each $\varepsilon>0$ and each $A\in\mathcal{O}(O)$, we denote the class of countable families $\{B_i:=B_{\rho_i}(x_i)\}_{i\in I}$ of disjoint open balls of A with $x_i\in A$, $\rho_i\in]0,\varepsilon[$ and $\mu(\partial B_i)=0$ such that $\mu(A\backslash\cup_{i\in I}B_i)=0$

by $\mathscr{V}_\varepsilon(A)$, and we consider $\overline{\mathrm{m}}^\varepsilon_{u,\omega} : \mathscr{O}(O) \to [0,\infty]$ given by

$$\overline{\mathrm{m}}^\varepsilon_{u,\omega}(A) := \inf\left\{\sum_{i\in I}\overline{\mathrm{m}}_{u,\omega}(B_i) : \{B_i\}_{i\in I} \in \mathscr{V}_\varepsilon(A)\right\},$$

and we define $\overline{\mathrm{m}}^*_{u,\omega} : \mathscr{O}(O) \to [0,\infty]$ by

$$\overline{\mathrm{m}}^*_{u,\omega}(A) := \sup_{\varepsilon>0}\overline{\mathrm{m}}^\varepsilon_{u,\omega}(A) = \lim_{\varepsilon\to 0}\overline{\mathrm{m}}^\varepsilon_{u,\omega}(A).$$

The set function $\overline{\mathrm{m}}^*_{u,\omega}$ is called the Vitali envelope of $\overline{\mathrm{m}}_{u,\omega}$ (see §3.3).

Step 1: link between Γ-$\overline{\lim}$ and Vitali envelope. Let $u \in H^{1,p}_\mu(O;\mathbb{R}^m)$. We are going to prove that

$$\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}} E_t(u,\omega) \leqslant \overline{\mathrm{m}}^*_{u,\omega}(O). \tag{4.29}$$

Without loss of generality we can assume that $\overline{\mathrm{m}}^*_{u,\omega}(O) < \infty$. Fix any $\varepsilon > 0$. By definition of $\overline{\mathrm{m}}^\varepsilon_{u,\omega}(O)$ there exists $\{B_i\}_{i\in I} \in \mathscr{V}_\varepsilon(O)$ such that

$$\sum_{i\in I}\overline{\mathrm{m}}_{u,\omega}(B_i) \leqslant \overline{\mathrm{m}}^\varepsilon_{u,\omega}(O) + \frac{\varepsilon}{2}. \tag{4.30}$$

Fix any $t > 0$. For each $i \in I$, by definition of $\mathrm{m}^t_{u,\omega}(B_i)$ there exists $v^i_t \in H^{1,p}_\mu(O;\mathbb{R}^m)$ such that $v^i_t - u \in H^{1,p}_{\mu,0}(B_i;\mathbb{R}^m)$ and

$$\int_{B_i} L_t\left(x, \nabla_\mu v^i_t(x), \omega\right) d\mu(x) \leqslant \mathrm{m}^t_{u,\omega}(B_i) + \frac{\varepsilon\mu(B_i)}{2\mu(O)}. \tag{4.31}$$

Define $u^\varepsilon_t : O \to \mathbb{R}^m$ by

$$u^\varepsilon_t := \begin{cases} u & \text{in } O\setminus \underset{i\in I}{\cup} B_i \\ v^i_t & \text{in } B_i. \end{cases}$$

Then $u^\varepsilon_t - u \in H^{1,p}_{\mu,0}(O;\mathbb{R}^m)$. Moreover, because of Proposition 3.4(ii), $\nabla_\mu u^\varepsilon_t(x) = \nabla_\mu v^i_t(x)$ for μ-a.a. $x \in B_i$. From (4.31) we see that

$$E_t(u^\varepsilon_t,\omega) \leqslant \sum_{i\in I}\mathrm{m}^t_{u,\omega}(B_i) + \frac{\varepsilon}{2},$$

hence $\overline{\lim}_{t\to\infty} E(u^\varepsilon_t,\omega) \leqslant \overline{\mathrm{m}}^\varepsilon_{u,\omega}(O) + \varepsilon$ by using (4.30), and consequently

$$\overline{\lim_{\varepsilon\to 0}}\,\overline{\lim_{t\to\infty}} E_t(u^\varepsilon_t,\omega) \leqslant \overline{\mathrm{m}}^*_{u,\omega}(O). \tag{4.32}$$

On the other hand, we have

$$\|u^\varepsilon_t - u\|^p_{L^p_\mu(O;\mathbb{R}^m)} = \int_O \left|u^\varepsilon_t - u\right|^p d\mu = \sum_{i\in I}\int_{B_i}\left|v^i_t - u\right|^p d\mu.$$

As O supports a p-Sobolev inequality, see Proposition 3.4(iii), and $\rho_i \in]0,\varepsilon[$ for all $i \in I$, we have

$$\sum_{i\in I}\int_{Q_i}\left|v^i_t - u\right|^p d\mu \leqslant \varepsilon^p C^p_S \sum_{i\in I}\int_{Q_i}\left|\nabla_\mu v^i_t - \nabla_\mu u\right|^p d\mu$$

with $C_S > 0$, and so

$$\|u_t^\varepsilon - u\|^p_{L^p_\mu(O;\mathbb{R}^m)} \leqslant 2^p \varepsilon^p C_S^p \left(\sum_{i\in I} \int_{B_i} |\nabla_\mu v_t^i|^p d\mu + \int_O |\nabla_\mu u|^p d\mu \right). \tag{4.33}$$

Taking (C_6), (4.30) and (4.31) into account, from (4.33) we deduce that

$$\overline{\lim_{t\to\infty}} \|u_t^\varepsilon - u\|^p_{L^p_\mu(O;\mathbb{R}^m)} \leqslant 2^p \varepsilon^p C_S^p \left(\frac{1}{c}\big(\overline{\mathrm{m}}^\varepsilon_{u,\omega}(O) + \varepsilon\big) + \int_O |\nabla_\mu u|^p d\mu \right)$$

with $c > 0$, which gives

$$\overline{\lim_{\varepsilon\to 0}} \overline{\lim_{t\to\infty}} \|u_t^\varepsilon - u\|^p_{L^p_\mu(O;\mathbb{R}^m)} = 0 \tag{4.34}$$

because $\lim_{\varepsilon\to 0} \overline{\mathrm{m}}^\varepsilon_{u,\omega}(O) = \overline{\mathrm{m}}^*_{u,\omega}(O) < \infty$. According to (4.32) and (4.34), by diagonalization there exists a mapping $t \mapsto \varepsilon_t$, with $\varepsilon_t \to 0$ as $t \to \infty$, such that:

$$\lim_{t\to\infty} \|w_t - u\|^p_{L^p_\mu(O;\mathbb{R}^m)} = 0; \tag{4.35}$$

$$\overline{\lim_{t\to\infty}} E_t(w_t, \omega) \leqslant \overline{\mathrm{m}}^*_{u,\omega}(O) \tag{4.36}$$

with $w_t := u_t^{\varepsilon_t}$. By (4.35) we have $\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,\omega) \leqslant \overline{\lim}_{t\to\infty} E_t(w_t,\omega)$, and (4.29) follows from (4.36).

Step 2: differentiation with respect to $\boldsymbol{\mu}$. Let $u \in H^{1,p}_\mu(O;\mathbb{R}^m)$ be such that $\mathscr{G}(u) := \int_O G(\nabla_\mu u(x)) d\mu(x) < \infty$. We are going to prove that

$$\overline{\mathrm{m}}^*_{u,\omega}(O) = \int_O \lim_{\rho\to 0} \frac{\overline{\mathrm{m}}_{u,\omega}(B_\rho(x))}{\mu(B_\rho(x))} d\mu(x). \tag{4.37}$$

According to Theorem 3.17, to prove (4.37) it suffices to establish that $\overline{\mathrm{m}}_{u,\omega}$ is subadditive and there exists a finite Radon measure ν on O which is absolutely continuous with respect to μ such that

$$\overline{\mathrm{m}}_{u,\omega}(A) \leqslant \nu(A) \tag{4.38}$$

for all $A \in \mathscr{O}(O)$. For each $t > 0$, from the definition of $\mathrm{m}^t_{u,\omega}$ in (4.28), it is easy to see that for every $A, B, C \in \mathscr{O}(O)$ with $B, C \subset A$, $B \cap C = \varnothing$ and $\mu(A\backslash B \cup C) = 0$, one has

$$\mathrm{m}^t_{u,\omega}(A) \leqslant \mathrm{m}^t_{u,\omega}(B) + \mathrm{m}^t_{u,\omega}(C),$$

and so

$$\overline{\lim_{t\to\infty}} \mathrm{m}^t_{u,\omega}(A) \leqslant \overline{\lim_{t\to\infty}} \mathrm{m}^t_{u,\omega}(B) + \overline{\lim_{t\to\infty}} \mathrm{m}^t_{u,\omega}(C),$$

i.e.

$$\overline{\mathrm{m}}_{u,\omega}(A) \leqslant \overline{\mathrm{m}}_{u,\omega}(B) + \overline{\mathrm{m}}_{u,\omega}(C),$$

which shows the subadditivity of $\overline{\mathrm{m}}_{u,\omega}$. On the other hand, given any $t > 0$, by using the right inequality in (C_7) we have

$$\mathrm{m}^t_{u,\omega}(A) \leqslant \int_A \beta\big(1 + G(\nabla_\mu u(x))\big) d\mu(x).$$

Thus (4.38) holds with the Radon measure $\nu := \beta\big(1 + G(\nabla_\mu u(\cdot))\big)\mu$ which is necessarily finite since $\mathscr{G}(u) < \infty$.

Step 3: cut-off method. Let $\tau \in]0,1[$, let $\sigma \in]\tau,1[$ and let $u \in H^{1,p}_\mu(O;\mathbb{R}^m)$ be such that $\mathscr{G}(\sigma u) < \infty$. We are going to prove that for μ-a.e. $x \in O$,

$$\overline{\lim_{\rho\to 0}} \frac{\overline{\mathrm{m}}_{\tau u,\omega}(B_\rho(x))}{\mu(B_\rho(x))} \leqslant \overline{\lim_{\rho\to 0}} \frac{\overline{\mathrm{m}}_{\tau u_x,\omega}(B_\rho(x))}{\mu(B_\rho(x))}, \tag{4.39}$$

where $u_x \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ is given by Proposition 3.4(iv).

Remark 4.1. For μ-a.e. $x \in O$, one has

$$\overline{\lim_{\rho\to 0}} \frac{\overline{\mathrm{m}}_{\tau u_x,\omega}(B_\rho(x))}{\mu(B_\rho(x))} = \overline{\lim_{\rho\to 0}}\, \overline{\lim_{t\to\infty}}\, \mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega).$$

Remark 4.2. If $\mathscr{G}(\tau u) < \infty$ then $\mathscr{G}(\tau u_x) < \infty$ for μ-a.a. $x \in O$, and so, by the step 2,

$$\overline{\lim_{\rho\to 0}} \frac{\overline{\mathrm{m}}_{\tau u,\omega}(B_\rho(x))}{\mu(B_\rho(x))} = \lim_{\rho\to 0} \frac{\overline{\mathrm{m}}_{\tau u,\omega}(B_\rho(x))}{\mu(B_\rho(x))} \text{ and } \overline{\lim_{\rho\to 0}} \frac{\overline{\mathrm{m}}_{\tau u_x,\omega}(B_\rho(x))}{\mu(B_\rho(x))} = \lim_{\rho\to 0} \frac{\overline{\mathrm{m}}_{\tau u_x,\omega}(B_\rho(x))}{\mu(B_\rho(x))}.$$

Fix any $t > 0$, any $\lambda \in]0,1[$, any $\rho > 0$ and any $\varepsilon > 0$. By definition of $\mathrm{m}^t_{\tau u_x,\omega}(B_{\lambda\rho}(x))$ in (4.28), there exists $w \in H^{1,p}_\mu(O;\mathbb{R}^m)$ such that

$$\tau w - \tau u_x \in H^{1,p}_{\mu,0}(B_{\lambda\rho}(x);\mathbb{R}^m) \tag{4.40}$$

and

$$\int_{B_{\lambda\rho}(x)} L_t(y,\tau\nabla_\mu w(y),\omega)d\mu(y) \leqslant \mathrm{m}^t_{\tau u_x,\omega}(B_{\lambda\rho}(x)) + \varepsilon\mu(B_{\lambda\rho}(x)). \tag{4.41}$$

By Proposition 3.4(v) there is a Urysohn function $\varphi \in \mathrm{Lip}(\Omega)$ for the pair $(\Omega\backslash B_\rho(x), \overline{B}_{\lambda\rho}(x))$ such that

$$\|D_\mu\varphi\|_{L^\infty_\mu(\Omega;\mathbb{R}^N)} \leqslant \frac{\theta}{\rho(1-\lambda)}$$

for some $\theta > 0$ (which does not depend on ρ). Define $v \in H^{1,p}_\mu(O;\mathbb{R}^m)$ by

$$v := \varphi u_x + (1-\varphi)u = \varphi(u_x - u) + u.$$

Then

$$\tau v - \tau u \in H^{1,p}_{\mu,0}(B_\rho(x);\mathbb{R}^m). \tag{4.42}$$

and, using Proposition 3.4(iv), i.e. $\nabla_\mu u_x(y) = \nabla_\mu u(x)$ for μ-a.a. $y \in O$, and Proposition 3.4(vii),

$$\tau\nabla_\mu v = \begin{cases} \tau\nabla_\mu u(x) & \text{in } \overline{B}_{\lambda\rho}(x_0) \\ \frac{\tau}{\sigma}\big(\varphi\sigma\nabla_\mu u(x) + (1-\varphi)\sigma\nabla_\mu u\big) + \big(1-\frac{\tau}{\sigma}\big)\,\Psi_\rho & \text{in } B_\rho(x_0)\backslash\overline{B}_{\lambda\rho}(x_0) \end{cases} \tag{4.43}$$

with $\Psi_\rho := \frac{\tau}{1-\frac{\tau}{\sigma}} D_\mu\varphi \otimes (u_x - u)$. From (4.40) and (4.42) we have $\tau v + (\tau w - \tau u_x) - \tau u \in H^{1,p}_{\mu,0}(B_\rho(x);\mathbb{R}^m)$. Noticing that $\mu(\partial B_{\lambda\rho}(x)) = 0$ (see Remark 3.3) and, because of Proposition

(3.4)(ii), $\nabla_\mu(\tau w-\tau u_x)(y)=\tau\nabla_\mu w-\tau\nabla_\mu u_x=0$ for μ-a.a. $y\in B_\rho(x)\backslash\overline{B}_{\lambda\rho}(x)$, we see that

$$\begin{aligned}\frac{\mathrm{m}^t_{\tau u,\omega}(B_\rho(x))}{\mu(B_{\lambda\rho}(x))} &\leqslant \frac{1}{\mu(B_{\lambda\rho}(x))}\int_{B_\rho(x)} L_t\left(y,\tau\nabla_\mu v+\tau\nabla_\mu w-\tau\nabla_\mu u_x,\omega\right)d\mu\\ &= \frac{1}{\mu(B_{\lambda\rho}(x))}\int_{\overline{B}_{\lambda\rho}(x)} L_t\left(y,\tau\nabla_\mu u(x)+\tau\nabla_\mu w-\tau\nabla_\mu u(x),\omega\right)d\mu\\ &\quad+\frac{1}{\mu(B_{\lambda\rho}(x))}\int_{B_\rho(x)\backslash\overline{B}_{\lambda\rho}(x)} L_t(y,\tau\nabla_\mu v,\omega)d\mu\\ &= \frac{1}{\mu(B_{\lambda\rho}(x))}\int_{B_{\lambda\rho}(x)} L_t\left(y,\tau\nabla_\mu w,\omega\right)d\mu\\ &\quad+\frac{1}{\mu(B_{\lambda\rho}(x))}\int_{B_\rho(x)\backslash\overline{B}_{\lambda\rho}(x)} L_t(y,\tau\nabla_\mu v,\omega)d\mu.\end{aligned}$$

From (4.41) and the right inequality in (C_7) it follows that

$$\begin{aligned}\frac{\mathrm{m}^t_{\tau u,\omega}(B_\rho(x))}{\mu(B_\rho(x))}\leqslant\frac{\mathrm{m}^t_{\tau u,\omega}(B_\rho(x))}{\mu(B_{\lambda\rho}(x))} &\leqslant \frac{\mathrm{m}^t_{\tau u_x,\omega}(B_{\lambda\rho}(x))}{\mu(B_{\lambda\rho}(x))}+\varepsilon+\beta\frac{\mu(B_\rho(x)\backslash\overline{B}_{\lambda\rho}(x))}{\mu(B_{\lambda\rho}(x))}\\ &\quad+\frac{\beta}{\mu(B_{\lambda\rho}(x))}\int_{B_\rho(x)\backslash\overline{B}_{\lambda\rho}(x)} G(\tau\nabla_\mu v)d\mu. \qquad (4.44)\end{aligned}$$

On the other hand, taking (4.43) into account and using (C_1) and the left inequality in (C_7), we have

$$G(\tau\nabla_\mu v) \leqslant c_1\left(1+G(\sigma\nabla_\mu u(x))+G(\sigma\nabla_\mu u)+G(\Psi_\rho)\right) \qquad (4.45)$$

with $c_1:=2(\gamma+\gamma^2)>0$. Moreover, it is easy to see that

$$\|\Psi_\rho\|_{L^\infty_\mu(B_\rho(x);\mathbb{M})} \leqslant \frac{\theta\tau}{(1-\frac{\tau}{\sigma})(1-\lambda)}\frac{1}{\rho}\|u-u_x\|_{L^\infty_\mu(B_\rho(x);\mathbb{R}^m)},$$

where

$$\lim_{\rho\to0}\frac{\theta\tau}{(1-\frac{\tau}{\sigma})(1-\lambda)}\frac{1}{\rho}\|u-u_x\|_{L^\infty_\mu(B_\rho(x);\mathbb{R}^m)}=0 \qquad (4.46)$$

by Proposition 3.4(iv), i.e., $\lim_{\rho\to0}\frac{1}{\rho}\|u-u_x\|_{L^\infty_\mu(B_\rho(x);\mathbb{R}^m)}=0$. From ($C_1$) and ($C_2$) there exists $r>0$ such that

$$c_2:=\sup_{|\xi|\leqslant r}G(\xi)<\infty$$

(see Remark 2.2). By (4.46) there exists $\bar\rho>0$ such that $\frac{\theta\tau}{(1-\frac{\tau}{\sigma})(1-\lambda)}\frac{1}{\rho}\|u-u_x\|_{L^\infty_\mu(B_\rho(x);\mathbb{R}^m)}<r$ for all $\rho\in]0,\bar\rho[$. Fix any $\rho\in]0,\bar\rho[$. We then have

$$G(\Psi_\rho)\leqslant c_2. \qquad (4.47)$$

From (4.45) and (4.47) it follows that

$$\begin{aligned}\frac{\beta}{\mu(B_{\lambda\rho}(x))}\int_{B_\rho(x)\setminus\overline{B}_{\lambda\rho}(x)} G(\tau\nabla_\mu v)d\mu &\leqslant \beta c_1\big(1+G(\sigma\nabla_\mu u(x))+c_2\big)\frac{\mu(B_\rho(x)\setminus\overline{B}_{\lambda\rho}(x))}{\mu(B_{\lambda\rho}(x))}\\ &\quad+\frac{\beta c_1}{\mu(B_{\lambda\rho}(x))}\int_{B_\rho(x)\setminus\overline{B}_{\lambda\rho}(x)} G(\sigma\nabla_\mu u(y))d\mu(y).\end{aligned}$$

But

$$\begin{aligned}\int_{B_\rho(x)\setminus\overline{B}_{\lambda\rho}(x)} G(\sigma\nabla_\mu u(y))d\mu(y) &\leqslant \mu(B_\rho(x))\fint_{B_\rho(x)}|G(\sigma\nabla_\mu u(y))-G(\sigma\nabla_\mu u(x))|d\mu(y)\\ &\quad+\mu(B_\rho(x)\setminus\overline{B}_{\lambda\rho}(x))G(\sigma\nabla_\mu u(x)),\end{aligned}$$

hence

$$\begin{aligned}\frac{\beta}{\mu(B_{\lambda\rho}(x))}\int_{B_\rho(x)\setminus\overline{B}_{\lambda\rho}(x)} G(\tau\nabla_\mu v)d\mu &\leqslant \beta c_1\big(1+2G(\sigma\nabla_\mu u(x))+c_2\big)\frac{\mu(B_\rho(x)\setminus\overline{B}_{\lambda\rho}(x))}{\mu(B_{\lambda\rho}(x))}\\ &\quad+\beta c_1\frac{\mu(B_\rho(x))}{\mu(B_{\lambda\rho}(x))}\fint_{B_\rho(x)}|G(\sigma\nabla_\mu u(y))-G(\sigma\nabla_\mu u(x))|d\mu(y). \quad (4.48)\end{aligned}$$

From (4.44) and (4.48) we deduce that

$$\begin{aligned}\frac{\mathrm{m}^t_{\tau u,\omega}(B_\rho(x))}{\mu(B_\rho(x))} &\leqslant \frac{\mathrm{m}^t_{\tau u_x,\omega}(B_{\lambda\rho}(x))}{\mu(B_{\lambda\rho}(x))}+\varepsilon\\ &\quad+\beta c_1\left(1+\frac{1}{c_1}+2G(\sigma\nabla_\mu u(x))+c_2\right)\frac{\mu(B_\rho(x)\setminus\overline{B}_{\lambda\rho}(x))}{\mu(B_{\lambda\rho}(x))}\\ &\quad+\beta c_1\frac{\mu(B_\rho(x))}{\mu(B_{\lambda\rho}(x))}\fint_{B_\rho(x)}|G(\sigma\nabla_\mu u(y))-G(\sigma\nabla_\mu u(x))|d\mu(y). \quad (4.49)\end{aligned}$$

As $\mathscr{G}(\sigma u)<\infty$, i.e., $G(\sigma\nabla_\mu u(\cdot))\in L^1_\mu(O)$, (and μ is a doubling measure) we can assert that:

$$G(\sigma\nabla_\mu u(x))<\infty; \quad (4.50)$$

$$\lim_{\rho\to 0}\fint_{B_\rho(x)}|G(\sigma\nabla_\mu u(y))-G(\sigma\nabla_\mu u(x))|d\mu(y)=0. \quad (4.51)$$

As the boundary of any ball is of zero measure (see Remark 3.3), we have

$$\overline{\lim_{\rho\to 0}}\frac{\mu(B_\rho(x)\setminus\overline{B}_{\lambda\rho}(x))}{\mu(B_{\lambda\rho}(x))}=\overline{\lim_{\rho\to 0}}\left(1-\frac{\mu(B_\rho(x))}{\mu(B_{\lambda\rho}(x))}\right)=1-\underline{\lim_{\rho\to 0}}\frac{\mu(B_\rho(x))}{\mu(B_{\lambda\rho}(x))},$$

and so, by using Proposition (3.4)(vi),

$$\lim_{\lambda\to 1^-}\overline{\lim_{\rho\to 0}}\frac{\mu(B_\rho(x)\setminus\overline{B}_{\lambda\rho}(x))}{\mu(B_{\lambda\rho}(x))}=0. \quad (4.52)$$

Moreover, we have:

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\frac{\mathrm{m}^t_{\tau u,\omega}(B_\rho(x))}{\mu(B_\rho(x))}=\overline{\lim_{\rho\to 0}}\,\frac{\overline{\mathrm{m}}_{\tau u,\omega}(B_\rho(x))}{\mu(B_\rho(x))} \tag{4.53}$$

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\frac{\mathrm{m}^t_{\tau u_x,\omega}(B_{\lambda\rho}(x))}{\mu(B_{\lambda\rho}(x))}\leqslant\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\frac{\mathrm{m}^t_{\tau u_x,\omega}(B_\rho(x))}{\mu(B_\rho(x))}=\overline{\lim_{\rho\to 0}}\,\frac{\overline{\mathrm{m}}_{\tau u_x,\omega}(B_\rho(x))}{\mu(B_\rho(x))}. \tag{4.54}$$

Letting $t\to\infty$, $\rho\to 0$ and $\lambda\to 1^-$ in (4.49) and using (4.50), (4.51), (4.52), (4.53) and (4.54) we conclude that

$$\overline{\lim_{\rho\to 0}}\,\frac{\overline{\mathrm{m}}_{\tau u,\omega}(B_\rho(x))}{\mu(B_\rho(x))}\leqslant\overline{\lim_{\rho\to 0}}\,\frac{\overline{\mathrm{m}}_{\tau u_x,\omega}(B_\rho(x))}{\mu(B_\rho(x))}+\varepsilon,$$

and (4.39) follows by letting $\varepsilon\to 0$.

Conclusion of the steps 1, 2 and 3. As a direct consequence of (4.29), (4.37) and (4.39) together with Remarks 4.1 and 4.2, we have the following lemma.

Lemma 4.3. *For every $\tau\in]0,1[$ and every $u\in H^{1,p}_\mu(O;\mathbb{R}^m)$ such that $\mathscr{G}(\tau u)<\infty$ and $\mathscr{G}(\sigma u)<\infty$ for some $\sigma\in]\tau,1[$, one has*

$$\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}}\,E_t(\tau u,\omega)\leqslant\int_O\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x).$$

Step 4: end of the proof. Let $u\in H^{1,p}_\mu(O;\mathbb{R}^m)$. We have to prove that

$$\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}}\,E_t(u,\omega)\leqslant\int_O\underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x). \tag{4.55}$$

Without loss of generality we can assume that

$$\int_O\underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x)=:\overline{E}_{\lim}(u,\omega)<\infty. \tag{4.56}$$

Then, by Proposition 2.8(i) we have

$$\nabla_\mu u(x)\in\overline{\mathscr{Q}_\mu\mathbb{G}_x}\text{ for }\mu\text{-a.a. }x\in O \tag{4.57}$$

and

$$\underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)=\underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)\text{ for }\mu\text{-a.a. }x\in O. \tag{4.58}$$

Substep 4-1: proving (4.55) under the constraint $\nabla_\mu u(x)\in\operatorname{int}(\mathscr{Q}_\mu\mathbb{G}_x)$ for μ-a.a. $x\in O$. Assume that

$$\nabla_\mu u(x)\in\operatorname{int}(\mathscr{Q}_\mu\mathbb{G}_x)\text{ for }\mu\text{-a.a. }x\in O. \tag{4.59}$$

Then, since (C_3) implies that $\tau\nabla_\mu u(x)\in\operatorname{int}(\mathscr{Q}_\mu\mathbb{G}_x)$ for all $\tau\in]0,1[$ and for μ-a.a. $x\in O$, by (C_5) we have

$$\underline{\lim_{\tau\to 1^-}}\,\mathscr{Q}_\mu G(x,\tau\nabla_\mu u(x))\geqslant\mathscr{Q}_\mu G(x,\nabla_\mu u(x))\text{ for }\mu\text{-a.a. }x\in O. \tag{4.60}$$

Using (4.60) and the left inequality in (C_7) we see that

$$\frac{1}{\alpha}\overline{E}_{\lim}(u,\omega) \geqslant \int_O \underline{\lim}_{\tau\to 1^-} \mathscr{Q}_\mu G(x,\tau\nabla_\mu u(x))d\mu \geqslant \int_O \mathscr{Q}_\mu G(x,\nabla_\mu u(x))d\mu =: \mathscr{Q}_\mu\mathscr{G}(u),$$

hence, by (4.56),

$$\mathscr{Q}_\mu\mathscr{G}(u) < \infty. \tag{4.61}$$

Taking (C_4) into account, from (4.59) and (4.61) it follows that

$$\mathscr{G}(u) < \infty. \tag{4.62}$$

But, by (C_1) we see that for every $\tau \in]0,1[$, $\mathscr{G}(\tau u) \leqslant \gamma\mu(O)(1+G(0)) + \gamma\mathscr{G}(u)$, hence, by ($C_2$) and (4.62), $\mathscr{G}(\tau u) < \infty$ for all $\tau \in]0,1[$, and so, by Lemma 4.3 we have

$$\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}} E_t(\tau u,\omega) \leqslant \int_O \overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x) \text{ for all } \tau\in]0,1[. \tag{4.63}$$

On the other hand, from the right inequality in (C_7) we see that for every $\tau \in]0,1[$,

$$\begin{aligned}\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(\cdot),\omega) &\leqslant \beta\big(1+\mathscr{Q}_\mu G(x,\tau\nabla u(\cdot))\big)\\ &\leqslant \beta\big(1+G(\tau\nabla u(\cdot))\big),\end{aligned}$$

and consequently, by using (C_1),

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(\cdot),\omega) \leqslant \beta\big(1+G(0)+G(\nabla_\mu u(\cdot)\big) := f(\cdot) \text{ for all } \tau\in]0,1[$$

with $f \in L^1_\mu(O)$ by (C_2) and (4.62). Taking (4.58) into account, from Lebesgue's dominated convergence theorem we deduce that

$$\underline{\lim_{\tau\to 1^-}} \int_O \overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x) = \int_O \underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x).$$

From (4.63) we conclude that

$$\underline{\lim_{\tau\to 1^-}} \Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}} E_t(\tau u,\omega) \leqslant \int_O \underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x),$$

and (4.55) follows because $\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(\cdot,\omega)$ is L^p_μ-lsc and $\tau u \to u$ in $L^p_\mu(O;\mathbb{R}^m)$ as $\tau\to 1^-$.

Substep 4-2: proof of (4.55). First of all, from (C_8) and Proposition 3.14 we can assert that $L_\infty := \overline{\lim}_{\rho\to 0}\overline{\lim}_{t\to\infty}\mathscr{H}^\rho_\mu L_t$ is ru-usc at ω with $a_\infty(\cdot,\omega)$ given by (2.2). Moreover, by (C_7) we see that for every $x\in O$, the effective domain of $L_\infty(x,\cdot,\omega)$ is equal to $\mathscr{Q}_\mu\mathbb{G}_x$. Taking ($C_3$) into account, from Theorem 3.12(ii) it follows that

$$\widehat{L}_\infty := \underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \mathscr{H}^\rho_\mu L_t \text{ is ru-usc at } \omega \text{ with } a_\infty(\cdot,\omega). \tag{4.64}$$

From (4.56) we see that $\nabla_\mu u(x) \in \widehat{\mathbb{L}}_{\infty,x,\omega}$ for μ-a.a. $x \in O$, where $\widehat{\mathbb{L}}_{\infty,x,\omega}$ denotes the effective domain of $\widehat{L}_\infty(x,\cdot,\omega)$. Hence, for every $\tau \in]0,1[$,

$$\begin{aligned}\int_O \widehat{L}_\infty(x,\tau\nabla_\mu u(x),\omega)d\mu \;&\leqslant\; \big(1+\Delta^{a_\infty}_{\widehat{L}_\infty}(\tau,\omega)\big)\int_O \widehat{L}_\infty(x,\nabla_\mu u(x),\omega)d\mu\\ &\quad+\Delta^{a_\infty}_{\widehat{L}_\infty}(\tau,\omega)\int_O a_\infty(x,\omega)d\mu\end{aligned}$$

with $\Delta^{a_\infty}_{\widehat{L}_\infty}(\tau,\omega) := \sup_{x\in O}\sup_{\xi\in\widehat{\mathbb{L}}_{\infty,x,\omega}} \frac{\widehat{L}_\infty(x,\tau\xi,\omega)-\widehat{L}_\infty(x,\xi,\omega)}{a_\infty(x,\omega)+\widehat{L}_\infty(x,\xi,\omega)}$, i.e.

$$\overline{E}_{\lim}(\tau u,\omega) \leqslant \big(1+\Delta^{a_\infty}_{\widehat{L}_\infty}(\tau,\omega)\big)\overline{E}_{\lim}(u,\omega) + \Delta^{a_\infty}_{\widehat{L}_\infty}(\tau,\omega)\int_O a_\infty(x,\omega)d\mu \tag{4.65}$$

for all $\tau \in]0,1[$. Using (4.56) and (2.2), i.e. $a_\infty \in L^1_\mu(O)$, we see that

$$\overline{E}_{\lim}(\tau u,\omega) < \infty \text{ for all } \tau \in]0,1[. \tag{4.66}$$

On the other hand, from (4.57) and (C_3) we deduce that

$$\nabla_\mu(\tau u)(x) \in \operatorname{int}(\mathscr{Q}_\mu\mathbb{G}_x) \text{ for all } \tau \in]0,1[\text{ and } \mu\text{-a.a. } x \in O. \tag{4.67}$$

According to (4.67) and (4.66), from the substep 4-1 we can assert that

$$\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}}\, E_t(\tau u,\omega) \leqslant \overline{E}_{\lim}(\tau u,\omega)$$

for all $\tau \in]0,1[$, and so, taking (4.65) into account,

$$\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}}\, E_t(\tau u,\omega) \leqslant \big(1+\Delta^{a_\infty}_{\widehat{L}_\infty}(\tau,\omega)\big)\overline{E}_{\lim}(u,\omega) + \Delta^{a_\infty}_{\widehat{L}_\infty}(\tau,\omega)\int_O a_\infty(x,\omega)d\mu \tag{4.68}$$

for all $\tau \in]0,1[$. Moreover, by (4.64) we have $\overline{\lim}_{\tau\to1^-}\Delta^{a_\infty}_{\widehat{L}_\infty}(\tau,\omega) \leqslant 0$. Hence, letting $\tau \to 1^-$ in (4.68) we conclude that

$$\underline{\lim_{\tau\to1^-}}\,\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}}\, E_t(\tau u,\omega) \leqslant \overline{E}_{\lim}(u,\omega),$$

and (4.55) follows because $\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(\cdot,\omega)$ is L^p_μ-lsc and $\tau u \to u$ in $L^p_\mu(O;\mathbb{R}^m)$ as $\tau \to 1^-$. ■

4.3. **Proof of the Γ-convergence result.** Here we prove Theorem 2.5

Proof of Theorem 2.5. Fix $\omega \in \Omega'$. By (C_7) we see that

$$\alpha\mathscr{Q}_\mu G(x,\xi) \leqslant \overline{\lim_{\rho\to0}}\,\underline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\xi,\omega) \leqslant \overline{\lim_{\rho\to0}}\,\overline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\xi,\omega) \leqslant \beta\big(1+\mathscr{Q}_\mu G(x,\xi)\big)$$

for all $x \in O$ and all $\xi \in \mathbb{M}$. So, for every $x \in O$, one has

$$\operatorname{dom}\left(\overline{\lim_{\rho\to0}}\,\underline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\cdot,\omega)\right) = \operatorname{dom}\left(\overline{\lim_{\rho\to0}}\,\overline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\cdot,\omega)\right) = \mathscr{Q}_\mu\mathbb{G}_x,$$

where $\operatorname{dom}\big(\overline{\lim}_{\rho\to 0}\underline{\lim}_{t\to\infty}\mathscr{H}^\rho_\mu L_t(x,\cdot,\omega)\big)$ (resp. $\operatorname{dom}\big(\overline{\lim}_{\rho\to 0}\overline{\lim}_{t\to\infty}\mathscr{H}^\rho_\mu L_t(x,\cdot,\omega)\big)$) denotes the effective domain of $\overline{\lim}_{\rho\to 0}\underline{\lim}_{t\to\infty}\mathscr{H}^\rho_\mu L_t(x,\cdot,\omega)$ (resp. $\overline{\lim}_{\rho\to 0}\overline{\lim}_{t\to\infty}\mathscr{H}^\rho_\mu L_t(x,\cdot,\omega)$). From (C_9) it follows that

$$\overline{\lim_{\rho\to 0}}\,\underline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\xi,\omega)\geqslant\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\xi,\omega)$$

for all $x\in O$ and all $\xi\in\mathbb{M}$, and consequently

$$\underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\underline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\tau\xi,\omega)\geqslant\underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\tau\xi,\omega)$$

for all $x\in O$ and all $\xi\in\mathbb{M}$. From Propositions 2.6 and 2.7 we deduce that

$$\begin{aligned}\Gamma(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}}\,E_t(u,\omega)&\geqslant\int_O\underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\underline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x)\\&\geqslant\int_O\underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x)\\&\geqslant\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}}\,E_t(u,\omega).\end{aligned}$$

for all $u\in H^{1,p}(O;\mathbb{R}^m)$. Hence

$$\Gamma(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}}\,E_t(u,\omega)=\int_O\underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\mathscr{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x)$$

for all $u\in H^{1,p}_\mu(O;\mathbb{R}^m)$. ■

5. Applications

In this section we give some applications of Theorem 2.5 and Proposition 2.8.

5.1. **Relaxation.** In case $L_t(x,\xi,\omega)=L(x,\xi)$, and so $E_t(\cdot,\omega)=E(\cdot)$, we retrieve the relaxation theorem established in [AHM18, Theorem 2.7]. More precisely, denoting the lsc envelope of E with respect to the strong topology of $L^p_\mu(\Omega;\mathbb{R}^m)$ by $\overline{E}$, as a direct consequence of Theorem 2.5 and Proposition 2.8 we have the following result.

Corollary 5.1. *Assume that the hypotheses of Theorem* 2.5 *are satisfied with* $L_t(x,\xi,\omega)=L(x,\xi)$. *Then*

$$\overline{E}(u)=\int_O\widehat{\mathscr{Q}_\mu L}(x,\nabla_\mu u(x))d\mu(x)$$

for all $u\in H^{1,p}_\mu(O;\mathbb{R}^m)$ *with* $\widehat{\mathscr{Q}_\mu L}:O\times\mathbb{M}\to[0,\infty]$ *given by*

$$\widehat{\mathscr{Q}_\mu L}(x,\xi)=\begin{cases}\lim\limits_{\tau\to 1^-}\mathscr{Q}_\mu L(x,\tau\xi) & \text{if } \xi\in\overline{\mathscr{Q}_\mu\mathbb{G}_x}\\ \infty & \text{otherwise,}\end{cases}$$

where $\mathscr{Q}_\mu L:O\times\mathbb{M}\to[0,\infty]$ *is defined by*

$$\mathscr{Q}_\mu L(x,\xi):=\overline{\lim_{\rho\to 0}}\inf\left\{\fint_{B_\rho(x)}L(y,\xi+\nabla_\mu w(y))d\mu(y):w\in H^{1,p}_{\mu,0}(B_\rho(x);\mathbb{R}^m)\right\}.$$

If moreover $\mathscr{Q}_\mu L(x,\cdot)$ *is lsc on* $\mathrm{int}(\mathscr{Q}_\mu\mathbb{G}_x)$ *for all* $x \in O$ *then*

$$\widehat{\mathscr{Q}_\mu L}(x,\xi) = \overline{\mathscr{Q}_\mu L}(x,\xi) = \begin{cases} \mathscr{Q}_\mu L(x,\xi) & \text{if } \xi \in \mathrm{int}(\mathscr{Q}_\mu\mathbb{G}_x) \\ \lim\limits_{\tau\to 1^-} \mathscr{Q}_\mu L(x,\tau\xi) & \text{if } \xi \in \partial\mathscr{Q}_\mu\mathbb{G}_x \\ \infty & \text{otherwise,} \end{cases}$$

where, for each $x \in O$, $\overline{\mathscr{Q}_\mu L}(x,\cdot)$ *denotes the lsc envelope of* $\mathscr{Q}_\mu L(x,\cdot)$.

5.2. **Homogenization.** Homogenization of integrals of the calculus of variations in non-euclidean settings has been studied for the first time in [AHM17] (see also [DDMM20, AHM20b]) for integrands having p-growth. In this paragraph, we attempt to develop a framework to deal with integrands which have not necessarily p-growth and can take infinite values, by using (the Γ-convergence result) Theorem 2.5 together with Proposition 2.8 and subadditive theorems that we proved in [AHM20b] (see Theorems 5.3 and 5.4).

5.2.1. **Homogenization framework.** Roughly speaking, once we have established a Γ-convergence result as Theorem 2.5, we can deduce homogenization theorems (see §5.2.3) by the use of suitable subadditive theorems (see §5.2.2) allowing to establish the condition (C_9) in Theorem 2.5. To apply this process, we need an appropriate framework with which we can work in the setting of metric measure spaces. In what follows, we adopt the following notation:

- we denote by $\mathscr{B}(X)$ the class of Borel subsets of X;
- we denote by $\mathscr{B}_0$ the class of $A \in \mathscr{B}(X)$ such that $\mu(A) < \infty$ and $\mu(\partial A) = 0$ with $\partial A = \overline{A}\backslash\mathring{A}$, where $\overline{A}$ (resp. $\mathring{A}$) is the closure (resp. the interior) of A;
- we denote by $\mathrm{Homeo}(X)$ the group of homeomorphisms on X;
- we denote by $\mathrm{Ba}(X)$ the class of open ball of X.

Note that by Remark 3.3 we have $\mu(\partial B) = 0$ for all $B \in \mathrm{Ba}(X)$, and so $\mathrm{Ba}(X) \subset \mathscr{B}_0$. In order to deal with homogenization in the framework of metric measure spaces, we need to introduce the quadruple $\big(\mathcal{G}, \{\tau_g\}_{g\in\mathcal{G}}, \mathbb{U}, \{h_t\}_{t>0}\big)$ with:

(O_1) $\mathcal{G}$ a subgroup of $\mathrm{Homeo}(X)$ such that μ is $\mathcal{G}$-invariant, i.e. $g^\sharp\mu = \mu$ for all $g \in \mathcal{G}$ which means that for every $g \in \mathcal{G}$ and every $A \in \mathscr{B}(X)$,

$$\mu(g^{-1}(A)) = \mu(A);$$

(O_2) $\{\tau_g\}_{g\in\mathcal{G}}$ a group of $\mathbb{P}$-preserving transformations on $(\Omega, \mathscr{F}, \mathbb{P})$, i.e.

$$\begin{cases} \tau_g \text{ is } \mathscr{F}\text{-mesurable for all } g \in \mathcal{G} \\ \tau_g \circ \tau_f = \tau_{g\circ f} \text{ and } \tau_{g^{-1}} = \tau_g^{-1} \text{ for all } g, f \in \mathcal{G} \\ \mathbb{P}(\tau_g(A)) = \mathbb{P}(A) \text{ for all } A \in \mathscr{F} \text{ and all } g \in \mathcal{G}; \end{cases}$$

(O_3) $\mathbb{U} \in \mathscr{B}_0$ such that $\mu(\mathbb{U}) > 0$;

(O_4) $\{h_t\}_{t>0} \subset \mathrm{Homeo}(X)$,

where $\mathbb{U}$ can be interpretated as the "unit cell" with respect to X and $\{h_t\}_{t>0}$ as a family of "dilations" in X. Let $\mathscr{J} \subset \mathscr{B}_0$ be given by

$$\mathscr{J} := \left\{ \underset{g\in H}{\cup} g^{-1}\big(h_k(\mathbb{U})\big) : k \in \mathbb{N}^*,\ H \in \mathscr{D}\big(h_k(\mathbb{U})\big) \text{ and } |H| < \infty \right\}$$

with

$$\mathscr{D}\big(h_k(\mathbb{U})\big) := \Big\{ H \subset \mathcal{G} : \big\{g^{-1}(h_k(\mathbb{U}))\big\}_{g\in H} \text{ is disjoint} \Big\}.$$

(Theses sets are of interest for the development of subadditive theorems in the setting on measure spaces with acting group, see [AHM20b] for more details.) To obtain homogenization results in the framework of metric measure spaces, we need to refine it by assuming that:

(F_1) $(X, \mathscr{B}(X), \mu)$ is $\mathcal{G}$-meshable with respect to $\big\{h_k(\mathbb{U})\big\}_{k\in\mathbb{N}^*}$, i.e. for each $k \in \mathbb{N}^*$ there is $\mathcal{G}_k \in \mathscr{D}(h_k(\mathbb{U}))$ with the property that for each $q \in \mathbb{N}^*$ there exist $G^-_{q,k} \subset G^+_{q,k} \in \mathscr{P}_{\mathrm{f}}(\mathcal{G}_k)$ such that

$$\begin{cases} \underset{g\in G^-_{q,k}}{\cup} g^{-1}(h_k(\mathbb{U})) \subset h_q(\mathbb{U}) \subset \underset{g\in G^+_{q,k}}{\cup} g^{-1}(h_k(\mathbb{U})) \\ \lim\limits_{q\to\infty} \big|G^+_{q,k}\backslash G^-_{q,k}\big| \dfrac{\mu(h_k(\mathbb{U}))}{\mu(h_q(\mathbb{U}))} = 0; \end{cases}$$

($\mathrm{F}^{\mathrm{s}}_1$) $(X, \mathscr{B}(X), \mu)$ is strongly $\mathcal{G}$-meshable with respect to $\big\{h_k(\mathbb{U})\big\}_{k\in\mathbb{N}^*}$, i.e. $(X, \mathscr{B}(X), \mu)$ is $\mathcal{G}$-meshable with respect to $\big\{h_k(\mathbb{U})\big\}_{k\in\mathbb{N}^*}$ with the additional property that $\mathcal{G} = \cup_{k\in\mathbb{N}^*} \mathcal{G}_k$ where every $\mathcal{G}_k$ is a countable discrete amenable[6] subgroup of $\mathcal{G}$;

(F_2) for every $B \in \mathrm{Ba}(X)$, $\{h_t(B)\}_{t>0}$ is asymptotically $\mathcal{G}$-regular, i.e. there exist two $\mathcal{G}$-regular families[7] $\{\underline{J}_t\}_{t>0}, \{\overline{J}_t\}_{t>0} \subset \mathscr{J}$ such that

$$\begin{cases} \underline{J}_t \subset h_t(B) \subset \overline{J}_t \text{ for all } t > 0 \\ \lim\limits_{t\to 0} \dfrac{\mu\left(\overline{J}_t \backslash \underline{J}_t\right)}{\mu(h_t(B))} = 0; \end{cases} \tag{5.1}$$

($\mathrm{F}^{\mathrm{s}}_2$) for every $B \in \mathrm{Ba}(X)$, $\{h_t(B)\}_{t>0}$ is asymptotically strongly $\mathcal{G}$-regular, i.e. there exist two strongly $\mathcal{G}$-regular families[8] $\{\underline{J}_t\}_{t>0}, \{\overline{J}_t\}_{t>0} \subset \mathscr{J}$ satisfying (5.1);

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[6]Let $\mathcal{H} \subset \mathcal{G}$ be a subgroup and let $\mathscr{P}_f(\mathcal{H})$ denote the class of finite subsets of $\mathcal{H}$. We say that $\mathcal{H}$ is amenable if for each $E \in \mathscr{P}_{\mathrm{f}}(\mathcal{H})$ and each $\delta > 0$ there exists $F \in \mathscr{P}_{\mathrm{f}}(\mathcal{H})$ such that,$|F\Delta EF| \leqslant \delta|F|$, where $|\cdot|$ denotes the counting measure on $\mathcal{G}$, Δ is the symmetric difference of sets and $EF = \{gof : (g, f) \in E \times F\}$. (For more details on the theory of amenable groups, see [Kre85, §6.4] and [DZ15, §2].)

[7]Given $\{J_t\}_{t>0} \subset \mathscr{J}$, for each $t > 0$, $J_t = \cup_{g\in H_t} g^{-1}(h_{q_t}(\mathbb{U}))$ with $H_t \in \mathscr{D}(h_{q_t}(\mathbb{U}))$. We say that $\{J_t\}_{t>0}$ is $\mathcal{G}$-regular if $\lim_{t\to\infty} q_t = \infty$ and for every $t > 0$ and every $k \in \mathbb{N}^*$ there exist $\bar{q}_{t,k} \in \mathbb{N}^*$, $\bar{g}_{t,k} \in \mathcal{G}$ and $F_{t,k} \in \mathscr{P}_{\mathrm{f}}(\mathcal{G}_k)$ such that $G^+_{q_t,k} H_t \subset F_{t,k}$, $\bar{g}^{-1}_{t,k}(h_{\bar{q}_{t,k}}(\mathbb{U})) = \cup_{r\in G^+_{q_t,k} H_t} r^{-1}(h_k(\mathbb{U}))$ and $\overline{\lim}_{t\to\infty} |F_{t,k}||G^+_{q_t,k} H_t|^{-1} = 1$.

[8]We say that $\{J_t\}_{t>0}$ is strongly $\mathcal{G}$-regular if it is $\mathcal{G}$-regular with the additional assumption that for each $k \in \mathbb{N}^*$ and each $i \in \{-, +\}$, $\{G^i_{q_t,k} H_t\}_{t>0}$ is of Følner-Tempelman type with respect to $\mathcal{G}_k$, i.e. $\lim_{t\to\infty} |gG^i_{q_t,k} H_t \Delta G^i_{q_t,k} H_t||G^i_{q_t,k} H_t|^{-1} = 0$ for all $g \in \mathcal{G}_k$ (Følner's condition) and $|\cup_{0<s\leqslant t} (G^i_{q_s,k} H_s)(G^i_{q_t,k} H_t)| \leqslant M|G^i_{q_t,k} H_t|$ for all $t > 0$ and some $M > 0$ (Tempelman's condition). (For more details on these conditions, see [Lin99, Lin01, DGZ14].)

(F_3) for each $t>0$, $((h_t)^{-1})^\sharp\mu=\mu(h_t(\mathbb{U}))\mu$, i.e. for every $A\in\mathscr{B}(X)$,

$$\mu(h_t(A))=\mu(h_t(\mathbb{U}))\mu(A);$$

(F_4) for every $g\in\mathcal{G}$ and every $B\in\mathrm{Ba}(X)$, there exists a bijective map $\mathscr{T}_{g,B}$ from $H^{1,p}_{\mu,0}(g^{-1}(B);\mathbb{R}^m)$ to $H^{1,p}_{\mu,0}(B;\mathbb{R}^m)$ such that $\nabla_\mu\mathscr{T}_{g,B}(w)=\nabla_\mu w\circ g^{-1}$ for all $w\in H^{1,p}_{\mu,0}(g^{-1}(B);\mathbb{R}^m)$ and $\nabla_\mu(\mathscr{T}_{g,B})^{-1}(v)=\nabla_\mu v\circ g$ for all $v\in H^{1,p}_{\mu,0}(B;\mathbb{R}^m)$;

(F_5) for every $t>0$ and every $B\in\mathrm{Ba}(X)$, there exists a bijective map $\mathscr{H}_{t,B}$ from $H^{1,p}_{\mu,0}(h_t(B);\mathbb{R}^m)$ to $H^{1,p}_{\mu,0}(B;\mathbb{R}^m)$ such that $\nabla_\mu\mathscr{H}_{t,B}(w)=\nabla_\mu w\circ h_t$ for all $w\in H^{1,p}_{\mu,0}(h_t(B);\mathbb{R}^m)$ and $\nabla_\mu(\mathscr{H}_{t,B})^{-1}(v)=\nabla_\mu v\circ(h_t)^{-1}$ for all $v\in H^{1,p}_{\mu,0}(B;\mathbb{R}^m)$.

Remark 5.2. From (F_3) we see that $\mu(h_t(\mathbb{U}))>0$ and $(h_t)^\sharp\mu=(\mu(h_t(\mathbb{U})))^{-1}\mu$ for all $t>0$. Moreover, as $\mu(\overline{\mathbb{U}}\backslash\mathring{\mathbb{U}})=0$ we have $\mu(\mathring{\mathbb{U}})=\mu(\mathbb{U})$ and so $\mu(h_t(\mathring{\mathbb{U}}))=\mu(h_t(\mathbb{U}))$ for all $t>0$.

5.2.2. **Subadditive theorems.** In what follows, we recall subadditive theorems that we proved in [AHM20b, Theorem 2.19]. Let $\mathcal{S}:\mathscr{B}_0\times\Omega\to[0,\infty]$ be such that $\mathcal{S}(A,\cdot)\in L^1(\Omega,\mathscr{F},\mathbb{P})$ for all $A\in\mathscr{B}_0$. In Theorems 5.3 and 5.4 below we need the following properties on $\mathcal{S}$:

(S_1) there exists $C>0$ such that for every $A\in\mathscr{B}_0$ and every $\omega\in\Omega$,

$$\mathcal{S}(A,\omega)\leqslant C\mu(A);$$

(S_2) $\mathcal{S}$ is subadditive, i.e. for every $A,B\in\mathscr{B}_0$ with $A\cap B=\varnothing$ and every $\omega\in\Omega$,

$$\mathcal{S}(A\cup B,\omega)\leqslant\mathcal{S}(A,\omega)+\mathcal{S}(B,\omega);$$

($\mathrm{S}_3^{\mathrm{s}}$) $\mathcal{S}$ is $\mathcal{G}$-covariant, i.e. for every $A\in\mathscr{B}_0$, every $g\in\mathcal{G}$ and every $\omega\in\Omega$,

$$\mathcal{S}\left(g^{-1}(A),\omega\right)=\mathcal{S}\big(A,\tau_g(\omega)\big)$$

and, when $\mathcal{S}$ is deterministic, i.e. $\mathcal{S}(\cdot,,\omega)=S(\cdot)$,

(S_3) $\mathcal{S}$ is $\mathcal{G}$-invariant, i.e. for every $A\in\mathscr{B}_0$ and every $g\in\mathcal{G}$,

$$\mathcal{S}\left(g^{-1}(A)\right)=\mathcal{S}(A).$$

Theorem 5.3 (deterministic case)**.** *Under* (O_1)*,* (O_3)*,* (O_4)*,* (F_1)*,* (F_2) *and* (S_1)–(S_3)*, for every* $B\in\mathrm{Ba}(X)$*, one has*

$$\lim_{t\to\infty}\frac{\mathcal{S}(h_t(B))}{\mu(h_t(B))}=\inf_{k\in\mathbb{N}^*}\frac{\mathcal{S}(h_k(\mathbb{U}))}{\mu(h_k(\mathbb{U}))}.$$

Let $\mathscr{I}:=\big\{A\in\mathscr{F}:\mathbb{P}(\tau_g(A)\Delta A)=0\text{ for all }g\in\mathcal{G}\big\}$ be the σ-algebra of invariant sets with respect to $(\Omega,\mathscr{F},\mathbb{P},\{\tau_g\}_{g\in\mathcal{G}})$. Recall that $(\Omega,\mathscr{F},\mathbb{P},\{\tau_g\}_{g\in\mathcal{G}})$ is said to be ergodic if $\mathbb{P}(A)\in\{0,1\}$ for all $A\in\mathscr{I}$.

Theorem 5.4 (stochastic case)**.** *Under* (O_1)*–*(O_4)*,* $(\mathrm{F}_1^{\mathrm{s}})$*,* $(\mathrm{F}_2^{\mathrm{s}})$*,* (S_1)*,* (S_2) *and* $(\mathrm{S}_3^{\mathrm{s}})$*, there exists* $\Omega' \in \mathscr{F}$ *with* $\mathbb{P}(\Omega') = 1$ *such that for every* $\omega \in \Omega'$ *and every* $B \in \mathrm{Ba}(X)$*, one has*

$$\lim_{t\to\infty} \frac{\mathcal{S}(h_t(B),\omega)}{\mu(h_t(B))} = \inf_{k\in\mathbb{N}^*} \frac{\mathbb{E}^{\mathscr{I}}[\mathcal{S}(h_k(\mathbb{U}),\cdot)](\omega)}{\mu(h_k(\mathbb{U}))},$$

where $\mathbb{E}^{\mathscr{I}}[\mathcal{S}(h_k(\mathbb{U}),\cdot)]$ *denotes the conditional expectation of* $\mathcal{S}(h_k(\mathbb{U}),\cdot)$ *over* $\mathscr{I}$ *with respect to* $\mathbb{P}$*. If moreover* $(\Omega,\mathscr{F},\mathbb{P},\{\tau_g\}_{g\in\mathcal{G}})$ *is ergodic then*

$$\lim_{t\to\infty} \frac{\mathcal{S}(h_t(B),\omega)}{\mu(h_t(B))} = \inf_{k\in\mathbb{N}^*} \frac{\mathbb{E}[\mathcal{S}(h_k(\mathbb{U}),\cdot)]}{\mu(h_k(\mathbb{U}))},$$

where $\mathbb{E}[\mathcal{S}(h_k(\mathbb{U}),\cdot)]$ *denotes the expectation of* $\mathcal{S}(h_k(\mathbb{U}),\cdot)$ *with respect to* $\mathbb{P}$*.*

5.2.3. **Homogenization theorems.** In what follows, we establish deterministic and stochastic homogenization theorems of nonconvex unbounded integrals in the setting of metric measure spaces according to our framework in §5.2.1 (see Theorems 5.9 and 5.10). Let $L : X \times \mathbb{M} \times \Omega \to [0,\infty]$ be a Borel measurable stochastic integrand such that:

- (H_1) L is p-coercive, i.e. there exists $c > 0$ such that for every $x \in X$, every $\xi \in \mathbb{M}$ and every $\omega \in \Omega$,
$$L(x,\xi,\omega) \geqslant c|\xi|^p;$$
- (H_2) L has G-growth, i.e. there exist $\alpha,\beta > 0$ such that for every $x \in X$, every $\xi \in \mathbb{M}$ and every $\omega \in \Omega$,
$$\alpha G(\xi) \leqslant L(x,\xi,\omega) \leqslant \beta(1+G(\xi))$$
with $G : \mathbb{M} \to [0,\infty]$ satisfying (C_1)-(C_5);
- $(\mathrm{H}_3^{\mathrm{s}})$ L is $\mathcal{G}$-stationary with respect to $\{\tau_g\}_{g\in\mathcal{G}}$, i.e. for every $x \in X$, every $\xi \in \mathbb{M}$, every $g \in \mathcal{G}$ and every $\omega \in \Omega$,
$$L(g^{-1}(x),\xi,\omega) = L(x,\xi,\tau_g(\omega))$$

and, when L is deterministic, i.e. $L(x,\xi,\omega) = L(x,\xi)$,

- (H_3) L is $\mathcal{G}$-periodic, i.e. for every $x \in X$, every $\xi \in \mathbb{M}$ and every $g \in \mathcal{G}$,
$$L(g^{-1}(x),\xi) = L(x,\xi).$$

For each $t > 0$, we consider $L_t : X \times \mathbb{M} \times \Omega \to [0,\infty]$ given by

$$L_t(x,\xi,\omega) := L(h_t(x),\xi,\omega). \tag{5.2}$$

Then, under (H_1) and (H_2), it is easy to see that $\{L_t\}_{t>0}$ satisfies (C_6) and (C_7). Homogenization of integrals was already studied in [AHM17] and [AHM20b, §3] in the p-growth case, i.e. when $G(\xi) = |\xi|^p$. Here we are concerned with the G-growth case. For this, we need to suppose that

(H$_4$) for every $\omega \in \Omega$, L is ru-usc at ω with respect to $\{h_t\}_{t>0}$, i.e. for every $\omega \in \Omega$, there exists $a(\cdot,\omega) \in L^1_\mu(X;]0,\infty])$ with

$$\overline{\lim_{t\to\infty}} \int_X a(h_t(x),\omega)d\mu(x) < \infty$$

and

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}} \fint_{B_\rho(\cdot)} a(h_t(y),\omega)d\mu(y) \in L^1_\mu(X)$$

such that

$$\overline{\lim_{\tau\to 1^-}}\ \Delta^a_L(\tau,\omega) \leqslant 0,$$

where $\Delta^a_L : [0,1]\times\Omega \to]-\infty,\infty]$ is given by

$$\Delta^a_L(\tau,\omega) := \sup_{x\in X}\sup_{\xi\in\mathbb{L}_{x,\omega}} \frac{L(x,\tau\xi,\omega)-L(x,\xi,\omega)}{a(x,\omega)+L(x,\xi,\omega)}$$

with $\mathbb{L}_{x,\omega}$ denoting the effective domain of $L(x,\cdot,\omega)$.

Lemma 5.5. *Let $\omega\in\Omega$. If L is ru-usc at ω with respect to $\{h_t\}_{t>0}$ with $a(\cdot,\omega)$, then $\{L_t\}_{t>0}$ is ru-usc at ω with $\{a(h_t(\cdot),\omega)\}_{t>0}$.*

Proof of Lemma 5.5. Set $a_t(\cdot,\omega) := a(h_t(\cdot),\omega)$ for all $t>0$. For any $\tau\in[0,1]$, any $t>0$, any $x\in\Omega$ and any $\xi\in\mathbb{L}_{t,x,\omega}$, one has

$$\frac{L_t(x,\tau\xi,\omega)-L_t(x,\xi,\omega)}{a_t(x,\omega)+L_t(x,\xi,\omega)} = \frac{L(h_t(x),\tau\xi,\omega)-L(h_t(x),\xi,\omega)}{a(h_t(x),\omega)+L(h_t(x),\xi,\omega)}. \tag{5.3}$$

As $\mathbb{L}_{t,x,\omega}=\mathbb{L}_{h_t(x),\omega}$ and $h_t(x)\in X$ we see that

$$\frac{L(h_t(x),\tau\xi,\omega)-L(h_t(x),\xi,\omega)}{a(h_t(x),\omega)+L(h_t(x),\xi,\omega)} \leqslant \sup_{y\in X}\sup_{\xi\in\mathbb{L}_{y,\omega}} \frac{L(y,\tau\xi,\omega)-L(y,\xi,\omega)}{a(y,\omega)+L(y,\xi,\omega)} = \Delta^a_L(\tau,\omega),$$

and from (5.3) we deduce that

$$\sup_{t>0}\Delta^{a_t}_{L_t}(\tau,\omega) \leqslant \Delta^a_L(\tau,\omega) \tag{5.4}$$

for all $\tau\in[0,1]$. But L is ru-usc at ω with $a(\cdot,\omega)$, i.e. $\overline{\lim}_{\tau\to1^-}\Delta^a_L(\tau,\omega)\leqslant 0$, and so, letting $\tau\to1^-$ in (5.4), we obtain $\overline{\lim}_{\tau\to1^-}\sup_{t>0}\Delta^{a_t}_{L_t}(\tau,\omega)\leqslant 0$ which means that $\{L_t\}_{t>0}$ is ru-usc at ω with $\{a_t(\cdot,\omega)\}_{t>0}=\{a(h_t(\cdot),\omega)\}_{t>0}$. ■

Lemma 5.5 shows that (H$_4$) implies (C$_8$). So, according to Theorem 2.5 and Proposition 2.8, to prove Theorems 5.9 and 5.10 below, it is sufficient to establish the condition (C$_9$) in Theorem 2.5. For this, we consider the following assumption:

(H$_5$) for every $\xi\in\cup_{y\in O}\mathcal{Q}_\mu\mathbb{G}_y$, there exists $C_\xi>0$ such that for every $A\in\mathscr{B}_0$,

$$\inf\left\{\int_{\mathring{A}} G(\xi+\nabla_\mu w(y))d\mu(y) : w\in H^{1,p}_{\mu,0}(\mathring{A};\mathbb{R}^m)\right\} \leqslant C_\xi\mu(A).$$

Remark 5.6. It is clear that if $\mathcal{Q}_\mu\mathbb{G}_y=\mathbb{G}$ for all $y\in O$, then (H$_5$) is satisfied with $C_\xi=G(\xi)$. (In particular, (H$_5$) holds when G is $H^{1,p}_\mu$-quasiconvex.)

For each $\xi \in \mathbb{M}$, let $\mathscr{S}_L^\xi : \mathscr{B}_0 \times \Omega \to [0,\infty]$ be defined by

$$\mathscr{S}_L^\xi(A,\omega) := \inf\left\{\int_{\mathring{A}} L(y,\xi+\nabla_\mu w(y),\omega)d\mu(y) : w \in H^{1,p}_{\mu,0}(\mathring{A};\mathbb{R}^m)\right\}.$$

As $\mathbb{M}$ is separable, also is $\cup_{y\in O}\mathscr{Q}_\mu\mathbb{G}_y$. Let $\mathbb{D} \subset \cup_{y\in O}\mathscr{Q}_\mu\mathbb{G}_y$ be a countable set such that $\overline{\mathbb{D}} = \overline{\cup_{y\in O}\mathscr{Q}_\mu\mathbb{G}_y}$. In the stochastic case, we need the following two additional assumptions:

($\mathrm{H}_6^{\mathrm{s}}$) for every $\xi \in \cup_{y\in O}\mathscr{Q}_\mu\mathbb{G}_y$ and every $A \in \mathscr{B}_0$, $\mathscr{S}_L^\xi(A,\cdot)$ is $\mathscr{F}$-mesurable;

($\mathrm{H}_7^{\mathrm{s}}$) there exist $\phi : [0,\infty[\to [0,\infty]$ and $\theta : \cup_{y\in O}\mathscr{Q}_\mu\mathbb{G}_y \times \mathbb{D} \to [0,\infty]$ with

$$\begin{cases} \lim\limits_{r\to 0}\phi(r) = 0 \\ \sup\left\{\theta(\xi,\zeta) : \zeta \in \mathbb{D} \text{ and } |\zeta| \leqslant M\right\} < \infty \end{cases}$$

such that for every $A \in \mathscr{B}_0$, every $\omega \in \Omega$, every $\xi \in \cup_{y\in O}\mathscr{Q}_\mu\mathbb{G}_y$ and every $\zeta \in \mathbb{D}$,

$$\left|\frac{\mathscr{S}_L^\xi(A,\omega)}{\mu(A)} - \frac{\mathscr{S}_L^\zeta(A,\omega)}{\mu(A)}\right| \leqslant \theta(\xi,\zeta)\phi(|\xi-\zeta|).$$

The following two propositions are consequences of Theorems 5.3 and 5.4 respectively. We only give the proof of the stochastic proposition. The deterministic proposition can be proved by the same method.

Proposition 5.7 (deterministic case)**.** *Under* (O_1), (O_3), (O_4), (F_1)–(F_5)*, the right inequality in* (H_2) *and* (H_3)–(H_5)*, for every* $\rho > 0$*, every* $x \in O$ *and every* $\xi \in \mathscr{Q}_\mu\mathbb{G}_x$*, one has*

$$\lim_{t\to\infty}\mathscr{H}_\mu^\rho L_t(x,\xi) = \lim_{t\to\infty}\frac{\mathscr{S}_L^\xi\left(h_t\left(B_\rho(x)\right)\right)}{\mu\left(h_t\left(B_\rho(x)\right)\right)} = \inf_{k\in\mathbb{N}^*}\frac{\mathscr{S}_L^\xi(h_k(\mathbb{U}))}{\mu(h_k(\mathbb{U}))},$$

which implies (C_9)*.*

Proposition 5.8 (Stochastic case)**.** *Under* (O_1)–(O_4), ($\mathrm{F}_1^{\mathrm{s}}$), ($\mathrm{F}_2^{\mathrm{s}}$), ($\mathrm{F}_3$)–($\mathrm{F}_5$)*, the right inequality in* (H_2), ($\mathrm{H}_3^{\mathrm{s}}$), ($\mathrm{H}_4$), ($\mathrm{H}_5$), ($\mathrm{H}_6^{\mathrm{s}}$) *and* ($\mathrm{H}_7^{\mathrm{s}}$)*, there exists* $\Omega' \in \mathscr{F}$ *with* $\mathbb{P}(\Omega') = 1$ *such that for every* $\omega \in \Omega'$*, every* $\rho > 0$*, every* $x \in O$ *and every* $\xi \in \mathscr{Q}_\mu\mathbb{G}_x$*, one has*

$$\lim_{t\to\infty}\mathscr{H}_\mu^\rho L_t(x,\xi,\omega) = \lim_{t\to\infty}\frac{\mathscr{S}_L^\xi\left(h_t\left(B_\rho(x)\right),\omega\right)}{\mu\left(h_t\left(B_\rho(x)\right)\right)} = \inf_{k\in\mathbb{N}^*}\frac{\mathbb{E}^{\mathscr{I}}[\mathscr{S}_L^\xi(h_k(\mathbb{U}),\cdot)](\omega)}{\mu(h_k(\mathbb{U}))},$$

where $\mathbb{E}^{\mathscr{I}}$ *denotes the conditional expectation over* $\mathscr{I}$ *with respect to* $\mathbb{P}$*, with* $\mathscr{I}$ *being the* σ*-algebra of invariant sets with respect to* $(\Omega,\mathscr{F},\mathbb{P},\{\tau_g\}_{g\in\mathcal{G}})$*. If moreover* $(\Omega,\mathscr{F},\mathbb{P},\{\tau_g\}_{g\in\mathcal{G}})$ *is ergodic then*

$$\lim_{t\to\infty}\mathscr{H}_\mu^\rho L_t(x,\xi,\omega) = \lim_{t\to\infty}\frac{\mathscr{S}_L^\xi\left(h_t\left(B_\rho(x)\right),\omega\right)}{\mu\left(h_t\left(B_\rho(x)\right)\right)} = \inf_{k\in\mathbb{N}^*}\frac{\mathbb{E}[\mathscr{S}_L^\xi(h_k(\mathbb{U}),\cdot)]}{\mu(h_k(\mathbb{U}))},$$

where $\mathbb{E}$ *denotes the expectation with respect to* $\mathbb{P}$*. Consequently* (C_9) *holds.*

Proof of Proposition 5.8. First of all, from $(\mathrm{H}_7^{\mathrm{s}})$ we see that for every $\omega\in\Omega$, every $k\in\mathbb{N}^*$, every $\xi\in\cup_{y\in O}\mathcal{Q}_\mu\mathbb{G}_y$ and every $\zeta\in\mathbb{D}$,

$$\frac{\mathcal{S}_L^\xi(h_k(\mathbb{U}),\omega)}{\mu(h_k(\mathbb{U}))}-\theta(\xi,\zeta)\phi(|\xi-\zeta|)\leqslant\frac{\mathcal{S}_L^\zeta(h_k(\mathbb{U}),\omega)}{\mu(h_k(\mathbb{U}))}\leqslant\frac{\mathcal{S}_L^\xi(h_k(\mathbb{U}),\omega)}{\mu(h_k(\mathbb{U}))}+\theta(\xi,\zeta)\phi(|\xi-\zeta|),$$

and so, passing to the conditional expectation $\mathbb{E}^{\mathscr{J}}$ and then to the infimum on k,

$$\left|\inf_{k\in\mathbb{N}^*}\frac{\mathbb{E}^{\mathscr{J}}[\mathcal{S}_L^\xi(h_k(\mathbb{U}),\cdot)](\omega)}{\mu(h_k(\mathbb{U}))}-\inf_{k\in\mathbb{N}^*}\frac{\mathbb{E}^{\mathscr{J}}[\mathcal{S}_L^\zeta(h_k(\mathbb{U}),\cdot)](\omega)}{\mu(h_k(\mathbb{U}))}\right|\leqslant\theta(\xi,\zeta)\phi(|\xi-\zeta|).$$

Taking the properties of ϕ and θ in $(\mathrm{H}_7^{\mathrm{s}})$ into account, we obtain

$$\lim_{\mathbb{D}\ni\zeta\to\xi}\inf_{k\in\mathbb{N}^*}\frac{\mathbb{E}^{\mathscr{J}}[\mathcal{S}_L^\zeta(h_k(\mathbb{U}),\cdot)](\omega)}{\mu(h_k(\mathbb{U}))}=\inf_{k\in\mathbb{N}^*}\frac{\mathbb{E}^{\mathscr{J}}[\mathcal{S}_L^\xi(h_k(\mathbb{U}),\cdot)](\omega)}{\mu(h_k(\mathbb{U}))}\tag{5.5}$$

for all $\omega\in\Omega$ and all $\xi\in\cup_{y\in O}\mathcal{Q}_\mu\mathbb{G}_y$. Fix any $\zeta\in\mathbb{D}$. From the right inequality in (H_2) and (H_5) we have $\mathcal{S}_L^\zeta(A,\omega)\leqslant\beta(1+C_\zeta)\mu(A)$ for all $\omega\in\Omega$ and all $A\in\mathscr{B}_0$, and so, by $(\mathrm{H}_6^{\mathrm{s}})$, $\mathcal{S}_L^\zeta(A,\cdot)\in L^1(\Omega,\mathscr{F},\mathbb{P})$ for all $A\in\mathscr{B}_0$. Moreover, from (O_1), (F_3) and $(\mathrm{H}_3^{\mathrm{s}})$ it easily seen that the set function $\mathcal{S}_L^\zeta$ is $\mathcal{G}$-covariant, and $\mathcal{S}_L^\zeta$ is also subadditive because, for each $A,B\in\mathscr{B}_0$, $\mu\big(\widehat{A\cup B}\backslash(\mathring{A}\cup\mathring{B})\big)=0$ since $\widehat{A\cup B}\backslash(\mathring{A}\cup\mathring{B})\subset\partial A\cup\partial B$ and $\mu(\partial A)=\mu(\partial B)=0$. Then, by Theorem 5.4, there exists $\Omega'_\zeta\in\mathscr{F}$ with $\mathbb{P}(\Omega'_\zeta)=1$ such that for every $\omega\in\Omega'_\zeta$ and every $B\in\mathrm{Ba}(X)$, one has

$$\lim_{t\to\infty}\frac{\mathcal{S}_L^\zeta(h_t(B),\omega)}{\mu(h_t(B))}=\inf_{k\in\mathbb{N}^*}\frac{\mathbb{E}^{\mathscr{J}}[\mathcal{S}_L^\zeta(h_k(\mathbb{U}),\cdot)](\omega)}{\mu(h_k(\mathbb{U}))}.\tag{5.6}$$

Now, set $\Omega':=\cap_{\zeta\in\mathbb{D}}\Omega'_\zeta$. Since $\mathbb{D}$ is countable, $\mathbb{P}(\Omega')=1$. Fix any $\omega\in\Omega'$, any $\xi\in\cup_{y\in O}\mathcal{Q}_\mu\mathbb{G}_y$ and any $B\in\mathrm{Ba}(X)$. From $(\mathrm{H}_7^{\mathrm{s}})$ we see that for every $t>0$ and every $\zeta\in\mathbb{D}$,

$$\frac{\mathcal{S}_L^\zeta(h_t(B),\omega)}{\mu(h_t(B))}-\theta(\xi,\zeta)\phi(|\xi-\zeta|)\leqslant\frac{\mathcal{S}_L^\xi(h_t(B),\omega)}{\mu(h_t(B))}\leqslant\frac{\mathcal{S}_L^\zeta(h_t(B),\omega)}{\mu(h_t(B))}+\theta(\xi,\zeta)\phi(|\xi-\zeta|),$$

and so, letting $t\to\infty$ and using (5.6), we obtain:

$$\begin{aligned}&\inf_{k\in\mathbb{N}^*}\frac{\mathbb{E}^{\mathscr{J}}[\mathcal{S}_L^\zeta(h_k(\mathbb{U}),\cdot)](\omega)}{\mu(h_k(\mathbb{U}))}-\theta(\xi,\zeta)\phi(|\xi-\zeta|)\leqslant\varliminf_{t\to\infty}\frac{\mathcal{S}_L^\xi(h_t(B),\omega)}{\mu(h_t(B))};\\&\varlimsup_{t\to\infty}\frac{\mathcal{S}_L^\xi(h_t(B),\omega)}{\mu(h_t(B))}\leqslant\inf_{k\in\mathbb{N}^*}\frac{\mathbb{E}^{\mathscr{J}}[\mathcal{S}_L^\zeta(h_k(\mathbb{U}),\cdot)](\omega)}{\mu(h_k(\mathbb{U}))}+\theta(\xi,\zeta)\phi(|\xi-\zeta|).\end{aligned}$$

By the properties of ϕ and θ in $(\mathrm{H}_7^{\mathrm{s}})$ and (5.5), letting $\mathbb{D}\ni\zeta\to\xi$, it follows that:

$$\begin{aligned}&\inf_{k\in\mathbb{N}^*}\frac{\mathbb{E}^{\mathscr{J}}[\mathcal{S}_L^\xi(h_k(\mathbb{U}),\cdot)](\omega)}{\mu(h_k(\mathbb{U}))}\leqslant\varliminf_{t\to\infty}\frac{\mathcal{S}_L^\xi(h_t(B),\omega)}{\mu(h_t(B))};\\&\varlimsup_{t\to\infty}\frac{\mathcal{S}_L^\xi(h_t(B),\omega)}{\mu(h_t(B))}\leqslant\inf_{k\in\mathbb{N}^*}\frac{\mathbb{E}^{\mathscr{J}}[\mathcal{S}_L^\xi(h_k(\mathbb{U}),\cdot)](\omega)}{\mu(h_k(\mathbb{U}))}.\end{aligned}$$

Consequently, for every $\omega \in \Omega'$, every $\xi \in \cup_{y\in O}\mathcal{Q}_\mu\mathbb{G}_y$ and every $B \in \mathrm{Ba}(X)$,

$$\lim_{t\to\infty} \frac{\mathcal{S}_L^\xi(h_t(B),\omega)}{\mu(h_t(B))} = \inf_{k\in\mathbb{N}^*} \frac{\mathbb{E}^{\mathcal{I}}[\mathcal{S}_L^\xi(h_k(\mathbb{U}),\cdot)](\omega)}{\mu(h_k(\mathbb{U}))}. \tag{5.7}$$

Finally, fix $x \in O$ and $\xi \in \mathcal{Q}_\mu\mathbb{G}_x$ (then $\xi \in \cup_{y\in O}\mathcal{Q}_\mu\mathbb{G}_y$). Taking (5.2) into account, from (O_3) and (F_3), we see that for every $\omega \in \Omega'$, every $B \in \mathrm{Ba}(X)$, and every $t > 0$, one has

$$\begin{aligned}
\mathcal{S}_L^\xi(h_t(B),\omega) &= \inf\left\{\int_{h_t(B)} L(y,\xi+\nabla_\mu w(y),\omega)d\mu(y) : w \in H^{1,p}_{\mu,0}(h_t(B);\mathbb{R}^m)\right\}\\
&= \inf\left\{\int_B L(h_t(y),\xi+\nabla_\mu w(h_t(y)),\omega)d(h_t^{-1})^\sharp\mu(y) : w \in H^{1,p}_{\mu,0}(h_t(B);\mathbb{R}^m)\right\}\\
&= \mu(h_t(\mathbb{U}))\inf\left\{\int_B L_t(y,\xi+\nabla_\mu w(h_t(y)),\omega)d\mu(y) : w \in H^{1,p}_{\mu,0}(h_t(B);\mathbb{R}^m)\right\}.
\end{aligned}$$

But $\mu(h_t(\mathbb{U}))\mu(B) = (h_t^{-1})^\sharp\mu(B) = \mu(h_t(B))$ by using (F_3), and so from (F_5) we obtain

$$\mathcal{S}_L^\xi(h_t(B),\omega) = \mu(h_t(B))\inf\left\{\fint_B L_t(y,\xi+\nabla_\mu w(y),\omega)d\mu(y) : w \in H^{1,p}_{\mu,0}(B;\mathbb{R}^m)\right\}$$

for all $\omega \in \Omega'$, all $B \in \mathrm{Ba}(X)$ and all $t > 0$. Consequently, for every $\omega \in \Omega'$ and every $\rho > 0$, we have:

$$\begin{aligned}
\underline{\lim_{t\to\infty}}\, \mathscr{H}_\mu^\rho L_t(x,\xi,\omega) &= \underline{\lim_{t\to\infty}} \frac{\mathcal{S}_L^\xi(h_t(B_\rho(x)),\omega)}{\mu(h_t(B_\rho(x))};\\
\overline{\lim_{t\to\infty}}\, \mathscr{H}_\mu^\rho L_t(x,\xi,\omega) &= \overline{\lim_{t\to\infty}} \frac{\mathcal{S}_L^\xi(h_t(B_\rho(x)),\omega)}{\mu(h_t(B_\rho(x)))},
\end{aligned}$$

and the proposition follows by using (5.7) with $B = B_\rho(x)$. ■

For each $t > 0$, let $E_t : H^{1,p}_\mu(O;\mathbb{R}^m)\times\Omega \to [0,\infty]$ be defined by (1.1) with L_t given by (5.2). Taking Theorem 2.5 and Proposition 2.8 into account, from Propositions 5.7 and 5.8 respectively, we deduce the following two homogenization theorems.

Theorem 5.9 (deterministic case)**.** *Assume that $p > \kappa$. Under* (O_1)*,* (O_3)*,* (O_4)*,* (F_1)–(F_5) *and* (H_1)–(H_5)*, one has*

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u) = \int_O L_{\mathrm{hom}}\big(\nabla_\mu u(x)\big)d\mu(x)$$

for all $u \in H^{1,p}_\mu(O;\mathbb{R}^m)$ with $L_{\mathrm{hom}} : \mathbb{M} \to [0,\infty]$ given by

$$L_{\mathrm{hom}}(\xi) = \begin{cases} \lim\limits_{\tau\to 1^-}\inf\limits_{k\in\mathbb{N}^*} \dfrac{\mathcal{S}_L^{\tau\xi}(h_k(\mathbb{U}))}{\mu(h_k(\mathbb{U}))} & \text{if } \xi \in \overline{\cup\limits_{y\in O}\mathcal{Q}_\mu\mathbb{G}_y}\\ \infty & \text{otherwise.}\end{cases}$$

Theorem 5.10 (stochastic case)**.** *Assume that $p > \kappa$. Under* (O_1)–(O_4), $(\mathrm{F}_1^{\mathrm{s}})$, $(\mathrm{F}_2^{\mathrm{s}})$, (F_3)–(F_5), (H_1), (H_2), $(\mathrm{H}_3^{\mathrm{s}})$, (H_4), (H_5), $(\mathrm{H}_6^{\mathrm{s}})$ *and* $(\mathrm{H}_7^{\mathrm{s}})$*, there exists $\Omega' \in \mathscr{F}$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega \in \Omega'$, one has*

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,\omega) = \int_O L_{\mathrm{hom}}\big(\nabla_\mu u(x),\omega\big)d\mu(x)$$

for all $u \in H^{1,p}_\mu(O;\mathbb{R}^m)$ with $L_{\mathrm{hom}} : \mathbb{M}\times\Omega \to [0,\infty]$ given by

$$L_{\mathrm{hom}}(\xi,\omega) = \begin{cases} \lim\limits_{\tau\to 1^-}\inf\limits_{k\in\mathbb{N}*} \dfrac{\mathbb{E}^{\mathscr{I}}[\mathscr{S}_L^{\tau\xi}(h_k(\mathbb{U}),\cdot)](\omega)}{\mu(h_k(\mathbb{U}))} & \text{if } \xi \in \overline{\bigcup\limits_{y\in O} \mathscr{Q}_\mu\mathbb{G}_y} \\ \infty & \text{otherwise,} \end{cases}$$

where $\mathbb{E}^{\mathscr{I}}$ denotes the conditional expectation over $\mathscr{I}$ with respect to $\mathbb{P}$, with $\mathscr{I}$ being the σ-algebra of invariant sets with respect to $(\Omega,\mathscr{F},\mathbb{P},\{\tau_g\}_{g\in\mathcal{G}})$. If moreover $(\Omega,\mathscr{F},\mathbb{P},\{\tau_g\}_{g\in\mathcal{G}})$ is ergodic then L_{hom} is deterministic and is given by

$$L_{\mathrm{hom}}(\xi) = \begin{cases} \lim\limits_{\tau\to 1^-}\inf\limits_{k\in\mathbb{N}*} \dfrac{\mathbb{E}[\mathscr{S}_L^{\tau\xi}(h_k(\mathbb{U}),\cdot)]}{\mu(h_k(\mathbb{U}))} & \text{if } \xi \in \overline{\bigcup\limits_{y\in O} \mathscr{Q}_\mu\mathbb{G}_y} \\ \infty & \text{otherwise,} \end{cases}$$

where $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$.

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(Omar Anza Hafsa) UNIVERSITE DE NIMES, Laboratoire MIPA, Site des Carmes, Place Gabriel Péri, 30021 Nîmes, France.
Email address: omar.anza-hafsa@unimes.fr

(Jean-Philippe Mandallena) UNIVERSITE DE NIMES, Laboratoire MIPA, Site des Carmes, Place Gabriel Péri, 30021 Nîmes, France.
Email address: jean-philippe.mandallena@unimes.fr