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ON Γ-CONVERGENCE AND HOMOGENIZATION OF NONCONVEX UNBOUNDED INTEGRALS IN CHEEGER-SOBOLEV SPACES

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Abstract. We study Γ-convergence of nonconvex integrals of the calculus of variations in the setting of Cheeger-Sobolev spaces when the integrands have not polynomial growth and can take infinite values. Homogenization in such a framework is also developed.

Contents

1. Introduction

Let (X, d, μ) be a metric measure space, where (X, d) is complete, supporting a weak $(1, p)$-Poincaré inequality with $p > 1$ and such that μ is a doubling positive Radon measure on X which satisfies the annular decay property (see §2.1). Let $m \geqslant 1$ be an integer, let $\Omega \subset X$ be a bounded open set such that $\mu(\overline{\Omega}\backslash\Omega) = 0$, let $\mathcal{O}(\Omega)$ be the class of open subsets of Ω

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and let $(\Sigma, \mathcal{F}, \mathbb{P})$ be a probability space. In this paper we consider a family of variational stochastic integrals $E_t : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \times \Sigma \to [0,\infty]$ defined by

$$E_t(u, A, \omega) := \int_A L_t\big(x, \nabla_\mu u(x), \omega\big) d\mu(x), \tag{1.1}$$

where $L_t : \Omega \times \mathbb{M} \times \Sigma \to [0,\infty]$ is a Borel measurable stochastic integrand[1] depending on a parameter $t > 0$ and not necessarily convex with respect to $\xi \in \mathbb{M}$, where $\mathbb{M}$ denotes the space of real $m \times N$ matrices. The space $H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ denotes the class of p-Cheeger-Sobolev functions from Ω to $\mathbb{R}^m$ and $\nabla_\mu u$ is the μ-gradient of u (see §3.1).

We are concerned with the problem of computing the almost sure Γ-convergence (see Definitions 2.3 and 2.4) of the stochastic family $\{E_t\}_{t>0}$, as $t \to \infty$, to a variational stochastic integral $E_\infty : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \times \Sigma \to [0,\infty]$ of the type

$$E_\infty(u, A, \omega) = \int_A L_\infty\big(x, \nabla_\mu u(x), \omega\big) d\mu(x) \tag{1.2}$$

with $L_\infty : \Omega \times \mathbb{M} \times \Sigma \to [0,\infty]$ not depending on the parameter t. When L_∞ is independent of the variable x, the procedure of passing from (1.1) to (1.2) is referred as stochastic homogenization. If furthermore L_∞ is independent of the variable ω then E_∞ is said to be deterministic, otherwise E_∞ is said to be stochastic. When $\{L_t\}_{t>0}$ is deterministic, i.e. L_t is independent of the variable ω for all $t > 0$, the procedure of passing from (1.1) to (1.2) is referred as deterministic homogenization.

Our motivation for developing Γ-convergence, and more generally calculus of variations, in the setting of metric measure spaces comes from applications to hyperelasticity. In fact, the interest of considering a general measure is that its support can be interpretated as a hyperelastic structure together with its singularities like for example thin dimensions, corners, junctions, etc. Such mechanical singular objects naturally lead to develop calculus of variations in the setting of metric measure spaces. Indeed, for example, a low multi-dimensional structures can be described by a finite number of smooth compact manifolds S_i of dimension k_i on which a superficial measure $\mu_i = \mathcal{H}^{k_i}|_{S_i}$ is attached. Such a situation leads to deal with the finite union of manifolds S_i, i.e. $X = \cup_i S_i$, together with the finite sum of measures μ_i, i.e. $\mu = \sum_i \mu_i$, whose mathematical framework is that of metric measure spaces (for more examples, we refer the reader to [BBS97, Zhi02, CJLP02] and [CPS07, Chapter 2, §10] and the references therein). In this way, having in mind the two basic conditions of hyperelasticity, i.e. "the non-interpenetration of the matter" and "the necessity of an infinite amount of energy to compress a finite piece of matter into a point", it is then of interest to study Γ-convergence of nonconvex integrals of type (1.1) when the integrands do not have p-growth and can take infinite values: this is the general purpose of the present paper. Note that although our framework needs some "convexity" assumptions (see especially (2.7) which implies that domain of $L(x,\cdot,\omega)$ is convex) it is consistent with the two above conditions of hyperelasticity (see [AHM11, §2.2] and [AHMZ15, §9]). Nevertheless, this dose of convexity

[1]Throughout the paper, by a Borel measurable stochastic integrand $L : \Omega \times \mathbb{M} \times \Sigma \to [0,\infty]$ we mean that L is $(\mathcal{B}(X) \otimes \mathcal{B}(\mathbb{M}) \otimes \mathcal{F}, \mathcal{B}(\overline{\mathbb{R}}))$-measurable, where $\mathcal{B}(X)$, $\mathcal{B}(\mathbb{M})$ and $\mathcal{B}(\overline{\mathbb{R}})$ denote the Borel σ-algebra on X, $\mathbb{M}$ and $\overline{\mathbb{R}}$ respectively.

makes our framework not consistent with another condition of hyperelasticity that is "frame-indifference" (see [AHMZ15, Remark 9.1]). (For more details on the theory of hyperelasticity we refer the reader to [MH94].)

Such a Γ-convergence problem in such a metric measure setting was studied for the first time in [AHM17] when the family $\{L_t\}_{t>0}$ is deterministic and has p-growth, i.e. there exist $\alpha, \beta > 0$ such that

$$\alpha|\xi|^p \leqslant L_t(x,\xi) \leqslant \beta(1+|\xi|^p) \tag{1.3}$$

for all $t > 0$, all $\xi \in \mathbb{M}$ and all $x \in \Omega$, where it is proved (see [AHM17, Theorem 2.2]) that if (1.3) holds then:

$$\Gamma(L^p_\mu)\text{-}\underline{\lim}_{t\to\infty} E_t(u,A) \geqslant \int_A \overline{\lim_{\rho\to 0}} \underline{\lim_{t\to\infty}} \mathcal{H}^\rho_\mu L_t(x,\nabla_\mu u(x))d\mu(x);$$
$$\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,A) = \int_A \lim_{\rho\to 0} \overline{\lim_{t\to\infty}} \mathcal{H}^\rho_\mu L_t(x,\nabla_\mu u(x))d\mu(x)$$

for all $u \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ and all $A \in \mathcal{O}(\Omega)$, where $\Gamma(L^p_\mu)\text{-}\underline{\lim}$ and $\Gamma(L^p_\mu)\text{-}\overline{\lim}$ denote respectively the Γ-liminf and the Γ-limsup with respect to the strong convergence of $L^p_\mu(\Omega;\mathbb{R}^m)$ (see Definition 2.3) and, for each $t > 0$ and each $\rho > 0$, $\mathcal{H}^\rho_\mu L_t : \Omega \times \mathbb{M} \to [0,\infty]$ is given by

$$\mathcal{H}^\rho_\mu L_t(x,\xi) := \inf\left\{ \fint_{Q_\rho(x)} L_t(y,\xi+\nabla_\mu w(y))d\mu(y) : w \in H^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m)\right\} \tag{1.4}$$

where $Q_\rho(x)$ denotes the open ball with radius $\rho > 0$ and the space $H^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m)$ is the closure of

$$\mathrm{Lip}_0(Q_\rho(x);\mathbb{R}^m) := \Big\{ u \in \mathrm{Lip}(\Omega;\mathbb{R}^m) : u = 0 \text{ on } \Omega\backslash Q_\rho(x)\Big\}$$

with respect to the $H^{1,p}_\mu$-norm, where $\mathrm{Lip}(\Omega;\mathbb{R}^m) := [\mathrm{Lip}(\Omega)]^m$ with $\mathrm{Lip}(\Omega)$ denoting the algebra of Lipschitz functions from Ω to $\mathbb{R}$. In particular (see [AHM17, Corollary 2.3]), if moreover, for every $x \in \Omega$, every $\rho > 0$ and every $\xi \in \mathbb{M}$, one has

$$\underline{\lim_{t\to\infty}} \mathcal{H}^\rho_\mu L_t(x,\xi) = \overline{\lim_{t\to\infty}} \mathcal{H}^\rho_\mu L_t(x,\xi) \tag{1.5}$$

then

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,A) = \int_A L_\infty(x,\nabla_\mu u(x))d\mu(x) \tag{1.6}$$

for all $u \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ and all $A \in \mathcal{O}(\Omega)$, where $L_\infty : \Omega \times \mathbb{M} \to [0,\infty]$ is given by

$$L_\infty(x,\xi) := \lim_{\rho\to 0} \lim_{t\to\infty} \mathcal{H}^\rho_\mu L_t(x,\xi). \tag{1.7}$$

This was illustrated in [AHM17] in the case of deterministic homogenization where it is proved (see [AHM17, Theorem 2.20]) that in the p-growth context and under additional assumptions on the metric measure space (X,d,μ), the equality (1.5) is verified independently of the open ball $Q_\rho(x)$ and so (1.6) holds with the integrand L_∞ in (1.7) which does not depend on the variable x, i.e. $L_\infty(x,\xi) = L_{\mathrm{hom}}(\xi)$ with

$$L_{\mathrm{hom}}(\xi) := \inf_{k\in\mathbb{N}^*} \inf\left\{ \fint_{h_k(\mathring{\mathbb{U}})} L(y,\xi+\nabla_\mu w(y))d\mu(y) : w \in H^{1,p}_{\mu,0}\left(h_k(\mathring{\mathbb{U}});\mathbb{R}^m\right)\right\}, \tag{1.8}$$

where $\mathbb{U} \subset X$ is the "unit cell" (with $\mathring{\mathbb{U}}$ denoting the interior of $\mathbb{U}$) and, for each $k \in \mathbb{N}^*$, h_k is a homeomorphism on X (see §2.4 for more details).

In this paper, we extend our previous results in [AHM17] to the unbounded case, i.e. to the case where the integrands L_t in (1.1) do not have p-growth and can take infinite values (see §2.1, §2.2, §2.3 and §2.4 for more details).

Our main contribution (see Theorem 2.11) is to prove that for $p > \kappa$, with $\kappa := \frac{\ln(C_d)}{\ln(2)}$ where $C_d \geqslant 1$ is the doubling constant, see (2.1), if, for $\mathbb{P}$-a.e. $\omega \in \Sigma$, $\{L_t\}_{t>0}$ is radially uniformly upper semicontinuous (ru-usc) at ω, i.e. there exists $\{a_t(\cdot,\omega)\}_{t>0} \subset L^1_\mu(\Omega;]0,\infty])$, with $\overline{\lim}_{t\to\infty} \int_\Omega a_t(x,\omega)d\mu(x) < \infty$ and $\overline{\lim}_{\rho\to 0} \overline{\lim}_{t\to\infty} \fint_{Q_\rho(x)} a_t(y,\omega)d\mu(y) < \infty$ for μ-a.a. $x \in \Omega$, such that

$$\overline{\lim_{\tau\to 1^-}} \sup_{t>0} \sup_{x\in\Omega} \sup_{\xi\in\mathbb{L}_{t,x,\omega}} \frac{L_t(x,\tau\xi,\omega) - L_t(x,\xi,\omega)}{a_t(x,\omega) + L_t(x,\xi,\omega)} \leqslant 0,$$

where $\mathbb{L}_{t,x,\omega}$ denotes the effective domain of $L_t(x,\cdot,\omega)$ and if $\{L_t\}_{t>0}$ has G-growth, i.e. there exist $\alpha,\beta > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Sigma$,

$$\alpha G(\xi) \leqslant L_t(x,\xi,\omega) \leqslant \beta(1 + G(\xi)),$$

for all $t > 0$, all $x \in \Omega$ and all $\xi \in \mathbb{M}$, where $G : \mathbb{M} \to [0,\infty]$ is Borel measurable, p-coercive and verifies some "convexity" assumptions, see (2.6) and (2.7), then for $\mathbb{P}$-a.e. $\omega \in \Sigma$, one has:

$$\Gamma(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}} E_t(u,A,\omega) \geqslant \int_A \underline{\lim_{\tau\to 1^-}} \overline{\lim_{\rho\to 0}} \underline{\lim_{t\to\infty}} \mathcal{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x);$$

$$\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}} E_t(u,A,\omega) = \int_A \underline{\lim_{\tau\to 1^-}} \lim_{\rho\to 0} \overline{\lim_{t\to\infty}} \mathcal{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x)$$

for all $u \in \mathfrak{G}$ and all $A \in \mathcal{O}(\Omega)$, where $\mathfrak{G}$ denotes the effective domain of the functional $u \mapsto \int_\Omega G(\nabla_\mu u(x))d\mu(x)$. This establishes (see Corollary 2.15) that if moreover

$$\underline{\lim_{t\to\infty}} \mathcal{H}^\rho_\mu L_t(x,\xi,\omega) = \overline{\lim_{t\to\infty}} \mathcal{H}^\rho_\mu L_t(x,\xi,\omega)$$

for $\mathbb{P}$-a.a. $\omega \in \Sigma$, all $x \in \Omega$, all $\rho > 0$ and all $\xi \in \mathbb{G}$, where $\mathbb{G}$ denotes the effective domain of G and $\mathcal{H}^\rho_\mu L_t(x,\xi,\omega)$ is given by (1.4) with "$L_t(x,\xi,\omega)$" instead of "$L_t(x,\xi)$", then for $\mathbb{P}$-a.e. $\omega \in \Sigma$, one has

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,\Omega,\omega) = \int_\Omega \widehat{L}_\infty(x,\nabla_\mu u(x),\omega)d\mu(x) \tag{1.9}$$

for all $u \in \mathfrak{G}$, where $\widehat{L}_\infty : \Omega \times \mathbb{M} \times \Sigma \to [0,\infty]$ is defined by

$$\widehat{L}_\infty(x,\xi,\omega) := \underline{\lim_{\tau\to 1^-}} L_\infty(x,\tau\xi,\omega)$$

with $L_\infty(x,\tau\xi,\omega)$ given by (1.7) with "$L_t(x,\tau\xi,\omega)$" instead of "$L_t(x,\xi)$". We also show that under suitable assumptions the equality (1.9) can be extended to the whole space $H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ (see Corollaries 2.16, 2.17 and 2.18).

Our Γ-convergence results apply to homogenization (see Theorems 2.25 and 2.34 and Corollaries 2.26, 2.27, 2.28, 2.35, 2.36 and 2.37). Generally speaking, for a measurable dynamical $\mathbb{G}$-system $(\Sigma,\mathcal{T},\mathbb{P},\{\tau_g\}_{g\in\mathbb{G}})$ (see Definition 3.38), where $\mathbb{G}$ is a subgroup of Homeo(X) with

Homeo(X) denoting the group of homeomorphisms on X, and under some additional assumptions on the triple $\big((X,d,\mu),\mathbb{G},\{h_t\}_{t>0}\big)$ (see §2.4 for more details), when

$$L_t(x,\xi,\omega)=L(h_t(x),\xi,\omega),$$

where $\{h_t\}_{t>0}\subset \text{Homeo}(X)$ and $L:X\times\mathbb{M}\times\Sigma\to[0,\infty]$ is ru-usc and satisfies

$$L(g^{-1}(x),\cdot,\omega)=L(x,\cdot,\tau_g(\omega))$$

for all $g\in\mathbb{G}$, we prove that (1.9) holds with

$$L_\infty(x,\xi,\omega)=L_{\rm hom}(\xi,\omega)$$

with $L_{\rm hom}:\mathbb{M}\times\Sigma\to[0,\infty]$ given by

$$L_{\rm hom}(\xi,\omega):=\inf_{k\in\mathbb{N}*}\mathbb{E}^{\mathcal{I}}\left[\inf\left\{\fint_{h_k(\mathring{\mathbb{U}})}L(y,\xi+\nabla_\mu w(y),\cdot)d\mu(y):w\in H^{1,p}_{\mu,0}\Big(h_k\big(\mathring{\mathbb{U}}\big);\mathbb{R}^m\Big)\right\}\right](\omega),$$

where $\mathbb{E}^{\mathcal{I}}$ denotes the conditional expectation over $\mathcal{I}$ with respect to $\mathbb{P}$, with $\mathcal{I}$ being the σ-algebra of invariant sets with respect to $(\Sigma,\mathcal{T},\mathbb{P},\{\tau_g\}_{g\in\mathbb{G}})$. If in addition $(\Sigma,\mathcal{T},\mathbb{P},\{\tau_g\}_{g\in\mathbb{G}})$ is ergodic (see Definition 3.40), then $L_{\rm hom}$ is deterministic and is given by

$$L_{\rm hom}(\xi):=\inf_{k\in\mathbb{N}*}\mathbb{E}\left[\inf\left\{\fint_{h_k(\mathring{\mathbb{U}})}L(y,\xi+\nabla_\mu w(y),\cdot)d\mu(y):w\in H^{1,p}_{\mu,0}\Big(h_k\big(\mathring{\mathbb{U}}\big);\mathbb{R}^m\Big)\right\}\right],$$

where $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$. When L is deterministic, $L_{\rm hom}$ is given by (1.8).

For related works in the Euclidean case, i.e. when $(X,d,\mu)=(\mathbb{R}^N,|\cdot-\cdot|,\mathcal{L}_N)$ where $\mathcal{L}_N$ is the Lebesgue measure on $\mathbb{R}^N$, we refer the reader to [Mar78, Bra85, DMM86, Mül87, JKO94, MM94, BG95, BD98, AM02, AM04, AHM11, AHLM11, AHMZ15, DG16, AHCM17] and the references therein.

The plan of the paper is as follows. In Sect. 2 we state the main results of the paper (see Theorems 2.11 in §2.3 for Γ-convergence and Theorems 2.26 and 2.27 in §2.4 for homogenization) and their consequences (see Corollaries 2.15, 2.16, 2.17 and 2.18 in §2.3 for Γ-convergence and Corollaries 2.26, 2.27, 2.28, 2.35, 2.36 and 2.37 in §2.4 for homogenization). Sect. 4 is dedicated to the proof of Theorem 2.11 and Theorems 2.26 and 2.27 are proved in Sect. 5, whereas the proofs of their corollaries are given in §2.3 and §2.4 respectively. Sect. 3 is devoted to several auxiliary results needed for proving Theorems 2.11, 2.26 and 2.27. In the appendix we give the proof of the integral representation of the Vitali envelope of a set function, that is used in the proof of Theorem 2.11, as well as the proofs of subadditive results in the setting of metric measure spaces, that are used to establish Theorems 2.26 and 2.27.

Notation. The open and closed balls centered at $x\in X$ with radius $\rho>0$ are denoted by:

$$Q_\rho(x):=\Big\{y\in X:d(x,y)<\rho\Big\};$$
$$\overline{Q}_\rho(x):=\Big\{y\in X:d(x,y)\leqslant\rho\Big\}.$$

For $x \in X$ and $\rho > 0$ we set

$$\partial Q_\rho(x) := \overline{Q}_\rho(x) \backslash Q_\rho(x) = \Big\{ y \in X : d(x,y) = \rho \Big\}.$$

For $A \subset X$, the diameter of A (resp. the distance from a point $x \in X$ to the subset A) is defined by $\operatorname{diam}(A) := \sup_{x,y \in A} d(x,y)$ (resp. $\operatorname{dist}(x,A) := \inf_{y \in A} d(x,y)$).
The symbol $⨍$ stands for the mean-value integral

$$⨍_B f d\mu = \frac{1}{\mu(B)} \int_B f d\mu.$$

2. Main results

2.1. **Setting of the problem.** Let (X, d, μ) be a separable and complete metric measure space. Here and subsequently, we assume that μ is doubling on X, i.e. there exists a constant $C_d \geqslant 1$ (called doubling constant) such that

$$\mu\left(Q_\rho(x)\right) \leqslant C_d \mu\left(Q_{\frac{\rho}{2}}(x)\right) \tag{2.1}$$

for μ-a.a. $x \in X$ and all $\rho > 0$, and X supports a weak $(1,p)$-Poincaré inequality with $1 < p < \infty$, i.e. there exist $C_P > 0$ and $\sigma \geqslant 1$ such that for μ-a.e. $x \in X$ and every $\rho > 0$,

$$⨍_{Q_\rho(x)} \left| f - ⨍_{Q_\rho(x)} f d\mu \right| d\mu \leqslant \rho C_P \left(⨍_{Q_{\sigma\rho}(x)} g^p d\mu \right)^{\frac{1}{p}} \tag{2.2}$$

for every $f \in L^p_\mu(\Omega)$, every p-weak upper gradient $g \in L^p_\mu(\Omega)$ for f and every open set $\Omega \subset X$ such that $Q_{\sigma\rho}(x) \subset \Omega$. (For the definition of the concept of p-weak upper gradient, see Definition 3.2.) As μ is doubling, for μ-a.e. $\bar{x} \in X$ and each $r > 0$, we have

$$\frac{\mu(Q_\rho(x))}{\mu(Q_r(\bar{x}))} \geqslant 4^{-\kappa} \left(\frac{\rho}{r}\right)^\kappa \tag{2.3}$$

for all $x \in Q_r(\bar{x})$ and all $0 < \rho \leqslant r$, where $\kappa := \frac{\ln(C_d)}{\ln(2)}$ (see [Haj03, Lemma 4.7]). We further assume that (X, d, μ) satisfies the annular decay property, i.e. there exist $\delta > 0$ and $C_A \geqslant 1$ such that

$$\mu\left(Q_{\sigma r}(x) \backslash Q_r(x)\right) \leqslant C_A \left(1 - \frac{1}{\sigma}\right)^\delta \mu(Q_{\sigma r}(x)) \tag{2.4}$$

for all $x \in X$, all $r > 0$ and all $\sigma \in]1, \infty[$.

Remark 2.1. From [Buc99, Corollary 2.2] and [CM98, Lemma 3.3] (see also [Che99, Proposition 6.12] and [HKST15, Proposition 11.5.3 pp. 328]), if moreover (X, d) is a length space then (2.4) holds.

Remark 2.2. If (2.4) holds then $\mu\big(\overline{Q_r(x)} \backslash Q_r(x)\big) = 0$ for all $x \in X$ and all $r > 0$, i.e. the boundary of balls is of zero measure. Indeed, given $x \in X$ and $r > 0$, we have $1 \geqslant \mu\big(Q_r(x)\big)/\mu\big(\overline{Q_r(x)}\big) \geqslant \mu(Q_r(x))/\mu(Q_{\sigma r}(x)) \geqslant 1 - C_A(1 - \frac{1}{\sigma})^\delta$ for all $\sigma \in]1, \infty[$. Hence, by letting $\sigma \to 1$, we obtain $\mu(Q_r(x))/\mu\big(\overline{Q_r(x)}\big) = 1$, i.e. $\mu(Q_r(x)) = \mu\big(\overline{Q_r(x)}\big)$.

From now on, we suppose $p > \kappa$, we fix a bounded open set $\Omega \subset X$ such that $\mu(\overline{\Omega}\backslash\Omega) = 0$ and an integer $m \geqslant 1$, and we denote the class of open subsets of Ω by $\mathcal{O}(\Omega)$.
Let us recall the definition of Γ-convergence and a.s Γ-convergence. (For more details on the theory of Γ-convergence we refer to [DM93].)

Definition 2.3. For each $t > 0$, let $E_t : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ and let $E_\infty : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$. We say that $\{E_t\}_{t>0}$ Γ-converges with respect to the strong convergence of $L^p_\mu(\Omega;\mathbb{R}^m)$, or simply $\Gamma(L^p_\mu)$-converges, to E_∞ as $t \to \infty$ if

$$\Gamma(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}} E_t(u,A) \geqslant E_\infty(u,A) \geqslant \Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}} E_t(u,A)$$

for any $u \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ and any $A \in \mathcal{O}(\Omega)$, with:

$$\Gamma(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}} E_t(u,A) := \inf\left\{\underline{\lim_{t\to\infty}} E_t(u_t,A) : u_t \overset{L^p_\mu}{\to} u\right\};$$
$$\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}} E_t(u,A) := \inf\left\{\overline{\lim_{t\to\infty}} E_t(u_t,A) : u_t \overset{L^p_\mu}{\to} u\right\}.$$

Then we write

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,A) = E_\infty(u,A).$$

Let $(\Sigma,\mathcal{F},\mathbb{P})$ be a probability space. Almost sure Γ-convergence is defined from Definition 2.3 as follows.

Definition 2.4. For each $t > 0$, let $E_t : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \times \Sigma \to [0,\infty]$ and let $E_\infty : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \times \Sigma \to [0,\infty]$. We say that $\{E_t\}_{t>0}$ almost sure Γ-converges with respect to the strong convergence of $L^p_\mu(\Omega;\mathbb{R}^m)$, or simply almost sure $\Gamma(L^p_\mu)$-converges, to E_∞ as $t \to \infty$ if for $\mathbb{P}$-a.e. $\omega \in \Sigma$, one has

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,A,\omega) = E_\infty(u,A,\omega).$$

for any $u \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ and any $A \in \mathcal{O}(\Omega)$.

For each $t > 0$, let $E_t : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \times \Sigma \to [0,\infty]$ be defined by

$$E_t(u,A,\omega) := \int_A L_t\big(x,\nabla_\mu u(x),\omega\big)d\mu(x).$$

The object of the paper is to compute the almost sure $\Gamma(L^p_\mu)$-convergence of $\{E_t\}_{t>0}$ as $t \to \infty$ in the case where the family $\{L_t\}_{t>0}$ does not have p-growth and can take infinite values.

2.2. **Growth and ru-usc conditions.** Let $G : \mathbb{M} \to [0,\infty]$ be a Borel measurable integrand such that G is p-coercive, i.e. there exists $c > 0$ such that for every $x \in \Omega$ and every $\xi \in \mathbb{M}$,

$$G(\xi) \geqslant c|\xi|^p. \tag{2.5}$$

We also assume that there exists $r > 0$ such that

$$\sup_{|\xi|\leqslant r} G(\xi) < \infty, \tag{2.6}$$

and there exists $\gamma > 0$ such that for every $x \in \Omega$, every $\tau \in]0,1[$ and every $\xi, \zeta \in \mathbb{M}$,

$$G(\tau\xi + (1-\tau)\zeta) \leqslant \gamma(1 + G(\xi) + G(\zeta)). \tag{2.7}$$

Remark 2.5. If (2.7) holds then $\mathbb{G}$ is convex, where $\mathbb{G}$ denotes the effective domain of G.

Remark 2.6. If (2.7) is satisfied and if $0 \in \mathrm{int}(\mathbb{G})$ then (2.6) holds, see [AHM12b, Lemma 4.1].

Let $\mathcal{G}, \overline{\mathcal{G}} : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \to [0,\infty]$ be the functionals defined by:

$$\mathcal{G}(u) := \int_\Omega G(\nabla_\mu u(x))d\mu(x); \tag{2.8}$$

$$\overline{\mathcal{G}}(u) := \inf\left\{ \varliminf_{n\to\infty} \mathcal{G}(u_n) : u_n \overset{L^p_\mu}{\to} u \right\}. \tag{2.9}$$

(The functional $\overline{\mathcal{G}}$ is the lower semi-continuous envelope of $\mathcal{G}$ with respect to the strong convergence of $L^p_\mu(\Omega;\mathbb{R}^m)$.) Let us denote the effective domains of $\mathcal{G}$ and $\overline{\mathcal{G}}$ by $\mathfrak{G}$ and $\mathfrak{G}^{\mathrm{lsc}}$ respectively. It is clear that $\mathfrak{G} \subset \mathfrak{G}^{\mathrm{lsc}}$. We futhermore assume that

$$\tau\mathfrak{G}^{\mathrm{lsc}} \subset \mathfrak{G} \text{ for all } \tau \in]0,1[. \tag{2.10}$$

Remark 2.7. If $\mathfrak{G}^{\mathrm{lsc}} \subset \mathfrak{G}$ and if (2.6) holds (and so $0 \in \mathfrak{G}$) and (2.7) is satisfied (and so $\mathfrak{G}$ is convex) then (2.10) holds.

Remark 2.8. If G is p-coercive, i.e. (2.5) holds, and if (2.7) is satisfied then $\mathfrak{G}^{\mathrm{lsc}} \subset \overline{\mathfrak{G}}$, where $\overline{\mathfrak{G}}$ denotes the closure of $\mathfrak{G}$ with respect to the norm of $H^{1,p}_\mu(\Omega;\mathbb{R}^m)$. Thus, if moreover $0 \in \mathrm{int}(\mathfrak{G})$, where $\mathrm{int}(\mathfrak{G})$ denotes the interior of $\mathfrak{G}$ with respect to the norm of $H^{1,p}_\mu(\Omega;\mathbb{R}^m)$, then (2.10) holds.

Throughout the paper, we assume that $\{L_t\}_{t>0}$ has G-growth, i.e. there exist $\alpha, \beta > 0$ such that for every $x \in \Omega$, every $\xi \in \mathbb{M}$ and $\mathbb{P}$-a.e. $\omega \in \Sigma$,

$$\alpha G(\xi) \leqslant L_t(x,\xi,\omega) \leqslant \beta(1 + G(\xi)). \tag{2.11}$$

Remark 2.9. Given $\omega \in \Sigma$, if (2.7) and (2.11) hold then the effective domain $\mathbb{L}_{t,x,\omega}$ of $L_t(x,\cdot,\omega)$ is equal to $\mathbb{G}$ and so is convex.

Remark 2.10. Given $\omega \in \Sigma$, if (2.11) is satisfied then the effective domains of the functionals $\Gamma(L^p_\mu)\text{-}\varliminf_{t\to\infty} E_t(\cdot,\Omega,\omega)$ and $\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(\cdot,\Omega,\omega)$ are both equal to $\mathfrak{G}^{\mathrm{lsc}}$.

When $G(\xi) = |\xi|^p$, we say that $\{L_t\}_{t>0}$ has p-growth. The p-growth case was already studied in [AHM17]. The object of this paper is to deal with the G-growth case. For this, in addition, we need to suppose that for $\mathbb{P}$-a.e. $\omega \in \Sigma$, $\{L_t\}_{t>0}$ is radially uniformly upper semicontinuous (ru-usc) at ω with $\{a_t(\cdot,\omega)\}_{t>0} \subset L^1_\mu(\Omega;]0,\infty])$, i.e.

$$\overline{\lim_{\tau\to 1^-}} \sup_{t>0} \Delta_{L_t}^{a_t(\cdot,\omega)}(\tau) \leqslant 0 \tag{2.12}$$

with $\Delta_{L_t}^{a_t(\cdot,\omega)} : [0,1] \to]-\infty,\infty]$ given by

$$\Delta_{L_t}^{a_t(\cdot,\omega)}(\tau) := \sup_{x\in\Omega} \sup_{\xi\in\mathbb{L}_{t,x,\omega}} \frac{L_t(x,\tau\xi,\omega) - L_t(x,\xi,\omega)}{a_t(x,\omega) + L_t(x,\xi,\omega)},$$

with the additional assumptions that

$$\overline{\lim_{t\to\infty}}\int_\Omega a_t(x,\omega)d\mu(x)<\infty \tag{2.13}$$

and

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\fint_{Q_\rho(x)} a_t(y,\omega)d\mu(y)<\infty \tag{2.14}$$

for μ-a.a. $x\in\Omega$. (For more details on the concept of ru-usc, see §3.2.)

2.3. **Γ-convergence.** In what follows $\mu\big(\overline{\Omega}\backslash\Omega\big)=0$, $p>\kappa$, where $\kappa:=\frac{\ln(C_d)}{\ln(2)}$ with $C_d\geqslant 1$ given by the inequality (2.1), and $m\geqslant 1$. For each $t>0$ and each $\rho>0$, let $\mathcal{H}^\rho_\mu L_t:\Omega\times\mathbb{M}\times\Sigma\to[0,\infty]$ be defined by

$$\mathcal{H}^\rho_\mu L_t(x,\xi,\omega):=\inf\left\{\fint_{Q_\rho(x)} L_t(y,\xi+\nabla_\mu w(y),\omega)d\mu(y):w\in H^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m)\right\}$$

where $Q_\rho(x)$ denotes the open ball with radius $\rho>0$ and the space $H^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m)$ is the closure of

$$\mathrm{Lip}_0(Q_\rho(x);\mathbb{R}^m):=\Big\{u\in\mathrm{Lip}(\Omega;\mathbb{R}^m):u=0\text{ on }\Omega\backslash Q_\rho(x)\Big\}$$

with respect to the $H^{1,p}_\mu$-norm, where $\mathrm{Lip}(\Omega;\mathbb{R}^m):=[\mathrm{Lip}(\Omega)]^m$ with $\mathrm{Lip}(\Omega)$ denoting the algebra of Lipschitz functions from Ω to $\mathbb{R}$ (see §3.1 for more details). The main result of the paper is the following.

Theorem 2.11. *If* (2.5), (2.6), (2.7), (2.11), (2.12), (2.13) *and* (2.14) *hold then for* $\mathbb{P}$*-a.e.* $\omega\in\Sigma$*, one has*:

$$\Gamma(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}}\,E_t(u,A,\omega)\geqslant\int_A\underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\underline{\lim_{t\to\infty}}\,\mathcal{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x); \tag{2.15}$$

$$\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}}\,E_t(u,A,\omega)=\int_A\underline{\lim_{\tau\to 1^-}}\,\lim_{\rho\to 0}\,\overline{\lim_{t\to\infty}}\,\mathcal{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)d\mu(x) \tag{2.16}$$

for all $u\in\mathfrak{G}$ *and all* $A\in\mathcal{O}(\Omega)$.

Assume furthermore that for $\mathbb{P}$-a.e. $\omega\in\Sigma$, every $x\in\Omega$ and every $\rho>0$, one has

$$\underline{\lim_{t\to\infty}}\,\mathcal{H}^\rho_\mu L_t(x,\xi,\omega)=\overline{\lim_{t\to\infty}}\,\mathcal{H}^\rho_\mu L_t(x,\xi,\omega)\text{ for all }\xi\in\mathbb{G} \tag{2.17}$$

and let $\widehat{L}_\infty:\Omega\times\mathbb{M}\times\Sigma\to[0,\infty]$ be defined by

$$\widehat{L}_\infty(x,\xi,\omega):=\underline{\lim_{\tau\to 1^-}}\,L_\infty(x,\tau\xi,\omega)$$

with $L_\infty:\Omega\times\mathbb{M}\times\Sigma\to[0,\infty]$ given by

$$L_\infty(x,\xi,\omega):=\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\,\mathcal{H}^\rho_\mu L_t(x,\xi,\omega). \tag{2.18}$$

Let $\mathcal{Q}_\mu G : \mathbb{M} \to [0,\infty]$ be defined by

$$\mathcal{Q}_\mu G(\xi) := \overline{\lim_{\rho\to 0}} \inf \left\{ \fint_{Q_\rho(x)} G(\xi + \nabla_\mu w(y)) d\mu(y) : w \in H^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m) \right\}$$

(Note that $\mathcal{Q}_\mu G$ is in fact given by (2.18) with "G" instead of "L_t".)

Remark 2.12. The integrand $\mathcal{Q}_\mu G$ is called the $H^{1,p}_\mu$-quasiconvexification of G. (For more details on the notion of $H^{1,p}_\mu$-quasiconvexity, we refer to [AHM19a].)

The following proposition, which make precise the representation of $\widehat{L}_\infty$, will be useful in our framework.

Proposition 2.13. *Given $\omega \in \Sigma$, assume that* (2.11) *is verified and $\{L_t\}_{t>0}$ is ru-usc at ω with $\{a_t(\cdot,\omega)\}_{t>0}$ satisfying* (3.20). *If $\mathcal{Q}_\mu\mathbb{G}$ is convex and $0 \in \operatorname{int}(\mathcal{Q}_\mu\mathbb{G})$ with $\operatorname{int}(\mathcal{Q}_\mu\mathbb{G})$ denoting the interior of $\mathcal{Q}_\mu\mathbb{G}$ and $\mathcal{Q}_\mu\mathbb{G}$ being the effective domain of $\mathcal{Q}_\mu G$, then*:

(a) $\widehat{L}_\infty$ *is ru-usc at ω*;
(b) $\widehat{L}_\infty(x,\xi,\omega) := \lim_{\tau\to 1^-} L_\infty(x,\tau\xi,\omega)$ *for all $x \in \Omega$ and all $\xi \in \mathbb{M}$.*

If moreover $L_\infty(x,\cdot,\omega)$ is lsc on $\operatorname{int}(\mathcal{Q}_\mu\mathbb{G})$ for all $x \in \Omega$ then:

(c) $$\widehat{L}_\infty(x,\xi,\omega) = \begin{cases} L_\infty(x,\xi,\omega) & \text{if } x \in \Omega \text{ and } \xi \in \operatorname{int}(\mathcal{Q}_\mu\mathbb{G}) \\ \lim\limits_{\tau\to 1^-} L_\infty(x,\tau\xi,\omega) & \text{if } x \in \Omega \text{ and } \xi \in \partial\mathcal{Q}_\mu\mathbb{G} \\ \infty & \text{otherwise;} \end{cases}$$

(d) *for every $x \in \Omega$, $\widehat{L}_\infty(x,\cdot,\omega)$ is the lsc envelope of $L_\infty(x,\cdot,\omega)$. In particular $\widehat{L}_\infty(\cdot,\cdot,\omega) \leqslant L_\infty(\cdot,\cdot,\omega)$.*

Proof of Proposition 2.13. From Proposition 3.17 we can assert that L_∞ is ru-usc at ω, and from (2.11) we see that $\mathbb{L}_{\infty,x,\omega} = \mathcal{Q}_\mu\mathbb{G}$ for all $x \in \Omega$. On the other hand, $\mathcal{Q}_\mu\mathbb{G}$ is convex and $0 \in \operatorname{int}(\mathcal{Q}_\mu\mathbb{G})$, hence $\tau\overline{\mathcal{Q}_\mu\mathbb{G}} \subset \operatorname{int}(\mathcal{Q}_\mu\mathbb{G})$ for all $\tau \in]0,1[$, where $\overline{\mathcal{Q}_\mu\mathbb{G}}$ denotes the closure of $\mathcal{Q}_\mu\mathbb{G}$, and the proposition follows from Theorem 3.14. ■

Remark 2.14. Let $\omega \in \Sigma$ be satisfying all the assumptions of Proposition 2.13. By Proposition 2.13(a) we see that $\widehat{L}_\infty$ is ru-usc at ω, and by Proposition 2.13(d) we can assert that for every $x \in \Omega$, $\widehat{L}_\infty(x,\cdot,\omega)$ is lsc and $\mathbb{L}_{\infty,x,\omega} \subset \widehat{\mathbb{L}}_{\infty,x,\omega} \subset \overline{\mathbb{L}}_{\infty,x,\omega}$. But, for each $x \in \Omega$, $\mathbb{L}_{\infty,x,\omega} = \mathcal{Q}_\mu\mathbb{G}$ and, for each $\tau \in]0,1[$, $\tau\overline{\mathcal{Q}_\mu\mathbb{G}} \subset \operatorname{int}(\mathcal{Q}_\mu\mathbb{G})$, hence $\tau\widehat{\mathbb{L}}_{\infty,x,\omega} \subset \operatorname{int}(\widehat{\mathbb{L}}_{\infty,x,\omega})$. Applying Theorem 3.14(a) and (d) with $L(\cdot,\cdot,\omega) = \widehat{L}_\infty(\cdot,\cdot,\omega)$ we deduce that

$$\widehat{\widehat{L}}_\infty(x,\xi,\omega) = \lim_{\tau\to 1^-} \widehat{L}_\infty(x,\tau\xi,\omega) = \widehat{L}_\infty(x,\xi,\omega)$$

for all $x \in \Omega$ and all $\xi \in \mathbb{M}$, where $\widehat{\widehat{L}}_\infty : \Omega \times \mathbb{M} \times \Sigma \to [0,\infty]$ is defined by

$$\widehat{\widehat{L}}_\infty(x,\xi,\omega) := \underline{\lim_{\tau\to 1^-}} \widehat{L}_\infty(x,\tau\xi,\omega).$$

The following result is a consequence of Theorem 2.11.

Corollary 2.15. *Under the assumptions of Theorem* 2.11*, if* (2.17) *is satisfied then for* $\mathbb{P}$*-a.e.* $\omega \in \Sigma$*, one has*

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,\Omega,\omega) = \int_\Omega \widehat{L}_\infty(x,\nabla_\mu u(x),\omega)d\mu(x) \tag{2.19}$$

for all $u \in \mathfrak{G}$*.*

Proof of Corollary 2.15. Let $\omega \in \Sigma$ be suitably fixed and let $u \in \mathfrak{G}$. Then, for μ-a.e. $x \in \Omega$, $\nabla_\mu u(x) \in \mathbb{G}$. But G satisfies (2.7) and so $\mathbb{G}$ is convex. Moreover, by (2.6) we have $0 \in \mathbb{G}$. Hence $\tau\nabla_\mu u(x) \in \mathbb{G}$ for all $\tau \in]0,1[$. From (2.17) it follows that

$$\underline{\lim_{t\to\infty}}\, \mathcal{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega) = \overline{\lim_{t\to\infty}}\, \mathcal{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)$$

for all $\rho > 0$ and all $\tau \in]0,1[$, and so, taking (2.18) into account,

$$\begin{aligned}
\underline{\lim_{\tau\to 1^-}}\, \overline{\lim_{\rho\to 0}}\, \underline{\lim_{t\to\infty}}\, \mathcal{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega) &= \underline{\lim_{\tau\to 1^-}}\, \lim_{\rho\to 0}\, \overline{\lim_{t\to\infty}}\, \mathcal{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega) \\
&= \underline{\lim_{\tau\to 1^-}}\, \lim_{\rho\to 0}\, \lim_{t\to\infty}\, \mathcal{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega) \\
&= \widehat{L}_\infty(x,\nabla_\mu u(x),\omega)
\end{aligned}$$

for μ-a.a. $x \in \Omega$, and (2.19) follows by using (2.15) and (2.16). ■

From Corollary 2.15 we deduce the following two results.

Corollary 2.16. *Under the assumptions of Corollary* 2.15*, if* $\mathfrak{G}^{\rm lsc} \subset \mathfrak{G}$ *then for* $\mathbb{P}$*-a.e.* $\omega \in \Sigma$*, one has*

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,\Omega,\omega) = \begin{cases} \displaystyle\int_\Omega \widehat{L}_\infty(x,\nabla_\mu u(x),\omega)d\mu(x) & \text{if } u \in \mathfrak{G}^{\rm lsc} \\ \infty & \text{if } u \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)\backslash\mathfrak{G}^{\rm lsc}. \end{cases} \tag{2.20}$$

Proof of Corollary 2.16. Let $\omega \in \Sigma$ be suitably fixed. Since $\mathfrak{G}^{\rm lsc} \subset \mathfrak{G}$, from Corollary 2.15 we deduce that

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,\Omega,\omega) = \int_\Omega \widehat{L}_\infty(x,\nabla_\mu u(x),\omega)d\mu(x) \text{ for all } u \in \mathfrak{G}^{\rm lsc}.$$

On the other hand, from (2.11) we see that:

$$\begin{aligned}
\alpha\overline{\mathcal{G}}(u) \leqslant \Gamma(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}}\, E_t(u,\Omega,\omega) \leqslant \beta\big(1+\overline{\mathcal{G}}(u)\big); \\
\alpha\overline{\mathcal{G}}(u) \leqslant \Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}}\, E_t(u,\Omega,\omega) \leqslant \beta\big(1+\overline{\mathcal{G}}(u)\big)
\end{aligned}$$

for all $u \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$, where $\overline{\mathcal{G}}$ is defined by (2.9). Hence

$$\Gamma(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}}\, E_t(u,\Omega,\omega) = \Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}}\, E_t(u,\Omega,\omega) = \infty \text{ for all } u \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)\backslash\mathfrak{G}^{\rm lsc},$$

and the proof is complete. ■

Corollary 2.17. *Under the assumptions of Corollary* 2.15*, if* (2.10) *is satisfied then for* $\mathbb{P}$*-a.e.* $\omega \in \Sigma$*, one has*

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,\Omega,\omega) = \begin{cases} \widehat{\mathcal{I}}(u,\omega) & \text{if } u \in \mathfrak{G}^{\rm lsc} \\ \infty & \text{if } u \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)\backslash\mathfrak{G}^{\rm lsc} \end{cases} \tag{2.21}$$

with $\widehat{\mathcal{I}}(\cdot,\omega) : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \to [0,\infty]$ *given by*

$$\widehat{\mathcal{I}}(u,\omega) := \varliminf_{\tau\to 1^-} \mathcal{I}(\tau u,\omega),$$

where $\mathcal{I}(\cdot,\omega) : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \to [0,\infty]$ *is defined by*

$$\mathcal{I}(u,\omega) := \int_\Omega \widehat{L}_\infty(x,\nabla_\mu u(x),\omega)d\mu(x).$$

Proof of Corollary 2.17. Let $\omega \in \Sigma$ be suitably fixed. From Corollary 2.15 we see that $\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,\Omega,\omega) = \mathcal{I}(u,\omega)$ for all $u \in \mathfrak{G}$. As $\{L_t\}_{t>0}$ is ru-usc at ω it is easily seen that $\{E_t(\cdot,\Omega,\cdot)\}_{t>0}$ is ru-usc on $\mathfrak{G}$ at ω. Hence, since (2.10) holds, from Corollary 3.23 we deduce that

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,\Omega,\omega) = \widehat{\mathcal{I}}(u,\omega) \text{ for all } u \in \mathfrak{G}^{\rm lsc}.$$

On the other hand, taking (2.11) into account, it is clear that if $u \notin \mathfrak{G}^{\rm lsc}$ then

$$\Gamma(L^p_\mu)\text{-}\varliminf_{t\to\infty} E_t(u,\Omega,\omega) = \Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}} E_t(u,\Omega,\omega) = \infty,$$

and (2.21) follows. ■

As a consequence of Corollary 2.17 we have the following result.

Corollary 2.18. *Under the assumptions of Corollary* 2.17*, if* (3.20) *is satisfied and if* $\mathcal{Q}_\mu\mathbb{G}$ *is convex,* $0 \in \operatorname{int}(\mathcal{Q}_\mu\mathbb{G})$ *and* $L_\infty(x,\cdot,\omega)$ *is lsc on* $\operatorname{int}(\mathcal{Q}_\mu\mathbb{G})$ *for* $\mathbb{P}$*-a.a.* $\omega \in \Sigma$ *and all* $x \in \Omega$*, then* (2.20) *holds for* $\mathbb{P}$*-a.a.* $\omega \in \Sigma$*.*

Proof of Corollary 2.18. By Corollary 2.17, (2.21) holds, and so it suffices to prove that $\widehat{\mathcal{I}}(u,\omega) = \mathcal{I}(u,\omega)$ for all $u \in \mathfrak{G}^{\rm lsc}$. First of all, we claim that $\mathfrak{G} \subset \mathfrak{I}_\omega$, where $\mathfrak{I}_\omega$ denotes the effective domain of $\mathcal{I}(\cdot,\omega)$. Indeed, let $u \in \mathfrak{G}$. Using the right inequality in (2.11) we have

$$\begin{aligned} \int_\Omega L_\infty(x,\nabla_\mu u(x),\omega)d\mu(x) &\leqslant \beta\left(|\Omega| + \int_\Omega \mathcal{Q}_\mu G(\nabla_\mu u(x))d\mu(x)\right) \\ &\leqslant \beta\left(|\Omega| + \int_\Omega G(\nabla_\mu u(x))d\mu(x)\right) \\ &= \beta\left(|\Omega| + \mathcal{G}(u)\right) < \infty, \end{aligned}$$

and the claim follows because $\widehat{L}_\infty(\cdot,\cdot,\omega) \leqslant L_\infty(\cdot,\cdot,\omega)$ by Proposition 2.13(d). On the other hand, as (3.20) holds, from Proposition 3.17 we deduce that L_∞ is ru-usc at ω, hence $\widehat{L}_\infty$ is ru-usc at ω by Proposition 2.13(a), and so $\widehat{\mathcal{I}}$ is ru-usc at ω. Consequently, we can assert

that $\widehat{\mathcal{I}}$ is ru-usc at ω on $\mathfrak{G}$ because $\mathfrak{G} \subset \mathfrak{I}_\omega$. From the second part of Theorem 3.22 it follows that

$$\widehat{\mathcal{I}}(u,\omega) = \begin{cases} \mathcal{I}(u,\omega) & \text{if } u \in \mathfrak{G} \\ \lim\limits_{\tau\to 1^-} \mathcal{I}(\tau u,\omega) & \text{if } u \in \mathfrak{G}^{\mathrm{lsc}}\backslash\mathfrak{G}. \end{cases}$$

We are thus reduced to show that $\mathcal{I}(u,\omega) = \lim_{\tau\to 1^-} \mathcal{I}(\tau u,\omega)$ for all $u \in \mathfrak{G}^{\mathrm{lsc}}\backslash\mathfrak{G}$. Let $u \in \mathfrak{G}^{\mathrm{lsc}}\backslash\mathfrak{G}$. Taking Remark 2.14 into account and using Fatou's lemma we see that

$$\begin{aligned} \lim_{\tau\to 1^-} \mathcal{I}(\tau u,\omega) &= \lim_{\tau\to 1^-} \int_\Omega \widehat{L}_\infty(x,\tau\nabla_\mu u(x),\omega)d\mu(x) \\ &\geqslant \int_\Omega \lim_{\tau\to 1^-} \widehat{L}_\infty(x,\tau\nabla_\mu u(x),\omega)d\mu(x) \\ &= \int_\Omega \widehat{L}_\infty(x,\tau\nabla_\mu u(x),\omega)d\mu(x) \\ &= \mathcal{I}(u,\omega). \end{aligned}$$

Hence, if $\mathcal{I}(u,\omega) = \infty$ then $\lim_{\tau\to 1^-} \mathcal{I}(\tau u,\omega) = \infty$. Assume that $\mathcal{I}(u,\omega) < \infty$. Then

$$\widehat{L}_\infty(\cdot,\nabla_\mu u(\cdot),\omega) \in L^1_\mu(\Omega). \tag{2.22}$$

As $\widehat{L}_\infty$ is ru-usc at ω we have

$$\Delta(\omega) := \overline{\lim_{\tau\to 1^-}} \Delta^{\widehat{a}_\infty(\cdot,\omega)}_{\widehat{L}_\infty}(\tau) \leqslant 0 \tag{2.23}$$

with $\Delta^{\widehat{a}_\infty(\cdot,\omega)}_{\widehat{L}_\infty}(\tau) := \sup_{x\in\Omega} \sup_{\xi\in\widehat{\mathbb{L}}_{\infty,x,\omega}} \frac{\widehat{L}_\infty(x,\tau\xi,\omega)-\widehat{L}_\infty(x,\xi,\omega)}{\widehat{a}_\infty(x,\omega)+\widehat{L}_\infty(x,\xi,\omega)}$ and

$$\widehat{a}_\infty(\cdot,\omega) \in L^1_\mu(\Omega;]0,\infty[). \tag{2.24}$$

By (2.23) there exists $\tau_0 \in]0,1[$ such that $\Delta^{\widehat{a}_\infty(\cdot,\omega)}_{\widehat{L}_\infty}(\tau) \leqslant \Delta(\omega) + 1$ for all $\tau \in [\tau_0,1[$. Consequently, we have

$$\begin{aligned} \widehat{L}_\infty(\cdot,\tau\nabla_\mu u(\cdot),\omega) &\leqslant \widehat{L}_\infty(\cdot,\nabla_\mu u(\cdot),\omega) + \Delta^{\widehat{a}_\infty(\cdot,\omega)}_{\widehat{L}_\infty}(\tau)\left(\widehat{a}_\infty(\cdot,\omega) + \widehat{L}_\infty(\cdot,\nabla_\mu u(\cdot),\omega)\right) \\ &\leqslant \widehat{L}_\infty(\cdot,\nabla_\mu u(\cdot),\omega) + (\Delta(\omega)+1)\left(\widehat{a}_\infty(\cdot,\omega) + \widehat{L}_\infty(\cdot,\nabla_\mu u(\cdot),\omega)\right) =: f(\cdot,\omega) \end{aligned}$$

for all $\tau \in [\tau_0,1[$. Moreover, $f(\cdot,\omega) \in L^1_\mu(\Omega)$ by (2.22) and (2.24) and from Remark 2.14 we see that for every $x \in \Omega$, $\lim_{\tau\to 1^-} \widehat{L}_\infty(x,\tau\nabla_\mu u(x),\omega) = \widehat{L}_\infty(x,\nabla_\mu u(x),\omega)$, and so by using Lebesgue's dominated convergence theorem we conclude that

$$\begin{aligned} \lim_{\tau\to 1^-} \mathcal{I}(\tau u,\omega) &= \lim_{\tau\to 1^-} \int_\Omega \widehat{L}_\infty(x,\tau\nabla_\mu u(x),\omega)d\mu(x) \\ &= \int_\Omega \widehat{L}_\infty(x,\tau\nabla_\mu u(x),\omega)d\mu(x) \\ &= \mathcal{I}(u,\omega), \end{aligned}$$

which completes the proof. ■

Remark 2.19. In case $L_t(x,\xi,\omega) = L(x,\xi)$, and so $E_t(\cdot,\Omega,\omega) = E(\cdot,\Omega)$, we retrieve the relaxation theorem established in [AHM18, Theorem 2.7]. More precisely, denoting the lower semi-continuous envelope of $E(\cdot,\Omega)$ with respect to the strong topology of $L^p_\mu(\Omega;\mathbb{R}^m)$ by $\overline{E}(\cdot,\Omega)$, as a direct consequence of Corollaries 2.15, 2.16 and 2.17 we have the following result.

Corollary 2.20. *Assume that the hypotheses of Theorem* 2.11 *are satisfied with* $L_t(x,\xi,\omega) = L(x,\xi)$.

(a) *For every* $u \in \mathfrak{G}$, *one has*

$$\overline{E}(u,\Omega) = \int_\Omega \widehat{\mathcal{Q}_\mu L}(x,\nabla_\mu u(x))d\mu(x),$$

where $\widehat{\mathcal{Q}_\mu L} : \Omega\times\mathbb{M} \to [0,\infty]$ *is defined by*

$$\widehat{\mathcal{Q}_\mu L}(x,\xi) = \varliminf_{\tau\to 1^-} \mathcal{Q}_\mu L(x,\tau\xi).$$

with $\mathcal{Q}_\mu L : \Omega\times\mathbb{M}\to[0,\infty]$ *given by*

$$\mathcal{Q}_\mu L(x,\xi) := \varlimsup_{\rho\to 0} \inf\left\{ \fint_{Q_\rho(x)} L(y,\xi+\nabla_\mu w(y))d\mu(y) : w \in H^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m)\right\}.$$

(b) *If* $\mathfrak{G}^{\rm lsc} \subset \mathfrak{G}$ *then*

$$\overline{E}(u,\Omega) = \begin{cases} \displaystyle\int_\Omega \widehat{\mathcal{Q}_\mu L}(x,\nabla_\mu u(x))d\mu(x) & \text{if } u\in\mathfrak{G}^{\rm lsc} \\ \infty & \text{if } u\in H^{1,p}_\mu(\Omega;\mathbb{R}^m)\backslash\mathfrak{G}^{\rm lsc}. \end{cases} \tag{2.25}$$

(c) *If* (2.10) *holds then*

$$\overline{E}(u,\Omega) = \begin{cases} \displaystyle\varliminf_{\tau\to 1^-}\int_\Omega \widehat{\mathcal{Q}_\mu L}(x,\tau\nabla_\mu u(x))d\mu(x) & \text{if } u\in\mathfrak{G}^{\rm lsc} \\ \infty & \text{if } u\in H^{1,p}_\mu(\Omega;\mathbb{R}^m)\backslash\mathfrak{G}^{\rm lsc}. \end{cases}$$

If moreover $\mathcal{Q}_\mu\mathbb{G}$ *is convex,* $0\in\operatorname{int}(\mathcal{Q}_\mu\mathbb{G})$ *and* $\mathcal{Q}_\mu L(x,\cdot)$ *is lsc on* $\operatorname{int}(\mathcal{Q}_\mu\mathbb{G})$ *for all* $x\in\Omega$, *then* (2.25) *holds.*

2.4. **Homogenization.** In order to deal with homogenization, it is necessary to make some refinements on our general setting, see (H_1)-(H_2)-(H_3)-$(H_4^{\rm w})$ for the deterministic case and (H_1)-(H_2)-(H_3)-$(H_4^{\rm s})$-(H_5) for the stochastic case. These refinements are an attempt to develop a framework for dealing with homogenization in the setting of metric measure spaces. (Such a development was attempted for the first time in [AHM17].)

Let $\mathcal{B}(X)$ be the class of Borel subsets of X, let $\mathcal{B}_{\mu,0}(X)$ denote the class of $Q\in\mathcal{B}(X)$ such that $\mu(Q)<\infty$ and $\mu(\partial Q)=0$ with $\partial Q = \overline{Q}\backslash\mathring{Q}$ and let $\mathrm{Ba}(X)$ be the class of open balls Q of X. As (X,d,μ) satisfies the annular decay property, i.e. (2.4), we have $\mu(\partial Q)=0$ for all $Q\in\mathrm{Ba}(X)$ (see Remark 2.2). Hence $\mathrm{Ba}(X)\subset\mathcal{B}_{\mu,0}(X)$.

Let $\mathrm{Homeo}(X)$ be the group of homeomorphisms on X, let $\mathbb{G}$ be a subgroup of $\mathrm{Homeo}(X)$ such that

(H_1) the measure μ is $\mathbb{G}$-invariant, i.e. $g^\sharp\mu = \mu$ for all $g \in \mathbb{G}$,

where $g^\sharp\mu$ denotes the image measure of μ by g, and let $\{h_t\}_{t>0} \subset \mathrm{Homeo}(X)$ be satisfying the following two conditions.

(H_2) There exists $\mathbb{U} \in \mathcal{B}_{\mu,0}(X)$ with $\mu(\mathbb{U}) > 0$ such that $(h_t^{-1})^\sharp\mu = \mu(h_t(\mathbb{U}))\mu$ for all $t > 0$.

(H_3) For each $t > 0$ and each open set $A \subset X$ with $\mu(A) > 0$, there exists a bijective map $H_{t,A} : H^{1,p}_{\mu,0}(h_t(A);\mathbb{R}^m) \to H^{1,p}_{\mu,0}(A;\mathbb{R}^m)$ such that $\nabla_\mu H_{t,A}(w) = \nabla_\mu w \circ h_t$ (resp. $\nabla_\mu H^{-1}_{t,A}(v) = \nabla_\mu v \circ h_t^{-1}$) for all $w \in H^{1,p}_{\mu,0}(h_t(A);\mathbb{R}^m)$ (resp. $v \in H^{1,p}_{\mu,0}(A;\mathbb{R}^m)$).

Remark 2.21. From (H_2) it is easy to see that for each $t > 0$, $\mu(h_t(\mathbb{U})) > 0$ and $h_t^\sharp\mu = \frac{1}{\mu(h_t(\mathbb{U}))}\mu$.

Remark 2.22. As $\mu(\overline{\mathbb{U}}\backslash\mathring{\mathbb{U}}) = 0$ we have $\mu(\mathring{\mathbb{U}}) = \mu(\mathbb{U})$ and, under (H_2), for each $t > 0$, $\mu(h_t(\mathring{\mathbb{U}})) = \mu(h_t(\mathbb{U}))$ because $h_t \in \mathrm{Homeo}(X)$ and $h_t^\sharp\mu = \frac{1}{\mu(h_t(\mathbb{U}))}\mu$.

As in §2.3 , we suppose that $\mu\big(\overline{\Omega}\backslash\Omega\big) = 0$, $p > \kappa$, where $\kappa := \frac{\ln(C_d)}{\ln(2)}$ with $C_d \geqslant 1$ given by the inequality (2.1), and $m \geqslant 1$.

2.4.1. *The deterministic case.* Let $G : \mathbb{M} \to [0,\infty]$ be a Borel measurable integrand satisfying (2.5), (2.6) and (2.7) and let $L : X \times \mathbb{M} \to [0,\infty]$ be a Borel measurable integrand having G-growth, i.e. there exist $\alpha, \beta > 0$ such that

$$\alpha G(\xi) \leqslant L(x,\xi) \leqslant \beta(1 + G(\xi)) \tag{2.26}$$

for all $x \in X$ and all $\xi \in \mathbb{M}$, and assumed to be $\mathbb{G}$-invariant, i.e.

$$L(g^{-1}(x),\xi) = L(x,\xi) \tag{2.27}$$

for all $x \in X$, all $\xi \in \mathbb{M}$ and all $g \in \mathbb{G}$. For each $t > 0$, let $L_t : X \times \mathbb{M} \to [0,\infty]$ be given by

$$L_t(x,\xi) = L(h_t(x),\xi). \tag{2.28}$$

(Then, we have $L_t\left((h_t^{-1}\mathrm{o}g^{-1}\mathrm{o}h_t)(x),\xi\right) = L_t(x,\xi)$ for all $x \in X$, all $\xi \in \mathbb{M}$, all $t > 0$ and all $g \in \mathbb{G}$.)

Definition 2.23. Such a $\{L_t\}_{t>0}$, defined by (2.27)-(2.28), is called a $(\mathbb{G}, \{h_t\}_{t>0})$-periodic family of integrands modelled on L.

Remark 2.24. If (2.26) holds then (2.11) is satisfied with L_t given by (2.28).

We further assume that L is ru-usc, i.e.

$$\overline{\lim_{\tau\to 1^-}}\ \Delta_L^a(\tau) \leqslant 0 \tag{2.29}$$

with $\Delta_L^a(\tau) := \sup_{x\in X}\sup_{\xi\in\mathbb{L}_x}\frac{L(x,\tau\xi)-L(x,\xi)}{a+L(x,\xi)}$, where $a \in L^1_\mu(X;]0,\infty])$, and we consider the following condition on the triple $\big((X,d,\mu),\mathbb{G},\{h_t\}_{t>0}\big)$.

($\mathrm{H}_4^{\mathrm{w}}$) For each $Q \in \mathrm{Ba}(X)$, $\{h_t(Q)\}_{t>0}$ is weakly $\mathbb{G}$-asymptotic with respect to $\{h_k(\mathbb{U})\}_{k\in\mathbb{N}^*}$ (see Definition 3.31).

The following result is a consequence of Corollary 2.15 and Theorem 3.33.

Theorem 2.25. *Assume that* (X,d,μ) *satisfies* (H_1), (H_2), (H_3) *and* $(\mathrm{H}_4^{\mathrm{w}})$, *and consider* $\{L_t\}_{t>0}$ *a* $(\mathbb{G},\{h_t\}_{t>0})$*-periodic family of integrands modelled on* L. *If* (2.5), (2.6), (2.7) (2.26) *and* (2.29) *are satisfied and if* (2.13) *and* (2.14) *hold with* $a_t = a\circ h_t$, *where* $a\in L^1_\mu(X;]0,\infty])$ *is given by* (2.29), *then*

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,\Omega) = \int_\Omega \widehat{L}_{\mathrm{hom}}(\nabla_\mu u(x))d\mu(x)$$

for all $u\in\mathfrak{G}$, *where* $\widehat{L}_{\mathrm{hom}}:\mathbb{M}\to[0,\infty]$ *is defined by*

$$\widehat{L}_{\mathrm{hom}}(\xi) := \varliminf_{\tau\to 1^-} L_{\mathrm{hom}}(\tau\xi)$$

with $L_{\mathrm{hom}}:\mathbb{M}\to[0,\infty]$ *given by*

$$L_{\mathrm{hom}}(\xi) := \inf_{k\in\mathbb{N}^*}\inf\left\{\fint_{h_k(\mathring{\mathbb{U}})} L(x,\xi+\nabla_\mu w(x))d\mu(x) : w\in H^{1,p}_{\mu,0}\left(h_k(\mathring{\mathbb{U}});\mathbb{R}^m\right)\right\}.$$

From Theorem 2.25 we deduce the following two results.

Corollary 2.26. *Let assumptions of Theorem* 2.25 *hold. If* $\mathfrak{G}^{\mathrm{lsc}}\subset\mathfrak{G}$ *then*

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,\Omega) = \begin{cases} \displaystyle\int_\Omega \widehat{L}_{\mathrm{hom}}(\nabla_\mu u(x))d\mu(x) & \text{if } u\in\mathfrak{G}^{\mathrm{lsc}} \\ \infty & \text{if } u\in H^{1,p}_\mu(\Omega;\mathbb{R}^m)\backslash\mathfrak{G}^{\mathrm{lsc}}. \end{cases} \tag{2.30}$$

Proof of Corollary 2.26. This follows by the same method as in the proof of Corollary 2.16 by using Theorem 2.25 instead of Corollary 2.15 and replacing "L_∞ by "L_{hom}". ■

Corollary 2.27. *Let assumptions of Theorem* 2.25 *hold. If* (2.10) *is satisfied then*

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,\Omega) = \begin{cases} \displaystyle\varliminf_{\tau\to 1^-}\int_\Omega \widehat{L}_{\mathrm{hom}}(\tau\nabla_\mu u(x))d\mu(x) & \text{if } u\in\mathfrak{G}^{\mathrm{lsc}} \\ \infty & \text{if } u\in H^{1,p}_\mu(\Omega;\mathbb{R}^m)\backslash\mathfrak{G}^{\mathrm{lsc}}. \end{cases}$$

Proof of Corollary 2.27. This follows by the same method as in the proof of Corollary 2.17 by using Theorem 2.25 instead of Corollary 2.15 and replacing "L_∞ by "L_{hom}", and by remarking that, since L is ru-usc, $\{L(h_t(\cdot),\cdot)\}_{t>0}$ is ru-usc (see Remark 3.16). ■

Let $\mathcal{Z}G:\mathbb{M}\to[0,\infty]$ be defined by

$$\mathcal{Z}G(\xi) := \inf\left\{\int_{\mathring{U}} G(\xi+\nabla_\mu w(y))d\mu(y) : w\in H^{1,p}_{\mu,0}(\mathring{\mathbb{U}};\mathbb{R}^m)\right\}. \tag{2.31}$$

As a consequence of Corollary 2.27 we have the following result.

Corollary 2.28. *Under the assumptions of Corollary* 2.27, *if* (3.26) *holds with* $\{A_k\}_{k\in\mathbb{N}^*} = \{h_k(\mathbb{U})\}_{k\in\mathbb{N}^*}$ *and* $a\in L^1_\mu(X;]0,\infty])$ *given by* (2.29), *and if* $\mathcal{Z}\mathbb{G}$ *is convex,* $0\in\mathrm{int}(\mathcal{Z}\mathbb{G})$ *and* L_{hom} *is lsc on* $\mathrm{int}(\mathcal{Z}\mathbb{G})$, *where* $\mathcal{Z}\mathbb{G}$ *denotes the effective domain of* $\mathcal{Z}G$, *then* (2.30) *holds.*

Proof of Corollary 2.28. From (2.11) we see that $\mathbb{L}_{\rm hom} = \mathbb{G}_{\rm hom}$ with $\mathbb{L}_{\rm hom}$ and $\mathbb{G}_{\rm hom}$ denoting the effective domain of $L_{\rm hom}$ and $G_{\rm hom}$ respectively, where $G_{\rm hom} : \mathbb{M} \to [0,\infty]$ is given by

$$G_{\rm hom}(\xi) := \inf_{k\in\mathbb{N}^*} \inf \left\{ \fint_{h_k(\mathring{\mathbb{U}})} G(\xi + \nabla_\mu w(x)) d\mu(x) : w \in H^{1,p}_{\mu,0}\left(h_k(\mathring{\mathbb{U}}); \mathbb{R}^m\right) \right\}.$$

But, for each $\xi \in \mathbb{M}$, by using (H_2) and (H_3), we see that

$$\begin{aligned} G_{\rm hom}(\xi) &= \inf_{k\in\mathbb{N}^*} \frac{1}{\mu(h_k(\mathring{\mathbb{U}}))} \inf \left\{ \int_{\mathring{\mathbb{U}}} G(\xi + \nabla_\mu w(h_k(x))) d(h_k^{-1})^\sharp\mu(x) : w \in H^{1,p}_{\mu,0}\left(h_k(\mathring{\mathbb{U}}); \mathbb{R}^m\right) \right\} \\ &= \inf_{k\in\mathbb{N}^*} \inf \left\{ \int_{\mathring{\mathbb{U}}} G(\xi + \nabla_\mu w(h_k(x))) d\mu(x) : w \in H^{1,p}_{\mu,0}\left(h_k(\mathring{\mathbb{U}}); \mathbb{R}^m\right) \right\} \\ &= \inf_{k\in\mathbb{N}^*} \inf \left\{ \int_{\mathring{\mathbb{U}}} G(\xi + \nabla_\mu w(x)) d\mu(x) : w \in H^{1,p}_{\mu,0}(\mathring{\mathbb{U}}; \mathbb{R}^m) \right\}, \end{aligned}$$

hence $G_{\rm hom} = \mathcal{Z}G$, and consequently $\mathbb{L}_{\rm hom} = \mathcal{Z}\mathbb{G}$. Arguing as in the proof of Proposition 2.13 with "$L_{\rm hom}$" instead of "L_∞" and "$\mathcal{Z}\mathbb{G}$" instead of "$\mathcal{Q}_\mu\mathbb{G}$", and by using Proposition 3.18 instead of Proposition 3.17, we see that Proposition 2.13 is valid with "$L_{\rm hom}$" instead of "L_∞" and "$\mathcal{Z}\mathbb{G}$" instead of "$\mathcal{Q}_\mu\mathbb{G}$". Thus, by the same method as in Remark 2.14 we can assert that

$$\lim_{\tau\to 1^-} \widehat{L}_{\rm hom}(\tau\xi) = \widehat{L}_{\rm hom}(\xi),$$

and the rest of the proof runs as in the proof of Corollary 2.18 with "$L_{\rm hom}$" instead of "L_∞" and by using Corollary 2.27 instead of Corollary 2.17. ■

Remark 2.29. To prove Theorem 2.25 (see Sect. 5), by using Theorem 3.33, we establish that for all $x \in \Omega$ and all $\rho > 0$, one has

$$\lim_{t\to\infty} \mathcal{H}^\rho_\mu L_t(x,\xi) = L_{\rm hom}(\xi) \text{ for all } \xi \in \mathbb{G}.$$

Hence $L_{\rm hom}(\xi) = L_\infty(x,\xi)$ for all $x \in \Omega$ and all $\xi \in \mathbb{G}$. Thus, if $\mathcal{Q}_\mu\mathbb{G} = \mathbb{G}$ then $L_{\rm hom} = L_\infty$, and so $\mathbb{L}_\infty = \mathbb{L}_{\rm hom} = \mathbb{G}_{\rm hom} = \mathcal{Z}\mathbb{G} = \mathbb{G}$. So, in such a case, Corollaries 2.26 and 2.27 are direct applications of Corollaries 2.16 and 2.17 respectively, and Corollary 2.28 can be restated as the following result which is a direct application of Corollary 2.18.

Corollary 2.30. *Under the assumptions of Corollary* 2.27*, if* (3.20) *holds with* $a_t = a \circ h_t$*, where* $a \in L^1_\mu(X;]0,\infty])$ *is given by* (2.29)*, and if* $\mathcal{Q}_\mu\mathbb{G} = \mathbb{G}$*,* $0 \in \mathrm{int}(\mathbb{G})$ *and* $L_{\rm hom}$ *is lsc on* $\mathrm{int}(\mathbb{G})$*, then* (2.30) *holds.*

2.4.2. *The stochastic case.* In what follows, we assume that $(\Sigma, \mathrm{T}, \mathbb{P}, \{\tau_g\}_{g\in\mathbb{G}})$ is a measurable dynamical $\mathbb{G}$-system. Let $L : X \times \mathbb{M} \times \Sigma \to [0,\infty]$ be a Borel measurable stochastic integrand having G-growth, i.e. there exist $\alpha, \beta > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Sigma$,

$$\alpha G(\xi) \leqslant L(x,\xi,\omega) \leqslant \beta(1 + G(\xi)) \tag{2.32}$$

for all $x \in X$ and all $\xi \in \mathbb{M}$ with $G : \mathbb{M} \to [0,\infty]$ satisfying (2.5), (2.6) and (2.7), and assumed to be $\mathbb{G}$-covariant, i.e.

$$L(g^{-1}(x),\xi,\omega) = L(x,\xi,\tau_g(\omega)) \tag{2.33}$$

for all $x \in X$, all $\xi \in \mathbb{M}$ and all $g \in \mathbb{G}$. For each $t > 0$, let $L_t : X \times \mathbb{M} \times \Sigma \to [0, \infty]$ be given by

$$L_t(x, \xi, \omega) = L(h_t(x), \xi, \omega). \tag{2.34}$$

(Then , we have $L_t\left((h_t^{-1} \mathrm{o} g^{-1} \mathrm{o} h_t)(x), \xi, \omega\right) = L_t(x, \xi, \tau_g(\omega))$ for all $x \in X$, all $\xi \in \mathbb{M}$, all $t > 0$, all $g \in \mathbb{G}$ and $\mathbb{P}$-a.a. $\omega \in \Sigma$.)

Definition 2.31. Such a $\{L_t\}_{t>0}$, defined by (2.33)-(2.34), is called a $(\mathbb{G}, \{h_t\}_{t>0})$-stochastic family of integrands modelled on L.

Remark 2.32. If (2.32) holds then (2.11) is satisfied with L_t given by (2.34).

We further assume that for $\mathbb{P}$-a.e. $\omega \in \Sigma$, L is ru-usc at ω, i.e.

$$\overline{\lim_{\tau \to 1^-}} \, \Delta_L^{a(\cdot,\omega)}(\tau) \leqslant 0 \tag{2.35}$$

with $\Delta_L^{a(\cdot,\omega)}(\tau) := \sup_{x \in X} \sup_{\xi \in \mathbb{L}_{x,\omega}} \frac{L(x,\tau\xi,\omega) - L(x,\xi,\omega)}{a(\cdot,\omega) + L(x,\xi,\omega)}$, where $a(\cdot, \omega) \in L^1_\mu(X;]0, \infty])$, and we consider the following conditions on the triple $\big((X, d, \mu), \mathbb{G}, \{h_t\}_{t>0}\big)$.

- ($\mathrm{H}_4^{\mathrm{s}}$) For each $Q \in \mathrm{Ba}(X)$, $\{h_t(Q)\}_{t>0}$ is strongly $\mathbb{G}$-asymptotic with respect to $\{h_k(\mathbb{U})\}_{k \in \mathbb{N}^*}$ (see Definition 3.37).
- (H_5) The metric measure space (X, d, μ) is meshable with respect to $\{h_k(\mathbb{U})\}_{k \in \mathbb{N}^*}$ (see Definition 3.34).

Remark 2.33. From Definitions 3.31 and 3.37 we see that ($\mathrm{H}_4^{\mathrm{s}}$) implies ($\mathrm{H}_4^{\mathrm{w}}$).

The following result is a consequence of Corollary 2.15 and Theorem 3.42.

Theorem 2.34. *Assume that* (X, d, μ) *satisfies* (H_1), (H_2), (H_3), ($\mathrm{H}_4^{\mathrm{s}}$) *and* ($\mathrm{H}_5$), *and consider* $\{L_t\}_{t>0}$ *a* $(\mathbb{G}, \{h_t\}_{t>0})$*-stochastic family of integrands modelled on* L. *If* (2.5), (2.6), (2.7) (2.32) *and* (2.35) *are satisfied and if* (2.13) *and* (2.14) *hold with* $a_t(\cdot, \omega) = a(h_t(\cdot), \omega)$, *where* $a(\cdot, \omega) \in L^1_\mu(X;]0, \infty])$ *is given by* (2.35), *then for* $\mathbb{P}$*-a.e.* $\omega \in \Sigma$, *one has*

$$\Gamma(L^p_\mu)\text{-}\lim_{t \to \infty} E_t(u, \Omega, \omega) = \int_\Omega \widehat{L}_{\mathrm{hom}}(\nabla_\mu u(x), \omega) d\mu(x)$$

for all $u \in \mathfrak{G}$, *where* $\widehat{L}_{\mathrm{hom}} : \mathbb{M} \times \Sigma \to [0, \infty]$ *is defined by*

$$\widehat{L}_{\mathrm{hom}}(\xi, \omega) := \underline{\lim_{\tau \to 1^-}} \, L_{\mathrm{hom}}(\tau\xi, \omega)$$

with $L_{\mathrm{hom}} : \mathbb{M} \times \Sigma \to [0, \infty]$ *given by*

$$L_{\mathrm{hom}}(\xi, \omega) := \inf_{k \in \mathbb{N}^*} \mathbb{E}^{\mathcal{I}} \left[\inf \left\{ \fint_{h_k(\mathring{\mathbb{U}})} L(y, \xi + \nabla_\mu w(y), \cdot) d\mu(y) : w \in H^{1,p}_{\mu,0}\left(h_k(\mathring{\mathbb{U}}); \mathbb{R}^m\right) \right\} \right] (\omega),$$

where $\mathbb{E}^{\mathcal{I}}$ *denotes the conditional expectation over* $\mathcal{I}$ *with respect to* $\mathbb{P}$, *with* $\mathcal{I}$ *being the* σ*-algebra of invariant sets with respect to* $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in \mathbb{G}})$. *If in addition* $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g \in \mathbb{G}})$ *is ergodic, see Definition* 3.40, *then* L_{hom} *is deterministic and is given by*

$$L_{\mathrm{hom}}(\xi) := \inf_{k \in \mathbb{N}^*} \mathbb{E} \left[\inf \left\{ \fint_{h_k(\mathring{\mathbb{U}})} L(y, \xi + \nabla_\mu w(y), \cdot) d\mu(y) : w \in H^{1,p}_{\mu,0}\left(h_k(\mathring{\mathbb{U}}); \mathbb{R}^m\right) \right\} \right], \tag{2.36}$$

where $\mathbb{E}$ *denotes the expectation with respect to* $\mathbb{P}$.

As in the deterministic case (see §2.4.1) we can establish the following three results. Corollaries 2.35 and 2.36 below are consequences of Theorem 2.34.

Corollary 2.35. *Let assumptions of Theorem* 2.34 *hold. If* $\mathfrak{G}^{\rm lsc} \subset \mathfrak{G}$ *then for* $\mathbb{P}$*-a.e.* $\omega \in \Sigma$*, one has*

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,\Omega,\omega) = \begin{cases} \displaystyle\int_\Omega \widehat{L}_{\rm hom}(\nabla_\mu u(x),\omega)d\mu(x) & \text{if } u \in \mathfrak{G}^{\rm lsc} \\ \infty & \text{if } u \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)\backslash\mathfrak{G}^{\rm lsc}. \end{cases} \tag{2.37}$$

If in addition $(\Sigma,\mathcal{T},\mathbb{P},\{\tau_g\}_{g\in\mathbb{G}})$ *is ergodic, then* $L_{\rm hom}$ *is deterministic and is given by* (2.36)*.*

Corollary 2.36. *Let assumptions of Theorem* 2.34 *hold. If* (2.10) *is satisfied then for* $\mathbb{P}$*-a.e.* $\omega \in \Sigma$*, one has*

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u,\Omega,\omega) = \begin{cases} \displaystyle\lim_{\tau\to 1^-}\int_\Omega \widehat{L}_{\rm hom}(\tau\nabla_\mu u(x),\omega)d\mu(x) & \text{if } u \in \mathfrak{G}^{\rm lsc} \\ \infty & \text{if } u \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)\backslash\mathfrak{G}^{\rm lsc}. \end{cases}$$

If in addition $(\Sigma,\mathcal{T},\mathbb{P},\{\tau_g\}_{g\in\mathbb{G}})$ *is ergodic, then* $L_{\rm hom}$ *is deterministic and is given by* (2.36)*.*

From Corollary 2.36 we deduce the following result.

Corollary 2.37. *Under the assumptions of Corollary* 2.36*, if* (3.26) *holds with* $\{A_k\}_{k\in\mathbb{N}^*} = \{h_k(\mathbb{U})\}_{k\in\mathbb{N}^*}$ *and* $a(\cdot,\omega) \in L^1_\mu(X;]0,\infty])$ *given by* (2.35)*, and if* $\mathcal{Z}\mathbb{G}$ *is convex,* $0 \in \text{int}(\mathcal{Z}\mathbb{G})$ *and* $L_{\rm hom}(\cdot,\omega)$ *is lsc on* $\text{int}(\mathcal{Z}\mathbb{G})$ *for* $\mathbb{P}$*-a.a.* $\omega \in \Sigma$*, where* $\mathcal{Z}\mathbb{G}$ *denotes the effective domain of* $\mathcal{Z}G : \mathbb{M} \to [0,\infty]$ *given by* (2.31)*, then* (2.37) *holds. If in addition* $(\Sigma,\mathcal{T},\mathbb{P},\{\tau_g\}_{g\in\mathbb{G}})$ *is ergodic, then* $L_{\rm hom}$ *is deterministic and is given by* (2.36)*.*

Remark 2.38. As in the deterministic case (see Remark 2.29), when $\mathcal{Q}_\mu\mathbb{G} = \mathbb{G}$, Corollaries 2.35 and 2.36 are direct applications of Corollaries 2.16 and 2.17, and Corollary 2.37 can be restated as the following result which is a direct application of Corollary 2.18.

Corollary 2.39. *Under the assumptions of Corollary* 2.36*, if* (3.20) *holds with* $a_t(\cdot,\omega) = a(h_t(\cdot),\omega)$*, where* $a(\cdot,\omega) \in L^1_\mu(X;]0,\infty])$ *is given by* (2.35)*, and if* $\mathcal{Q}_\mu\mathbb{G} = \mathbb{G}$*,* $0 \in \text{int}(\mathbb{G})$ *and* $L_{\rm hom}(\cdot,\omega)$ *is lsc on* $\text{int}(\mathbb{G})$ *for* $\mathbb{P}$*-a.a.* $\omega \in \Sigma$*, then* (2.37) *holds. If in addition* $(\Sigma,\mathcal{T},\mathbb{P},\{\tau_g\}_{g\in\mathbb{G}})$ *is ergodic, then* $L_{\rm hom}$ *is deterministic and is given by* (2.36)*.*

3. Auxiliary results

In this section we give the auxiliary results that we need for proving the Γ-convergence and homogenization theorems.

3.1. **The p-Cheeger-Sobolev space.** Let $p > 1$ be a real number, let (X,d,μ) be a metric measure space, where (X,d) is complete, supporting a weak $(1,p)$-Poincaré inequality, see (2.2), and such that μ is a doubling positive Radon measure on X, see (2.1), which satisfies the annular decay property, see (2.4), and let $\Omega \subset X$ be a bounded open set. We begin with the concept of upper gradient introduced by Heinonen and Koskela (see [HK98]).

Definition 3.1. A Borel function $g : \Omega \to [0,\infty]$ is said to be an upper gradient for $f : \Omega \to \mathbb{R}$ if $|f(c(1)) - f(c(0))| \leqslant \int_0^1 g(c(s))ds$ for all continuous rectifiable curves $c : [0,1] \to \Omega$.

The concept of upper gradient has been generalized by Cheeger as follows (see [Che99, Definition 2.8]).

Definition 3.2. A function $g \in L^p_\mu(\Omega)$ is said to be a p-weak upper gradient for $f \in L^p_\mu(\Omega)$ if there exist $\{f_n\}_n \subset L^p_\mu(\Omega)$ and $\{g_n\}_n \subset L^p_\mu(\Omega)$ such that for each $n \geqslant 1$, g_n is an upper gradient for f_n, $f_n \to f$ in $L^p_\mu(\Omega)$ and $g_n \to g$ in $L^p_\mu(\Omega)$.

Denote the algebra of Lipschitz functions from Ω to $\mathbb{R}$ by $\mathrm{Lip}(\Omega)$. (Note that, by Hopf-Rinow's theorem (see [BH99, Proposition 3.7, pp. 35]), the closure of Ω is compact, and so every Lipschitz function from Ω to $\mathbb{R}$ is bounded.) From Cheeger and Keith (see [Che99, Theorem 4.38] and [Kei04, Definition 2.1.1 and Theorem 2.3.1]) we have the following result.

Theorem 3.3. *There exists a countable family $\{(\Omega_k, \xi^k)\}_k$ of μ-measurable disjoint subsets Ω_k of Ω with $\mu(\Omega \setminus \cup_k \Omega_k) = 0$ and of functions $\xi^k = (\xi^k_1, \cdots, \xi^k_{N(k)}) : \Omega \to \mathbb{R}^{N(k)}$ with $\xi^k_i \in \mathrm{Lip}(\Omega)$ satisfying the following properties*:

(a) *there exists an integer $N \geqslant 1$ such that $N(k) \in \{1, \cdots, N\}$ for all k;*

(b) *for every k and every $f \in \mathrm{Lip}(\Omega)$ there is a unique $D^k_\mu f \in L^\infty_\mu(\Omega_k; \mathbb{R}^{N(k)})$ such that for μ-a.e. $x \in \Omega_k$,*

$$\lim_{\rho \to 0} \frac{1}{\rho} \|f - f_x\|_{L^\infty_\mu(Q_\rho(x))} = 0,$$

where $f_x \in \mathrm{Lip}(\Omega)$ is given by $f_x(y) := f(x) + D^k_\mu f(x) \cdot (\xi^k(y) - \xi^k(x))$; in particular

$$D^k_\mu f_x(y) = D^k_\mu f(x) \text{ for } \mu\text{-a.e. } y \in \Omega_k;$$

(c) *the operator $D_\mu : \mathrm{Lip}(\Omega) \to L^\infty_\mu(\Omega; \mathbb{R}^N)$ given by*

$$D_\mu f := \sum_k \mathbb{1}_{X_k} D^k_\mu f,$$

where $\mathbb{1}_{\Omega_k}$ denotes the characteristic function of Ω_k, is linear and, for each $f, g \in \mathrm{Lip}(\Omega)$, one has

$$D_\mu(fg) = f D_\mu g + g D_\mu f;$$

(d) *for every $f \in \mathrm{Lip}(\Omega)$, $D_\mu f = 0$ μ-a.e. on every μ-measurable set where f is constant.*

Let $\mathrm{Lip}(\Omega; \mathbb{R}^m) := [\mathrm{Lip}(\Omega)]^m$ and let $\nabla_\mu : \mathrm{Lip}(\Omega; \mathbb{R}^m) \to L^\infty_\mu(\Omega; \mathbb{M})$ given by

$$\nabla_\mu u := \begin{pmatrix} D_\mu u_1 \\ \vdots \\ D_\mu u_m \end{pmatrix} \text{ with } u = (u_1, \cdots, u_m).$$

From Theorem 3.3(c) we see that for every $u \in \mathrm{Lip}(\Omega; \mathbb{R}^m)$ and every $f \in \mathrm{Lip}(\Omega)$, one has

$$\nabla_\mu(fu) = f \nabla_\mu u + D_\mu f \otimes u. \tag{3.1}$$

Definition 3.4. The p-Cheeger-Sobolev space $H^{1,p}_\mu(\Omega; \mathbb{R}^m)$ is defined as the completion of $\mathrm{Lip}(\Omega; \mathbb{R}^m)$ with respect to the norm

$$\|u\|_{H^{1,p}_\mu(\Omega;\mathbb{R}^m)} := \|u\|_{L^p_\mu(\Omega;\mathbb{R}^m)} + \|\nabla_\mu u\|_{L^p_\mu(\Omega;\mathbb{M})}. \tag{3.2}$$

Taking Proposition 3.6(a) below into account, since $\|\nabla_\mu u\|_{L^p_\mu(\Omega;\mathbb{M})} \leqslant \|u\|_{H^{1,p}_\mu(\Omega;\mathbb{R}^m)}$ for all $u \in \mathrm{Lip}(\Omega;\mathbb{R}^m)$ the linear map ∇_μ from $\mathrm{Lip}(\Omega;\mathbb{R}^m)$ to $L^p_\mu(\Omega;\mathbb{M})$ has a unique extension to $H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ which will still be denoted by ∇_μ and will be called the μ-gradient.

Remark 3.5. When Ω is a bounded open subset of $X = \mathbb{R}^N$ and μ is the Lebesgue measure on $\mathbb{R}^N$, we retrieve the (classical) Sobolev spaces $H^{1,p}(\Omega;\mathbb{R}^m)$. For more details on the various possible extensions of the classical theory of the Sobolev spaces to the setting of metric measure spaces, we refer to [Hei07, §10-14] (see also [Che99, Sha00, GT01, Haj03]).

The following proposition (whose proof is given below, see also [AHM15, AHM17, AHM18]) provides useful properties for dealing with calculus of variations in the metric measure setting.

Proposition 3.6. *We have the following results*:

(a) *the μ-gradient is closable in $H^{1,p}_\mu(\Omega;\mathbb{R}^m)$, i.e. for every $u \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ and every $A \in \mathcal{O}(\Omega)$, if $u(x) = 0$ for μ-a.a. $x \in A$ then $\nabla_\mu u(x) = 0$ for μ-a.a. $x \in A$;*

(b) *Ω supports a p-Sobolev inequality, i.e. there exists $C_S > 0$ such that*

$$\left(\int_{Q_\rho(x)} |v|^p d\mu\right)^{\frac{1}{p}} \leqslant \rho C_S \left(\int_{Q_\rho(x)} |\nabla_\mu v|^p d\mu\right)^{\frac{1}{p}} \tag{3.3}$$

for all $0 < \rho \leqslant \rho_0$, with $\rho_0 > 0$, and all $v \in H^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m)$, where, for each $A \in \mathcal{O}(\Omega)$, $H^{1,p}_{\mu,0}(A;\mathbb{R}^m)$ is the closure of $\mathrm{Lip}_0(A;\mathbb{R}^m)$ with respect to $H^{1,p}_\mu$-norm defined in (3.2) with

$$\mathrm{Lip}_0(A;\mathbb{R}^m) := \big\{u \in \mathrm{Lip}(\Omega;\mathbb{R}^m) : u = 0 \text{ on } \Omega\backslash A\big\};$$

(c) *Ω satisfies the Vitali covering theorem, i.e. for every $A \subset \Omega$ and every family $\mathcal{F}$ of closed balls in Ω, if $\inf\{\rho > 0 : \overline{Q}_\rho(x) \in \mathcal{F}\} = 0$ for all $x \in A$ then there exists a countable disjoint subfamily $\mathcal{G}$ of $\mathcal{F}$ such that $\mu(A\backslash \cup_{Q\in\mathcal{G}} Q) = 0$; in other words, $A \subset \big(\cup_{Q\in\mathcal{G}} Q\big) \cup N$ with $\mu(N) = 0$;*

(d) *for every $u \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ and μ-a.e. $x \in \Omega$ there exists $u_x \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ such that:*

$$\nabla_\mu u_x(y) = \nabla_\mu u(x) \text{ for } \mu\text{-a.a. } y \in \Omega; \tag{3.4}$$

$$\lim_{\rho\to 0} \frac{1}{\rho}\|u - u_x\|_{L^\infty_\mu(Q_\rho(x);\mathbb{R}^m)} = 0 \text{ if } p > \kappa, \tag{3.5}$$

where $\kappa := \frac{\ln(C_d)}{\ln(2)}$ with $C_d \geqslant 1$ given by the inequality (2.1);

(e) *for every $x \in \Omega$, every $\rho > 0$ and every $\lambda \in]0,1[$ there exists a Urysohn function[2] $\varphi \in \mathrm{Lip}(\Omega)$ for the pair $(\Omega\backslash Q_\rho(x), \overline{Q}_{\lambda\rho}(x))$ such that*

$$\|D_\mu\varphi\|_{L^\infty_\mu(\Omega;\mathbb{R}^N)} \leqslant \frac{\theta}{\rho(1-\lambda)}$$

for some $\theta > 0$;

[2]Given a metric space (Ω, d), by a Urysohn function from Ω to $\mathbb{R}$ for the pair $(\Omega\backslash V, K)$, where $K \subset V \subset \Omega$ with K compact and V open, we mean a continuous function $\varphi : \Omega \to \mathbb{R}$ such that $\varphi(x) \in [0,1]$ for all $x \in \Omega$, $\varphi(x) = 0$ for all $x \in \Omega\backslash V$ and $\varphi(x) = 1$ for all $x \in K$.

(f) *for μ-a.e. $x \in \Omega$,*

$$\lim_{\lambda\to 1^-} \underline{\lim_{\rho\to 0}} \frac{\mu(Q_{\lambda\rho}(x))}{\mu(Q_\rho(x))} = \lim_{\lambda\to 1^-} \overline{\lim_{\rho\to 0}} \frac{\mu(Q_{\lambda\rho}(x))}{\mu(Q_\rho(x))} = 1. \tag{3.6}$$

Remark 3.7. As μ is a Radon measure, if Ω satisfies the Vitali covering theorem, i.e. Proposition 3.6(c) holds, then for every $A \in \mathcal{O}(\Omega)$ and every $\varepsilon > 0$ there exists a countable family $\{Q_{\rho_i}(x_i)\}_{i\in I}$ of disjoint open balls of A with $x_i \in A$, $\rho_i \in]0, \varepsilon[$ such that $\mu\big(A \backslash \cup_{i\in I} Q_{\rho_i}(x_i)\big) = 0$. By the annular decay property, see (2.4), we also have $\mu(\partial Q_{\rho_i}(x_i)) = 0$ for all $i \in I$ (see Remark 2.2).

Proof of Proposition 3.6. Firstly, Ω satisfies the Vitali covering theorem, i.e. the property (c) holds, because μ is doubling on Ω (see [Fed69, Theorem 2.8.18]). Secondly, the closability of the μ-gradient in $\mathrm{Lip}(\Omega;\mathbb{R}^m)$, given by Theorem 3.3(d), can be extended from $\mathrm{Lip}(\Omega;\mathbb{R}^m)$ to $H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ by using the closability theorem of Franchi, Hajłasz and Koskela (see [FHK99, Theorem 10]). Thus, the property (a) is satisfied. Thirdly, according to [BB11, Corollary 4.24 pp. 93], since μ is doubling on Ω and Ω supports a weak $(1,p)$-Poincaré inequality, we can assert that Ω supports a weak (p,p)-Poincaré inequality, i.e. there exist $c_p > 0$ and $\lambda \geqslant 1$ such that for μ-a.e. $x \in \Omega$ and every $\rho > 0$,

$$\left(\fint_{Q_\rho(x)} \left| f - \fint_{Q_\rho(x)} f d\mu \right|^p d\mu \right)^{\frac{1}{p}} \leqslant \rho c_p \left(\fint_{Q_{\lambda\rho}(x)} g^p d\mu \right)^{\frac{1}{p}}$$

for all $f \in L^p_\mu(\Omega)$ and all p-weak upper gradient $g \in L^p_\mu(\Omega)$ for f. Hence, by using the Sobolev inequality in [BB11, Theorem 5.51 pp. 142], it follows that there exists $c > 0$ such that for every $0 < \rho \leqslant \rho_0$, with $\rho_0 \geqslant 0$ and every $v \in H^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m)$,

$$\left(\fint_{Q_\rho(x)} |v|^p d\mu \right)^{\frac{1}{p}} \leqslant \rho c \left(\fint_{Q_\rho(x)} |g_v|^p d\mu \right)^{\frac{1}{p}}, \tag{3.7}$$

where g_v is the minimal p-weak upper gradient[3] for v. Moreover (see [Che99, §4] and also [BB11, §B.2, pp. 363], [Bjö00] and [GH13, Remark 2.15]), there exists $\theta \geqslant 1$ such that for every $w \in H^{1,p}_\mu(\Omega)$ and μ-a.e. $x \in \Omega$,

$$\frac{1}{\theta} g_w(x) \leqslant |D_\mu w(x)| \leqslant \theta g_w(x),$$

where g_w is the minimal p-weak upper gradient for w. As for $v = (v_i)_{i=1,\cdots,m} \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ we have $\nabla_\mu v = (D_\mu v_i)_{i=1,\cdots,m}$, it follows that

$$\frac{1}{\theta} |g_v(x)| \leqslant |\nabla_\mu v(x)| \leqslant \theta |g_v(x)| \tag{3.8}$$

[3]From Cheeger (see [Che99, Theorems 2.10 and 2.18]), for each $w \in H^{1,p}_\mu(\Omega)$ there exists a unique p-weak upper gradient for w, denoted by $g_w \in L^p_\mu(\Omega)$ and called the minimal p-weak upper gradient for w, such that for every p-weak upper gradient $g \in L^p_\mu(\Omega)$ for w, $g_w(x) \leqslant g(x)$ for μ-a.a. $x \in \Omega$. For $v = (v_i)_{i=1,\cdots,m} \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$, $g_v := (g_{v_i})_{i=1,\cdots,m}$ is naturally called the minimal p-weak upper gradient for v.

for μ-a.a. $x \in \Omega$. Combining (3.7) with (3.8) we obtain the property (b). Fourthly, from Björn (see [Bjö00, Corollary 4.6(ii)] we see that for every k, every $u \in H^{1,p}_{\mu}(\Omega;\mathbb{R}^m)$ and μ-a.e. $x \in \Omega_k$,

$$\nabla_\mu u_x(y) = \nabla_\mu u(x) \text{ for } \mu\text{-a.a. } y \in \Omega_k,$$

where $u_x \in H^{1,p}_{\mu}(\Omega;\mathbb{R}^m)$ is given by

$$u_x(y) := u(x) + \nabla_\mu u(x) \cdot (\xi^k(y) - \xi^k(x)),$$

and if $p > \kappa$ then u is L^∞_μ-differentiable at x, i.e.

$$\lim_{\rho\to 0} \frac{1}{\rho} \|u(y) - u_x(y)\|_{L^\infty_\mu(Q_\rho(x);\mathbb{R}^m)} = 0.$$

Hence the property (d) is verified. Fifthly, given $\rho > 0$, $\lambda \in]0,1[$ and $x \in \Omega$, there exists a Urysohn function $\varphi \in \mathrm{Lip}(\Omega)$ for the pair $(X\backslash Q_\rho(x)), \overline{Q}_{\lambda\rho}(x))$ such

$$\|\mathrm{Lip}\varphi\|_{L^\infty_\mu(\Omega)} \leqslant \frac{1}{\rho(1-\lambda)},$$

where for every $y \in \Omega$,

$$\mathrm{Lip}\varphi(y) := \overline{\lim_{d(y,z)\to 0}} \frac{|\varphi(y) - \varphi(z)|}{d(y,z)}.$$

But, since μ is doubling and Ω supports a weak $(1,p)$-Poincaré inequality, from Cheeger (see [Che99, Theorem 6.1]) we have $\mathrm{Lip}\varphi(y) = g_\varphi(y)$ for μ-a.a. $y \in \Omega$, where g_φ is the minimal p-weak upper gradient for φ. Hence

$$\|D_\mu\varphi\|_{L^\infty_\mu(\Omega;\mathbb{R}^N)} \leqslant \frac{\theta}{\rho(1-\lambda)}$$

because $|D_\mu\varphi(y)| \leqslant \theta |g_\varphi(y)|$ for μ-a.a. $y \in \Omega$. Consequently the property (e) holds. Finally, given $x \in \Omega$, by using the annular decay property (2.4) with $r = \lambda\rho$ and $\sigma = \frac{1}{\lambda}$, where $\rho > 0$ and $\lambda \in]0,1[$, we see that

$$\mu(Q_\rho(x)\backslash Q_{\lambda\rho}(x)) \leqslant C_A(1-\lambda)^\delta \mu(Q_\rho(x))$$

for all $\rho > 0$ and all $\lambda \in]0,1[$ with $C_A \geqslant 1$ given by (2.4), and the property (f) follows. ■

In the framework of the p-Cheeger-Sobolev spaces with $p > \kappa$, we have the following L^∞_μ-compactness result.

Theorem 3.8. *Assume that $\mu\big(\overline{\Omega}\backslash\Omega\big) = 0$. If $p > \kappa$ and if $u \in H^{1,p}_{\mu}(\Omega;\mathbb{R}^m)$ and $\{u_n\}_n \subset H^{1,p}_{\mu}(\Omega;\mathbb{R}^m)$ are such that*

$$\lim_{n\to\infty} \|u_n - u\|_{L^p_\mu(\Omega;\mathbb{R}^m)} = 0 \text{ and } \sup_{n\geqslant 1} \|\nabla_\mu u_n\|_{L^p_\mu(\Omega;\mathbb{M})} < \infty, \tag{3.9}$$

then, up to a subsequence,

$$\lim_{n\to\infty} \|u_n - u\|_{L^\infty_\mu(\Omega;\mathbb{R}^m)} = 0. \tag{3.10}$$

Proof of Theorem 3.8. Since (X,d,μ) is a complete doubling metric space, (X,d,μ) is proper, i.e. every closed ball is compact (see [HKST15, Lemma 4.1.14]), and so $(\overline{\Omega},d|_{\overline{\Omega}\times\overline{\Omega}})$ is compact. Thus, as $\mu\big(\overline{\Omega}\backslash\Omega\big)=0$ we can assert that $(\overline{\Omega},d|_{\overline{\Omega}\times\overline{\Omega}},\mu|_{\overline{\Omega}})$ is a compact doubling metric measure space supporting a weak $(1,p)$-Poincaré inequality. In what follows, to simplify the notation we set $(Y,\delta,\nu):=(\overline{\Omega},d|_{\overline{\Omega}\times\overline{\Omega}},\mu|_{\overline{\Omega}})$.

Step 1: two auxiliary lemmas. We need the following two lemmas (cf. Lemmas 3.9 and 3.10).

Lemma 3.9. *If $p>\kappa$ then for every $r>0$ and every $\bar{x}\in Y$ there exists $C(r,\bar{x})>0$ such that*

$$|u(y)-u(z)|\leqslant C(r,\bar{x})\delta(y,z)^{1-\frac{\kappa}{p}}\left(\int_{Q_{6\sigma r}(\bar{x})}|\nabla_\nu u|^p d\nu\right)^{\frac{1}{p}}$$

for all $u\in H^{1,p}_\nu(Y;\mathbb{R}^m)$ and all $y,z\in Q_r(\bar{x})$, where $\sigma\geqslant 1$ is given by (2.2).

Proof of Lemma 3.9. From [Haj03, Theorem 9.7] we can assert that there exists $c>0$ such that

$$|w(y)-w(z)|\leqslant cr^{\frac{\kappa}{p}}\delta(y,z)^{1-\frac{\kappa}{p}}\left(\fint_{Q_{6\sigma r}(\bar{x})}g_w^p d\nu\right)^{\frac{1}{p}} \tag{3.11}$$

for all $w\in H^{1,p}_\nu(Y)$, all $\bar{x}\in Y$, all $r>0$ and all $y,z\in Q_r(\bar{x})$, where $\sigma\geqslant 1$ is given by (2.2) and $g_w\in L^p_\nu(Y)$ denotes the minimal p-weak upper gradient for w. On the other hand, from (2.3) it is easy to see that for every $r>0$ and every $\bar{x}\in Y$ there exists $\theta(r,\bar{x})>0$ such that

$$\mu(Q_r(\bar{x}))\geqslant\theta(r,\bar{x})r^\kappa.$$

But $g_w\leqslant\alpha|D_\nu w|$ with $\alpha\geqslant 1$ (see [Che99, §4]) and so $\fint_{Q_{6\sigma r}(\bar{x})}g_w^p d\nu\leqslant\alpha^p\fint_{Q_{6\sigma r}(\bar{x})}|D_\nu w|^p d\nu$. Thus, for each $r>0$, each $\bar{x}\in Y$ and each $y,z\in Q_r(\bar{x})$, (3.11) can be rewritten as follows

$$|w(y)-w(z)|\leqslant C(r,\bar{x})\delta(y,z)^{1-\frac{\kappa}{p}}\left(\int_{Q_{6\sigma r}(\bar{x})}|D_\nu w|^p d\nu\right)^{\frac{1}{p}}$$

with $C(r,\bar{x})=\frac{c\alpha}{\theta(r,\bar{x})}>0$. It follows that for every $r>0$ and every $\bar{x}\in Y$, we have

$$\begin{aligned}
|u(y)-u(z)| &\leqslant C(r,\bar{x})\delta(y,z)^{1-\frac{\kappa}{p}}\sum_{i=1}^m\left(\int_{Q_{6\sigma r}(\bar{x})}|D_\nu u_i|^p d\nu\right)^{\frac{1}{p}}\\
&\leqslant C(r,\bar{x})\delta(y,z)^{1-\frac{\kappa}{p}}\left(\int_{Q_{6\sigma r}(\bar{x})}\sum_{i=1}^m|D_\nu u_i|^p d\nu\right)^{\frac{1}{p}}\\
&= C(r,\bar{x})\delta(y,z)^{1-\frac{\kappa}{p}}\left(\int_{Q_{6\sigma r}(\bar{x})}|\nabla_\nu u|^p d\nu\right)^{\frac{1}{p}}
\end{aligned}$$

for all $u\in H^{1,p}_\nu(Y;\mathbb{R}^m)$ and all $y,z\in Q_r(\bar{x})$, and the proof of Lemma 3.9 is complete. ■

Denote the space of continuous functions from Y to $\mathbb{R}^m$ by $C(Y;\mathbb{R}^m)$. As a consequence of Lemma 3.9 we have the following result.

Lemma 3.10. *If $p > \kappa$ then $H^{1,p}_\nu(Y;\mathbb{R}^m)$ continuously embeds into $C(Y;\mathbb{R}^m)$, i.e.*

$$H^{1,p}_\nu(Y;\mathbb{R}^m) \subset C(Y;\mathbb{R}^m)$$

and there exists $K_0 > 0$ such that

$$\|u\|_{C(Y;\mathbb{R}^m)} \leqslant K_0 \|u\|_{H^{1,p}_\nu(Y;\mathbb{R}^m)} \tag{3.12}$$

for all $u \in H^{1,p}_\nu(X;\mathbb{R}^m)$. Moreover, there exists $K_1 > 0$ such that

$$|u(y) - u(z)| \leqslant K_1 \delta(y,z)^{1-\frac{\kappa}{p}} \|\nabla_\nu u\|_{L^p_\nu(Y;\mathbb{M})} \tag{3.13}$$

for all $u \in H^{1,p}_\nu(Y;\mathbb{R}^m)$ and all $y, z \in Y$.

Proof of Lemma 3.10. Applying Lemma 3.9 with $r = \operatorname{diam}(Y)$ and for a fixed $\bar{x} = x_0 \in Y$, where $\operatorname{diam}(Y) = \sup\{\delta(y,z) : y, z \in Y\} < \infty$ because (Y,δ) is compact, we see that

$$\begin{aligned} |u(y) - u(z)| &\leqslant C\left(\operatorname{diam}(Y), x_0\right) \delta(y,z)^{1-\frac{\kappa}{p}} \|\nabla_\nu u\|_{L^p_\nu(Y;\mathbb{M})} \\ &\leqslant C\left(\operatorname{diam}(Y), x_0\right) \operatorname{diam}(Y)^{1-\frac{\kappa}{p}} \|\nabla_\nu u\|_{L^p_\nu(Y;\mathbb{M})} \end{aligned} \tag{3.14}$$

for all $u \in H^{1,p}_\nu(Y;\mathbb{R}^m)$ and all $y, z \in Y$. Hence (3.13) holds with $K_1 = C\left(\operatorname{diam}(Y), x_0\right)$ and every $u \in H^{1,p}_\nu(Y;\mathbb{R}^m)$ is $(1 - \frac{\kappa}{p})$-Hölder continuous. In particular, it follows that $H^{1,p}_\nu(Y;\mathbb{R}^m) \subset C(Y;\mathbb{R}^m)$. On the other hand, given any $u \in H^{1,p}_\nu(Y;\mathbb{R}^m)$ and any $y \in Y$, we have $|u(y)|^p \leqslant 2^p \left(|u(y) - u(z)|^p + |u(z)|^p\right)$ for all $z \in Y$, and consequently

$$\nu(Y)^{\frac{1}{p}} |u(y)| \leqslant 2^{1+\frac{1}{p}} \left(\int_Y |u(y) - u(z)|^p d\nu(z) \right)^{\frac{1}{p}} + 2^{1+\frac{1}{p}} \|u\|_{L^p_\nu(Y;\mathbb{R}^m)}. \tag{3.15}$$

But, by (3.14) we have

$$\left(\int_Y |u(y) - u(z)|^p d\nu(z) \right)^{\frac{1}{p}} \leqslant \nu(Y)^{\frac{1}{p}} C\left(\operatorname{diam}(Y), x_0\right) \operatorname{diam}(Y)^{1-\frac{\kappa}{p}} \|\nabla_\nu u\|_{L^p_\nu(Y;\mathbb{M})}. \tag{3.16}$$

Hence, combining (3.15) and (3.16) we deduce that for every $y \in Y$,

$$\begin{aligned} |u(y)| &\leqslant 2^{1+\frac{1}{p}} C\left(\operatorname{diam}(Y), x_0\right) \operatorname{diam}(Y)^{1-\frac{\kappa}{p}} \|\nabla_\nu u\|_{L^p_\nu(Y;\mathbb{M})} + \frac{2^{1+\frac{1}{p}}}{\nu(Y)^{\frac{1}{p}}} \|u\|_{L^p_\nu(Y;\mathbb{R}^m)} \\ &\leqslant K_0 \|u\|_{H^{1,p}_\nu(Y;\mathbb{R}^m)} \end{aligned}$$

with $K_0 = \sup\left\{ 2^{1+\frac{1}{p}} C\left(\operatorname{diam}(Y), x_0\right) \operatorname{diam}(Y)^{1-\frac{\kappa}{p}}, \frac{2^{1+\frac{1}{p}}}{\nu(Y)^{\frac{1}{p}}} \right\}$, and (3.12) follows. ■

Step 2: end of the proof of Theorem 3.8. As $\mu\big(\overline{\Omega}\backslash\Omega\big) = 0$, from (3.9) we deduce that

$$\lim_{n\to\infty} \|u_n - u\|_{L^p_\nu(Y;\mathbb{R}^m)} = 0 \text{ and } \sup_{n\geqslant 1} \|\nabla_\nu u_n\|_{L^p_\nu(Y;\mathbb{M})} < \infty,$$

and so $\sup_{n\geqslant 1} \|u_n\|_{H^{1,p}_\nu(Y;\mathbb{R}^m)} < \infty$. By Lemma 3.10 we can assert that $\sup_{n\geqslant 1} \|u_n\|_{C(Y;\mathbb{R}^m)} < \infty$, i.e. $\{u_n\}_n$ is bounded in $C(Y;\mathbb{R}^m)$ with (Y,δ) a compact metric space. Moreover, using (3.13) we see that $\{u_n\}_n$ is equicontinuous. Consequently, up to a subsequence,

$$\lim_{n\to\infty} \|u_n - u\|_{L^\infty_\nu(Y;\mathbb{R}^m)} = 0$$

by Arzelà-Ascoli's theorem, and (3.10) follows because $\mu\big(\overline{\Omega}\backslash\Omega\big) = 0$. ■

3.2. **Ru-usc integrands.** Let (X, d, μ) be a metric measure space, let $\Omega \subset X$ be an open set, let $(\Sigma, \mathcal{F}, \mathbb{P})$ be a probability space and let $L : \Omega \times \mathbb{M} \times \Sigma \to [0, \infty]$ be a Borel measurable stochastic integrand. For each $\omega \in \Sigma$ and each $x \in \Omega$, we denote the effective domain of $L(x, \cdot, \omega)$ by $\mathbb{L}_{x,\omega}$ and, for each $a(\cdot, \omega) \in L^1_\mu(\Omega;]0, \infty])$, we define $\Delta_L^{a(\cdot,\omega)} : [0, 1] \to]-\infty, \infty]$ by

$$\Delta_L^{a(\cdot,\omega)}(\tau) := \sup_{x\in\Omega} \sup_{\xi\in\mathbb{L}_{x,\omega}} \frac{L(x, \tau\xi, \omega) - L(x, \xi, \omega)}{a(x, \omega) + L(x, \xi, \omega)}.$$

Definition 3.11. Let $\omega \in \Sigma$. We say that L is radially uniformly upper semicontinuous (ru-usc) at ω if there exists $a(\cdot, \omega) \in L^1_\mu(\Omega;]0, \infty])$ such that

$$\overline{\lim_{\tau\to 1^-}} \Delta_L^{a(\cdot,\omega)}(\tau) \leqslant 0.$$

The concept of ru-usc integrand was introduced in [AH10] and then developed in [AHM11, AHM12a, AHM12b, Man13, AHM14, AHMZ15, AHM18].

Remark 3.12. If L is ru-usc at $\omega \in \Sigma$ then $\overline{\lim}_{\tau\to 1^-} L(x, \tau\xi, \omega) \leqslant L(x, \xi, \omega)$ for all $x \in \Omega$ and all $\xi \in \mathbb{L}_{x,\omega}$. On the other hand, given $\omega \in \Sigma$, if there exist $x \in \Omega$ and $\xi \in \mathbb{L}_{x,\omega}$ such that $L(x, \cdot, \omega)$ is lsc at ξ then, for each $a(\cdot, \omega) \in L^1_\mu(\Omega;]0, \infty])$, $\underline{\lim}_{\tau\to 1^-} \Delta_L^{a(\cdot,\omega)}(\tau) \geqslant 0$, and so if in addition L is ru-usc at ω then $\lim_{\tau\to 1^-} \Delta_L^{a(\cdot,\omega)}(\tau) = 0$ for some $a(\cdot, \omega) \in L^1_\mu(\Omega;]0, \infty])$.

Remark 3.13. Given $\omega \in \Sigma$, if, for every $x \in \Omega$, $L(x, \cdot, \omega)$ is convex and $0 \in \mathbb{L}_{x,\omega}$, then L is ru-usc at ω.

The interest of Definition 3.11 comes from the following theorem. (For a proof we refer to [AHM11, Theorem 3.5] and also [AHM12b, §4.2].) Let $\widehat{L} : \Omega \times \mathbb{M} \times \Sigma \to [0, \infty]$ be defined by

$$\widehat{L}(x, \xi, \omega) := \underline{\lim_{\tau\to 1^-}} L(x, \tau\xi, \omega).$$

Theorem 3.14. *Let $\omega \in \Sigma$. If L is ru-usc at ω and if for every $x \in \Omega$,*

$$\tau\overline{\mathbb{L}}_{x,\omega} \subset \operatorname{int}(\mathbb{L}_{x,\omega}) \text{ for all } \tau \in]0, 1[,$$

then:

(a) $\widehat{L}$ *is ru-usc at* ω;

(b) $\widehat{L}(x, \xi, \omega) := \lim_{\tau\to 1^-} L(x, \tau\xi, \omega)$ *for all* $x \in \Omega$ *and all* $\xi \in \mathbb{M}$.

If moreover $L(x, \cdot, \omega)$ *is lsc on* $\operatorname{int}(\mathbb{L}_{x,\omega})$ *then*:

(c) $\widehat{L}(x, \xi, \omega) = \begin{cases} L(x, \xi, \omega) & \text{if } \xi \in \operatorname{int}(\mathbb{L}_{x,\omega}) \\ \lim\limits_{\tau\to 1^-} L(x, \tau\xi, \omega) & \text{if } \xi \in \partial\mathbb{L}_{x,\omega} \\ \infty & \text{otherwise;} \end{cases}$

(d) *for every* $x \in \Omega$, $\widehat{L}(x, \cdot, \omega)$ *is the lsc envelope of* $L(x, \cdot, \omega)$.

The following definition extends Definition 3.11 to a family $\{L_t\}_{t>0}$ of Borel measurable stochastic integrands $L_t : \Omega \times \mathbb{M} \times \Sigma \to [0, \infty]$. (When $L_t = L$ for all $t > 0$ we retrieve Definition 3.11.)

Definition 3.15. Let $\omega \in \Sigma$. We say that $\{L_t\}_{t>0}$ is ru-usc at ω if there exists $\{a_t(\cdot,\omega)\}_{t>0} \subset L^1_\mu(\Omega;]0,\infty])$ such that

$$\overline{\lim_{\tau\to 1^-}} \sup_{t>0} \Delta_{L_t}^{a_t(\cdot,\omega)}(\tau) \leqslant 0.$$

Remark 3.16. Let $L : X \times \mathbb{M} \times \Sigma \to [0,\infty]$ be a Borel measurable stochastic integrand and, for each $t > 0$, let $L_t : \Omega \times \mathbb{M} \times \Sigma \to [0,\infty]$ be given by $L_t(x,\xi,\omega) = L(h_t(x),\xi,\omega)$ with $h_t : X \to X$. Given $\omega \in \Sigma$, if L is ru-usc at ω with $a(\cdot,\omega) \in L^1_\mu(X;]0,\infty])$ then $\{L_t\}_{t>0}$ is ru-usc at ω with $\{a_t(\cdot,\omega)\}_{t>0} = \{a(h_t(\cdot),\omega)\}_{t>0}$. Indeed, for any $\tau \in [0,1]$, any $t > 0$, any $x \in \Omega$ and any $\xi \in \mathbb{L}_{t,x,\omega}$ with $\mathbb{L}_{t,x,\omega}$ denoting the effective domain of $L_t(x,\cdot,\omega)$, one has

$$\frac{L_t(x,\tau\xi,\omega) - L_t(x,\xi,\omega)}{a_t(x,\omega) + L_t(x,\xi,\omega)} = \frac{L(h_t(x),\tau\xi,\omega) - L(h_t(x),\xi,\omega)}{a(h_t(x),\omega) + L(h_t(x),\xi,\omega)}. \tag{3.17}$$

As $\mathbb{L}_{t,x,\omega} = \mathbb{L}_{h_t(x),\omega}$ where, for each $y \in X$, $\mathbb{L}_{y,\omega}$ denotes the effective domain of $L(y,\cdot,\omega)$, and $h_t(x) \in X$, we see that

$$\frac{L(h_t(x),\tau\xi,\omega) - L(h_t(x),\xi,\omega)}{a(h_t(x),\omega) + L(h_t(x),\xi,\omega)} \leqslant \sup_{y\in X} \sup_{\xi\in\mathbb{L}_{y,\omega}} \frac{L(y,\tau\xi,\omega) - L(y,\xi,\omega)}{a(y,\omega) + L(y,\xi,\omega)} = \Delta_L^{a(\cdot,\omega)}(\tau),$$

and from (3.17) we deduce that

$$\sup_{t>0} \Delta_{L_t}^{a_t(\cdot,\omega)}(\tau) \leqslant \Delta_L^{a(\cdot,\omega)}(\tau) \tag{3.18}$$

for all $\tau \in [0,1]$. But L is ru-usc at ω with $a(\cdot,\omega)$, i.e. $\overline{\lim}_{\tau\to 1^-} \Delta_L^{a(\cdot,\omega)}(\tau) \leqslant 0$, and so, letting $\tau \to 1^-$ in (3.18), we get $\overline{\lim}_{\tau\to 1^-} \sup_{t>0} \Delta_{L_t}^{a_t(\cdot,\omega)}(\tau) \leqslant 0$ which means that $\{L_t\}_{t>0}$ is ru-usc at ω with $\{a_t(\cdot,\omega)\}_{t>0} = \{a(h_t(\cdot),\omega)\}_{t>0}$.

For each $t > 0$ and each $\rho > 0$, let $\mathcal{H}^\rho_\mu L_t : \Omega \times \mathbb{M} \times \Sigma \to [0,\infty]$ be defined by

$$\mathcal{H}^\rho_\mu L_t(x,\xi,\omega) := \inf\left\{ \fint_{Q_\rho(x)} L_t(y,\xi + \nabla_\mu w(y),\omega) d\mu(y) : w \in H^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m) \right\}.$$

Let $L_\infty : \Omega \times \mathbb{M} \times \Sigma \to [0,\infty]$ be given by

$$L_\infty(x,\xi,\omega) := \overline{\lim_{\rho\to 0}}\, \overline{\lim_{t\to\infty}}\, \mathcal{H}^\rho_\mu L_t(x,\xi,\omega). \tag{3.19}$$

The following proposition shows that, under a suitable condition, ru-usc is conserved under the operation characterized by (3.19).

Proposition 3.17. *Let $\omega \in \Sigma$ and let $\{a_t(\cdot,\omega)\}_{t>0} \subset L^1_\mu(\Omega;]0,\infty])$ be such that*

$$\overline{\lim_{\rho\to 0}}\, \overline{\lim_{t\to\infty}} \fint_{Q_\rho(\cdot)} a_t(y,\omega) d\mu(y) =: a_\infty(\cdot,\omega) \in L^1_\mu(\Omega;]0,\infty]). \tag{3.20}$$

If $\{L_t\}_{t>0}$ is ru-usc at ω with $\{a_t(\cdot,\omega)\}_{t>0}$ then L_∞ is ru-usc at ω with $a_\infty(\cdot,\omega)$.

Proof of Proposition 3.17. Fix any $\tau \in [0,1]$, any $x \in \Omega$ and any $\xi \in \mathbb{L}_{\infty,x,\omega}$, where $\mathbb{L}_{\infty,x,\omega}$ is the effective domain of $L_\infty(x,\cdot,\omega)$. Then $L_\infty(x,\xi,\omega) = \overline{\lim}_{\rho\to 0}\, \overline{\lim}_{t\to\infty}\, \mathcal{H}^\rho_\mu L_t(x,\xi,\omega) < \infty$ and without loss of generality we can suppose that $\mathcal{H}^\rho_\mu L_t(x,\xi,\omega) < \infty$ for all $\rho > 0$ and all $t > 0$.

Fix any $\rho>0$ and any $t>0$. By definition, there exists $\{w_n\}_n \subset H^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m)$ such that:

$$\mathcal{H}^\rho_\mu L_t(x,\xi,\omega) = \lim_{n\to\infty} \fint_{Q_\rho(x)} L_t(y,\xi+\nabla_\mu w_n(y),\omega)d\mu(y); \tag{3.21}$$

$$\xi+\nabla_\mu w_n(y) \in \mathbb{L}_{t,y,\omega} \text{ for all } n\geqslant 1 \text{ and } \mu\text{-a.a. } y\in Q_\rho(x), \tag{3.22}$$

where $\mathbb{L}_{t,y,\omega}$ denotes the effective domain of $L_t(y,\cdot,\omega)$. Moreover, for every $n\geqslant 1$,

$$\mathcal{H}^\rho_\mu L_t(x,\tau\xi,\omega) \leqslant \fint_{Q_\rho(x)} L_t\big(y,\tau(\xi+\nabla_\mu w_n(y)),\omega\big)d\mu(y)$$

since $\tau w_n \in H^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m)$, and so

$$\delta^\tau_{\rho,t}(x,\xi,\omega) \leqslant \varliminf_{n\to\infty} \fint_{Q_\rho(x)} \big(L_t(y,\tau(\xi+\nabla_\mu w_n(y)),\omega) - L_t(y,\xi+\nabla_\mu w_n(y),\omega)\big)d\mu(y) \tag{3.23}$$

with $\delta^\tau_{\rho,t}(x,\xi,\omega) := \mathcal{H}^\rho_\mu L_t(x,\tau\xi,\omega) - \mathcal{H}^\rho_\mu L_t(x,\xi,\omega)$. Taking (3.22) into account, for every $n\geqslant 1$ and μ-a.e. $y\in Q_\rho(x)$, one has

$$\lambda^\tau_{t,n}(y,\xi,\omega) \leqslant \Delta^{a_t(\cdot,\omega)}_{L_t}(\tau)\big(a_t(y,\omega)+L_t(y,\xi+\nabla_\mu w_n(y),\omega)\big),$$

with $\lambda^\tau_{t,n}(y,\xi,\omega) := L_t\big(y,\tau(\xi+\nabla_\mu w_n(y)),\omega\big) - L_t\big(y,\xi+\nabla_\mu w_n(y),\omega\big)$, hence

$$\fint_{Q_\rho(x)} \lambda^\tau_{t,n}(y,\xi,\omega)d\mu \leqslant \Delta^{a_t(\cdot,\omega)}_{L_t}(\tau)\left(\fint_{Q_\rho(x)} a_t(y,\omega)d\mu + \fint_{Q_\rho(x)} L_t(y,\xi+\nabla_\mu w_n(y),\omega)d\mu\right)$$

for all $n\geqslant 1$. Letting $n\to\infty$ and using (3.21) and (3.23), it follows that

$$\begin{aligned}\delta^\tau_{\rho,t}(x,\xi,\omega) &\leqslant \Delta^{a_t(\cdot,\omega)}_{L_t}(\tau)\left(\fint_{Q_\rho(x)} a_t(y,\omega)d\mu(y) + \mathcal{H}^\rho_\mu L_t(x,\xi,\omega)\right)\\ &\leqslant \Delta_\omega(\tau)\left(\fint_{Q_\rho(x)} a_t(y,\omega)d\mu(y) + \mathcal{H}^\rho_\mu L_t(x,\xi,\omega)\right)\end{aligned} \tag{3.24}$$

for all $\rho>0$ and all $t>0$, where $\Delta_\omega(\tau) := \sup_{s>0}\Delta^{a_s(\cdot,\omega)}_{L_s}(\tau)$. By letting $t\to\infty$ and $\rho\to 0$ in (3.24), we get

$$L_\infty(x,\tau\xi,\omega) - L_\infty(x,\xi,\omega) \leqslant \Delta_\omega(\tau)\big(a_\infty(x,\omega)+L_\infty(x,\xi,\omega)\big)$$

with $a_\infty(\cdot,\omega)\in L^1_\mu(\Omega;]0,\infty])$ given by (3.20), which implies that $\Delta^{a_\infty(\cdot,\omega)}_{L_\infty}(\tau) \leqslant \Delta_\omega(\tau)$ for all $\tau\in[0,1]$. As $\{L_t\}_{t>0}$ is ru-usc at ω with $\{a_t(\cdot,\omega)\}_{t>0}$, i.e. $\overline{\lim}_{\tau\to 1^-}\Delta_\omega(\tau)\leqslant 0$, we conclude that $\overline{\lim}_{\tau\to 1^-}\Delta^{a_\infty(\cdot,\omega)}_{L_\infty}(\tau)\leqslant 0$ which means that L_∞ is ru-usc at ω with $a_\infty(\cdot,\omega)$. ■

Given $L: X\times\mathbb{M}\times\Sigma\to[0,\infty]$ and $\{A_k\}_{k\in\mathbb{N}*}$ a sequence of open subsets of X such that $\mu(A_k)>0$, let $L_{\rm hom}:\mathbb{M}\times\Sigma\to[0,\infty]$ be defined by

$$L_{\rm hom}(\xi,\omega) := \inf_{k\in\mathbb{N}^*}\inf\left\{\fint_{A_k} L(x,\xi+\nabla_\mu w(x),\omega)d\mu(x) : w\in H^{1,p}_{\mu,0}(A_k;\mathbb{R}^m)\right\}. \tag{3.25}$$

The following result shows that, under a suitable condition, ru-usc is conserved under the operation characterized by (3.25).

Proposition 3.18. *Let $\omega \in \Sigma$ and let $a(\cdot,\omega) \in L^1_\mu(\Omega;]0,\infty])$ be such that*

$$⨍_{A_k} a(x,\omega)d\mu(x) = \overline{a}(\omega) \in]0,\infty[\text{ for all } k \in \mathbb{N}^*. \tag{3.26}$$

If L is ru-usc at ω with $a(\cdot,\omega)$ then $L_{\rm hom}$ is ru-usc at ω with $\overline{a}(\omega)$.

Proof of Proposition 3.18. Fix any $\tau \in [0,1]$ and any $\xi \in \mathbb{L}_{{\rm hom},\omega}$, where $\mathbb{L}_{{\rm hom},\omega}$ denotes the effective domain of $L_{\rm hom}(\cdot,\omega)$. By definition, there exist $\{k_n\}_n \subset \mathbb{N}^*$ and $\{w_n\}_n \subset H^{1,p}_{\mu,0}(A_{k_n};\mathbb{R}^m)$ such that:

$$L_{\rm hom}(\xi,\omega) = \lim_{n\to\infty} ⨍_{A_{k_n}} L(x,\xi+\nabla_\mu w_n(x),\omega)d\mu(x); \tag{3.27}$$

$$\xi + \nabla_\mu w_n(x) \in \mathbb{L}_{x,\omega} \text{ for all } n \geqslant 1 \text{ and } \mu\text{-a.a. } x \in A_{k_n}. \tag{3.28}$$

Moreover, for every $n \geqslant 1$,

$$L_{\rm hom}(\tau\xi,\omega) \leqslant ⨍_{A_{k_n}} L(x,\tau(\xi+\nabla_\mu w_n(x)),\omega)d\mu(x)$$

because $\tau w_n \in H^{1,p}_{\mu,0}(A_{k_n};\mathbb{R}^m)$, hence

$$L_{\rm hom}(\tau\xi,\omega) - L_{\rm hom}(\xi,\omega) \leqslant \varlimsup_{n\to\infty} ⨍_{A_{k_n}} \big(L(x,\tau(\xi+\nabla_\mu w_n(x)),\omega) - L(x,\xi+\nabla_\mu w_n(x),\omega)\big)d\mu(x).$$

But, taking (3.28) into account, since L is ru-usc with $a \in L^1_\mu(X;]0,\infty])$, for every $n \geqslant 1$ and μ-a.e. $x \in A_{k_n}$, one has

$$L(x,\tau(\xi+\nabla_\mu w_n(x)),\omega) - L(x,\xi+\nabla_\mu w_n(x),\omega) \leqslant \Delta_L^{a(\cdot,\omega)}(\tau)\left(a(x,\omega) + L(x,\xi+\nabla_\mu w_n(x),\omega\right),$$

and so, by using (3.26) and (3.27), we deduce that

$$L_{\rm hom}(\tau\xi,\omega) - L_{\rm hom}(\xi,\omega) \leqslant \Delta_L^{a(\cdot,\omega)}(\tau)\big(\overline{a}(\omega) + L_{\rm hom}(\xi,\omega)\big),$$

which implies that $\Delta^{\overline{a}(\omega)}_{L_{\rm hom}}(\tau) \leqslant \Delta_L^{a(\cdot,\omega)}(\tau)$ for all $\tau \in [0,1]$, and the proof is complete. ■

3.3. **Ru-usc functionals.** Let (X,d,μ) be a metric measure space with the same properties as in §3.1, let $\Omega \subset X$ be a bounded open set, let $(\Sigma,\mathcal{F},\mathbb{P})$ be a probability space and let $\mathcal{J} : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \times \Sigma \to [0,\infty]$ be a functional. For each $\omega \in \Sigma$, we denote the effective domain of $\mathcal{J}(\cdot,\omega)$ by $\mathfrak{J}_\omega$. As for the case of integrands, we have the following definition.

Definition 3.19. Let $\omega \in \Sigma$. Given $\mathfrak{D} \subset \mathfrak{J}_\omega$, we say that $\mathcal{J}$ is ru-usc on $\mathfrak{D}$ at ω if there exists $a(\omega) \in]0,\infty[$ such that

$$\overline{\lim_{\tau\to 1^-}}\ \Delta^{a(\omega)}_{\mathcal{J},\mathfrak{D}}(\tau) \leqslant 0$$

with $\Delta^{a(\omega)}_{\mathcal{J},\mathfrak{D}} : [0,1] \to]-\infty,\infty]$ defined by

$$\Delta^{a(\omega)}_{\mathcal{J},\mathfrak{D}}(\tau) := \sup_{u\in\mathfrak{D}} \frac{\mathcal{J}(\tau u,\omega) - \mathcal{J}(u,\omega)}{a(\omega) + \mathcal{J}(u,\omega)}.$$

(For more details on the notion of ru-usc functional we refer to [AHM12b, §4.2] and [AHM14].) As for the case of integrands, the interest of definition 3.19 comes from the following theorem which is the analogue of Theorem 3.14. Let $\widehat{\mathcal{J}} : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \times \Sigma \to [0,\infty]$ be defined by

$$\widehat{\mathcal{J}}(u,\omega) := \varlimsup_{\tau\to 1^-} \mathcal{J}(\tau u,\omega).$$

When $\mathfrak{D} = \mathfrak{J}_\omega$ we simply say that $\mathcal{J}$ is ru-usc at ω.

Theorem 3.20. *Let $\omega \in \Sigma$. Given $\mathfrak{D} \subset \mathfrak{J}_\omega$ and $\mathfrak{E} \supset \mathfrak{D}$ such that*

$$\tau\mathfrak{E} \subset \mathfrak{D} \text{ for all } \tau \in]0,1[, \tag{3.29}$$

if $\mathcal{J}$ is ru-usc on $\mathfrak{D}$ at ω and if $\mathcal{J}(\cdot,\omega)$ is L^p_μ-lsc on $\mathfrak{D}$, i.e. $\varliminf_{n\to\infty} \mathcal{J}(u_n,\omega) \geqslant \mathcal{J}(u,\omega)$ for all $u \in \mathfrak{D}$ and all $\{u_n\}_n \subset \mathfrak{D}$ such that $u_n \to u$ in $L^p_\mu(\Omega;\mathbb{R}^m)$, then:

(a) $\overline{\mathcal{J}}^{\mathfrak{D}}(u,\omega) = \widehat{\mathcal{J}}(u,\omega)$ *for all $u \in \mathfrak{E}$, where $\overline{\mathcal{J}}^{\mathfrak{D}}(\cdot,\omega) : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \to [0,\infty]$ is defined by*

$$\overline{\mathcal{J}}^{\mathfrak{D}}(u,\omega) := \inf\left\{ \varliminf_{n\to\infty} \mathcal{J}(u_n,\omega) : \mathfrak{D} \ni u_n \overset{L^p_\mu}{\to} u \right\};$$

(b) $$\widehat{\mathcal{J}}(u,\omega) = \begin{cases} \mathcal{J}(u,\omega) & \text{if } u \in \mathfrak{D} \\ \varlimsup_{\tau\to 1^-} \mathcal{J}(\tau u,\omega) & \text{if } u \in \mathfrak{E}\backslash\mathfrak{D}. \end{cases}$$

(For a proof of Theorem 3.20 we refer to [AHM12b, Theorem 4.1], see also [AHM14].) For each $t > 0$, let $\mathcal{E}_t : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \times \Sigma \to [0,\infty]$ be a functional depending on a parameter t and, for each $\omega \in \Sigma$, let $\mathfrak{S}_{t,\omega}$ denote the effective domain of $\mathcal{E}_t(\cdot,\omega)$. As for the case of integrands, the following definition extends Definition 3.19.

Definition 3.21. Let $\omega \in \Sigma$ and, for each $t > 0$, let $\mathfrak{D}_t \subset \mathfrak{S}_{t,\omega}$. We say that $\{\mathcal{E}_t\}_{t>0}$ is ru-usc on $\{\mathfrak{D}_t\}_{t>0}$ at ω if there exists $\{a_t(\omega)\}_{t>0} \subset]0,\infty[$ with $\overline{\lim}_{t\to\infty} a_t(\omega) < \infty$ such that

$$\varlimsup_{\tau\to 1^-} \sup_{t>0} \Delta^{a_t(\omega)}_{\mathcal{E}_t,\mathfrak{D}_t}(\tau) \leqslant 0.$$

When $\mathfrak{D}_t = \mathfrak{D}$ for all $t > 0$ (and so $\mathfrak{D} \subset \cap_{t>0}\mathfrak{S}_{t,\omega}$) we say that $\{\mathcal{E}_t\}_{t>0}$ is ru-usc on $\mathfrak{D}$ at ω, and when $\mathfrak{D}_t = \mathfrak{S}_{t,\omega}$ for all $t > 0$ we simply say that $\{\mathcal{E}_t\}_{t>0}$ is ru-usc at ω.

The following result is an extension of Theorem 3.20.

Theorem 3.22. *Let $\omega \in \Sigma$ and let $\mathfrak{D} \subset \cap_{t>0}\mathfrak{S}_{t,\omega}$ and $\mathfrak{E} \supset \mathfrak{D}$ be such that (3.29) holds. Assume that $\{\mathcal{E}_t\}_{t>0}$ is ru-usc on $\mathfrak{D}$ at ω and there exists $\mathcal{I}(\cdot,\omega) : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \to [0,\infty]$ such that $\{\mathcal{E}_t(\cdot,\omega)\}_{t>0}$ $\Gamma(L^p_\mu)$-converges to $\mathcal{I}(\cdot,\omega)$ on $\mathfrak{D}$, i.e.*

$$\Gamma(L^p_\mu)\text{-}\varlimsup_{t\to\infty} \mathcal{E}_t(u,\omega) = \mathcal{I}(u,\omega) \text{ for all } u \in \mathfrak{D} \tag{3.30}$$

Then

$$\Gamma(L^p_\mu)\text{-}\overline{\varlimsup_{t\to\infty}} \mathcal{E}_t(u,\omega) \leqslant \widehat{\mathcal{I}}(u,\omega) \leqslant \Gamma_{\mathfrak{D}}(L^p_\mu)\text{-}\varliminf_{t\to\infty} \mathcal{E}_t(u,\omega) \text{ for all } u \in \mathfrak{E} \tag{3.31}$$

with

$$\Gamma_{\mathfrak{D}}(L^p_\mu)\text{-}\varliminf_{t\to\infty} \mathcal{E}_t(u,\omega) := \inf\left\{ \varliminf_{t\to\infty} \mathcal{E}_t(u_t) : \mathfrak{D} \ni u_t \overset{L^p_\mu}{\to} u \right\}.$$

If moreover $\mathcal{I}$ is ru-usc on $\mathfrak{D}$ at ω then

$$\widehat{\mathcal{I}}(u,\omega) = \begin{cases} \mathcal{I}(u,\omega) & \text{if } u \in \mathfrak{D} \\ \lim\limits_{\tau\to 1^-} \mathcal{I}(\tau u,\omega) & \text{if } u \in \mathfrak{E}\backslash\mathfrak{D}. \end{cases} \tag{3.32}$$

Proof of Theorem 3.22. Fix $u \in \mathfrak{E}$. By (3.29), for any $\tau \in]0,1[$, we have $\tau u \in \mathfrak{D}$. From (3.30) it follows that $\mathcal{I}(\tau u,\omega) = \Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} \mathcal{E}_t(\tau u,\omega) = \Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} \mathcal{E}_t(\tau u,\omega)$ for all $\tau \in]0,1[$, and consequently

$$\widehat{\mathcal{I}}(u,\omega) = \underline{\lim_{\tau\to 1^-}}\, \mathcal{I}(\tau u,\omega) = \underline{\lim_{\tau\to 1^-}}\, \Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}}\, \mathcal{E}_t(\tau u,\omega) \geqslant \Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}}\, \mathcal{E}_t(u,\omega),$$

which gives the left inequality in (3.31). Let us now prove the right inequality in (3.31). Let $\{u_t\}_{t>0} \subset \mathfrak{D}$ be such that:

$$u_t \overset{L^p_\mu}{\to} u; \tag{3.33}$$

$$\lim_{t\to\infty} \mathcal{E}_t(u_t,\omega) = \Gamma_{\mathfrak{D}}(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}}\, \mathcal{E}_t(u,\omega). \tag{3.34}$$

By (3.29), for any $\tau \in]0,1[$, we have $\tau u_t \in \mathfrak{D}$ for all $t > 0$. Hence $\tau u_t \overset{L^p_\mu}{\to} \tau u$ by (3.33), and so, by using (3.30),

$$\underline{\lim_{t\to\infty}}\, \mathcal{E}_t(\tau u_t,\omega) \geqslant \Gamma(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}}\, \mathcal{E}_t(\tau u,\omega) = \mathcal{I}(\tau u,\omega)$$

for all $\tau \in]0,1[$. It follows that

$$\underline{\lim_{\tau\to 1^-}}\, \underline{\lim_{t\to\infty}}\, \mathcal{E}_t(\tau u_t,\omega) \geqslant \widehat{\mathcal{I}}(u,\omega). \tag{3.35}$$

On the other hand, since $\{u_t\}_{t>0} \subset \mathfrak{D}$, for every $\tau \in]0,1[$ and every $t > 0$, we have

$$\begin{aligned} \mathcal{E}_t(\tau u_t,\omega) &\leqslant \big(1 + \Delta^{a_t(\omega)}_{\mathcal{E}_t,\mathfrak{D}}(\tau)\big)\mathcal{E}_t(u_t,\omega) + a_t(\omega)\Delta^{a_t(\omega)}_{\mathcal{E}_t,\mathfrak{D}}(\tau) \\ &\leqslant \left(1 + \sup_{s>0} \Delta^{a_s(\omega)}_{\mathcal{E}_s,\mathfrak{D}}(\tau)\right) \mathcal{E}_t(u_t,\omega) + a_t(\omega) \sup_{s>0} \Delta^{a_s(\omega)}_{\mathcal{E}_s,\mathfrak{D}}(\tau), \end{aligned}$$

and so, by letting $t \to \infty$ and by using (3.34), we get

$$\begin{aligned} \underline{\lim_{t\to\infty}}\, \mathcal{E}_s(\tau u_s,\omega) &\leqslant \left(1 + \sup_{s>0} \Delta^{a_s(\omega)}_{\mathcal{E}_s,\mathfrak{D}}(\tau)\right) \lim_{s\to\infty} \mathcal{E}_t(u_t,\omega) + \overline{\lim_{t\to\infty}}\, a_t(\omega) \sup_{s>0} \Delta^{a_s(\omega)}_{\mathcal{E}_s,\mathfrak{D}}(\tau) \\ &= \left(1 + \sup_{s>0} \Delta^{a_s(\omega)}_{\mathcal{E}_s,\mathfrak{D}}(\tau)\right) \Gamma_{\mathfrak{D}}(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}}\, \mathcal{E}_t(u,\omega) + \overline{\lim_{t\to\infty}}\, a_t(\omega) \sup_{s>0} \Delta^{a_s(\omega)}_{\mathcal{E}_s,\mathfrak{D}}(\tau). \end{aligned}$$

As $\{\mathcal{E}_t\}_{t>0}$ is ru-usc on $\mathfrak{D}$ at ω, i.e. $\overline{\lim}_{\tau\to 1^-} \sup_{s>0} \Delta^{a_s(\omega)}_{\mathcal{E}_s,\mathfrak{D}}(\tau) \leqslant 0$ (and $\overline{\lim}_{t\to\infty} a_t(\omega) < \infty$), letting $\tau \to 1^-$ we conclude that

$$\overline{\lim_{\tau\to 1^-}}\, \underline{\lim_{t\to\infty}}\, \mathcal{E}_s(\tau u_s,\omega) \leqslant \Gamma_{\mathfrak{D}}(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}}\, \mathcal{E}_t(u,\omega), \tag{3.36}$$

and the right inequality in (3.31) follows by combining (3.35) with (3.36).
From (3.30) we see that $\mathcal{I}(\cdot,\omega)$ is L^p_μ-lsc on $\mathfrak{D}$, and (3.32) follows from Theorem 3.20(b). ■

The following result is a consequence of the first part of Theorem 3.22.

Corollary 3.23. *Let $\omega \in \Sigma$, let $\mathfrak{D} \subset H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ be such that $\mathfrak{D} = \mathfrak{S}_{t,\omega}$ for all $t > 0$ and let $\mathfrak{E} \supset \mathfrak{D}$ be such that (3.29) holds. Assume that $\{\mathcal{E}_t\}_{t>0}$ is ru-usc on $\mathfrak{D}$ at ω and there exists $\mathcal{I}(\cdot,\omega) : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \to [0,\infty]$ satisfying (3.30). Then*

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} \mathcal{E}_t(u,\omega) = \widehat{\mathcal{I}}(u,\omega) \text{ for all } u \in \mathfrak{E}.$$

Proof of Corollary 3.23. As $\mathfrak{D} = \mathfrak{S}_{t,\omega}$ for all $t > 0$ we have

$$\Gamma_{\mathfrak{D}}(L^p_\mu)\text{-}\underline{\lim}_{t\to\infty} \mathcal{E}_t(\cdot,\omega) = \Gamma(L^p_\mu)\text{-}\underline{\lim}_{t\to\infty} \mathcal{E}_t(\cdot,\omega),$$

and the corollary follows from the first part of Theorem 3.22. ■

3.4. **The De Giorgi-Letta lemma.** Let $\Omega = (\Omega, d)$ be a metric space, let $\mathcal{O}(\Omega)$ be the class of open subsets of X and let $\mathcal{B}(\Omega)$ be the class of Borel subsets of Ω, i.e. the smallest σ-algebra containing the open (or equivalently the closed) subsets of Ω. The following result is due to De Giorgi and Letta (see [DGL77] and also [But89, Lemma 3.3.6 pp. 105]).

Lemma 3.24. *Let $\mathcal{S} : \mathcal{O}(\Omega) \to [0,\infty]$ be an increasing set function, i.e. $\mathcal{S}(A) \leqslant \mathcal{S}(B)$ for all $A, B \in \mathcal{O}(\Omega)$ such $A \subset B$, satisfying the following four conditions*:

(a) $\mathcal{S}(\varnothing) = 0$;
(b) *$\mathcal{S}$ is superadditive, i.e. $\mathcal{S}(A \cup B) \geqslant \mathcal{S}(A) + \mathcal{S}(B)$ for all $A, B \in \mathcal{O}(\Omega)$ such that $A \cap B = \varnothing$;*
(c) *$\mathcal{S}$ is subadditive, i.e. $\mathcal{S}(A \cup B) \leqslant \mathcal{S}(A) + \mathcal{S}(B)$ for all $A, B \in \mathcal{O}(\Omega)$;*
(d) *there exists a finite Radon measure ν on Ω such that $\mathcal{S}(A) \leqslant \nu(A)$ for all $A \in \mathcal{O}(\Omega)$.*

Then, $\mathcal{S}$ can be uniquely extended to a finite positive Radon measure on Ω which is absolutely continuous with respect to ν.

3.5. **Integral representation of the Vitali envelope of a set function.** What follows was first developed in [BFM98, BB00] (see also [AHM16]). Here we only recall what is needed for proving Theorem 2.11. Let (Ω, d) be a metric space, let $\mathcal{O}(\Omega)$ be the class of open subsets of Ω and let μ be a positive finite Radon measure on Ω. We begin with the concept of differentiability with respect to μ of a set function.

Definition 3.25. We say that a set function $\Theta : \mathcal{O}(\Omega) \to \mathbb{R}$ is differentiable with respect to μ if

$$d_\mu\Theta(x) := \lim_{\rho\to 0} \frac{\Theta(Q_\rho(x))}{\mu(Q_\rho(x))} \tag{3.37}$$

exists and is finite for μ-a.e. $x \in \Omega$.

Remark 3.26. It is easy to see that the limit in (3.37) exists and is finite if and only if $-\infty < d^+_\mu\Theta \leqslant d^-_\mu\Theta < \infty$, where $d^-_\mu\Theta : \Omega \to [-\infty,\infty[$ and $d^+_\mu\Theta : \Omega \to]-\infty,\infty]$ are given by:

$$d^-_\mu\Theta(x) := \lim_{\rho\to 0} d^-_\mu\Theta(x,\rho) \text{ with } d^-_\mu\Theta(x,\rho) := \inf\left\{\frac{\Theta(Q)}{\mu(Q)} : Q \in \mathrm{Ba}(\Omega,x,\rho)\right\}; \tag{3.38}$$

$$d^+_\mu\Theta(x) := \lim_{\rho\to 0} d^+_\mu\Theta(x,\rho) \text{ with } d^+_\mu\Theta(x,\rho) := \sup\left\{\frac{\Theta(Q)}{\mu(Q)} : Q \in \mathrm{Ba}(\Omega,x,\rho)\right\}, \tag{3.39}$$

where $\mathrm{Ba}(\Omega,x,\rho)$ denotes the class of open balls Q of Ω such that $x \in Q$, $\mathrm{diam}(Q) \in]0,\rho[$ and $\mu(\partial Q) = 0$, where $\partial Q := \overline{Q}\backslash Q$. We then have $d_\mu\Theta = d^-_\mu\Theta = d^+_\mu\Theta$.

Remark 3.27. In (3.38) and (3.39) we can replace $\mathrm{Ba}(\Omega, x, \rho)$ by $\mathrm{Ba}(A, x, \rho)$ whenever $A \in \mathcal{O}(\Omega)$ and $x \in A$.

For each $\varepsilon > 0$ and each $A \in \mathcal{O}(\Omega)$, we denote the class of countable families $\{Q_i := Q_{\rho_i}(x_i)\}_{i\in I}$ of disjoint open balls of A with $x_i \in A$, $\rho_i = \mathrm{diam}(Q_i) \in]0, \varepsilon[$ and $\mu(\partial Q_i) = 0$ such that $\mu(A \setminus \cup_{i\in I} Q_i) = 0$ by $\mathcal{V}_\varepsilon(A)$.

Definition 3.28. Given $\Theta : \mathcal{O}(\Omega) \to \mathbb{R}$, for each $\varepsilon > 0$ we define $\Theta^\varepsilon : \mathcal{O}(\Omega) \to [-\infty, \infty]$ by

$$\Theta^\varepsilon(A) := \inf\left\{\sum_{i\in I} \Theta(Q_i) : \{Q_i\}_{i\in I} \in \mathcal{V}_\varepsilon(A)\right\}. \tag{3.40}$$

By the Vitali envelope of Θ we denote the set function $\Theta^* : \mathcal{O}(\Omega) \to [-\infty, \infty]$ defined by

$$\Theta^*(A) := \sup_{\varepsilon>0} \Theta^\varepsilon(A) = \lim_{\varepsilon\to 0} \Theta^\varepsilon(A). \tag{3.41}$$

The interest of Definition 3.28 comes from the following integral representation result whose proof is postponed in Appendix A.1.

Theorem 3.29. *Let $\Theta : \mathcal{O}(\Omega) \to \mathbb{R}$ be a set function satisfying the following two conditions*:

(a) *there exists a finite Radon measure ν on Ω which is absolutely continuous with respect to μ such that $|\Theta(A)| \leqslant \nu(A)$ for all $A \in \mathcal{O}(\Omega)$;*
(b) *Θ is subadditive, i.e. $\Theta(A) \leqslant \Theta(B) + \Theta(C)$ for all $A, B, C \in \mathcal{O}(\Omega)$ with $B, C \subset A$, $B \cap C = \varnothing$ and $\mu(A \setminus B \cup C) = 0$.*

Then Θ is differentiable with respect to μ, $d_\mu\Theta \in L^1_\mu(\Omega)$ and

$$\Theta^*(A) = \int_A d_\mu\Theta(x) d\mu(x)$$

for all $A \in \mathcal{O}(\Omega)$.

As a direct consequence, we have

Corollary 3.30. *Let $\Theta : \mathcal{O}(\Omega) \to \mathbb{R}$ be a set function satisfying the assumptions* (a) *and* (b) *of Theorem* 3.29. *Then Θ and Θ^* are differentiable with respect to μ and $d_\mu\Theta^* = d_\mu\Theta$.*

3.6. **Subadditive theorems.** What follows was first developed in [AHM17, AHM19b]. Let (X, d, μ) be a metric measure space with μ a positive Radon measure on X. Let $\mathcal{B}(X)$ be the class of Borel subsets of X and let $\mathcal{B}_{\mu,0}(X)$ denote the class of $Q \in \mathcal{B}(X)$ such that $\mu(Q) < \infty$ and $\mu(\partial Q) = 0$. Let $\mathrm{Homeo}(X)$ be the group of homeomorphisms on X and let $\mathbb{G}$ be a subgroup of $\mathrm{Homeo}(X)$ for which μ is $\mathbb{G}$-invariant, i.e. $g^\sharp\mu = \mu$ for all $g \in \mathbb{G}$, where $g^\sharp\mu$ denotes the image measure of μ by g. From now on, we consider $\{\mathbb{U}_k\}_{k\in\mathbb{N}^*} \subset \mathcal{B}_{\mu,0}(X)$ with $\mu(\mathbb{U}_k) > 0$ for all $k \in \mathbb{N}^*$ and, for each $k \in \mathbb{N}^*$, we consider the class $\mathcal{U}_k(\mathbb{G})$ defined by

$$\mathcal{U}_k(\mathbb{G}) := \Big\{\mathbb{H} \subset \mathbb{G} : \{g^{-1}(\mathbb{U}_k)\}_{g\in\mathbb{H}} \text{ is disjoint}\Big\}.$$

In what follows, $|\cdot|$ denotes the counting measure on $\mathbb{G}$ and, for any $\mathbb{H} \subset \mathbb{G}$, $\mathcal{P}_{\mathrm{f}}(\mathbb{H})$ denotes the class of finite subsets of $\mathbb{H}$.

3.6.1. *The deterministic case.* The following definition sets a framework, in the setting of metric measure spaces, for establishing a subadditive theorem in the deterministic case and (see Theorem 3.33).

Definition 3.31. Let $\{Q_t\}_{t>0} \subset \mathcal{B}_{\mu,0}(X)$. We say that $\{Q_t\}_{t>0}$ is weakly $\mathbb{G}$-asymptotic with respect to $\{\mathbb{U}_k\}_{k\in\mathbb{N}^*}$ if for each $k \in \mathbb{N}^*$ there exists $\mathbb{H}_k \in \mathcal{U}_k(\mathbb{G})$ with the property that for each $t > 0$ there exist $m_{t,k} \in \mathbb{N}^*$, $g_{t,k} \in \mathbb{G}$ and $F_{t,k}, H^-_{t,k}, H^+_{t,k} \in \mathcal{P}_{\mathrm{f}}(\mathbb{H}_k)$ such that:

$$\bigcup_{g\in H^-_{t,k}} g^{-1}(\mathbb{U}_k) \subset Q_t \subset \bigcup_{g\in H^+_{t,k}} g^{-1}(\mathbb{U}_k); \tag{3.42}$$

$$\lim_{t\to\infty} \frac{\mu\Big(\bigcup_{g\in H^+_{t,k}} g^{-1}(\mathbb{U}_k)\setminus \bigcup_{g\in H^-_{t,k}} g^{-1}(\mathbb{U}_k)\Big)}{\mu(Q_t)} = 0; \tag{3.43}$$

$$H^+_{t,k} \subset F_{t,k} \text{ and } \bigcup_{g\in F_{t,k}} g^{-1}(\mathbb{U}_k) = g^{-1}_{t,k}(\mathbb{U}_{m_{t,k}}); \tag{3.44}$$

$$\overline{\lim_{t\to\infty}} \frac{|F_{t,k}|}{|H^+_{t,k}|} \leqslant 1. \tag{3.45}$$

Let us recall the definition of a subadditive and $\mathbb{G}$-invariant set function.

Definition 3.32. Let $\mathcal{S} : \mathcal{B}_{\mu,0}(X) \to \mathbb{R}$ be a set function.

(a) The set function $\mathcal{S}$ is said to be subadditive if

$$\mathcal{S}(A \cup B) \leqslant \mathcal{S}(A) + \mathcal{S}(B).$$

for all $A, B \in \mathcal{B}_{\mu,0}(X)$ such that $A \cap B = \varnothing$.

(b) The set function $\mathcal{S}$ is said to be $\mathbb{G}$-invariant if

$$\mathcal{S}(g^{-1}(A)) = \mathcal{S}(A)$$

for all $A \in \mathcal{B}_{\mu,0}(X)$ and all $g \in \mathbb{G}$.

The following result is used in the proof of Theorem 2.25. It was established in [AHM19b, Theorem 2.3] (see also [AHM17, Theorem 2.17]). For the convenience of the reader its proof is given in §A.2.1.

Theorem 3.33. *Let $\mathcal{S} : \mathcal{B}_{\mu,0}(X) \to \mathbb{R}$ be a subadditive and $\mathbb{G}$-invariant set function with the following boundedness condition*:

$$|\mathcal{S}(Q)| \leqslant c\mu(Q) \tag{3.46}$$

for all $Q \in \mathcal{B}_{\mu,0}(X)$ and some $c > 0$, and assume that μ is $\mathbb{G}$-invariant. Then, for each $\{Q_t\}_{t>0} \subset \mathcal{B}_{\mu,0}(X)$ which is weakly $\mathbb{G}$-asymptotic with respect to $\{\mathbb{U}_k\}_{k\in\mathbb{N}^*}$, one has

$$\lim_{t\to\infty} \frac{\mathcal{S}(Q_t)}{\mu(Q_t)} = \inf_{k\in\mathbb{N}^*} \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)}.$$

3.6.2. *The stochastic case.* We begin with the following definition.

Definition 3.34. The metric mesaure space (X, d, μ) is said to be meshable with respect to $\{\mathbb{U}_k\}_{k\in\mathbb{N}^*}$ if for each $k \in \mathbb{N}^*$ there exists $\mathbb{H}_k \in \mathcal{U}_k(\mathbb{G})$ with the property that for each $n \in \mathbb{N}^*$ there exist $H^-_{n,k}, H^+_{n,k} \in \mathcal{P}_{\mathrm{f}}(\mathbb{H}_k)$ such that:

$$\bigcup_{g\in H^-_{n,k}} g^{-1}(\mathbb{U}_k) \subset \mathbb{U}_n \subset \bigcup_{g\in H^+_{n,k}} g^{-1}(\mathbb{U}_k); \tag{3.47}$$

$$\lim_{n\to\infty} \frac{\mu\left(\bigcup_{g\in H^+_{n,k}} g^{-1}(\mathbb{U}_k)\setminus \bigcup_{g\in H^-_{n,k}} g^{-1}(\mathbb{U}_k)\right)}{\mu(\mathbb{U}_n)} = 0. \tag{3.48}$$

The interest of Definition 3.34 comes from the following proposition (which is used in the proof of Theorem 3.42).

Proposition 3.35. *Let $\mathcal{S} : \mathcal{B}_{\mu,0}(X) \to \mathbb{R}$ be a subadditive and $\mathbb{G}$-invariant set function satisfying* (3.46). *If (X, d, μ) is meshable with respect to $\{\mathbb{U}_k\}_{k\in\mathbb{N}^*}$ then*

$$\lim_{n\to\infty} \frac{\mathcal{S}(\mathbb{U}_n)}{\mu(\mathbb{U}_n)} = \inf_{k\in\mathbb{N}^*} \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)}. \tag{3.49}$$

Proof of Proposition 3.35. First of all, it is clear that $\frac{\mathcal{S}(\mathbb{U}_n)}{\mu(\mathbb{U}_n)} \geqslant \inf_{k\in\mathbb{N}^*} \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)}$ for all $n \in \mathbb{N}^*$, and so

$$\varliminf_{n\to\infty} \frac{\mathcal{S}(\mathbb{U}_n)}{\mu(\mathbb{U}_n)} \geqslant \inf_{k\in\mathbb{N}^*} \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)}. \tag{3.50}$$

On the other hand, fix any $k \in \mathbb{N}^*$ and any $n \in \mathbb{N}^*$ and set:

$$\mathbb{U}^-_{n,k} := \bigcup_{g\in H^-_{n,k}} g^{-1}(\mathbb{U}_k);$$
$$\mathbb{U}^+_{n,k} := \bigcup_{g\in H^+_{n,k}} g^{-1}(\mathbb{U}_k),$$

where $H^-_{n,k}$ and $H^+_{n,k} \in \mathcal{P}_{\mathrm{f}}(\mathbb{H}_k)$ with $\mathbb{H}_k$ given by Definition 3.34. By the left inclusion in (3.47) we have $\mathbb{U}^-_{n,k} \subset \mathbb{U}_n$ and so $\mathbb{U}_n = \mathbb{U}^-_{n,k} \cup \left(\mathbb{U}_n \setminus \mathbb{U}^-_{n,k}\right)$. Hence

$$\mathcal{S}(\mathbb{U}_n) \leqslant \mathcal{S}\left(\mathbb{U}^-_{n,k}\right) + \mathcal{S}\left(\mathbb{U}_n\setminus\mathbb{U}^-_{n,k}\right)$$

because $\mathcal{S}$ is subadditive, and consequently

$$\frac{\mathcal{S}(\mathbb{U}_n)}{\mu(\mathbb{U}_n)} \leqslant \frac{\mathcal{S}\left(\mathbb{U}^-_{n,k}\right)}{\mu\left(\mathbb{U}^-_{n,k}\right)} \frac{\mu\left(\mathbb{U}^-_{n,k}\right)}{\mu(\mathbb{U}_n)} + \frac{\mathcal{S}\left(\mathbb{U}_n\setminus\mathbb{U}^-_{n,k}\right)}{\mu(\mathbb{U}_n)}.$$

Using again the subadditivity of $\mathcal{S}$ and its $\mathbb{G}$-invariance (resp. the $\mathbb{G}$-invariance of μ) we have

$$\mathcal{S}\left(\mathbb{U}^-_{n,k}\right) \leqslant \left|H^-_{n,k}\right| \mathcal{S}(\mathbb{U}_k)$$
$$\left(\text{resp. } \mu\left(\mathbb{U}^-_{n,k}\right) = \left|H^-_{n,k}\right| \mu(\mathbb{U}_k)\right).$$

Moreover, $\mathbb{U}_n \subset \mathbb{U}^+_{n,k}$ by the right inclusion in (3.47), which implies that $\mathbb{U}_n\setminus\mathbb{U}^-_{n,k} \subset \mathbb{U}^+_{n,k}\setminus\mathbb{U}^-_{n,k}$ and so

$$\mathcal{S}\left(\mathbb{U}_n\setminus\mathbb{U}^-_{n,k}\right) \leqslant c\mu\left(\mathbb{U}^+_{n,k}\setminus\mathbb{U}^-_{n,k}\right)$$

with $c > 0$ given by (3.46). It follows that

$$\begin{aligned}\frac{\mathcal{S}(\mathbb{U}_n)}{\mu(\mathbb{U}_n)} &\leqslant \frac{\mathcal{S}\left(\mathbb{U}_k\right)}{\mu\left(\mathbb{U}_k\right)}\frac{\mu\left(\mathbb{U}_{n,k}^-\right)}{\mu(\mathbb{U}_n)} + \frac{c\mu\left(\mathbb{U}_{n,k}^+\backslash\mathbb{U}_{n,k}^-\right)}{\mu(\mathbb{U}_n)} \\ &\leqslant \frac{\mathcal{S}\left(\mathbb{U}_k\right)}{\mu\left(\mathbb{U}_k\right)} + \frac{c\mu\left(\mathbb{U}_{n,k}^+\backslash\mathbb{U}_{n,k}^-\right)}{\mu(\mathbb{U}_n)}\end{aligned}$$

because $\mu\left(\mathbb{U}_{n,k}^-\right) \leqslant \mu(\mathbb{U}_n)$ since $\mathbb{U}_{n,k}^- \subset \mathbb{U}_n$. Letting $n \to \infty$ and using (3.48), and then passing to the infimum on k, we obtain

$$\overline{\lim_{n\to\infty}} \frac{\mathcal{S}(\mathbb{U}_n)}{\mu(\mathbb{U}_n)} \leqslant \inf_{k\in\mathbb{N}^*} \frac{\mathcal{S}\left(\mathbb{U}_k\right)}{\mu\left(\mathbb{U}_k\right)}, \tag{3.51}$$

and (3.49) follows by combining (3.50) with (3.51). ■

In what follows, Δ denotes the symmetric difference of sets, i.e. $E\Delta F := (E\backslash F) \cup (F\backslash E)$ for any $E, F \subset \mathbb{G}$, and we adopt the following notation: $EF := \{gof : (g, f) \in E \times F\}$ and $E^{-1}F := \{g^{-1}of : (g, f) \in E \times F\}$ and, for any $g \in \mathbb{G}$, $gF := \{gof : f \in F\}$. From now on, for each $k \in \mathbb{N}^*$, we consider the class $\mathcal{U}_k^a(\mathbb{G})$ defined by

$$\mathcal{U}_k^a(\mathbb{G}) := \Big\{\mathbb{H} \in \mathcal{U}_k(\mathbb{G}) : \mathbb{H} \text{ is countable, discrete and amenable group}\Big\},$$

where amenability of $\mathbb{H}$ means that for each $E \in \mathcal{P}_{\mathrm{f}}(\mathbb{H})$ and each $\delta > 0$ there exists $F \in \mathcal{P}_{\mathrm{f}}(\mathbb{H})$ such that

$$|F\Delta EF| \leqslant \delta|F|.$$

(For more details about the theory of amenability, we refer to [Gre69, OW87, Pat88, Tem92, AAB+10, DZ15] and the references therein, see also [Kre85, §6.4].)

The property of Følner-Tempelman stated in the definition below is needed to use both Lindenstrauss's ergodic theorem (see Theorem A.3) which is valid for general amenable groups, and a maximal inequality (see Lemma A.4) which is valid for countable discrete amenable groups. (These two results are used in the proof of Theorem 3.42.)

Definition 3.36. Let $\mathbb{H} \in \mathcal{U}_k^a(\mathbb{G})$ and let $\{G_t\}_{t>0} \subset \mathcal{P}_{\mathrm{f}}(\mathbb{H})$. We say that $\{G_t\}_{t>0}$ is of Følner-Tempelman type with respect to $\mathbb{H}$ if it satisfies the following two conditions:

(a) Følner's condition: for every $g \in \mathbb{H}$, one has

$$\lim_{t\to\infty} \frac{\left|gG_t\Delta G_t\right|}{\left|G_t\right|} = 0;$$

(b) Tempelman's condition: there exists $M > 0$, which called the Templeman constant associated with $\{G_t\}_{t>0}$, such that for every $t > 0$, one has

$$\Big|\bigcup_{0<s\leqslant t} G_sG_t\Big| \leqslant M|G_t|.$$

Together with Definition 3.34, the following definition set a framework for establishing a subadditive theorem in the stochastic case and in the setting of metric measure spaces (see Theorem 3.42).

Definition 3.37. Let $\{Q_t\}_{t>0} \subset \mathcal{B}_{\mu,0}(X)$. We say that $\{Q_t\}_{t>0}$ is strongly $\mathbb{G}$-asymptotic with respect to $\{\mathbb{U}_k\}_{k\in\mathbb{N}^*}$ if there exists $\{\mathbb{G}_k\}_{k\in\mathbb{N}^*}$ with $\mathbb{G}_k \in \mathcal{U}_k^a(\mathbb{G})$ for all $k \in \mathbb{N}^*$ and $\cup_{k\in\mathbb{N}^*}\mathbb{G}_k = \mathbb{G}$ such that for each $k \in \mathbb{N}^*$ and each $t > 0$ there exist $m_{t,k} \in \mathbb{N}^*$, $g_{t,k} \in \mathbb{G}$ and $F_{t,k}, G^-_{t,k}, G^+_{t,k} \in \mathcal{P}_{\mathrm{f}}(\mathbb{G}_k)$ such that (3.42), (3.43), (3.44) and (3.45) are satisfied with the additional assumption that $\{G^-_{t,k}\}_{t>0}$ and $\{G^+_{t,k}\}_{t>0}$ are of Følner-Tempelman type with respect to $\mathbb{G}_k$.

Let $(\Sigma, \mathcal{T}, \mathbb{P})$ be a probability space and let $\{\tau_g : \Sigma \to \Sigma\}_{g\in\mathbb{G}}$ be satisfying the following three properties:

(mesurability) τ_g is $\mathcal{T}$-mesurable for all $g \in \mathbb{G}$;
(group property) $\tau_g \circ \tau_f = \tau_{g\circ f}$ and $\tau_{g^{-1}} = \tau_g^{-1}$ for all $g, f \in \mathbb{G}$;
(mass invariance) $\mathbb{P}(\tau_g(E)) = \mathbb{P}(E)$ for all $E \in \mathcal{T}$ and all $g \in \mathbb{G}$.

Definition 3.38. Such a $\{\tau_g\}_{g\in\mathbb{G}}$ is said to be a group of $\mathbb{P}$-preserving transformation on $(\Sigma, \mathcal{T}, \mathbb{P})$ and the quadruplet $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g\in\mathbb{G}})$ is called a measurable dynamical $\mathbb{G}$-system.

Remark 3.39. If $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g\in\mathbb{G}})$ is a measurable dynamical $\mathbb{G}$-system then, for any subgroup $\mathbb{H}$ of $\mathbb{G}$, $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g\in\mathbb{H}})$ is a measurable dynamical $\mathbb{H}$-system.

Let $\mathcal{I} := \{E \in \mathcal{T} : \mathbb{P}(\tau_g(E)\Delta E) = 0 \text{ for all } g \in \mathbb{G}\}$ be the σ-algebra of invariant sets with respect to $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g\in\mathbb{G}})$. (For any subgroup $\mathbb{H}$ of $\mathbb{G}$, we denote the σ-algebra of invariant sets with respect to $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g\in\mathbb{H}})$ by $\mathcal{I}_{\mathbb{H}}$.)

Definition 3.40. When $\mathbb{P}(E) \in \{0, 1\}$ for all $E \in \mathcal{I}$, the measurable dynamical $\mathbb{G}$-system $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g\in\mathbb{G}})$ is said to be ergodic.

In what follows, we assume that $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g\in\mathbb{G}})$ is a measurable dynamical $\mathbb{G}$-system. Let us recall the definition of a subadditive process.

Definition 3.41. A set function $\mathcal{S} : \mathcal{B}_{\mu,0}(X) \to L^1(\Sigma, \mathcal{T}, \mathbb{P})$ is called a subadditive process if it is subadditive in the sense of Definition 3.32(a) and $\mathbb{G}$-covariant, i.e.

$$\mathcal{S}(g^{-1}(A)) = \mathcal{S}(A)\circ\tau_g$$

for all $A \in \mathcal{B}_{\mu,0}(X)$ and all $g \in \mathbb{G}$. If in addition the measurable dynamical $\mathbb{G}$-system $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g\in\mathbb{G}})$ is ergodic, then $\mathcal{S}$ is called an ergodic subadditive process.

The following result is used in the proof of Theorem 2.34. It was established in [AHM19b, Theorem 2.11]. For the convenience of the reader its proof is given in §A.2.2.

Theorem 3.42. *Assume that (X, d, μ) is meshable with respect to $\{\mathbb{U}_k\}_{k\in\mathbb{N}^*}$ and consider $\mathcal{S} : \mathcal{B}_{\mu,0}(X) \to L^1(\Sigma, \mathcal{T}, \mathbb{P})$ a subadditive process satisfying* (3.46). *Then, for each $\{Q_t\}_{t>0} \subset \mathcal{B}_{\mu,0}(X)$ which is strongly $\mathbb{G}$-asymptotic with respect to $\{\mathbb{U}_k\}_{k\in\mathbb{N}^*}$, one has*

$$\lim_{t\to\infty} \frac{\mathcal{S}(Q_t)(\omega)}{\mu\big(Q_t\big)} = \inf_{k\in\mathbb{N}^*} \frac{\mathbb{E}^{\mathcal{I}}[\mathcal{S}(\mathbb{U}_k)](\omega)}{\mu(\mathbb{U}_k)} \text{ for } \mathbb{P}\text{-a.a. } \omega \in \Sigma,$$

where $\mathbb{E}^{\mathcal{I}}[\mathcal{S}(\mathbb{U}_k)]$ denotes the conditional expectation of $\mathcal{S}(\mathbb{U}_k)$ over $\mathcal{I}$ with respect to $\mathbb{P}$. If in addition $(\Sigma, \mathcal{T}, \mathbb{P}, \{\tau_g\}_{g\in\mathbb{G}})$ is ergodic, then

$$\lim_{t\to\infty} \frac{\mathcal{S}(Q_t)(\omega)}{\mu\big(Q_t\big)} = \inf_{k\in\mathbb{N}^*} \frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_k)]}{\mu(\mathbb{U}_k)} \text{ for } \mathbb{P}\text{-a.a. } \omega \in \Sigma,$$

where $\mathbb{E}[\mathcal{S}(\mathbb{U}_k)]$ *denotes the expectation of* $\mathcal{S}(\mathbb{U}_k)$ *with respect to* $\mathbb{P}$.

4. Proof of the Γ-convergence theorem

This section is devoted to the proof of Theorem 2.11 which is divided into five steps.

Proof of Theorem 2.11. Let $\omega \in \Sigma$ be satisfying all the assumptions of Theorem 2.11.

Step 1: integral representation of the Γ-limit inf and the Γ-limit sup. For each $u \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ we consider the set functions $\mathcal{S}^-_{u,\omega}, \mathcal{S}^+_{u,\omega} : \mathcal{O}(\Omega) \to [0,\infty]$ given by:

$$\mathcal{S}^-_{u,\omega}(A) := \Gamma(L^p_\mu)\text{-}\varliminf_{t\to\infty} E_t(u,A,\omega);$$
$$\mathcal{S}^+_{u,\omega}(A) := \Gamma(L^p_\mu)\text{-}\varlimsup_{t\to\infty} E_t(u,A,\omega).$$

Recall that $\mathfrak{G}$ is the effective domain of the functional $u \mapsto \int_\Omega G(\nabla_\mu u(x))d\mu(x)$.

Step 1 consists of proving the following lemma.

Lemma 4.1. *If* (2.5), (2.6), (2.7), (2.11), (2.12) *and* (2.13) *hold then*:

$$\mathcal{S}^-_{u,\omega}(A) = \int_A \lambda^-_{u,\omega}(x)d\mu(x);$$
$$\mathcal{S}^+_{u,\omega}(A) = \int_A \lambda^+_{u,\omega}(x)d\mu(x)$$

for all $u \in \mathfrak{G}$ *and all* $A \in \mathcal{O}(\Omega)$ *with* $\lambda^-_{u,\omega}, \lambda^+_{u,\omega} \in L^1_\mu(\Omega)$ *given by*:

$$\lambda^-_{u,\omega}(x) = \lim_{\rho\to 0} \frac{\mathcal{S}^-_{u,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))};$$
$$\lambda^+_{u,\omega}(x) = \lim_{\rho\to 0} \frac{\mathcal{S}^+_{u,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))}.$$

Proof of Lemma 4.1. Fix $u \in \mathfrak{G}$. Using the right inequality in (2.11) we see that:

$$\mathcal{S}^-_{u,\omega}(A) \leqslant \beta\mu(A) + \beta\int_A G(\nabla_\mu u(x))d\mu(x); \tag{4.1}$$
$$\mathcal{S}^+_{u,\omega}(A) \leqslant \beta\mu(A) + \beta\int_A G(\nabla_\mu u(x))d\mu(x)$$

for all $A \in \mathcal{O}(\Omega)$. Thus, the condition (d) of Lemma 3.24 is satisfied with $\nu = \beta\big(1 + G(\nabla_\mu u(\cdot))\big)\mu$ (which is absolutely continuous with respect to μ). On the other hand, it is easily seen that the conditions (a) and (b) of Lemma 3.24 are satisfied. Hence, the proof is completed if we prove the condition (c) of Lemma 3.24, i.e.

$$\mathcal{S}^-_{u,\omega}(A\cup B) \leqslant \mathcal{S}^-_{u,\omega}(A) + \mathcal{S}^-_{u,\omega}(B) \text{ for all } A,B \in \mathcal{O}(\Omega); \tag{4.2}$$
$$\mathcal{S}^+_{u,\omega}(A\cup B) \leqslant \mathcal{S}^+_{u,\omega}(A) + \mathcal{S}^+_{u,\omega}(B) \text{ for all } A,B \in \mathcal{O}(\Omega). \tag{4.3}$$

Indeed, by Lemma 3.24, the set function $\mathcal{S}^-_{u,\omega}$ (resp. $\mathcal{S}^+_{u,\omega}$) can be (uniquely) extended to a (finite) positive Radon measure which is absolutely continuous with respect to μ, and

the theorem follows by using Radon-Nikodym's theorem and then Lebesgue's differentiation theorem.

Remark 4.2. Lemma 4.1 shows that $\Gamma(L^p_\mu)\text{-}\underline{\lim}_{t\to\infty} E_t(u,\cdot,\omega)$ and $\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,\cdot,\omega)$ can be uniquely extended to a finite positive Radon measure on Ω which is absolutely continuous with respect to μ.

Substep 1-1: an auxiliary result for proving Lemma 4.1. To show (4.2) (resp. (4.3)) we need the following lemma.

Lemma 4.3. *If $U,V,Z,T\in\mathcal{O}(\Omega)$ are such that $\overline{Z}\subset U$ and $T\subset V$, then*:

$$\mathcal{S}^-_{u,\omega}(Z\cup T)\leqslant \mathcal{S}^-_{u,\omega}(U)+\mathcal{S}^-_{u,\omega}(V); \tag{4.4}$$

$$\mathcal{S}^+_{u,\omega}(Z\cup T)\leqslant \mathcal{S}^+_{u,\omega}(U)+\mathcal{S}^+_{u,\omega}(V). \tag{4.5}$$

Proof of Lemma 4.3. As the proofs of (4.4) and (4.5) are the same, we only give the proof of (4.4). Let $\{u_t\}_{t>0}$ and $\{v_t\}_{t>0}$ be two sequences in $H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ such that:

$$\|u_t-u\|_{L^p_\mu(\Omega;\mathbb{R}^m)}\to 0; \tag{4.6}$$

$$\|v_t-u\|_{L^p_\mu(\Omega;\mathbb{R}^m)}\to 0; \tag{4.7}$$

$$\lim_{t\to\infty}\int_U L_t(x,\nabla_\mu u_t(x),\omega)d\mu(x)=\mathcal{S}^-_{u,\omega}(U)<\infty; \tag{4.8}$$

$$\lim_{t\to\infty}\int_V L_t(x,\nabla_\mu v_t(x),\omega)d\mu(x)=\mathcal{S}^-_{u,\omega}(V)<\infty. \tag{4.9}$$

Since $\{L_t\}_{t>0}$ is p-coercive (see (2.5) and the left inequality in (2.11)), from (4.8) and(4.9) we see that $\sup_{t>0}\|\nabla_\mu u_t\|_{L^p_\mu(\Omega;\mathbb{M})}<\infty$ and $\sup_{t>0}\|\nabla_\mu v_t\|_{L^p_\mu(\Omega;\mathbb{M})}<\infty$. As $p>\kappa$, taking (4.6) and (4.7) into account, by Corollary 3.8 we can assert, up to a subsequence, that:

$$\|u_t-u\|_{L^\infty_\mu(\Omega;\mathbb{R}^m)}\to 0; \tag{4.10}$$

$$\|v_t-u\|_{L^\infty_\mu(\Omega;\mathbb{R}^m)}\to 0. \tag{4.11}$$

Fix $\delta\in]0,\mathrm{dist}(Z,\partial U)[$ with $\partial U:=\overline{U}\backslash U$, fix any $q\geqslant 1$ and consider $W^-_i,W^+_i\subset\Omega$ given by:

$$W^-_i:=\left\{x\in\Omega:\mathrm{dist}(x,Z)\leqslant\tfrac{\delta}{3}+\tfrac{(i-1)\delta}{3q}\right\};$$
$$W^+_i:=\left\{x\in\Omega:\tfrac{\delta}{3}+\tfrac{i\delta}{3q}\leqslant\mathrm{dist}(x,Z)\right\},$$

where $i\in\{1,\cdots,q\}$. For every $i\in\{1,\cdots,q\}$ there exists a Urysohn function $\varphi_i\in\mathrm{Lip}(\Omega)$ for the pair (W^+_i,W^-_i). Fix any $t>0$ and define $w^i_t\in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ by

$$w^i_t:=\varphi_i u_t+(1-\varphi_i)v_t. \tag{4.12}$$

Fix any $\tau\in]0,1[$. Setting $W_i:=\Omega\backslash(W^-_i\cup W^+_i)$ and using Theorem 3.3(d) and (3.1) we have

$$\nabla_\mu(\tau w^i_t)=\tau\nabla_\mu w^i_t=\begin{cases}\tau\nabla_\mu u_t & \text{in } W^-_i\\ (1-\tau)\frac{t}{1-\tau}D_\mu\varphi_i\otimes(u_t-v_t)+\tau\big(\varphi_i\nabla_\mu u_t+(1-\varphi_i)\nabla_\mu v_t\big) & \text{in } W_i\\ \tau\nabla_\mu v_t & \text{in } W^+_i.\end{cases}$$

Noticing that $Z \cup T = ((Z \cup T) \cap W_i^-) \cup (W \cap W_i) \cup (T \cap W_i^+)$ with $(Z \cup T) \cap W_i^- \subset U$, $T \cap W_i^+ \subset V$ and $W := T \cap \{x \in U : \frac{\delta}{3} < \mathrm{dist}(x, Z) < \frac{2\delta}{3}\}$ we deduce that for every $i \in \{1, \cdots, q\}$,

$$\begin{aligned}\int_{Z\cup T} L_t(x, \tau\nabla_\mu w_t^i, \omega)d\mu &\leqslant \int_U L_t(x, \tau\nabla_\mu u_t, \omega)d\mu + \int_V L_t(x, \tau\nabla_\mu v_t, \omega)d\mu \\ &\quad + \int_{W\cap W_i} L_t(x, \tau\nabla_\mu w_t^i, \omega)d\mu. \end{aligned} \tag{4.13}$$

Fix any $i \in \{1, \cdots, q\}$. From the right inequality in (2.11) and the inequality (2.7) we see that

$$\begin{aligned}\int_{W\cap W_i} L_t(x, \tau\nabla_\mu w_t^i, \omega)d\mu &\leqslant \beta\mu(W \cap W_i) + \beta\int_{W\cap W_i} G(\tau\nabla_\mu w_t^i)d\mu \\ &\leqslant \beta(1+\gamma)\mu(W \cap W_i) \\ &\quad +\beta\gamma\int_{W\cap W_i} G(\varphi_i\nabla_\mu u_t + (1-\varphi_i)\nabla_\mu v_t)d\mu \\ &\quad +\beta\gamma\int_{W\cap W_i} G\left(\frac{\tau}{1-\tau}D_\mu\varphi_i \otimes (u_t - v_t)\right)d\mu,\end{aligned}$$

and by using again the inequality (2.7) and the left inequality in (2.11) we obtain

$$\begin{aligned}\int_{W\cap W_i} L_t(x, \tau\nabla_\mu w_t^i, \omega)d\mu &\leqslant \beta(1+\gamma+\gamma^2)\mu(W \cap W_i) \\ &\quad +\frac{\beta\gamma^2}{\alpha}\left(\int_{W\cap W_i} L_t(x, \nabla_\mu u_t, \omega)d\mu + \int_{W\cap W_i} L_t(x, \nabla_\mu v_t, \omega)d\mu\right) \\ &\quad +\beta\gamma\int_{W\cap W_i} G\left(\frac{\tau}{1-\tau}D_\mu\varphi_i \otimes (u_t - v_t)\right)d\mu. \end{aligned} \tag{4.14}$$

On the other hand, we have

$$\left|\frac{\tau}{1-\tau}D_\mu\varphi_i(x) \otimes (u_t(x) - v_t(x))\right| \leqslant \left|\frac{\tau}{1-\tau}\right| \|D_\mu\varphi_i\|_{L^\infty_\mu(\Omega)}\|u_t - v_t\|_{L^\infty_\mu(\Omega;\mathbb{R}^m)}$$

for μ-a.a. $x \in \Omega$. But $\lim_{t\to\infty}\|u_t - v_t\|_{L^\infty_\mu(\Omega;\mathbb{R}^m)} = 0$ by (4.10) and (4.11), hence for each $\tau \in]0, 1[$ and each $i \in \{1, \cdots, q\}$ there exists $t_{\tau,i} > 0$ such that

$$\left|\frac{\tau}{1-\tau}D_\mu\varphi_i(x) \otimes (u_t(x) - v_t(x))\right| \leqslant r$$

for μ-a.a. $x \in \Omega$ and all $t \geqslant t_{\tau,i}$ with $r > 0$ given by (2.6). Hence

$$\begin{aligned}\int_{W\cap W_i} G\left(\frac{\tau}{1-\tau}D_\mu\varphi_i \otimes (u_t - v_t)\right)d\mu &\leqslant \int_{W\cap W_i}\sup_{|\xi|\leqslant r} G(\xi)d\mu \\ &= \mu(W \cap W_i)\sup_{|\xi|\leqslant r} G(\xi) \end{aligned} \tag{4.15}$$

for all $t \geqslant T_{\tau,q}$ with $T_{\tau,q} = \max\{t_{\tau,i} : i \in \{1, \cdots, q\}\}$. Moreover, we have

$$\begin{aligned}\int_U L_t(x, \tau\nabla_\mu u_t, \omega)d\mu &\leqslant \int_U L_t(x, \nabla_\mu u_t, \omega)d\mu \\ &\quad + \Delta_{L_t}^{a_t(\cdot,\omega)}(\tau)\left(\int_U a_t(x,\omega)d\mu(x) + \int_U L_t(x, \nabla_\mu u_t, \omega)d\mu\right) \\ &\leqslant \int_U L_t(x, \nabla_\mu u_t, \omega)d\mu \\ &\quad + \Delta_\omega(\tau)\left(\int_U a_t(x,\omega)d\mu(x) + \int_U L_t(x, \nabla_\mu u_t, \omega)d\mu\right) \qquad (4.16)\end{aligned}$$

with $\Delta_\omega(\tau) := \sup_{s>0} \Delta_{L_s}^{a_s(\cdot,\omega)}(\tau)$, where $\{a_s(\cdot,\omega)\}_{s>0} \subset L^1_\mu(\Omega;]0, \infty])$ is given by (2.12). In the same way, we have

$$\begin{aligned}\int_V L_t(x, \tau\nabla_\mu v_t, \omega)d\mu &\leqslant \int_V L_t(x, \nabla_\mu v_t, \omega)d\mu \\ &\quad + \Delta_\omega(\tau)\left(\int_V a_t(x,\omega)d\mu(x) + \int_V L_t(x, \nabla_\mu v_t, \omega)d\mu\right). \qquad (4.17)\end{aligned}$$

Taking (4.15) into account and substituting (4.14), (4.16) and (4.17) into (4.13) and then averaging these inequalities, it follows that for every $q \geqslant 1$, every $\tau \in]0, 1[$ and every $t \geqslant T_{\tau,q}$, there exists $i_{t,\tau,q} \in \{1, \cdots, q\}$ such that

$$\begin{aligned}\int_{Z\cup T} L_t(x, \nabla_\mu(\tau w_t^{i_{t,\tau,q}}), \omega)d\mu &\leqslant \int_U L_t(x, \nabla_\mu u_t, \omega)d\mu + \int_V L_t(x, \nabla_\mu v_t, \omega)d\mu \\ &\quad + \Delta_\omega(\tau)\left(\int_U a_t(x,\omega)d\mu(x) + \int_U L_t(x, \nabla_\mu u_t, \omega)d\mu\right) \\ &\quad + \Delta_\omega(\tau)\left(\int_V a_t(x,\omega)d\mu(x) + \int_V L_t(x, \nabla_\mu v_t, \omega)d\mu\right) \\ &\quad + \frac{c}{q}\mu(\Omega)\sup_{|\xi|\leqslant r} G(\xi) \\ &\quad + \frac{c}{q}\left(\int_U L_t(x, \nabla_\mu u_t, \omega)d\mu + \int_V L_t(x, \nabla_\mu v_t, \omega)d\mu\right)\end{aligned}$$

with $c = \max\left\{\beta(1 + \gamma + \gamma^2) + 1, \frac{\beta\gamma^2}{\alpha}\right\}$, where $\overline{\lim}_{t\to\infty} \int_A a_t(x,\omega)d\mu < \infty$ by (2.13) and $\sup_{|\xi|\leqslant r} G(\xi) < \infty$ by (2.6). As $\overline{\lim}_{\tau\to 1^-} \Delta_\omega(\tau) \leqslant 0$, letting $t \to \infty$, $\tau \to 1^-$ and $q \to \infty$ and using (4.8) and (4.9), we get

$$\overline{\lim_{q\to\infty}}\ \overline{\lim_{\tau\to 1^-}}\ \overline{\lim_{t\to\infty}} \int_{Z\cup T} L_t(x, \nabla_\mu(\tau w_t^{i_{t,\tau,q}}), \omega)d\mu \leqslant \mathcal{S}^-_{u,\omega}(U) + \mathcal{S}^-_{u,\omega}(V). \qquad (4.18)$$

On the other hand, taking (4.12) into account and using (4.6) and (4.7) we see that

$$\lim_{q\to\infty} \lim_{\tau\to 1^-} \lim_{t\to\infty} \|\tau w_t^{i_{t,\tau,q}} - u\|_{L^p_\mu(\Omega;\mathbb{R}^m)} = 0.$$

By diagonalization, there exist increasing mappings $t \mapsto \tau_t$ and $t \mapsto q_t$ with $\tau_t \to 1^-$ and $q_t \to \infty$ such that:

$$\begin{aligned}\underline{\lim}_{t\to\infty} \int_{Z\cup T} L_t(x, \nabla_\mu \hat{w}_t, \omega) d\mu &\leqslant \overline{\lim}_{t\to\infty} \int_{Z\cup T} L_t(x, \nabla_\mu \hat{w}_t, \omega) d\mu \\ &\leqslant \overline{\lim}_{q\to\infty} \overline{\lim}_{\tau\to 1^-} \overline{\lim}_{t\to\infty} \int_{Z\cup T} L_t(x, \nabla_\mu(\tau w_t^{i_{t,\tau,q}}), \omega) d\mu; \\ \lim_{t\to\infty} \|\hat{w}_t - u\|_{L^p_\mu(\Omega;\mathbb{R}^m)} &= 0,\end{aligned}$$

where $\hat{w}_t := \tau_t w_t^{i_{t,\tau_t,q_t}}$. Hence

$$\mathcal{S}^-_{u,\omega}(Z\cup T) \leqslant \overline{\lim}_{q\to\infty} \overline{\lim}_{\tau\to 1^-} \overline{\lim}_{t\to\infty} \int_{Z\cup T} L_t(x, \nabla_\mu(\tau w_t^{i_{t,\tau,q}}), \omega) d\mu,$$

and (4.4) follows from (4.18). ■

Substep 1-2: end of the proof of Lemma 4.1. We now prove (4.2). Fix $A, B \in \mathcal{O}(\Omega)$. Fix any $\varepsilon > 0$ and consider $C, D \in \mathcal{O}(\Omega)$ such that $\overline{C} \subset A$, $\overline{D} \subset B$ and

$$\beta\mu(E) + \beta \int_E G(\nabla_\mu u(x)) d\mu(x) < \varepsilon$$

with $E := A \cup B \backslash \overline{C \cup D}$. Then $\mathcal{S}^-_{u,\omega}(E) \leqslant \varepsilon$ by (4.1). Let $\hat{C}, \hat{D} \in \mathcal{O}(\Omega)$ be such that $\overline{C} \subset \hat{C}$, $\overline{\hat{C}} \subset A$, $\overline{D} \subset \hat{D}$ and $\overline{\hat{D}} \subset B$. Applying Lemma 4.3 with $U = \hat{C} \cup \hat{D}$, $V = T = E$ and $Z = C \cup D$ (resp. $U = A$, $V = B$, $Z = \hat{C}$ and $T = \hat{D}$) we obtain

$$\mathcal{S}^-_{u,\omega}(A \cup B) \leqslant \mathcal{S}^-_{u,\omega}(\hat{C} \cup \hat{D}) + \varepsilon \ \big(\text{resp. } \mathcal{S}^-_{u,\omega}(\hat{C} \cup \hat{D}) \leqslant \mathcal{S}^-_{u,\omega}(A) + \mathcal{S}^-_{u,\omega}(B)\big),$$

and (4.2) follows by letting $\varepsilon \to 0$. ■

Step 2: other formulas for the Γ-limit inf and the Γ-limit sup. Consider the variational functionals $E^-_{0,\omega}, E^+_{0,\omega} : H^{1,p}_\mu(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty]$ given by:

$$\begin{aligned}E^-_{0,\omega}(u, A) &:= \inf\left\{ \underline{\lim}_{t\to\infty} E_t(u_t, A, \omega) : H^{1,p}_{\mu,0}(\Omega;\mathbb{R}^m) \ni u_t - u \overset{L^p_\mu}{\to} 0 \right\}; \\ E^+_{0,\omega}(u, A) &:= \inf\left\{ \overline{\lim}_{t\to\infty} E_t(u_t, A, \omega) : H^{1,p}_{\mu,0}(\Omega;\mathbb{R}^m) \ni u_t - u \overset{L^p_\mu}{\to} 0 \right\}.\end{aligned}$$

As $H^{1,p}_{\mu,0}(\Omega;\mathbb{R}^m) \subset H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ it is clear that:

$$\mathcal{S}^-_{u,\omega}(A) \leqslant E^-_{0,\omega}(u, A); \tag{4.19}$$

$$\mathcal{S}^+_{u,\omega}(A) \leqslant E^+_{0,\omega}(u, A) \tag{4.20}$$

for all $u \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ and all $A \in \mathcal{O}(\Omega)$. On the other hand, we have the following lemma.

Lemma 4.4. *If* (2.5), (2.6), (2.7), (2.11), (2.12) *and* (2.13) *hold then for every* $u \in \mathfrak{G}$, *every* $A \in \mathcal{O}(\Omega)$ *and every* $\tau \in]0,1[$, *one has*:

$$E^-_{0,\omega}(\tau u, A) \leqslant \big(1+\Delta_\omega(\tau)\big)\mathcal{S}^-_{u,\omega}(A) + \overline{\lim_{t\to\infty}}\,\Delta_\omega(\tau)\int_A a_t(x,\omega)d\mu(x); \tag{4.21}$$

$$E^+_{0,\omega}(\tau u, A) \leqslant \big(1+\Delta_\omega(\tau)\big)\mathcal{S}^+_{u,\omega}(A) + \overline{\lim_{t\to\infty}}\,\Delta_\omega(\tau)\int_A a_t(x,\omega)d\mu(x), \tag{4.22}$$

with $\Delta_\omega(\tau) := \sup_{s>0}\Delta_{L_s}^{a_s(\cdot,\omega)}(\tau)$, *where* $\{a_s(\cdot,\omega)\}_{s>0} \subset L^1_\mu(\Omega;]0,\infty])$ *is given by* (2.12) *and satisfies* (2.13). *As a consequence* (4.19)-(4.21) *and* (4.20)-(4.22) *we have*:

$$\begin{aligned}\Gamma(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}}\,E_t(u,A,\omega) &= \lim_{\tau\to1^-}E^-_{0,\omega}(\tau u,A);\\ \Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}}\,E_t(u,A,\omega) &= \lim_{\tau\to1^-}E^+_{0,\omega}(\tau u,A)\end{aligned} \tag{4.23}$$

for all $u \in \mathfrak{G}$ *and* $A \in \mathcal{O}(\Omega)$.

Proof of Lemma 4.4. Fix $u \in \mathfrak{G}$ and $A \in \mathcal{O}(\Omega)$. As the proofs of (4.21) and (4.22) are the same, we only prove (4.21). Let $\{u_t\}_{t>0} \subset H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ be such that:

$$\|u_t - u\|_{L^p_\mu(\Omega;\mathbb{R}^m)} \to 0; \tag{4.24}$$

$$\lim_{t\to\infty}\int_A L_t(x,\nabla_\mu u_t(x),\omega)d\mu(x) = \mathcal{S}^-_{u,\omega}(A) < \infty. \tag{4.25}$$

Since $\{L_t\}_{t>0}$ is p-coercive (see (2.5) and the left inequality in (2.11)), from (4.25) we see that $\sup_{t>0}\|\nabla_\mu u_t\|_{L^p_\mu(\Omega;\mathbb{M})} < \infty$. As $p > \kappa$, taking (4.24) into account, by Corollary 3.8 we can assert, up to a subsequence, that:

$$\|u_t - u\|_{L^\infty_\mu(\Omega;\mathbb{R}^m)} \to 0. \tag{4.26}$$

Fix $\delta > 0$ and set $A_\delta := \{x \in A : \mathrm{dist}(x,\partial A) > \delta\}$ with $\partial A := \overline{A}\backslash A$. Fix any $t > 0$ and any $q \geqslant 1$ and consider $W^-_i, W^+_i \subset \Omega$ given by

$$\begin{aligned}W^-_i &:= \Big\{x \in \Omega : \mathrm{dist}(x,A_\delta) \leqslant \tfrac{\delta}{3} + \tfrac{(i-1)\delta}{3q}\Big\};\\ W^+_i &:= \Big\{x \in \Omega : \tfrac{\delta}{3} + \tfrac{i\delta}{3q} \leqslant \mathrm{dist}(x,A_\delta)\Big\},\end{aligned}$$

where $i \in \{1,\cdots,q\}$. (Note that $W^-_i \subset A$.) For every $i \in \{1,\cdots,q\}$ there exists a Urysohn function $\varphi_i \in \mathrm{Lip}(\Omega)$ for the pair (W^+_i, W^-_i). Define $w^i_t : \Omega \to \mathbb{R}^m$ by

$$w^i_t := \varphi_i u_t + (1-\varphi_i)u. \tag{4.27}$$

Then $w^i_t - u \in H^{1,p}_{\mu,0}(A;\mathbb{R}^m)$. Fix any $\tau \in]0,1[$. Setting $W_i := X\backslash(W^-_i \cup W^+_i) \subset A$ and using Theorem 3.3(d) and (3.1) we have

$$\nabla_\mu(\tau w^i_t) = \tau\nabla_\mu w^i_t = \begin{cases}\tau\nabla_\mu u_t & \text{in } W^-_i\\ (1-\tau)\frac{\tau}{1-\tau}D_\mu\varphi_i\otimes(u_t-u) + \tau\big(\varphi_i\nabla_\mu u_t + (1-\varphi_i)\nabla_\mu u\big) & \text{in } W_i\\ \tau\nabla_\mu u & \text{in } W^+_i.\end{cases}$$

Fix any $t>0$. Noticing that $A=W_i^-\cup W_i\cup(A\cap W_i^+)$ we deduce that for every $i\in\{1,\cdots,q\}$,

$$\begin{aligned}\int_A L_t(x,\tau\nabla_\mu w_t^i,\omega)d\mu &\leqslant \int_A L_t(x,\tau\nabla_\mu u_t,\omega)d\mu+\int_{A\cap W_i^+} L_t(x,\tau\nabla_\mu u,\omega)d\mu\\ &\quad+\int_{W_i} L_t(x,\tau\nabla_\mu w_t^i,\omega)d\mu. \end{aligned}\tag{4.28}$$

Fix any $q\in\{1,\cdots,q\}$. From the right inequality in (2.11) and the inequality (2.7) we see that

$$\begin{aligned}\int_{W_i} L_t(x,\tau\nabla_\mu w_t^i,\omega)d\mu &\leqslant \beta\mu(W_i)+\beta\int_{W_i} G(\tau\nabla_\mu w_t^i)d\mu\\ &\leqslant \beta(1+\gamma)\mu(W_i)\\ &\quad+\beta\gamma\int_{W_i} G(\varphi_i\nabla_\mu u_t+(1-\varphi_i)\nabla_\mu u)d\mu\\ &\quad+\beta\gamma\int_{W_i} G\left(\frac{\tau}{1-\tau}D_\mu\varphi_i\otimes(u_t-u)\right)d\mu, \end{aligned}\tag{4.29}$$

and by using again the inequality (2.7) and the left inequality in (2.11) we obtain

$$\begin{aligned}\int_{W_i} L_t(x,\tau\nabla_\mu w_t^i,\omega)d\mu &\leqslant \beta(1+\gamma+\gamma^2)\mu(W_i)\\ &\quad+\frac{\beta\gamma^2}{\alpha}\left(\int_{W_i} L_t(x,\nabla_\mu u_t,\omega)d\mu+\int_{W_i} L_t(x,\nabla_\mu u,\omega)d\mu\right)\\ &\quad+\beta\gamma\int_{W_i} G\left(\frac{\tau}{1-\tau}D_\mu\varphi_i\otimes(u_t-u)\right)d\mu. \end{aligned}\tag{4.30}$$

Remark 4.5. Since $u\in\mathfrak{G}$, from (2.11) we see that $\overline{\lim}_{t\to\infty}\int_E L_t(x,\nabla_\mu u,\omega)d\mu<\infty$ for all $E\in\mathcal{O}(\Omega)$.

On the other hand, we have

$$\left|\frac{\tau}{1-\tau}D_\mu\varphi_i(x)\otimes(u_t(x)-u(x))\right|\leqslant\left|\frac{\tau}{1-\tau}\right|\|D_\mu\varphi_i\|_{L^\infty_\mu(\Omega)}\|u_t-u\|_{L^\infty_\mu(\Omega;\mathbb{R}^m)}$$

for μ-a.a. $x\in\Omega$. But $\lim_{t\to\infty}\|u_t-u\|_{L^\infty_\mu(\Omega;\mathbb{R}^m)}=0$ by (4.26), hence for each $i\in\{1,\cdots,q\}$ there exists $t_i>0$ such that

$$\left|\frac{\tau}{1-\tau}D_\mu\varphi_i(x)\otimes(u_t(x)-u(x))\right|\leqslant r$$

for μ-a.a. $x\in\Omega$ and all $t\geqslant t_i$ with $r>0$ given by (2.6). Hence

$$\begin{aligned}\int_{W_i} G\left(\frac{\tau}{1-\tau}D_\mu\varphi_i\otimes(u_t-u)\right)d\mu &\leqslant \int_{W_i}\sup_{|\xi|\leqslant r}G(\xi)d\mu\\ &= \mu(W_i)\sup_{|\xi|\leqslant r}G(\xi)\end{aligned}\tag{4.31}$$

for all $t \geqslant T_q$ with $T_q = \max\{t_i : i \in \{1, \cdots, q\}\}$. Moreover, we have:

$$\begin{aligned}\int_A L_t(x, \tau\nabla_\mu u_t, \omega)d\mu \;&\leqslant\; \int_A L_t(x, \nabla_\mu u_t, \omega)d\mu \\ &\quad +\Delta_\omega(\tau)\left(\int_A a_t(x,\omega)d\mu + \int_A L_t(x, \nabla_\mu u_t, \omega)d\mu\right); \qquad (4.32)\\ \int_{A\cap W_i^+} L_t(x, \tau\nabla_\mu u, \omega)d\mu \;&\leqslant\; \int_{A\cap W_i^+} L_t(x, \nabla_\mu u, \omega)d\mu \\ &\quad +\Delta_\omega(\tau)\left(\int_{A\cap W_i^+} a_t(x,\omega)d\mu + \int_{A\cap W_i^+} L_t(x, \nabla_\mu u, \omega)d\mu\right). \qquad (4.33)\end{aligned}$$

Taking (4.31) into account and substituting (4.30), (4.32) and (4.33) into (4.28) and then averaging these inequalities, it follows that for every $q \geqslant 1$ and every $t \geqslant T_q$, there exists $i_{t,q} \in \{1, \cdots, q\}$ such that

$$\begin{aligned}\int_A L_t(x, \nabla_\mu(\tau w_t^{i_{t,q}}), \omega)d\mu \;\leqslant\; &\int_A L_t(x, \nabla_\mu u_t, \omega)d\mu \\ &+\Delta_\omega(\tau)\left(\int_A a_t(x,\omega)d\mu + \int_A L_t(x, \nabla_\mu u_t, \omega)d\mu\right) \\ &+\frac{1}{q}\int_A L_t(x, \nabla_\mu u, \omega)d\mu \\ &+\frac{1}{q}\Delta_\omega(\tau)\left(\int_A a_t(x,\omega)d\mu + \int_A L_t(x, \nabla_\mu u, \omega)d\mu\right) \\ &+\frac{c}{q}\mu(A)\sup_{|\xi|\leqslant r} G(\xi) \\ &+\frac{c}{q}\left(\int_A L_t(x, \nabla_\mu u_t, \omega)d\mu + \int_A L_t(x, \nabla_\mu u, \omega)d\mu\right)\end{aligned}$$

with $c = \max\left\{\beta(1+\gamma+\gamma^2)+1, \frac{\beta\gamma^2}{\alpha}\right\}$, where $\overline{\lim}_{t\to\infty}\int_A a_t(x,\omega)d\mu < \infty$ by (2.13) and $\sup_{|\xi|\leqslant r} G(\xi) < \infty$ by (2.6). Thus, letting $t \to \infty$ and $q \to \infty$ and using (4.25), we get

$$\overline{\lim_{q\to\infty}}\,\overline{\lim_{t\to\infty}}\int_A L_t(x, \nabla_\mu(\tau w_t^{i_{t,q}}), \omega)d\mu \leqslant \big(1+\Delta_\omega(\tau)\big)\mathcal{S}^-_{u,\omega}(A) + \overline{\lim_{t\to\infty}}\,\Delta_\omega(\tau)\int_A a_t(x,\omega)d\mu. \quad (4.34)$$

On the other hand, taking (4.27) into account and using (4.24) we see that

$$\lim_{q\to\infty}\lim_{t\to\infty}\|\tau w_t^{i_{t,q}} - \tau u\|_{L^p_\mu(\Omega;\mathbb{R}^m)} = 0.$$

By diagonalization, there exists an increasing mapping $t \mapsto q_t$ with $q_t \to \infty$ such that:

$$\begin{aligned}&\underline{\lim_{t\to\infty}}\int_A L_t(x, \nabla_\mu \hat{w}_t, \omega)d\mu \leqslant \overline{\lim_{t\to\infty}}\int_A L_t(x, \nabla_\mu \hat{w}_t, \omega)d\mu \leqslant \overline{\lim_{q\to\infty}}\,\overline{\lim_{t\to\infty}}\int_A L_t(x, \nabla_\mu(\tau w_t^{i_{t,q}}), \omega)d\mu;\\ &\lim_{t\to\infty}\|\hat{w}_t - u\|_{L^p_\mu(\Omega;\mathbb{R}^m)} = 0,\end{aligned}$$

where $\hat{w}_t := \tau w_t^{i_{t,q_t}}$ is such that $\hat{w}_t - \tau u \in H^{1,p}_{\mu,0}(A;\mathbb{R}^m)$. Hence

$$E^-_{0,\omega}(\tau u, A) \leqslant \overline{\lim_{q\to\infty}}\, \overline{\lim_{t\to\infty}} \int_A L_t(x, \nabla_\mu(\tau w_t^{i_{t,q}}), \omega) d\mu,$$

and (4.21) follows from (4.34).
As $\overline{\lim}_{\tau\to 1^-} \Delta_\omega(\tau) \leqslant 0$ we have $\overline{\lim}_{\tau\to 1^-} \overline{\lim}_{t\to\infty} \Delta_\omega(\tau) \int_A a_t(x,\omega) d\mu(x) \leqslant 0$, and so from (4.21) we deduce that

$$\overline{\lim_{\tau\to 1^-}}\, E^-_{0,\omega}(\tau u, A) \leqslant \mathcal{S}^+_{u,\omega}(A).$$

Moreover, from (4.19) we have

$$\mathcal{S}^+_{u,\omega}(A) \leqslant \underline{\lim_{\tau\to 1^-}}\, \mathcal{S}^+_{\tau u,\omega}(A) \leqslant \underline{\lim_{t\to 1^-}}\, E^-_{0,\omega}(\tau u, A),$$

which gives (4.23). ■

Step 3: using the Vitali envelope. For each $u \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ we consider the set functions $\breve{\mathrm{m}}^-_{u,\omega}, \breve{\mathrm{m}}^+_{u,\omega} : \mathcal{O}(\Omega) \to [0,\infty]$ defined by:

$$\begin{aligned} \underline{\breve{\mathrm{m}}}_{u,\omega}(A) &:= \overline{\lim_{\tau\to 1^-}}\, \underline{\mathrm{m}}_{\tau u,\omega}(A); \\ \overline{\breve{\mathrm{m}}}_{u,\omega}(A) &:= \overline{\lim_{\tau\to 1^-}}\, \overline{\mathrm{m}}_{\tau u,\omega}(A). \end{aligned} \tag{4.35}$$

where, for each $z \in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$, $\underline{\mathrm{m}}_{z,\omega}, \overline{\mathrm{m}}_{z,\omega} : \mathcal{O}(\Omega) \to [0,\infty]$ are given by:

$$\begin{aligned} \underline{\mathrm{m}}_{z,\omega}(A) &:= \underline{\lim_{t\to\infty}} \inf\Big\{ E_t(v,A,\omega) : v - z \in H^{1,p}_{\mu,0}(A;\mathbb{R}^m)\Big\}; \\ \overline{\mathrm{m}}_{z,\omega}(A) &:= \overline{\lim_{t\to\infty}} \inf\Big\{ E_t(v,A,\omega) : v - z \in H^{1,p}_{\mu,0}(A;\mathbb{R}^m)\Big\}. \end{aligned}$$

For each $\varepsilon > 0$ and each $A \in \mathcal{O}(\Omega)$, we denote the class of countable families $\{Q_i := Q_{\rho_i}(x_i)\}_{i\in I}$ of disjoint open balls of A with $x_i \in A$ and $\rho_i = \operatorname{diam}(Q_i) \in]0,\varepsilon[$ such that $\mu(A \setminus \cup_{i\in I} Q_i) = 0$ by $\mathcal{V}_\varepsilon(A)$, we consider $\overline{\breve{\mathrm{m}}}^\varepsilon_{u,\omega} : \mathcal{O}(\Omega) \to [0,\infty]$ given by

$$\overline{\breve{\mathrm{m}}}^\varepsilon_{u,\omega}(A) := \inf\left\{ \sum_{i\in I} \overline{\breve{\mathrm{m}}}_{u,\omega}(Q_i) : \{Q_i\}_{i\in I} \in \mathcal{V}_\varepsilon(A)\right\},$$

and we define $\overline{\breve{\mathrm{m}}}^*_{u,\omega} : \mathcal{O}(\Omega) \to [0,\infty]$ by

$$\overline{\breve{\mathrm{m}}}^*_{u,\omega}(A) := \sup_{\varepsilon>0} \overline{\breve{\mathrm{m}}}^\varepsilon_{u,\omega}(A) = \lim_{\varepsilon\to 0} \overline{\breve{\mathrm{m}}}^\varepsilon_{u,\omega}(A).$$

The set function $\overline{\breve{\mathrm{m}}}^*_{u,\omega}$ is called the Vitali envelope of $\overline{\breve{\mathrm{m}}}_{u,\omega}$, see §3.5 for more details.

Remark 4.6. For any $\{Q_i\}_{i\in I} \in \mathcal{V}_\varepsilon(A)$, as the annular decay property, see (2.4), holds we have $\mu(\partial Q_i) = 0$ for all $i \in I$, see Remark 2.2.

Remark 4.7. As Ω satisfies the Vitali covering theorem, see Proposition 3.6(c), we have $\mathcal{V}_\varepsilon(A) \neq \varnothing$ for all $A \in \mathcal{O}(\Omega)$ and all $\varepsilon > 0$.

Step 3 consists of proving the following lemma.

Lemma 4.8. *If* (2.5), (2.6), (2.7), (2.11), (2.12) *and* (2.13) *hold then*:

$$\Gamma(L^p_\mu)\text{-}\varliminf_{t\to\infty} E_t(u,A,\omega) \geqslant \breve{\underline{\mathrm{m}}}_{u,\omega}(A); \tag{4.36}$$

$$\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,A,\omega) = \breve{\overline{\mathrm{m}}}^{\,*}_{u,\omega}(A) \tag{4.37}$$

for all $u \in \mathfrak{G}$ *and all* $A \in \mathcal{O}(\Omega)$.

Proof of Lemma 4.8. Fix $u \in \mathfrak{G}$. Given any $A \in \mathcal{O}(\Omega)$, it is easy to see that:

$$\underline{\mathrm{m}}_{\tau u,\omega}(A) \leqslant E^-_{0,\omega}(\tau u, A);$$
$$\overline{\mathrm{m}}_{\tau u,\omega}(A) \leqslant E^+_{0,\omega}(\tau u, A)$$

for all $\tau \in]0,1[$, hence:

$$\breve{\underline{\mathrm{m}}}_{u,\omega}(A) = \overline{\lim_{\tau\to 1^-}}\, \underline{\mathrm{m}}_{\tau u,\omega}(A) \leqslant \lim_{\tau\to 1^-} E^-_{0,\omega}(\tau u, A) = \Gamma(L^p_\mu)\text{-}\varliminf_{t\to\infty} E_t(u,A,\omega);$$
$$\breve{\overline{\mathrm{m}}}_{u,\omega}(A) = \overline{\lim_{\tau\to 1^-}}\, \overline{\mathrm{m}}_{\tau u,\omega}(A) \leqslant \lim_{\tau\to 1^-} E^+_{0,\omega}(\tau u, A) = \Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,A,\omega)$$

by Lemma 4.4, and consequently

$$\breve{\overline{\mathrm{m}}}^{\,*}_{u,\omega}(A) \leqslant \Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,A,\omega)$$

because in the proof of Lemma 4.4 it is established that $\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,\cdot,\omega)$ can be uniquely extended to a finite positive Radon measure on Ω, see Remark 4.2. Hence (4.36) holds and, to establish (4.37), it remains to prove that

$$\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,A,\omega) \leqslant \breve{\overline{\mathrm{m}}}^{\,*}_{u,\omega}(A) \tag{4.38}$$

with $\breve{\overline{\mathrm{m}}}^{\,*}_{u,\omega}(A) < \infty$. Fix any $\varepsilon > 0$. By definition of $\breve{\overline{\mathrm{m}}}^{\,\varepsilon}_{u,\omega}(A)$ there exists $\{Q_i\}_{i\in I} \in \mathcal{V}_\varepsilon(A)$ such that

$$\sum_{i\in I} \breve{\overline{\mathrm{m}}}_{u,\omega}(Q_i) \leqslant \breve{\overline{\mathrm{m}}}^{\,\varepsilon}_{u,\omega}(A) + \frac{\varepsilon}{2}. \tag{4.39}$$

Fix any $t > 0$ and define $\mathrm{m}^t_{u,\omega} : \mathcal{O}(\Omega) \to [0,\infty]$ by

$$\mathrm{m}^t_{u,\omega}(U) := \inf\Big\{E_t(v,U,\omega) : v - z \in H^{1,p}_{\mu,0}(U;\mathbb{R}^m)\Big\}. \tag{4.40}$$

(Thus $\overline{\mathrm{m}}_{u,\omega}(\cdot) = \overline{\lim}_{t\to\infty} \mathrm{m}^t_{u,\omega}(\cdot)$.) Fix any $\tau \in]0,1[$. For each $i \in I$, by definition of $\mathrm{m}^{\mathrm{t}}_{\tau\mathrm{u},\omega}(\mathrm{Q_i})$ there exists $v^i_{t,\tau} \in H^{1,p}_\mu(Q_i;\mathbb{R}^m)$ such that $v^i_{t,\tau} - \tau u \in H^{1,p}_{\mu,0}(Q_i;\mathbb{R}^m)$ and

$$E(v^i_{t,\tau}, Q_i, \omega) \leqslant \mathrm{m}^t_{\tau u,\omega}(Q_i) + \frac{\varepsilon\mu(Q_i)}{2\mu(A)}. \tag{4.41}$$

Define $u^\varepsilon_{t,\tau} : \Omega \to \mathbb{R}^m$ by

$$u^\varepsilon_{t,\tau} := \begin{cases} \tau u & \text{in } \Omega\backslash A \\ v^i_{t,\tau} & \text{in } Q_i. \end{cases}$$

Then $u^\varepsilon_{t,\tau} - \tau u \in H^{1,p}_{\mu,0}(A;\mathbb{R}^m)$. Moreover, because of Proposition 3.6(a), $\nabla_\mu u^\varepsilon_{t,\tau}(x) = \nabla_\mu v^i_{t,\tau}(x)$ for μ-a.e. $x \in Q_i$. From (4.41) we see that

$$E_t(u^\varepsilon_{t,\tau}, A, \omega) \leqslant \sum_{i\in I} \mathrm{m}^t_{\tau u,\omega}(Q_i) + \frac{\varepsilon}{2},$$

hence $\overline{\lim}_{\tau\to 1^-} \overline{\lim}_{t\to\infty} E(u^\varepsilon_{t,\tau}, A, \omega) \leqslant \overline{\mathrm{m}}^\varepsilon_{u,\omega}(A) + \varepsilon$ by using (4.39), and consequently

$$\overline{\lim_{\varepsilon\to 0}}\ \overline{\lim_{\tau\to 1^-}}\ \overline{\lim_{t\to\infty}} E_t(u^\varepsilon_{t,\tau}, A, \omega) \leqslant \overline{\mathrm{m}}^{*}_{u,\omega}(A). \tag{4.42}$$

On the other hand, we have

$$\begin{aligned}
\|u^\varepsilon_{t,\tau} - u\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)} &\leqslant 2^p \left(\|u^\varepsilon_{t,\tau} - \tau u\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)} + \|\tau u - u\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)} \right) \\
&= 2^p \left(\int_A |u^\varepsilon_{t,\tau} - \tau u|^p d\mu + (1-\tau)^p \|u\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)} \right) \\
&= 2^p \left(\sum_{i\in I} \int_{Q_i} |v^i_{t,\tau} - \tau u|^p d\mu + (1-\tau)^p \|u\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)} \right).
\end{aligned}$$

As Ω supports a p-Sobolev inequality, see Proposition 3.6(b), and $\mathrm{diam}(Q_i) \in]0,\varepsilon[$ for all $i \in I$, we have

$$\sum_{i\in I} \int_{Q_i} |v^i_{t,\tau} - \tau u|^p d\mu \leqslant \varepsilon^p C_S^p \sum_{i\in I} \int_{Q_i} |\nabla_\mu v^i_{t,\tau} - \tau \nabla_\mu u|^p d\mu$$

with $C_S > 0$ given by (3.3), hence

$$\sum_{i\in I} \int_{Q_i} |v^i_{t,\tau} - \tau u|^p d\mu \leqslant 2^p \varepsilon^p C_S^p \left(\sum_{i\in I} \int_{Q_i} |\nabla_\mu v^i_{t,\tau}|^p d\mu + \tau^p \int_A |\nabla_\mu u|^p d\mu \right),$$

and consequently

$$\begin{aligned}
\|u^\varepsilon_{t,\tau} - u\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)} &\leqslant 2^{2p} \varepsilon^p C_S^p \left(\sum_{i\in I} \int_{Q_i} |\nabla_\mu v^i_{t,\tau}|^p d\mu + \tau^p \int_A |\nabla_\mu u|^p d\mu \right) \\
&\quad + 2^p (1-\tau)^p \|u\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)}.
\end{aligned} \tag{4.43}$$

Taking (2.5), the left inequality in (2.11), (4.39) and (4.41) into account, from (4.43) we deduce that

$$\overline{\lim_{\tau\to 1^-}}\ \overline{\lim_{t\to\infty}} \|u^\varepsilon_{t,\tau} - u\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)} \leqslant 2^{2p} C_S^p \varepsilon^p \left(\frac{1}{\alpha c} (\overline{\mathrm{m}}^\varepsilon_{u,\omega}(A) + \varepsilon) + \int_A |\nabla_\mu u|^p d\mu \right),$$

which gives

$$\lim_{\varepsilon\to 0}\ \overline{\lim_{\tau\to 1^-}}\ \overline{\lim_{t\to\infty}} \|u^\varepsilon_{t,\tau} - u\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)} = 0 \tag{4.44}$$

because $\lim_{\varepsilon\to 0} \overline{\mathrm{m}}^\varepsilon_{u,\omega}(A) = \overline{\mathrm{m}}^{*}_{u,\omega}(A) < \infty$. According to (4.42) and (4.44), by diagonalization there exist mappings $t \mapsto \tau_t$ and $t \to \varepsilon_t$, with $\tau_t \to 1^-$ and $\varepsilon_t \to 0$ as $t \to \infty$, such that:

$$\lim_{t\to\infty} \|w_t - u\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)} = 0; \tag{4.45}$$

$$\overline{\lim_{t\to\infty}} E_t(w_t, A, \omega) \leqslant \overline{\mathrm{m}}^{*}_{u,\omega}(A) \tag{4.46}$$

with $w_t := u^{\varepsilon_t}_{t,\tau_t}$. By (4.45) we have $\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,A,\omega) \leqslant \overline{\lim}_{t\to\infty} E_t(w_t,A,\omega)$, and inequality (4.38) follows from (4.46). ■

Step 4: differentiation with respect to $\boldsymbol{\mu}$. Using Lemma 4.1, Remark 4.2 and Lemma 4.8, it is easily seen that

$$\Gamma(L^p_\mu)\text{-}\underline{\lim}_{t\to\infty} E_t(u,A,\omega) \geqslant \int_A \overline{\lim_{\rho\to 0}} \frac{\breve{\underline{\mathrm{m}}}_{u,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))} d\mu(x) = \int_A \overline{\lim_{\rho\to 0}}\, \overline{\lim_{\tau\to 1^-}} \frac{\underline{\mathrm{m}}_{\tau u,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))} d\mu(x); \quad (4.47)$$

$$\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}} E_t(u,A,\omega) = \int_A \lim_{\rho\to 0} \frac{\breve{\overline{\mathrm{m}}}^*_{u,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))} d\mu(x) \quad (4.48)$$

for all $u \in \mathfrak{G}$ and all $A \in \mathcal{O}(\Omega)$. The goal of Step 4 is to apply Theorem 3.29 (with $\Theta = \breve{\overline{\mathrm{m}}}_{u,\omega}$ where $u \in \mathfrak{G}$) for proving the following lemma.

Lemma 4.9. *If* (2.5), (2.6), (2.7), (2.11), (2.12) *and* (2.13) *hold then*

$$\breve{\overline{\mathrm{m}}}^*_{u,\omega}(A) = \int_A \lim_{\rho\to 0} \frac{\breve{\overline{\mathrm{m}}}_{u,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))} d\mu(x) \quad (4.49)$$

for all $u \in \mathfrak{G}$ and all $A \in \mathcal{O}(\Omega)$. As a consequence, we have

$$\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}} E_t(u,A,\omega) = \int_A \lim_{\rho\to 0} \frac{\breve{\overline{\mathrm{m}}}_{u,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))} d\mu(x) = \int_A \lim_{\rho\to 0}\, \overline{\lim_{\tau\to 1^-}} \frac{\overline{\mathrm{m}}_{\tau u,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))} d\mu(x) \quad (4.50)$$

for all $u \in \mathfrak{G}$ and all $A \in \mathcal{O}(\Omega)$.

Proof of Lemma 4.9. Fix $u \in \mathfrak{G}$. The integral representation of $\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,\cdot,\omega)$ in (4.50) follows from (4.49), (4.48) and the definition of $\breve{\overline{\mathrm{m}}}_{u,\omega}$ in (4.35). So, we only need to establish (4.49). For this, it is sufficient to prove that $\breve{\overline{\mathrm{m}}}_{u,\omega}$ is subadditive and there exists a finite Radon measure ν on Ω which is absolutely continuous with respect to μ such that

$$\breve{\overline{\mathrm{m}}}_{u,\omega}(A) \leqslant \nu(A) \quad (4.51)$$

for all $A \in \mathcal{O}(\Omega)$, and then to apply Theorem 3.29. For each $t > 0$ and each $\tau \in]0,1[$, from the definition of $\mathrm{m}^t_{\tau u,\omega}$ in (4.40), it is easy to see that for every $A, B, C \in \mathcal{O}(\Omega)$ with $B, C \subset A$, $B \cap C = \varnothing$ and $\mu(A \backslash B \cup C) = 0$,

$$\mathrm{m}^t_{\tau u,\omega}(A) \leqslant \mathrm{m}^t_{\tau u,\omega}(B) + \mathrm{m}^t_{\tau u,\omega}(C),$$

and so

$$\overline{\lim_{\tau\to 1^-}}\, \overline{\lim_{t\to\infty}} \mathrm{m}^t_{\tau u,\omega}(A) \leqslant \overline{\lim_{\tau\to 1^-}}\, \overline{\lim_{t\to\infty}} \mathrm{m}^t_{\tau u,\omega}(B) + \overline{\lim_{\tau\to 1^-}}\, \overline{\lim_{t\to\infty}} \mathrm{m}^t_{\tau u,\omega}(C),$$

i.e.

$$\breve{\overline{\mathrm{m}}}_{u,\omega}(A) \leqslant \breve{\overline{\mathrm{m}}}_{u,\omega}(B) + \breve{\overline{\mathrm{m}}}_{u,\omega}(C), \quad (4.52)$$

which shows the subadditivity of $\breve{\overline{\mathrm{m}}}_{u,\omega}$.

Remark 4.10. As, in general, the limit inf of the sum is not smaller than the sum of the limit inf, we cannot assert that (4.52) holds for $\breve{\underline{\mathrm{m}}}_{u,\omega}$ instead of $\breve{\overline{\mathrm{m}}}_{u,\omega}$ and so that $\breve{\underline{\mathrm{m}}}_{u,\omega}$ is subadditive.

On the other hand, given any $t>0$ and any $\tau\in]0,1[$, by using the right inequality in (2.11) we have

$$\mathrm{m}^t_{\tau u,\omega}(A)\leqslant \beta\mu(A)+\beta\int_A G(\tau\nabla_\mu u(x))d\mu(x).$$

But, from (2.7) we see that $G(\tau\nabla_\mu u(x))\leqslant\gamma(1+G(\nabla_\mu u(x))+G(0))$ for μ-a.a. $x\in\Omega$, hence

$$\begin{aligned}\mathrm{m}^t_{\tau u,\omega}(A) &\leqslant \beta\mu(A)+\beta\gamma\mu(A)+\beta\gamma\left(\int_A G(\nabla_\mu u(x))d\mu(x)+\mu(A)G(0)\right)\\ &\leqslant \beta\mu(A)+\beta\gamma\mu(A)+\beta\gamma\mu(A)G(0)+\beta\gamma\int_A G(\nabla_\mu u(x))d\mu(x).\end{aligned}$$

Letting $t\to\infty$ and $\tau\to1^-$ we conclude that

$$\breve{\overline{\mathrm{m}}}_{u,\omega}(A)\leqslant c\left(\mu(A)+\int_A G(\nabla_\mu u(x))d\mu(x)\right)$$

with $c:=\beta(1+\gamma+\gamma G(0))$. Thus (4.51) holds with the Radon measure $\nu:=c\big(1+G(\nabla_\mu u(\cdot))\big)\mu$ which is necessarily finite since $u\in\mathfrak{G}$ and $G(0)<\infty$ by (2.6). ■

Step 5: establishing the Γ-limit inf and the Γ-limit sup formulas. According to (4.47) and (4.50), the proof of Theorem 2.11 will be completed (see Substep 5-2) if we prove that for each $u\in\mathfrak{G}$ and μ-a.e. $x\in\Omega$, we have

$$\varliminf_{\tau\to1^-}\varlimsup_{\rho\to0}\frac{\mathrm{m}_{\tau u_x,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))}\leqslant\varlimsup_{\rho\to0}\frac{\breve{\overline{\mathrm{m}}}_{u,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))};\tag{4.53}$$

and

$$\varliminf_{\tau\to1^-}\lim_{\rho\to0}\frac{\overline{\mathrm{m}}_{\tau u_x,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))}=\lim_{\rho\to0}\frac{\breve{\overline{\mathrm{m}}}_{u,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))},\tag{4.54}$$

i.e.

$$\varliminf_{\tau\to1^-}\varlimsup_{\rho\to0}\frac{\overline{\mathrm{m}}_{\tau u_x,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))}\leqslant\lim_{\rho\to0}\frac{\breve{\overline{\mathrm{m}}}_{u,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))};\tag{4.55}$$

$$\varliminf_{\tau\to1^-}\varlimsup_{\rho\to0}\frac{\overline{\mathrm{m}}_{\tau u_x,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))}\geqslant\lim_{\rho\to0}\frac{\breve{\overline{\mathrm{m}}}_{u,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))},\tag{4.56}$$

where $u_x\in H^{1,p}_\mu(\Omega;\mathbb{R}^m)$ is given by Proposition 3.6(d) (and satisfies (3.4) and (3.5)).

Substep 5-1: proofs of (4.53), (4.55) and (4.56). We only give the proof of (4.53). As the proofs of (4.55) and (4.56) use the same method, the details are left to the reader.
First of all, by diagonalization there exists a mapping $\sigma\mapsto\tau_\sigma$ with $\tau_\sigma\to1^-$ as $\sigma\to1^-$ such that:

$$\begin{aligned}&\lim_{\sigma\to1^-}\frac{\tau_\sigma}{\sigma}=1;\\ &\varlimsup_{\tau\to1^-}\varlimsup_{\sigma\to1^-}\Delta_\omega\left(\frac{\tau}{\sigma}\right)\leqslant\varlimsup_{\sigma\to1^-}\Delta_\omega\left(\frac{\tau_\sigma}{\sigma}\right),\end{aligned}$$

where $\Delta_\omega(\cdot) := \sup_{s>0} \Delta_{L_s}^{a_s(\cdot,\omega)}(\cdot)$ with $\{a_s(\cdot,\omega)\}_{s>0} \subset L^1_\mu(\Omega;]0,\infty])$ given by (2.12). But $\overline{\lim}_{r\to 1^-} \Delta_\omega(r) \leqslant 0$, hence

$$\overline{\lim_{\tau\to 1^-}}\ \overline{\lim_{\sigma\to 1^-}}\ \Delta_\omega\left(\frac{\tau}{\sigma}\right) \leqslant 0. \tag{4.57}$$

Fix any $\varepsilon > 0$. For each $\tau \in]0,1[$ there exists $\sigma_\tau \in]\tau,1[$ such that

$$\Delta_\omega\left(\frac{\tau}{\sigma}\right) \leqslant \overline{\lim_{\sigma\to 1^-}}\ \Delta_\omega\left(\frac{\tau}{\sigma}\right) + \frac{\varepsilon}{2} \tag{4.58}$$

for all $\sigma \in [\sigma_\tau, 1[$. In the same way, there exists $\tau_0 \in]0,1[$ such that

$$\overline{\lim_{\sigma\to 1^-}}\ \Delta_\omega\left(\frac{\tau}{\sigma}\right) \leqslant \overline{\lim_{\tau\to 1^-}}\ \overline{\lim_{\sigma\to 1^-}}\ \Delta_\omega\left(\frac{\tau}{\sigma}\right) + \frac{\varepsilon}{2} \tag{4.59}$$

for all $\tau \in [\tau_0, 1[$, and from (4.57), (4.58) and (4.59) we deduce that

$$\Delta_\omega\left(\frac{\tau}{\sigma}\right) \leqslant \varepsilon \tag{4.60}$$

for all $\tau \in [\tau_0, 1[$ and all $\sigma \in [\sigma_\tau, 1[$.
Fix $u \in \mathfrak{G}$. Fix any $t > 0$, any $\lambda \in]0,1[$, any $\rho > 0$, any $\tau \in [\tau_0, 1[$ and any $\sigma \in [\sigma_\tau, 1[$. By definition of $\mathrm{m}^t_{\sigma u,\omega}(Q_{\lambda\rho}(x))$ in (4.40), there exists $w : \Omega \to \mathbb{R}^m$ such that $w - \sigma u \in H^{1,p}_{\mu,0}(Q_{\lambda\rho}(x);\mathbb{R}^m)$ and

$$\int_{Q_{\lambda\rho}(x)} L_t(y, \nabla_\mu w(y), \omega) d\mu(y) \leqslant \mathrm{m}^t_{\sigma u,\omega}(Q_{\lambda\rho}(x)) + \varepsilon\mu(Q_{\lambda\rho}(x)). \tag{4.61}$$

By Proposition 3.6(e) there is a Urysohn function $\varphi \in \mathrm{Lip}(\Omega)$ for the pair $(\Omega\backslash Q_\rho(x), \overline{Q}_{\lambda\rho}(x))$ such that

$$\|D_\mu\varphi\|_{L^\infty_\mu(\Omega;\mathbb{R}^N)} \leqslant \frac{\theta}{\rho(1-\lambda)} \tag{4.62}$$

for some $\theta > 0$ (which does not depend on ρ). Define $v \in H^{1,p}_\mu(Q_\rho(x);\mathbb{R}^m)$ by

$$v := \varphi\frac{\tau}{\sigma}u + (1-\varphi)\frac{\tau}{\sigma}u_x.$$

Then $v - \frac{\tau}{\sigma}u_x \in H^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m)$. Using Theorem 3.3(d) and (3.1) we have

$$\begin{aligned}
\nabla_\mu(\sigma v) &= \begin{cases} \nabla_\mu(\tau u) & \text{in } \overline{Q}_{\lambda\rho}(x) \\ \tau D_\mu\varphi \otimes (u - u_x) + \sigma\big(\varphi\frac{\tau}{\sigma}\nabla_\mu u + (1-\varphi)\frac{\tau}{\sigma}\nabla_\mu u(x)\big) & \text{in } Q_\rho(x)\backslash\overline{Q}_{\lambda\rho}(x) \end{cases} \\
&= \begin{cases} \nabla_\mu(\tau u) & \text{in } \overline{Q}_{\lambda\rho}(x) \\ (1-\tau)\frac{\tau}{1-\tau}D_\mu\varphi \otimes (u - u_x) + \tau\big(\varphi\nabla_\mu u + (1-\varphi)\nabla_\mu u(x)\big) & \text{in } Q_\rho(x)\backslash\overline{Q}_{\lambda\rho}(x). \end{cases}
\end{aligned}$$

As $\frac{\tau}{\sigma}w - \tau u \in H^{1,p}_{\mu,0}(Q_{\lambda\rho}(x);\mathbb{R}^m)$ we have $\sigma v + (\frac{\tau}{\sigma}w - \tau u) - \tau u_x \in H^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m)$. Noticing that $\mu(\partial Q_{\lambda\rho}(x)) = 0$ (see Remark 2.2) and, because of Proposition (3.6)(a), $\nabla_\mu(\frac{\tau}{\sigma}w - \tau u)(y) = 0$

for μ-a.a. $y \in Q_\rho(x)\backslash\overline{Q}_{\lambda\rho}(x)$, we see that

$$\begin{aligned}
\frac{\mathrm{m}^t_{\tau u_x,\omega}(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))} &\leqslant \frac{1}{\mu(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)} L_t\left(y,\nabla_\mu(\sigma v)+\nabla_\mu\Big(\frac{\tau}{\sigma}w-\tau u\Big),\omega\right)d\mu\\
&= \frac{1}{\mu(Q_{\lambda\rho}(x))}\int_{\overline{Q}_{\lambda\rho}(x)} L_t\left(y,\nabla_\mu(\tau u)+\nabla_\mu\Big(\frac{\tau}{\sigma}w-\tau u\Big),\omega\right)d\mu\\
&\quad+\frac{1}{\mu(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)\backslash\overline{Q}_{\lambda\rho}(x)} L_t(y,\nabla_\mu(\sigma v),\omega)d\mu\\
&= \frac{1}{\mu(Q_{\lambda\rho}(x))}\int_{Q_{\lambda\rho}(x)} L_t\left(y,\frac{\tau}{\sigma}\nabla_\mu w,\omega\right)d\mu\\
&\quad+\frac{1}{\mu(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)\backslash Q_{\lambda\rho}(x)} L_t(y,\nabla_\mu(\sigma v),\omega)d\mu.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\mathrm{m}^t_{\tau u_x,\omega}(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))} \leqslant& \frac{1}{\mu(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)} L_t(y,\nabla_\mu w,\omega)d\mu\\
&+\Delta_\omega\left(\frac{\tau}{\sigma}\right)\left(\frac{\mu(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))}\fint_{Q_\rho(x)} a_t(y,\omega)d\mu+\frac{1}{\mu(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)} L_t(y,\nabla_\mu w,\omega)d\mu\right)\\
&+\frac{1}{\mu(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)\backslash Q_{\lambda\rho}(x)} L_t(y,\nabla_\mu(\sigma v),\omega)d\mu.
\end{aligned}$$

Taking (4.61), (2.7) and the right inequality in (2.11) into account we deduce that

$$\begin{aligned}
\frac{\mathrm{m}^t_{\tau u_x,\omega}(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))} \leqslant& \left(1+\Delta_\omega\left(\frac{\tau}{\sigma}\right)\right)\left(\frac{\mathrm{m}^t_{\sigma u,\omega}(Q_{\lambda\rho}(x))}{\mu(Q_{\lambda\rho}(x))}+\varepsilon\right)\\
&+\Delta_\omega\left(\frac{\tau}{\sigma}\right)\frac{\mu(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))}\fint_{Q_\rho(x)} a_t(y,\omega)d\mu\\
&+\frac{c}{\mu(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)\backslash Q_{\lambda\rho}(x)} G\left(\frac{\tau}{1-\tau}D_\mu\varphi\otimes(u-u_x)\right)d\mu\\
&+\frac{c}{\mu(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)\backslash Q_{\lambda\rho}(x)} \big(G(\nabla_\mu u(y))+G(\nabla_\mu u(x))\big)d\mu\\
&+c\left(\frac{\mu(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))}-1\right)
\end{aligned}$$

with $c := \beta + \beta\gamma + \beta\gamma^2$, where $\gamma > 0$ and $\beta > 0$ given by (2.7) and (2.11) respectively. Thus, taking (4.60) into account, noticing that $\mu(Q_\rho(x)) \geqslant \mu(Q_{\lambda\rho}(x))$, we obtain

$$\begin{aligned}\frac{\mathrm{m}^t_{\tau u_x,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))} &\leqslant (1+\varepsilon)\left(\frac{\mathrm{m}^t_{\sigma u,\omega}(Q_{\lambda\rho}(x))}{\mu(Q_{\lambda\rho}(x))}+\varepsilon\right)\\ &\quad+\varepsilon\frac{\mu(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))}\fint_{Q_\rho(x)} a_t(y,\omega)d\mu\\ &\quad+\frac{c}{\mu(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)\setminus Q_{\lambda\rho}(x)} G\left(\frac{\tau}{1-\tau}D_\mu\varphi\otimes(u-u_x)\right)d\mu\\ &\quad+\frac{c}{\mu(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)\setminus Q_{\lambda\rho}(x)} G(\nabla_\mu u(y))d\mu\\ &\quad+c\left(\frac{\mu(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))}-1\right)G(\nabla_\mu u(x))\\ &\quad+c\left(\frac{\mu(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))}-1\right). \end{aligned}\tag{4.63}$$

On the other hand, by (4.62) we have

$$\begin{aligned}\left|\frac{\tau}{1-\tau}D_\mu\varphi(y)\otimes(u(y)-u_x(y))\right| &\leqslant \left|\frac{\tau}{1-\tau}\right|\|D_\mu\varphi\|_{L^\infty_\mu(\Omega)}\|u-u_x\|_{L^\infty_\mu(Q_\rho(x);\mathbb{R}^m)}\\ &\leqslant \frac{\tau\theta}{(1-\tau)(1-\lambda)}\frac{1}{\rho}\|u-u_x\|_{L^\infty_\mu(Q_\rho(x);\mathbb{R}^m)}\end{aligned}$$

for μ-a.a. $y \in Q_\rho(x)\setminus Q_{\lambda\rho}(x)$. But, since $p > \kappa$, $\lim_{\rho\to 0}\frac{1}{\rho}\|u-u_x\|_{L^\infty_\mu(Q_\rho(x);\mathbb{R}^m)} = 0$ by (3.5), hence there exists $\rho_0 > 0$ (which depends on τ and λ) such that

$$\left|\frac{\tau}{1-\tau}D_\mu\varphi(y)\otimes(u(y)-u_x(y))\right| \leqslant r$$

for μ-a.a. $y \in Q_\rho(x)\setminus Q_{\lambda\rho}(x)$ and all $\rho \in]0,\rho_0[$ with $r > 0$ given by (2.6). Hence

$$\begin{aligned}\int_{Q_\rho(x)\setminus Q_{\lambda\rho}(x)} G\left(\frac{\tau}{1-\tau}D_\mu\varphi\otimes(u-u_x)\right)d\mu &\leqslant \int_{Q_\rho(x)\setminus Q_{\lambda\rho}(x)}\sup_{|\xi|\leqslant r}G(\xi)d\mu\\ &= \mu(Q_\rho(x)\setminus Q_{\lambda\rho}(x))\sup_{|\xi|\leqslant r}G(\xi)\end{aligned}\tag{4.64}$$

for all $\rho \in]0,\rho_0[$. Moreover, it easy to see that

$$\begin{aligned}\int_{Q_\rho(x)\setminus Q_{\lambda\rho}(x)} G(\nabla_\mu u(y))d\mu \leqslant \mu(Q_\rho(x))\fint_{Q_\rho(x)}\big|G(\nabla_\mu u(y))-G(\nabla_\mu u(x))\big|d\mu\\ +\mu\left(Q_\rho(x)\setminus Q_{\lambda\rho}(x)\right)G(\nabla_\mu u(x)).\end{aligned}\tag{4.65}$$

Taking (4.64) and (4.65) into account, from (4.63) we deduce that

$$\begin{aligned}\frac{\mathrm{m}^t_{\tau u_x,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))} &\leqslant (1+\varepsilon)\left(\frac{\mathrm{m}^t_{\sigma u,\omega}(Q_{\lambda\rho}(x))}{\mu(Q_{\lambda\rho}(x))}+\varepsilon\right)\\ &+\varepsilon\frac{\mu(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))}\fint_{Q_\rho(x)} a_t(y,\omega)d\mu\\ &+c\frac{\mu(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))}\fint_{Q_\rho(x)}\big|G(\nabla_\mu u(y))-G(\nabla_\mu u(x))\big|d\mu(y)\\ &+c\left(\frac{\mu(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))}-1\right)\sup_{|\xi|\leqslant r}G(\xi)\\ &+2c\left(\frac{\mu(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))}-1\right)G(\nabla_\mu u(x))\\ &+c\left(\frac{\mu(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))}-1\right). \end{aligned}\tag{4.66}$$

As $u\in\mathfrak{G}$, i.e. $G(\nabla_\mu u(\cdot))\in L^1_\mu(\Omega)$, (and μ is a doubling measure) we can assert that

$$\lim_{\rho\to 0}\fint_{Q_\rho(x)}\big|G(\nabla_\mu u(y))-G(\nabla_\mu u(x))\big|d\mu(y)=0,\tag{4.67}$$

and by (2.14) we have

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{t\to\infty}}\fint_{Q_\rho(x)} a_t(y,\omega)d\mu(y)=:a_\infty(x,\omega)\in[0,\infty[.\tag{4.68}$$

Letting $t\to\infty$, $\sigma\to 1^-$ and $\rho\to 0$ in (4.66) and using (4.67) and (4.68) we see that

$$\begin{aligned}\overline{\lim_{\rho\to 0}}\frac{\underline{\mathrm{m}}_{\tau u_x,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))} &\leqslant (1+\varepsilon)\left(\overline{\lim_{\rho\to 0}}\frac{\breve{\underline{\mathrm{m}}}_{u,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))}+\varepsilon\right)\\ &+\varepsilon\,\overline{\lim_{\rho\to 0}}\frac{\mu(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))}a_\infty(x,\omega)\\ &+c\left(\overline{\lim_{\rho\to 0}}\frac{\mu(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))}-1\right)\sup_{|\xi|\leqslant r}G(\xi)\\ &+2c\left(\overline{\lim_{\rho\to 0}}\frac{\mu(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))}-1\right)G(\nabla_\mu u(x))\\ &+c\left(\overline{\lim_{\rho\to 0}}\frac{\mu(Q_\rho(x))}{\mu(Q_{\lambda\rho}(x))}-1\right).\end{aligned}\tag{4.69}$$

Letting $\tau\to 1^-$ and $\lambda\to 1^-$ in (4.69) and using (3.6) we conclude that

$$\underline{\lim_{\tau\to 1^-}}\,\overline{\lim_{\rho\to 0}}\frac{\underline{\mathrm{m}}_{\tau u_x,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))} \leqslant (1+\varepsilon)\left(\lim_{\rho\to 0}\frac{\breve{\underline{\mathrm{m}}}_{u,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))}+\varepsilon\right)+\varepsilon a_\infty(x,\omega),\tag{4.70}$$

and (4.53) follows by letting $\varepsilon\to 0$.

Substep 5-2: end of the proof of Theorem 2.11. Combining (4.47) with (4.53) and (4.50) with (4.54) we get:

$$\Gamma(L^p_\mu)\text{-}\varliminf_{t\to\infty} E_t(u,A,\omega) \geqslant \int_A \varliminf_{\tau\to 1^-} \overline{\lim_{\rho\to 0}} \frac{\underline{\mathrm{m}}_{\tau u_x,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))} d\mu(x);$$
$$\Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}} E_t(u,A,\omega) = \int_A \varliminf_{\tau\to 1^-} \lim_{\rho\to 0} \frac{\overline{\mathrm{m}}_{\tau u_x,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))} d\mu(x)$$

for all $u \in \mathfrak{G}$ and all $A \in \mathcal{O}(\Omega)$. On the other hand, given any $u \in \mathfrak{G}$, it is easily seen that:

$$\varliminf_{\tau\to 1^-} \overline{\lim_{\rho\to 0}} \frac{\underline{\mathrm{m}}_{\tau u_x,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))} = \varliminf_{\tau\to 1^-} \overline{\lim_{\rho\to 0}} \varliminf_{t\to\infty} \mathcal{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega);$$
$$\varliminf_{\tau\to 1^-} \lim_{\rho\to 0} \frac{\overline{\mathrm{m}}_{\tau u_x,\omega}(Q_\rho(x))}{\mu(Q_\rho(x))} = \varliminf_{\tau\to 1^-} \lim_{\rho\to 0} \overline{\lim_{t\to\infty}} \mathcal{H}^\rho_\mu L_t(x,\tau\nabla_\mu u(x),\omega)$$

for μ-a.a. $x \in \Omega$, and (2.15) and (2.16) follow. ■

5. Proofs of the homogenization theorems

As the proof of Theorem 2.25 follows by the same method as in the proof of Theorem 2.34, by using Theorem 3.33 instead of Theorem 3.42, we only give the proof of Theorem 2.34.

Proof of Theorem 2.34. The proof consists of applying Corollary 2.15. First of all, taking Remarks 2.32 and 3.16 into account, it is easy to see that (2.5), (2.6), (2.7), (2.11), (2.12), (2.13) and (2.14) are satisfied. So, we only need to prove that for $\mathbb{P}$-a.e. $\omega \in \Sigma$ and every $x \in \Omega$, one has

$$\varliminf_{t\to\infty} \mathcal{H}^\rho_\mu L_t(x,\xi,\omega) = \overline{\lim_{t\to\infty}} \mathcal{H}^\rho_\mu L_t(x,\xi,\omega) = L_{\mathrm{hom}}(\xi,\omega) \text{ for all } \xi \in \mathbb{G}. \tag{5.1}$$

Let $\xi \in \mathbb{G}$ and let $\mathcal{S}^\xi : \mathcal{B}_{\mu,0}(X) \to L^1(\Sigma,\mathcal{T},\mathbb{P})$ be defined by

$$\mathcal{S}^\xi(A)(\omega) := \inf\left\{\int_{\mathring{A}} L(y,\xi+\nabla_\mu w(y),\omega)d\mu(y) : w \in H^{1,p}_{\mu,0}\big(\mathring{A};\mathbb{R}^m\big)\right\},$$

where by (2.32) we have $0 \leqslant \mathcal{S}^\xi(A)(\omega) \leqslant c\mu\big(\mathring{A}\big) \leqslant c\mu(A)$ for all $A \in \mathcal{B}_{\mu,0}(X)$ and all $\omega \in \Sigma$ with $c := \beta(1+G(\xi))$ ($c < \infty$ because $\xi \in \mathbb{G}$). In particular $\mathcal{S}^\xi$ satisfies the boundedness condition in (3.46). On the other hand, taking (2.34) into account, from (H_2), we see that for any $Q \in \mathrm{Ba}(X)$, any $t > 0$ and any $\omega \in \Sigma$, one has

$$\begin{aligned}
\mathcal{S}^\xi\left(h_t(Q)\right)(\omega) &= \inf\left\{\int_{h_t(Q)} L(y,\xi+\nabla_\mu w(y),\omega)d\mu(y) : w \in H^{1,p}_{\mu,0}(h_t(Q);\mathbb{R}^m)\right\} \\
&= \inf\left\{\int_Q L(h_t(y),\xi+\nabla_\mu w(h_t(y)),\omega)d(h_t^{-1})^\sharp\mu(y) : w \in H^{1,p}_{\mu,0}(h_t(Q);\mathbb{R}^m)\right\} \\
&= \mu(h_t(\mathbb{U}))\inf\left\{\int_Q L_t(y,\xi+\nabla_\mu w(h_t(y)),\omega)d\mu(y) : w \in H^{1,p}_{\mu,0}(h_t(Q);\mathbb{R}^m)\right\}.
\end{aligned}$$

But $\mu(h_t(\mathbb{U}))\mu(Q) = (h_t^{-1})^\sharp\mu(Q) = \mu(h_t(Q))$ by using again (H_2), and so from (H_3) we obtain

$$\mathcal{S}^\xi\left(h_t(Q)\right)(\omega) = \mu(h_t(Q))\inf\left\{\fint_Q L_t(y,\xi+\nabla_\mu w(y),\omega)d\mu(y) : w\in H^{1,p}_{\mu,0}(Q;\mathbb{R}^m)\right\}$$

for all $Q\in \mathrm{Ba}(X)$, all $t>0$ and all $\omega\in\Sigma$. Consequently, we have:

$$\underline{\lim_{t\to\infty}}\,\mathcal{H}^\rho_\mu L_t(x,\xi,\omega) = \underline{\lim_{t\to\infty}}\,\frac{\mathcal{S}^\xi\left(h_t(Q_\rho(x))\right)(\omega)}{\mu\left(h_t(Q_\rho(x))\right)}; \tag{5.2}$$

$$\overline{\lim_{t\to\infty}}\,\mathcal{H}^\rho_\mu L_t(x,\xi,\omega) = \overline{\lim_{t\to\infty}}\,\frac{\mathcal{S}^\xi\left(h_t(Q_\rho(x))\right)(\omega)}{\mu\left(h_t(Q_\rho(x))\right)} \tag{5.3}$$

for all $x\in\Omega$, all $\rho>0$ and $\mathbb{P}$-a.a. $\omega\in\Sigma$. Moreover, from (H_1) and (2.33) it easily seen that the set function $\mathcal{S}^\xi$ is $\mathbb{G}$-covariant, and $\mathcal{S}^\xi$ is also subadditive because, for each $A,B\in\mathcal{B}_{\mu,0}(X)$, $\mu\big(\widehat{A\cup B}\backslash(\mathring{A}\cup\mathring{B})\big)=0$ since $\widehat{A\cup B}\backslash(\mathring{A}\cup\mathring{B})\subset\partial A\cup\partial B$ and $\mu(\partial A)=\mu(\partial B)=0$. Thus, taking $(\mathrm{H}_4^{\mathrm{s}})$ and (H_5) into account, for every $x\in\Omega$ and every $\rho>0$, we can apply Theorem 3.42 with $\{\mathbb{U}_k\}_{k\in\mathbb{N}^*}=\{h_k(\mathbb{U})\}_{k\in\mathbb{N}^*}$ and $\{Q_t\}_{t>0}=\{h_t(Q_\rho(x))\}_{t>0}$, and, noticing that $\mu(h_k(\mathbb{U}))=\mu(\widehat{h_k(\mathbb{U})})=\mu(h_k(\mathring{\mathbb{U}}))$ for all $k\in\mathbb{N}^*$, we conclude that

$$\begin{aligned}\lim_{t\to\infty}\frac{\mathcal{S}^\xi\left(h_t(Q_\rho(x))\right)(\omega)}{\mu\left(h_t(Q_\rho(x))\right)} &= \inf_{k\in\mathbb{N}^*}\frac{\mathbb{E}^{\mathcal{I}}\left[S^\xi\left(h_k(\mathbb{U})\right)\right](\omega)}{\mu(h_k(\mathbb{U}))}\\ &= \inf_{k\in\mathbb{N}^*}\mathbb{E}^{\mathcal{I}}\left[\frac{S^\xi\left(h_k(\mathbb{U})\right)}{\mu(h_k(\mathring{\mathbb{U}}))}\right](\omega)\\ &= L_{\mathrm{hom}}(\xi,\omega),\end{aligned}$$

for $\mathbb{P}$-a.a. $\omega\in\Sigma$, and (5.1) follows from (5.2) and (5.3). ■

A. Appendix

A.1. **Proof of the integral representation of the Vitali envelope of a set function.** In this appendix we prove Theorem 3.29.

Proof of Theorem 3.29. First of all, from (a) we see that $-d_\mu\nu\leqslant d^-_\mu\Theta\leqslant d^+_\mu\Theta\leqslant d_\mu\nu$. Hence $d^-_\mu\Theta, d^+_\mu\Theta\in L^1_\mu(\Omega)$ because ν is a finite Radon measure which is absolutely continuous with respect to the finite Radon measure μ. So $\lambda^-(A),\lambda^+(A)\in\mathbb{R}$ for all $A\in\mathcal{O}(\Omega)$, where $\lambda^-,\lambda^+:\mathcal{O}(\Omega)\to\mathbb{R}$ are given by:

$$\lambda^-(A) := \int_A d^-_\mu\Theta(x)d\mu(x);$$
$$\lambda^+(A) := \int_A d^+_\mu\Theta(x)d\mu(x).$$

In what follows, we consider $\overline{\Theta}^*:\mathcal{O}(\Omega)\to\mathbb{R}$ defined by

$$\overline{\Theta}^*(A) := \inf_{\varepsilon>0}\sup\left\{\sum_{i\in I}\Theta(Q_i) : \{Q_i\}_{i\in I}\in\mathcal{V}_\varepsilon(A)\right\}. \tag{A.1}$$

(It is clear that $\Theta^* \leqslant \overline{\Theta}^*$. In fact, we are going to prove that under the assumptions (a) and (b) of Theorem 3.29 we have $\Theta^*(A) = \overline{\Theta}^*(A) = \int_A d_\mu\Theta(x)d\mu(x)$ for all $A \in \mathcal{O}(\Omega)$.) We divide the proof into three steps.

Step 1: proving that $\boldsymbol{\Theta^* = \lambda^-}$ and $\boldsymbol{\overline{\Theta}^* = \lambda^+}$. Define $\theta^-, \theta^+ : \mathcal{O}(\Omega) \to \mathbb{R}$ by:

$\theta^-(A) := \Theta(A) - \lambda^-(A)$;
$\theta^+(A) := \Theta(A) - \lambda^+(A)$.

In what follows, θ^* (resp. $\overline{\theta}^*$) is defined by (3.41) (resp. (A.1)) with Θ replaced by θ^- (resp. θ^+).

Substep 1-1: an auxiliary lemma.

Lemma A.1. *Under the assumption* (a) *of Theorem* 3.29 *we have* $\theta^* = \overline{\theta}^* = 0$.

Proof of Lemma A.1. We only prove that $\theta^* = 0$. (The proof of $\overline{\theta}^* = 0$ follows from similar arguments and is left to the reader.)
First of all, from the assumption (a) it is clear that

$$|\theta^-(A)| \leqslant \hat{\nu}(A) \tag{A.2}$$

for all $A \in \mathcal{O}(\Omega)$, where $\hat{\nu} := \nu + |\nu|$ is absolutely continuous with respect to μ (with $|\nu|$ denoting the total variation of ν).
Secondly, we can assert that

$$d_\mu^- \theta^- = 0, \tag{A.3}$$

where for any set function $\mathrm{s} : \mathcal{O}(\Omega) \to \mathbb{R}$, the function $d_\mu^-\mathrm{s} : \Omega \to [-\infty, \infty[$ (resp. $d_\mu^+\mathrm{s} : \Omega \to]-\infty, \infty]$) is defined by (3.38) (resp. (3.39)) with Θ replaced by s. Indeed, for any $x \in X$, it is easily seen that

$$d_\mu^-\Theta(x,\rho) - d_\mu^+\lambda^-(x,\rho) \leqslant d_\mu^-\theta^-(x,\rho) \leqslant d_\mu^-\Theta(x,\rho) - d_\mu^-\lambda^-(x,\rho).$$

for all $\rho > 0$, and letting $\rho \to 0$, we obtain

$$d_\mu^-\Theta(x) - d_\mu^+\lambda^-(x) \leqslant d_\mu^-\theta^-(x) \leqslant d_\mu^-\Theta(x) - d_\mu^-\lambda^-(x).$$

But $d_\mu^-\lambda^-(x) = d_\mu^+\lambda^-(x) = d_\mu^-\Theta(x)$, hence $d_\mu^-\theta^-(x) = 0$.
Finally, to conclude we prove that (A.2) and (A.3) imply $\theta^* = 0$. For this, we are going to prove the following two assertions:

$$\text{if } d_\mu^-\theta^- \leqslant 0 \text{ then } \theta^* \leqslant 0; \tag{A.4}$$

$$\text{under (A.2), if } d_\mu^-\theta^- \geqslant 0 \text{ then } \theta^* \geqslant 0. \tag{A.5}$$

Proof of (A.4). Fix $A \in \mathcal{O}(\Omega)$. Fix any $\varepsilon > 0$. Then $d_\mu^-\theta^- < \varepsilon$, and so in particular $\lim_{\rho\to 0} d_\mu^-\theta^-(x,\rho) < \varepsilon$ for all $x \in A$. Hence, for each $x \in A$ there exists $\{\rho_{x,n}\}_n \subset]0, \varepsilon[$ with $\rho_{x,n} \to 0$ as $n \to \infty$ such that $d_\mu^-\theta^-(x,\rho_{x,n}) < \varepsilon$ for all $n \geqslant 1$. Taking Remark 3.27 into account, it follows that for each $x \in A$ and each $n \geqslant 1$ there is $Q_{x,n} \in \mathrm{Ba}(A, x, \rho_{x,n})$ such that for each $x \in A$ and each $n \geqslant 1$,

$$\frac{\theta^-(Q_{x,n})}{\mu(Q_{x,n})} < \varepsilon. \tag{A.6}$$

Moreover, since $\operatorname{diam}\big(\overline{Q}_{x,n}\big) = \operatorname{diam}(Q_{x,n}) \leqslant \rho_{x,n}$ for all $x \in A$ and all $n \geqslant 1$, we have $\inf\big\{\operatorname{diam}\big(\overline{Q}_{x,n}\big) : n \geqslant 1\big\} = 0$ (where $\overline{Q}_{x,n}$ denotes the closed ball corresponding to the open ball $Q_{x,n}$). Let $\mathcal{F}_0$ be the family of closed balls of Ω given by

$$\mathcal{F}_0 := \Big\{\overline{Q}_{x,n} : x \in A \text{ and } n \geqslant 1\Big\}.$$

As Ω satisfies the Vitali covering theorem, from the above we deduce that there exists a disjoint countable subfamily $\{\overline{Q}_i\}_{i\in I_0}$ of closed balls of $\mathcal{F}_0$ (with $Q_i \subset A$, $\mu(\partial Q_i) = 0$ and $\operatorname{diam}(Q_i) \in]0,\varepsilon[$) such that $\mu\big(A\setminus \cup_{i\in I_0} \overline{Q}_i\big) = 0$, which means that $\{Q_i\}_{i\in I_0} \in \mathcal{V}_\varepsilon(A)$. From (A.6) we see that $\theta^-(Q_i) < \varepsilon\mu(Q_i)$ for all $i \in I_0$, hence

$$\sum_{i\in I_0} \theta^-(Q_i) \leqslant \varepsilon \sum_{i\in I_0} \mu(Q_i) = \varepsilon\mu(A).$$

Consequently $\theta^{-,\varepsilon}(A) \leqslant \varepsilon\mu(A)$ for all $\varepsilon > 0$, where $\theta^{-,\varepsilon}$ is defined by (3.40) with Θ replaced by θ^-, and letting $\varepsilon \to 0$ we obtain $\theta^*(A) \leqslant 0$.

Proof of (A.5). Fix $A \in \mathcal{O}(\Omega)$. By Egorov's theorem, there exists a sequence $\{B_n\}_n$ of Borel subsets of A such that:

$$\lim_{n\to\infty} \mu(A\setminus B_n) = 0; \tag{A.7}$$

$$\lim_{\varepsilon\to 0} \sup_{x\in B_n} \big|d_\mu^- \theta^-(x) - d_\mu^- \theta^-(x,\varepsilon)\big| = 0 \text{ for all } n \geqslant 1. \tag{A.8}$$

As $\hat{\nu}$ is absolutely continuous with respect to μ, by (A.7) we have

$$\lim_{n\to\infty} \hat{\nu}(A\setminus B_n) = 0. \tag{A.9}$$

Moreover, as $d_\mu^-\theta^- \geqslant 0$, from (A.8) we deduce that

$$\varliminf_{\varepsilon\to 0} \inf_{x\in B_n} d_\mu^- \theta^-(x,\varepsilon) \geqslant 0 \text{ for all } n \geqslant 1. \tag{A.10}$$

Fix any $n \geqslant 1$ and any $\varepsilon > 0$. By definition of $\theta^{-,\varepsilon}$, there exists $\{Q_i\}_{i\in I} \in \mathcal{V}_\varepsilon(A)$ such that

$$\theta^{-,\varepsilon}(A) > \sum_{i\in I} \theta^-(Q_i) - \varepsilon. \tag{A.11}$$

Set $I_n := \big\{i \in I : Q_i \cap B_n \neq \varnothing\big\}$. Using (A.2) we have

$$\begin{aligned}\sum_{i\in I} \theta^-(Q_i) = \sum_{i\in I_n} \theta^-(Q_i) + \sum_{i\in I\setminus I_n} \theta^-(Q_i) &\geqslant \sum_{i\in I_n} \theta^-(Q_i) - \sum_{i\in I\setminus I_n} \hat{\nu}(Q_i) \\ &\geqslant \sum_{i\in I_n} \frac{\theta^-(Q_i)}{\mu(Q_i)}\mu(Q_i) - \hat{\nu}\left(\underset{i\in I\setminus I_n}{\cup} Q_i\right),\end{aligned}$$

and, choosing $x_i \in Q_i \cap B_n$ for each $i \in I_n$ and noticing that $\cup_{i\in I\setminus I_n} Q_i \subset A\setminus B_n$, it follows that

$$\begin{aligned}\sum_{i\in I} \theta^-(Q_i) &\geqslant \sum_{i\in I_n} d_\mu^-\theta^-(x_i,\varepsilon)\mu(Q_i) - \hat{\nu}(A\setminus B_n) \\ &\geqslant \inf_{x\in B_n} d_\mu^-\theta^-(x,\varepsilon) \sum_{i\in I_n} \mu(Q_i) - \hat{\nu}(A\setminus B_n).\end{aligned}$$

Taking (A.11) into account, we conclude that

$$\theta^{-,\varepsilon}(A) \geqslant \inf_{x\in B_n} d_\mu^- \theta^-(x,\varepsilon) \sum_{i\in I_n} \mu(Q_i) - \hat{\nu}(A\backslash B_n) - \varepsilon$$

for all $\varepsilon > 0$ and all $n \geqslant 1$, which gives $\theta^*(A) \geqslant 0$ by letting $\varepsilon \to 0$ and using (A.10) and then by letting $n \to \infty$ and using (A.9). ■

Substep 1-2: using Lemma A.1. As λ^- and λ^+ are absolutely continuous with respect to μ, it is easy to see that:

$\theta^* = \Theta^* - \lambda^-$;
$\overline{\theta}^* = \overline{\Theta}^* - \lambda^+$.

Hence $\Theta^* = \lambda^-$ and $\overline{\Theta}^* = \lambda^+$ by Lemma A.1.

Step 2: proving that $\boldsymbol{\Theta^* = \overline{\Theta}^*}$. We only need to prove that $\overline{\Theta}^* \leqslant \Theta^*$. For this, it is sufficient to show that for each open ball Q of Ω with $\mu(\partial Q) = 0$, one has

$$\Theta(Q) \leqslant \Theta^*(Q). \tag{A.12}$$

Fix any $\varepsilon > 0$. By definition of Θ^ε, there exists $\{Q_i\}_{i\in I} \in \mathcal{V}_\varepsilon(Q)$ such that

$$\sum_{i\in I} \Theta(Q_i) \leqslant \Theta^\varepsilon(Q) + \varepsilon. \tag{A.13}$$

Since $\mu\big(Q\backslash \cup_{i\in I} Q_i\big) = 0$ there is a sequence $\{I_n\}_n$ of finite subsets of I such that

$$\lim_{n\to\infty} \mu\left(Q\backslash \bigcup_{i\in I_n} Q_i\right) = \lim_{n\to\infty} \mu\left(\bigcup_{i\in I\backslash I_n} Q_i\right) = 0. \tag{A.14}$$

Fix any $n \geqslant 1$. As Θ is subadditive by assumption (b), we have

$$\Theta\left(\bigcup_{i\in I_n} Q_i\right) \leqslant \sum_{i\in I_n} \Theta(Q_i).$$

Moreover, $\mu\left(Q\backslash\big[(\cup_{i\in I_n} Q_i) \cup (Q\backslash\overline{\cup_{i\in I_n} Q_i}\big)\right]) = 0$ because $\mu(\partial Q_i) = 0$ for all $i \in I_n$, so that

$$\Theta(Q) \leqslant \Theta\left(\bigcup_{i\in I_n} Q_i\right) + \Theta\left(Q\backslash\overline{\bigcup_{i\in I_n} Q_i}\right)$$

by using again the subadditivity of Θ, and consequently

$$\sum_{i\in I_n} \Theta(Q_i) \geqslant \Theta(Q) - \Theta\left(Q\backslash\overline{\bigcup_{i\in I_n} Q_i}\right).$$

Thus, using the assumption (a), we get

$$\begin{aligned}
\sum_{i\in I} \Theta(Q_i) &= \sum_{i\in I\backslash I_n} \Theta(Q_i) + \sum_{i\in I_n} \Theta(Q_i) \\
&\geqslant \sum_{i\in I\backslash I_n} \Theta(Q_i) + \Theta(Q) - \Theta\left(Q\backslash\overline{\bigcup_{i\in I_n} Q_i}\right) \\
&\geqslant \Theta(Q) - \nu\left(\bigcup_{i\in I\backslash I_n} Q_i\right) - \nu\left(Q\backslash\overline{\bigcup_{i\in I_n} Q_i}\right).
\end{aligned}$$

But, $\nu(\partial Q_i) = 0$ for all $i \in I_n$ because ν is absolutely with respect to μ, so that

$$\nu\left(Q\backslash\overline{\underset{i\in I_n}{\cup} Q_i}\right) = \nu\left(Q\backslash\underset{i\in I_n}{\cup} Q_i\right) = \nu\left(\underset{i\in I\backslash I_n}{\cup} Q_i\right),$$

and thus

$$\sum_{i\in I}\Theta(Q_i) \geqslant \Theta(Q) - 2\nu\left(\underset{i\in I\backslash I_n}{\cup} Q_i\right). \tag{A.15}$$

Combining (A.13) with (A.15) we conclude that

$$\Theta(Q) \leqslant \Theta^{\varepsilon}(Q) + 2\nu\left(\underset{i\in I\backslash I_n}{\cup} Q_i\right) + \varepsilon,$$

and (A.12) follows by letting $n \to \infty$ and using (A.14) and then by letting $\varepsilon \to 0$.

Step 3: end of the proof of Theorem 3.29. From steps 1 and 2 we have

$$\int_\Omega d_\mu^-\Theta(x)d\mu(x) = \Theta^*(\Omega) = \overline{\Theta}^*(\Omega) = \int_X d_\mu^+\Theta(x)d\mu(x).$$

Thus $\int_\Omega(d_\mu^+\Theta(x) - d_\mu^-\Theta(x))d\mu(x) = 0$. But $d_\mu^+\Theta \geqslant d_\mu^-\Theta$, i.e. $d_\mu^+\Theta - d_\mu^-\Theta \geqslant 0$, hence $d_\mu^+\Theta - d_\mu^-\Theta = 0$, i.e. $d_\mu^+\Theta = d_\mu^-\Theta$, and the proof of Theorem 3.29 is complete. ■

A.2. **Proofs of the subadditive theorems.** In this appendix we prove Theorem 3.33 (see §A.2.1) and Theorem 3.42 (see §A.2.2).

A.2.1. *The deterministic case.* Here we prove Theorem 3.33.

Proof of Theorem 3.33. First of all, let $\{k_j\}_{j\in\mathbb{N}^*}$ be such that

$$\lim_{j\to\infty}\frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})} = \inf_{k\in\mathbb{N}^*}\frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)}. \tag{A.16}$$

We divide the proof into three steps.

Step 1: establishing lower bound and upper bound. Fix any $j \in \mathbb{N}^*$ and any $t > 0$ and set:

$$Q_{t,j}^- := \underset{g\in G_{t,k_j}^-}{\cup} g^{-1}(\mathbb{U}_{k_j});$$
$$Q_{t,j}^+ := \underset{g\in G_{t,k_j}^+}{\cup} g^{-1}(\mathbb{U}_{k_j}),$$

where $G_{t,k_j}^-, G_{t,k_j}^+ \in \mathcal{P}_{\mathrm{f}}(\mathbb{G}_{k_j})$ with $\mathbb{G}_{k_j} \in \mathcal{U}_{k_j}(\mathbb{G})$ given by Definition 3.31.

Substep 1-1: lower bound. By the right inclusion in (3.42) we have $Q_t \subset Q_{t,j}^+$ and so $Q_{t,j}^+ = Q_t \cup (Q_{t,j}^+\backslash Q_t)$. Hence

$$\mathcal{S}\left(Q_{t,j}^+\right) \leqslant \mathcal{S}\left(Q_t\right) + \mathcal{S}\left(Q_{t,j}^+\backslash Q_t\right),$$

and consequently

$$\frac{\mathcal{S}\left(Q_{t,j}^+\right)}{\mu\left(Q_{t,j}^+\right)} \leqslant \frac{\mathcal{S}\left(Q_t\right)}{\mu\left(Q_t\right)} + \frac{\mathcal{S}\left(Q_{t,j}^+\backslash Q_t\right)}{\mu\left(Q_t\right)}.$$

As $Q^-_{t,j} \subset Q_t$ by the left inclusion in (3.42), we see that $Q^+_{t,j}\backslash Q_t \subset Q^+_{t,j}\backslash Q^-_{t,j}$ and so

$$\mathcal{S}\left(Q^+_{t,j}\backslash Q_t\right) \leqslant c\mu\left(Q^+_{t,j}\backslash Q^-_{t,j}\right)$$

with $c > 0$ given by (3.46). It follows that

$$\frac{\mathcal{S}\left(Q^+_{t,j}\right)}{\mu\left(Q^+_{t,j}\right)} \leqslant \frac{\mathcal{S}\left(Q_t\right)}{\mu\left(Q_t\right)} + \frac{c\mu\left(Q^+_{t,j}\backslash Q^-_{t,j}\right)}{\mu\left(Q_t\right)}.$$

Letting $t \to \infty$ and using (3.43) we obtain

$$\underline{l}_j := \varliminf_{t\to\infty} \frac{\mathcal{S}\left(Q^+_{t,j}\right)}{\mu\left(Q^+_{t,j}\right)} \leqslant \varliminf_{t\to\infty} \frac{\mathcal{S}(Q_t)}{\mu(Q_t)} =: \underline{l}. \tag{A.17}$$

Substep 1-2: upper bound. By the left inclusion in (3.42) we have $Q^-_{t,j} \subset Q_t$ and so $Q_t = Q^-_{t,j} \cup (Q_t\backslash Q^-_{t,j})$. Hence

$$\mathcal{S}(Q_t) \leqslant \mathcal{S}\left(Q^-_{t,j}\right) + \mathcal{S}\left(Q_t\backslash Q^-_{t,j}\right),$$

and consequently

$$\frac{\mathcal{S}(Q_t)}{\mu(Q_t)} \leqslant \frac{\mathcal{S}\left(Q^-_{t,j}\right)}{\mu\left(Q^-_{t,j}\right)}\frac{\mu\left(Q^-_{t,j}\right)}{\mu(Q_t)} + \frac{\mathcal{S}\left(Q_t\backslash Q^-_{t,j}\right)}{\mu(Q_t)}.$$

As $Q_t \subset Q^+_{t,j}$ by the right inclusion in (3.42), we see that $Q_t\backslash Q^-_{t,j} \subset Q^+_{t,j}\backslash Q^-_{t,j}$ and so

$$\mathcal{S}\left(Q_t\backslash Q^-_{t,j}\right) \leqslant c\mu\left(Q^+_{t,j}\backslash Q^-_{t,j}\right)$$

with $c > 0$ given by (3.46). It follows that

$$\begin{aligned}\frac{\mathcal{S}(Q_t)}{\mu(Q_t)} &\leqslant \frac{\mathcal{S}\left(Q^-_{t,j}\right)}{\mu\left(Q^-_{t,j}\right)}\frac{\mu\left(Q^-_{t,j}\right)}{\mu(Q_t)} + \frac{c\mu\left(Q^+_{t,j}\backslash Q^-_{t,j}\right)}{\mu(Q_t)} \\ &\leqslant \frac{\mathcal{S}\left(Q^-_{t,j}\right)}{\mu\left(Q^-_{t,j}\right)} + \frac{c\mu\left(Q^+_{t,j}\backslash Q^-_{t,j}\right)}{\mu(Q_t)}\end{aligned}$$

because $\mu\left(Q^-_{t,j}\right) \leqslant \mu(Q_t)$ since $Q^-_{t,j} \subset Q_t$. Letting $t \to \infty$ and using (3.43) we obtain

$$\bar{l} := \varlimsup_{t\to\infty} \frac{\mathcal{S}(Q_t)}{\mu(Q_t)} \leqslant \varlimsup_{t\to\infty} \frac{\mathcal{S}\left(Q^-_{t,j}\right)}{\mu\left(Q^-_{t,j}\right)} =: \bar{l}_j. \tag{A.18}$$

Step 2: we prove that $\underline{l} = \bar{l}$. It is sufficient to prove that for each $\varepsilon > 0$, one has

$$\bar{l} - \underline{l} < \varepsilon. \tag{A.19}$$

Fix $\varepsilon > 0$. From (A.17) and (A.18) we see that $\bar{l} - \underline{l} \leqslant \bar{l}_j - \underline{l}_j$. So, to prove (A.19) it suffices to show that there exists $j \in \mathbb{N}^*$ such that

$$\bar{l}_j - \underline{l}_j < \varepsilon. \tag{A.20}$$

Let $\mathcal{S}_j : \mathcal{P}_{\mathrm{f}}(\mathbb{G}_{k_j}) \to \mathbb{R}$ be defined by

$$\mathcal{S}_j(E) := \frac{1}{\mu(\mathbb{U}_{k_j})}\left[\mathcal{S}\Big(\bigcup_{g\in E} g^{-1}(\mathbb{U}_{k_j})\Big) - |E|\mathcal{S}(\mathbb{U}_{k_j})\right]. \tag{A.21}$$

As $\mathcal{S}$ is subadditive, we can assert that $\mathcal{S}_j$ is negative, i.e.

$$\mathcal{S}_j(E) = \frac{1}{\mu(\mathbb{U}_{k_j})}\left[\mathcal{S}\Big(\underset{g\in E}{\cup} g^{-1}(\mathbb{U}_{k_j})\Big) - |E|\mathcal{S}(\mathbb{U}_{k_j})\right] \leqslant 0 \tag{A.22}$$

for all $E \in \mathcal{P}_{\mathrm{f}}(\mathbb{G}_{k_j})$. Moreover, it is easily seen that $\mathcal{S}_j$ is decreasing, i.e. for all $E, F \in \mathcal{P}_{\mathrm{f}}(\mathbb{G}_{k_j})$, if $E \subset F$ then $\mathcal{S}_j(E) \geqslant \mathcal{S}_j(F)$. Consider $m_{t,k_j} \in \mathbb{N}^*$, $g_{t,k_j} \in \mathbb{G}$ and $F_{t,k_j} \in \mathcal{P}_{\mathrm{f}}(\mathbb{G}_{k_j})$ given by Definition 3.31. From (3.44) it follows that

$$\begin{aligned}\mathcal{S}_j\left(G^+_{t,k_j}\right) \geqslant \mathcal{S}_j\left(F_{t,k_j}\right) &= \frac{1}{\mu(\mathbb{U}_{k_j})}\left[\mathcal{S}\Big(\underset{g\in F_{t,k_j}}{\cup} g^{-1}(\mathbb{U}_{k_j})\Big) - \left|F_{t,k_j}\right|\mathcal{S}(\mathbb{U}_{k_j})\right]\\ &= \frac{1}{\mu(\mathbb{U}_{k_j})}\left[\mathcal{S}\big(g^{-1}_{t,k_j}(\mathbb{U}_{m_{t,k_j}})\big) - \left|F_{t,k_j}\right|\mathcal{S}(\mathbb{U}_{k_j})\right].\end{aligned}$$

Hence, since $\frac{1}{|G^+_{t,k_j}|} \geqslant \frac{1}{|F_{t,k_j}|}$ and $\mathcal{S}$ and μ are $\mathbb{G}$-invariant, we get

$$\begin{aligned}\frac{\mathcal{S}_j\left(G^+_{t,k_j}\right)}{\left|G^+_{t,k_j}\right|} &\geqslant \frac{1}{|G^+_{t,k_j}|\mu(\mathbb{U}_{k_j})}\left[\mathcal{S}\big(g^{-1}_{t,k_j}(\mathbb{U}_{m_{t,k_j}})\big) - \left|F_{t,k_j}\right|\mathcal{S}(\mathbb{U}_{k_j})\right]\\ &\geqslant \frac{\mathcal{S}\big(g^{-1}_{t,k_j}(\mathbb{U}_{m_{t,k_j}})\big)}{|F_{t,k_j}|\mu(\mathbb{U}_{k_j})} - \frac{\left|F_{t,k_j}\right|}{\left|G^+_{t,k_j}\right|}\frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})}\\ &= \frac{\mathcal{S}\big(g^{-1}_{t,k_j}(\mathbb{U}_{m_{t,k_j}})\big)}{\mu\big(g^{-1}_{t,k_j}(\mathbb{U}_{m_{t,k_j}})\big)} - \frac{\left|F_{t,k_j}\right|}{\left|G^+_{t,k_j}\right|}\frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})}\\ &= \frac{\mathcal{S}\big(\mathbb{U}_{m_{t,k_j}}\big)}{\mu\big(\mathbb{U}_{m_{t,k_j}}\big)} - \frac{\left|F_{t,k_j}\right|}{\left|G^+_{t,k_j}\right|}\frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})}\\ &\geqslant \inf_{k\in\mathbb{N}^*}\frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)} - \frac{\left|F_{t,k_j}\right|}{\left|G^+_{t,k_j}\right|}\frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})}.\end{aligned}$$

Letting $t \to \infty$ and taking (3.45) into account, we deduce that

$$\varliminf_{t\to\infty}\frac{\mathcal{S}_j\left(G^+_{t,k_j}\right)}{\left|G^+_{t,k_j}\right|} \geqslant \inf_{k\in\mathbb{N}^*}\frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)} - \frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})}. \tag{A.23}$$

By (A.16) we can assert that there exists $j_\varepsilon \in \mathbb{N}^*$ such that for all $j \geqslant j_\varepsilon$, one has

$$\frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})} - \inf_{k\in\mathbb{N}^*}\frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)} < \varepsilon. \tag{A.24}$$

Combining (A.23) with (A.24) we conclude that

$$\varliminf_{t\to\infty}\frac{\mathcal{S}_j\left(G^+_{t,k_j}\right)}{\left|G^+_{t,k_j}\right|} > -\varepsilon \tag{A.25}$$

for all $j \geqslant j_\varepsilon$. On the other hand, by using (A.21) with $E = G^+_{t,k_j}$ and (A.22) with $E = G^-_{t,k_j}$ we get:

$$\frac{\mathcal{S}\left(Q^+_{t,j}\right)}{\mu\left(Q^+_{t,j}\right)} - \frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})} = \frac{\mathcal{S}_j\left(G^+_{t,k_j}\right)}{\left|G^+_{t,k_j}\right|}; \tag{A.26}$$

$$\frac{\mathcal{S}\left(Q^-_{t,j}\right)}{\mu\left(Q^-_{t,j}\right)} - \frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})} \leqslant 0. \tag{A.27}$$

Letting $t \to \infty$ in (A.26) and (A.27) and taking (A.25) into account, we deduce that:

$$\underline{l}_j - \frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})} > -\varepsilon \text{ for all } j \geqslant j_\varepsilon; \tag{A.28}$$

$$\overline{l}_j - \frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})} \leqslant 0 \text{ for all } j \in \mathbb{N}^*, \tag{A.29}$$

and (A.20) follows with $j = j_\varepsilon$. We set $l := \underline{l} = \overline{l}$ and $\gamma := \inf_{k\in\mathbb{N}*} \frac{\mathcal{S}(\mathbb{U}_k)}{\mu(\mathbb{U}_k)}$.

Step 3: we prove that $l = \gamma$. Combining (A.18) with (A.29) we see that $l \leqslant \frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})}$ for all $j \in \mathbb{N}^*$, and so $l \leqslant \gamma$ by letting $j \to \infty$ and using (A.16). On the other hand, combining (A.17) with (A.28) we see that $l > -\varepsilon + \frac{\mathcal{S}(\mathbb{U}_{k_j})}{\mu(\mathbb{U}_{k_j})}$ for all $j \geqslant j_\varepsilon$. Letting $j \to \infty$ and using (A.16) we deduce that $l \geqslant -\varepsilon + \gamma$ for all $\varepsilon > 0$, and so $l \geqslant \gamma$ by letting $\varepsilon \to 0$. ■

A.2.2. *The stochastic case.* Here we prove Theorem 3.42.

Proof of Theorem 3.42. The proof is divided into four steps.

Step 1: establishing lower bound and upper bound. Fix any $k \in \mathbb{N}^*$ and any $t > 0$ and set:

$$Q^-_{t,k} := \underset{g\in G^-_{t,k}}{\cup} g^{-1}(\mathbb{U}_k);$$
$$Q^+_{t,k} := \underset{g\in G^+_{t,k}}{\cup} g^{-1}(\mathbb{U}_k),$$

where $G^-_{t,k}, G^+_{t,k} \in \mathcal{P}_{\mathrm{f}}(\mathbb{G}_k)$ with $\mathbb{G}_k \in \mathcal{U}^a_k(\mathbb{G})$ given by Definition 3.37. Arguing as in Step 1 of the proof of Theorem 3.33, for each $\omega \in \Sigma$, we get:

$$\underline{l}_k(\omega) := \varliminf_{t\to\infty} \frac{\mathcal{S}\left(Q^+_{t,k}\right)(\omega)}{\mu\left(Q^+_{t,k}\right)} \leqslant \varliminf_{t\to\infty} \frac{\mathcal{S}(Q_t)(\omega)}{\mu(Q_t)} =: \underline{l}(\omega) \tag{A.30}$$

$$\overline{l}(\omega) := \varlimsup_{t\to\infty} \frac{\mathcal{S}(Q_t)(\omega)}{\mu(Q_t)} \leqslant \varlimsup_{t\to\infty} \frac{\mathcal{S}\left(Q^-_{t,k}\right)(\omega)}{\mu\left(Q^-_{t,k}\right)} =: \overline{l}_k(\omega). \tag{A.31}$$

Remark A.2. Arguing as in Step 1-1 of the proof of Theorem 3.33, we see that we also have

$$\varlimsup_{t\to\infty} \frac{\mathcal{S}\left(Q^+_{t,k}\right)(\omega)}{\mu\left(Q^+_{t,k}\right)} \leqslant \overline{l}(\omega) \tag{A.32}$$

for all $\omega \in \Sigma$. (This is used in Step 3.)

Step 2: we prove that $\underline{l}(\omega) = \bar{l}(\omega)$ for $\mathbb{P}$-a.a. $\omega \in \Sigma$. It is sufficient to prove that for each $\alpha > 0$, one has

$$\mathbb{P}\left(\left\{\omega \in \Sigma : \bar{l}(\omega) - \underline{l}(\omega) > \alpha\right\}\right) = 0. \tag{A.33}$$

Fix $\alpha > 0$. From (A.30) and (A.31) we see that for each $k \in \mathbb{N}^*$, one has

$$\left\{\omega \in \Sigma : \bar{l}(\omega) - \underline{l}(\omega) > \alpha\right\} \subset \left\{\omega \in \Sigma : \bar{l}_k(\omega) - \underline{l}_k(\omega) > \alpha\right\} =: W_{k,\alpha}. \tag{A.34}$$

So, to prove (A.33) it suffices to show that for each $\varepsilon > 0$ there exists $k \in \mathbb{N}^*$ such that

$$\mathbb{P}(W_{k,\alpha}) \leqslant \frac{M_k}{\alpha}\varepsilon, \tag{A.35}$$

where $M_k > 0$ is the Tempelman constant associated with $\{G^+_{t,k}\}_{t>0}$. Fix $\varepsilon > 0$.

Substep 2-1: constructing a decreasing negative subadditive process on $\mathcal{P}_{\mathrm{f}}(\mathbb{G}_k)$. Let $\mathcal{A}_k : \mathcal{P}_{\mathrm{f}}(\mathbb{G}_k) \to L^1(\Sigma, \mathcal{T}, \mathbb{P})$ be defined by

$$\mathcal{A}_k(E) := \sum_{g\in E} \mathcal{S}\left(\mathbb{U}_k\right) \circ \tau_g,$$

where $\mathbb{G}_k \in \mathcal{U}^a_k(\mathbb{G})$ is (a countable discrete and amenable subgroup of $\mathbb{G}$) given by Definition 3.37, and let $\mathcal{S}_k : \mathcal{P}_{\mathrm{f}}(\mathbb{G}_k) \to L^1(\Sigma, \mathcal{T}, \mathbb{P})$ be defined by

$$\mathcal{S}_k(E) := \frac{1}{\mu\left(\mathbb{U}_k\right)}\left[\mathcal{S}\Big(\underset{g\in E}{\cup} g^{-1}\left(\mathbb{U}_k\right)\Big) - \mathcal{A}_k(E)\right]. \tag{A.36}$$

As $\mathcal{S}$ is subadditive and $\mathbb{G}$-covariant (and so $\mathbb{G}_k$-covariant) and $\mathcal{A}_k$ is additive and $\mathbb{G}_k$-covariant, we can assert that $\mathcal{S}_k$ is a subadditive process[4] on $\mathcal{P}_{\mathrm{f}}(\mathbb{G}_k)$ which is negative, i.e.

$$\mathcal{S}_k(E)(\omega) = \frac{1}{\mu\left(\mathbb{U}_k\right)}\left[\mathcal{S}\Big(\underset{g\in E}{\cup} g^{-1}\left(\mathbb{U}_k\right)\Big)(\omega) - \mathcal{A}_k(E)(\omega)\right] \leqslant 0 \tag{A.37}$$

for all $E \in \mathcal{P}_{\mathrm{f}}(\mathbb{G}_k)$ and all $\omega \in \Sigma$. Moreover, it is easily seen that $\mathcal{S}_k$ is decreasing, i.e. for all $E, F \in \mathcal{P}_{\mathrm{f}}(\mathbb{G}_k)$, if $E \subset F$ then $\mathcal{S}_k(E) \geqslant \mathcal{S}_k(F)$. Consider $m_{t,k} \in \mathbb{N}^*$, $g_{t,k} \in \mathbb{G}$ and $F_{t,k} \in \mathcal{P}_{\mathrm{f}}(\mathbb{G}_k)$ given by Definition 3.37. From (3.44) it follows that

$$\begin{aligned}
\mathcal{S}_k\left(G^+_{t,k}\right) \geqslant \mathcal{S}_k\left(F_{t,k}\right) &= \frac{1}{\mu(\mathbb{U}_k)}\left[\mathcal{S}\Big(\underset{g\in F_{t,k}}{\cup} g^{-1}\left(\mathbb{U}_k\right)\Big) - \mathcal{A}_k\left(F_{t,k}\right)\right] \\
&= \frac{1}{\mu\left(\mathbb{U}_k\right)}\left[\mathcal{S}\big(g^{-1}_{t,k}\big(\mathbb{U}_{m_{t,k}}\big)\big) - \mathcal{A}_k\left(F_{t,k}\right)\right].
\end{aligned}$$

[4]The set function $\mathcal{S}_k : \mathcal{P}_{\mathrm{f}}(\mathbb{G}_k) \to L^1(\Sigma, \mathcal{T}, \mathbb{P})$ is said to be a subadditive process on $\mathcal{P}_{\mathrm{f}}(\mathbb{G}_k)$ if it is subadditive, i.e. $\mathcal{S}_k(E \cup F) \leqslant \mathcal{S}_k(E) + \mathcal{S}_k(F)$ for all $E, F \in \mathcal{P}_{\mathrm{f}}(\mathbb{G}_k)$ such that $E \cap F = \varnothing$, and $\mathbb{G}_k$-covariant, i.e. $\mathcal{S}_k(Eg) = \mathcal{S}_k(E) \circ \tau_g$ for all $E \in \mathcal{P}_{\mathrm{f}}(\mathbb{G}_k)$ and all $g \in \mathbb{G}_k$.

By using the $\mathbb{G}$-covariance of $\mathcal{S}$ we see that

$$\begin{aligned}\int_\Sigma \mathcal{S}_k\left(G_{t,k}^+\right)(\omega)d\mathbb{P}(\omega) &\geqslant \frac{1}{\mu\left(\mathbb{U}_k\right)}\left[\int_\Sigma \mathcal{S}\big(g_{t,k}^{-1}\big(\mathbb{U}_{m_{t,k}}\big)\big)(\omega)d\mathbb{P}(\omega) - \int_\Sigma \mathcal{A}_k\left(F_{t,k}\right)(\omega)d\mathbb{P}(\omega)\right]\\ &= \frac{1}{\mu\left(\mathbb{U}_k\right)}\left[\int_\Sigma \mathcal{S}\big(\mathbb{U}_{m_{t,k}}\big)(\omega)d\mathbb{P}(\omega) - \big|F_{t,k}\big|\mathbb{E}[\mathcal{S}\left(\mathbb{U}_k\right)]\right]\\ &= \frac{\mathbb{E}\left[\mathcal{S}\big(\mathbb{U}_{m_{t,k}}\big)\right]}{\mu\left(\mathbb{U}_k\right)} - \big|F_{t,k}\big|\frac{\mathbb{E}[\mathcal{S}\left(\mathbb{U}_k\right)]}{\mu\left(\mathbb{U}_k\right)}.\end{aligned}$$

Consequently, since $\frac{1}{|G_{t,k}^+|} \geqslant \frac{1}{|F_{t,k}|}$ and μ is $\mathbb{G}$-invariant, we get

$$\begin{aligned}\frac{\mathbb{E}[\mathcal{S}_j\big(G_{t,k}^+\big)]}{\big|G_{t,k}^+\big|} &\geqslant \frac{\mathbb{E}\left[\mathcal{S}\big(\mathbb{U}_{m_{t,k}}\big)\right]}{\mu\big(\mathbb{U}_{m_{t,k}}\big)} - \frac{\big|F_{t,k}\big|}{\big|G_{t,k}^+\big|}\frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_k)]}{\mu(\mathbb{U}_k)}\\ &\geqslant \inf_{m\in\mathbb{N}^*}\frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_m)]}{\mu(\mathbb{U}_m)} - \frac{\big|F_{t,k}\big|}{\big|G_{t,k}^+\big|}\frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_k)]}{\mu(\mathbb{U}_k)}.\end{aligned}$$

Letting $t \to \infty$ and taking (3.45) into account, we deduce that

$$\varliminf_{t\to\infty}\frac{\mathbb{E}[\mathcal{S}_k\big(G_{t,k}^+\big)]}{\big|G_{t,k}^+\big|} \geqslant \inf_{m\in\mathbb{N}^*}\frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_m)]}{\mu(\mathbb{U}_m)} - \frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_k)]}{\mu(\mathbb{U}_k)}. \tag{A.38}$$

As $\mathcal{S}$ is subadditive and $\mathbb{G}$-covariant, we see that the set function $\mathbb{E}[\mathcal{S}(\cdot)]$ is subadditive and $\mathbb{G}$-invariant. From Proposition 3.35 it follows that there exists $k_\varepsilon \in \mathbb{N}^*$ such that for all $k \geqslant k_\varepsilon$, one has

$$\frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_k)]}{\mu(\mathbb{U}_k)} - \inf_{m\in\mathbb{N}^*}\frac{\mathbb{E}[\mathcal{S}(\mathbb{U}_m)]}{\mu(\mathbb{U}_m)} < \varepsilon. \tag{A.39}$$

Combining (A.38) with (A.39) we conclude that

$$\overline{\varlimsup_{t\to\infty}}\frac{\mathbb{E}\big[\mathcal{S}_k\big(G_{t,k}^+\big)\big]}{\big|G_{t,k}^+\big|} > -\varepsilon \tag{A.40}$$

for all $k \geqslant k_\varepsilon$.

Substep 2-2: using Lindenstrauss's ergodic theorem. We need the following pointwise additive ergodic theorem[5] due to Lindenstrauss (see [Lin01, Theorem 1.2] and also [DGZ14, Theorem 2.1]).

Theorem A.3. *Let $\Theta \in L^1(\Sigma,\mathcal{T},\mathbb{P})$ and let $\{G_t\}_{t>0} \subset \mathcal{P}_{\mathrm{f}}(\mathbb{G}_k)$. If $\{G_t\}_{t>0}$ is of Følner-Tempelman type with respect to $\mathbb{G}_k$ then*

$$\lim_{t\to\infty}\frac{1}{|G_t|}\sum_{g\in G_t}\Theta\big(\tau_g(\omega)\big) = \mathbb{E}^{\mathcal{I}_{\mathbb{G}_k}}[\Theta](\omega) \text{ for } \mathbb{P}\text{-a.a } \omega\in\Sigma,$$

where $\mathcal{I}_{\mathbb{G}_k}$ is the σ-algebra of invariant sets with respect to $(\Sigma,\mathcal{T},\mathbb{P},\{\tau_g\}_{g\in\mathbb{G}_k})$ and $\mathbb{E}^{\mathcal{I}_{\mathbb{G}_k}}[\Theta]$ denotes the conditional expectation over $\mathcal{I}_{\mathbb{G}_k}$ with respect to $\mathbb{P}$.

[5]Lindenstrauss's ergodic theorem is established under the weaker condition that $\{G_t\}_{t>0}$ is of tempered Følner type (see [Lin01, Definition 1.1] and [DGZ14, §2] for more details). The tempered Følner condition implies the Følner-Tempelman condition, but the converse is not true (see [Lin01, DGZ14]).

As $\{G_{t,k}^-\}_{t>0}$ and $\{G_{t,k}^+\}_{t>0}$ are of Følner-Tempelman type with respect to $\mathbb{G}_k$, applying Theorem A.3 with $\Theta = \mathcal{S}(\mathbb{U}_k)$ we deduce that there exists $\widehat{\Sigma} \in \mathcal{T}$ with $\mathbb{P}(\widehat{\Sigma}) = 1$ such that

$$\lim_{t\to\infty} \frac{\mathcal{A}_k\left(G_{t,k}^-\right)(\omega)}{\left|G_{t,k}^-\right|} = \lim_{t\to\infty} \frac{\mathcal{A}_k\left(G_{t,k}^+\right)(\omega)}{\left|G_{t,k}^+\right|} = \mathbb{E}^{\mathcal{I}_{\mathbb{G}_k}}[\mathcal{S}(\mathbb{U}_k)](\omega) \text{ for all } \omega \in \widehat{\Sigma}. \tag{A.41}$$

On the other hand, by using (A.36) with $E = G_{t,k}^+$ and (A.37) with $E = G_{t,k}^-$ we get:

$$\begin{aligned} &\frac{\mathcal{S}\big(Q_{t,k}^+\big)(\omega)}{\mu\big(Q_{t,k}^+\big)} - \frac{1}{\mu(\mathbb{U}_k)} \frac{\mathcal{A}_k\big(G_{t,k}^+\big)(\omega)}{\left|G_{t,k}^+\right|} = \frac{\mathcal{S}_k\big(G_{t,k}^+\big)(\omega)}{\left|G_{t,k}^+\right|} \geqslant \inf_{s>0} \frac{\mathcal{S}_k\big(G_{s,k}^+\big)(\omega)}{\left|G_{s,k}^+\right|}; \qquad (A.42)\\ &\frac{\mathcal{S}\big(Q_{t,k}^-\big)(\omega)}{\mu\big(Q_{t,k}^-\big)} - \frac{1}{\mu(\mathbb{U}_k)} \frac{\mathcal{A}_k\big(G_{t,k}^-\big)(\omega)}{\left|G_{t,k}^-\right|} \leqslant 0 \end{aligned}$$

for all $\omega \in \Sigma$. Letting $t \to \infty$ we deduce that:

$$\underline{l}_k(\omega) - \frac{\mathbb{E}^{\mathcal{I}_{\mathbb{G}_k}}[\mathcal{S}(\mathbb{U}_k)](\omega)}{\mu(\mathbb{U}_k)} \geqslant \inf_{t>0} \frac{\mathcal{S}_k\big(G_{t,k}^+\big)(\omega)}{\left|G_{t,k}^+\right|} \text{ for all } k \in \mathbb{N}^* \text{ and all } \omega \in \widehat{\Sigma}; \tag{A.43}$$

$$\bar{l}_k(\omega) - \frac{\mathbb{E}^{\mathcal{I}_{\mathbb{G}_k}}[\mathcal{S}(\mathbb{U}_k)](\omega)}{\mu(\mathbb{U}_k)} \leqslant 0 \text{ for all } k \in \mathbb{N}^* \text{ and all } \omega \in \widehat{\Sigma};. \tag{A.44}$$

In what follows, without loss of generality, we assume that $\widehat{\Sigma} = \Sigma$.

Substep 2-3: using a maximal inequality. We need the following lemma (see [DGZ14, Lemma 3.5] and also [AK81, Theorem 4.2]).

Lemma A.4. *Let $\mathcal{K} : \mathcal{P}_{\mathrm{f}}(\mathbb{G}_k) \to L^1(\Sigma, \mathcal{T}, \mathbb{P})$ be a negative subadditive process and let $\{G_t\}_{t>0} \subset \mathcal{P}_{\mathrm{f}}(\mathbb{G}_k)$. Fix $\alpha > 0$ and consider $V_\alpha^{\mathcal{K}} \in \mathcal{T}$ given by*

$$V_\alpha^{\mathcal{K}} := \left\{\omega \in \Sigma : \inf_{t>0} \frac{\mathcal{K}(G_t)(\omega)}{|G_t|} < -\alpha\right\}.$$

If $\{G_t\}_{t>0}$ is of Følner-Tempelman type with respect to $\mathbb{G}_k$ then

$$\mathbb{P}\left(V_\alpha^{\mathcal{K}}\right) \leqslant -\frac{M}{\alpha} \overline{\lim_{t\to\infty}} \frac{\mathbb{E}[\mathcal{K}(G_t)]}{|G_t|},$$

where $M > 0$ is the Templeman constant associated with $\{G_t\}_{t>0}$.

As $\mathcal{S}_k : \mathcal{P}_{\mathrm{f}}(\mathbb{G}_k) \to L^1(\Sigma, \mathcal{T}, \mathbb{P})$ defined by (A.36) is a negative subadditive process, we can apply Theorem A.4 with $\mathcal{K} = \mathcal{S}_k$. Hence, since $\{G_{t,k}^+\}_{t>0}$ is of Følner-Tempelman type with respect to $\mathbb{G}_k$, one has

$$\mathbb{P}\left(V_\alpha^{\mathcal{S}_k}\right) \leqslant -\frac{M_k}{\alpha} \overline{\lim_{t\to\infty}} \frac{\mathbb{E}[\mathcal{S}_k\big(G_{t,k}^+\big)]}{\left|G_{t,k}^+\right|},$$

where $M_k > 0$ is the Templeman constant associated with $\{G_{t,k}^+\}_{t>0}$. Consequently, taking (A.40) into account, we get

$$\mathbb{P}\left(V_\alpha^{\mathcal{S}_k}\right) \leqslant \frac{M_k}{\alpha}\varepsilon \text{ for all } k \geqslant k_\varepsilon. \tag{A.45}$$

Substep 2-4: end of Step 2. From (A.43) and (A.44) it follows that

$$\bar{l}_k - \underline{l}_k \leqslant -\inf_{t>0} \frac{\mathcal{S}_k\big(G^+_{t,k}\big)}{\big|G^+_{t,k}\big|}.$$

Hence $W_{k,\alpha} \subset V_\alpha^{\mathcal{S}_k}$, where $W_{k,\alpha}$ is defined in (A.34). From (A.45) we conclude that (A.35) is satisfied with $k = k_\varepsilon$.

In what follows we set $l := \bar{l} = \underline{l}$ and $\gamma := \inf_{k\in\mathbb{N}^*} \gamma_k$ with $\gamma_k := \frac{\mathbb{E}^{\mathcal{I}_{\mathbb{G}_k}}[\mathcal{S}(\mathbb{U}_k)]}{\mu(\mathbb{U}_k)}$ for all $k \in \mathbb{N}^*$.

Step 3: we prove that $l(\omega) = \gamma(\omega)$ for $\mathbb{P}$-a.a. $\omega \in \Sigma$. First of all, from (A.31) and (A.44) we see that $l(\omega) \leqslant \gamma_k(\omega)$ for $\mathbb{P}$-a.a. $\omega \in \Sigma$ and all $k \in \mathbb{N}^*$, and so

$$l(\omega) \leqslant \gamma(\omega) \text{ for } \mathbb{P}\text{-a.a. } \omega \in \Sigma. \tag{A.46}$$

On the other hand, letting $t \to \infty$ in (A.42) and using (A.41) we get

$$\overline{\lim_{t\to\infty}} \frac{\mathcal{S}\big(Q^+_{t,k}\big)(\omega)}{\mu\big(Q^+_{t,k}\big)} - \gamma_k(\omega) \geqslant \overline{\lim_{t\to\infty}} \frac{\mathcal{S}_k\big(G^+_{t,k}\big)(\omega)}{\big|G^+_{t,k}\big|} \text{ for } \mathbb{P}\text{-a.a. } \omega \in \Sigma$$

and so, taking (A.32) into account, one has

$$l(\omega) - \gamma_k \geqslant \overline{\lim_{t\to\infty}} \frac{\mathcal{S}_k\big(G^+_{t,k}\big)(\omega)}{\big|G^+_{t,k}\big|} \text{ for } \mathbb{P}\text{-a.a. } \omega \in \Sigma.$$

It follows that

$$\int_\Sigma \big[l(\omega) - \gamma_k\big]\, d\mathbb{P}(\omega) \geqslant \int_\Sigma \overline{\lim_{t\to\infty}} \frac{\mathcal{S}_k\big(G^+_{t,k}\big)(\omega)}{\big|G^+_{t,k}\big|} d\mathbb{P}(\omega).$$

But, by using Fatou's lemma and (A.40) we see that for any $k \geqslant k_\varepsilon$, one has

$$\int_\Sigma \overline{\lim_{t\to\infty}} \frac{\mathcal{S}_k\big(G^+_{t,k}\big)(\omega)}{\big|G^+_{t,k}\big|} d\mathbb{P}(\omega) > -\varepsilon, \tag{A.47}$$

and consequently

$$\begin{aligned} \int_\Sigma l(\omega) d\mathbb{P}(\omega) &\geqslant \int_\Sigma \gamma_k(\omega) d\mathbb{P}(\omega) - \varepsilon \\ &\geqslant \int_\Sigma \gamma(\omega) d\mathbb{P}(\omega) - \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ we deduce that

$$\int_\Sigma \big[l(\omega) - \gamma(\omega)\big]\, d\mathbb{P}(\omega) \geqslant 0, \tag{A.48}$$

and the result follows by combining (A.46) with (A.48).

In what follows, we set $\gamma^{\mathcal{I}} := \inf_{k\in\mathbb{N}^*} \gamma_k^{\mathcal{I}}$ with $\gamma_k^{\mathcal{I}} := \frac{\mathbb{E}^{\mathcal{I}}[\mathcal{S}(\mathbb{U}_k)]}{\mu(\mathbb{U}_k)}$ for all $k \in \mathbb{N}^*$.

Step 4: we prove that $l(\omega) = \gamma^{\mathcal{I}}(\omega)$ for $\mathbb{P}$-a.a. $\omega \in \Sigma$. Since γ_k is $\mathcal{I}_{\mathbb{G}_k}$-measurable for all $k \in \mathbb{N}^*$, $\gamma = \inf_{k\in\mathbb{N}^*} \gamma_k$ is $\cap_{k\in\mathbb{N}^*}\mathcal{I}_{\mathbb{G}_k}$-measurable. But $\cap_{k\in\mathbb{N}^*}\mathcal{I}_{\mathbb{G}_k} = \mathcal{I}$ because $\cup_{k\in\mathbb{N}^*}\mathbb{G}_k = \mathbb{G}$, hence γ is $\mathcal{I}$-measurable and so l is $\mathcal{I}$-measurable by Step 3. It follows that

$$\mathbb{E}^{\mathcal{I}}[l] = l. \tag{A.49}$$

As $\mathcal{I} \subset \mathcal{I}_{\mathbb{G}_k}$ for all $k \in \mathbb{N}^*$ we also have

$$\mathbb{E}^{\mathcal{I}}[\gamma_k] = \gamma_k^{\mathcal{I}} \text{ for all } k \in \mathbb{N}^*. \tag{A.50}$$

Arguing as in Step 3, for each $k \in \mathbb{N}^*$, we have $l \leqslant \gamma_k$ hence $\mathbb{E}^{\mathcal{I}}[l] \leqslant \mathbb{E}^{\mathcal{I}}[\gamma_k]$ and so $l \leqslant \gamma_k^{\mathcal{I}}$ by using (A.49) and (A.50). Consequently

$$l \leqslant \gamma^{\mathcal{I}}. \tag{A.51}$$

Fix any $E \in \mathcal{I}$. Arguing again as in Step 3 we see that for any $k \geqslant k_\varepsilon$, one has

$$\int_E l(\omega)d\mathbb{P}(\omega) \geqslant \int_E \gamma_k(\omega)d\mathbb{P}(\omega) - \varepsilon.$$

But $\int_E \gamma_k(\omega)d\mathbb{P}(\omega) = \int_E \mathbb{E}^{\mathcal{I}}[\gamma_k](\omega)d\mathbb{P}(\omega)$ by definition of the conditional expectation, hence $\int_E \gamma_k(\omega)d\mathbb{P}(\omega) = \int_E \gamma_k^{\mathcal{I}}(\omega)d\mathbb{P}(\omega)$ by (A.50), and so

$$\begin{aligned}\int_E l(\omega)d\mathbb{P}(\omega) &\geqslant \int_E \gamma_k^{\mathcal{I}}(\omega)d\mathbb{P}(\omega) - \varepsilon \\ &\geqslant \int_E \gamma^{\mathcal{I}}(\omega)d\mathbb{P}(\omega) - \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ we get

$$\int_E l(\omega)d\mathbb{P}(\omega) \geqslant \int_E \gamma^{\mathcal{I}}(\omega)d\mathbb{P}(\omega) \text{ for all } E \in \mathcal{I}. \tag{A.52}$$

Combining (A.51) with (A.52) we deduce that

$$\int_E l(\omega)d\mathbb{P}(\omega) = \int_E \gamma^{\mathcal{I}}(\omega)d\mathbb{P}(\omega) \text{ for all } E \in \mathcal{I},$$

which implies that $l = \mathbb{E}^{\mathcal{I}}[\gamma^{\mathcal{I}}]$ by unicity of the conditional expectation. But $\gamma^{\mathcal{I}}$ is $\mathcal{I}$-measurable because $\gamma_k^{\mathcal{I}}$ is $\mathcal{I}$-measurable for all $k \in \mathbb{N}^*$, hence $\mathbb{E}^{\mathcal{I}}[\gamma^{\mathcal{I}}] = \gamma^{\mathcal{I}}$ and consequently $l = \gamma^{\mathcal{I}}$. ■

References

[AAB+10] Claire Anantharaman, Jean-Philippe Anker, Martine Babillot, Aline Bonami, Bruno Demange, Sandrine Grellier, François Havard, Philippe Jaming, Emmanuel Lesigne, Patrick Maheux, Jean-Pierre Otal, Barbara Schapira, and Jean-Pierre Schreiber. *Théorèmes ergodiques pour les actions de groupes*, volume 41 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*. L'Enseignement Mathématique, Geneva, 2010. With a foreword in English by Amos Nevo.

[AH10] Omar Anza Hafsa. On the integral representation of relaxed functionals with convex bounded constraints. *ESAIM Control Optim. Calc. Var.*, 16(1):37–57, 2010.

[AHCM17] Omar Anza Hafsa, Nicolas Clozeau, and Jean-Philippe Mandallena. Homogenization of nonconvex unbounded singular integrals. *Ann. Math. Blaise Pascal*, 24(2):135–193, 2017.

[AHLM11] Omar Anza Hafsa, Mohamed Lamine Leghmizi, and Jean-Philippe Mandallena. On a homogenization technique for singular integrals. *Asymptot. Anal.*, 74(3-4):123–134, 2011.

[AHM11] Omar Anza Hafsa and Jean-Philippe Mandallena. Homogenization of nonconvex integrals with convex growth. *J. Math. Pures Appl. (9)*, 96(2):167–189, 2011.

[AHM12a] Omar Anza Hafsa and Jean-Philippe Mandallena. Homogenization of unbounded singular integrals in $W^{1,\infty}$. *Ric. Mat.*, 61(2):185–217, 2012.

[AHM12b] Omar Anza Hafsa and Jean-Philippe Mandallena. On the relaxation of unbounded multiple integrals. Preprint, arXiv:1207.2652, 2012.

[AHM14] Omar Anza Hafsa and Jean-Philippe Mandallena. Radial representation of lower semicontinuous envelope. *Boll. Unione Mat. Ital.*, 7(1):1–18, 2014.

[AHM15] Omar Anza Hafsa and Jean-Philippe Mandallena. On the relaxation of variational integrals in metric Sobolev spaces. *Adv. Calc. Var.*, 8(1):69–91, 2015.

[AHM16] Omar Anza Hafsa and Jean Philippe Mandallena. Γ-limits of functionals determined by their infima. *J. Convex Anal.*, 23(1):103–137, 2016.

[AHM17] Omar Anza Hafsa and Jean-Philippe Mandallena. Γ-convergence of nonconvex integrals in Cheeger-Sobolev spaces and homogenization. *Adv. Calc. Var.*, 10(4):381–405, 2017.

[AHM18] Omar Anza Hafsa and Jean-Philippe Mandallena. Relaxation of nonconvex unbounded integrals with general growth conditions in Cheeger-Sobolev spaces. *Bull. Sci. Math.*, 142:49–93, 2018.

[AHM19a] Omar Anza Hafsa and Jean-Philippe Mandallena. Lower semicontinuity of integrals of the calculus of variations. Preprint, 2019.

[AHM19b] Omar Anza Hafsa and Jean-Philippe Mandallena. Subadditive theorems in metric measure spaces and homogenization in Cheeger-Sobolev spaces. Preprint, 2019.

[AHMZ15] Omar Anza Hafsa, Jean-Philippe Mandallena, and Hamdi Zorgati. Homogenization of unbounded integrals with quasiconvex growth. *Ann. Mat. Pura Appl. (4)*, 194(6):1619–1648, 2015.

[AK81] M. A. Akcoglu and U. Krengel. Ergodic theorems for superadditive processes. *J. Reine Angew. Math.*, 323:53–67, 1981.

[AM02] Felipe Alvarez and Jean-Philippe Mandallena. Homogenization of multiparameter integrals. *Nonlinear Anal.*, 50(6, Ser. A: Theory Methods):839–870, 2002.

[AM04] Felipe Alvarez and Jean-Philippe Mandallena. Multi-parameter homogenization by localization and blow-up. *Proc. Roy. Soc. Edinburgh Sect. A*, 134(5):801–814, 2004.

[BB00] Guy Bouchitté and Michel Bellieud. Regularization of a set function—application to integral representation. *Ricerche Mat.*, 49(suppl.):79–93, 2000. Contributions in honor of the memory of Ennio De Giorgi (Italian).

[BB11] Anders Björn and Jana Björn. *Nonlinear potential theory on metric spaces*, volume 17 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2011.

[BBS97] Guy Bouchitte, Giuseppe Buttazzo, and Pierre Seppecher. Energies with respect to a measure and applications to low-dimensional structures. *Calc. Var. Partial Differential Equations*, 5(1):37–54, 1997.

[BD98] Andrea Braides and Anneliese Defranceschi. *Homogenization of multiple integrals*, volume 12 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1998.

[BFM98] Guy Bouchitté, Irene Fonseca, and Luisa Mascarenhas. A global method for relaxation. *Arch. Rational Mech. Anal.*, 145(1):51–98, 1998.

[BG95] Andrea Braides and Adriana Garroni. Homogenization of periodic nonlinear media with stiff and soft inclusions. *Math. Models Methods Appl. Sci.*, 5(4):543–564, 1995.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[Bjö00] Jana Björn. L^q-differentials for weighted Sobolev spaces. *Michigan Math. J.*, 47(1):151–161, 2000.

[Bra85] Andrea Braides. Homogenization of some almost periodic coercive functional. *Rend. Accad. Naz. Sci. XL Mem. Mat. (5)*, 9(1):313–321, 1985.

[Buc99] Stephen M. Buckley. Is the maximal function of a Lipschitz function continuous? *Ann. Acad. Sci. Fenn. Math.*, 24(2):519–528, 1999.

[But89] Giuseppe Buttazzo. *Semicontinuity, relaxation and integral representation in the calculus of variations*, volume 207 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1989.

[Che99] J. Cheeger. Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.*, 9(3):428–517, 1999.

[CJLP02] Gregory A. Chechkin, Vasili V. Jikov, Dag Lukkassen, and Andrey L. Piatnitski. On homogenization of networks and junctions. *Asymptot. Anal.*, 30(1):61–80, 2002.

[CM98] Tobias H. Colding and William P. Minicozzi, II. Liouville theorems for harmonic sections and applications. *Comm. Pure Appl. Math.*, 51(2):113–138, 1998.

[CPS07] G. A. Chechkin, A. L. Piatnitski, and A. S. Shamaev. *Homogenization*, volume 234 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2007. Methods and applications, Translated from the 2007 Russian original by Tamara Rozhkovskaya.

[DG16] Mitia Duerinckx and Antoine Gloria. Stochastic homogenization of nonconvex unbounded integral functionals with convex growth. *Arch. Ration. Mech. Anal.*, 221(3):1511–1584, 2016.

[DGL77] E. De Giorgi and G. Letta. Une notion générale de convergence faible pour des fonctions croissantes d'ensemble. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(1):61–99, 1977.

[DGZ14] Anthony H. Dooley, Valentyn Ya. Golodets, and Guohua Zhang. Sub-additive ergodic theorems for countable amenable groups. *J. Funct. Anal.*, 267(5):1291–1320, 2014.

[DM93] Gianni Dal Maso. *An introduction to* Γ*-convergence*. Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston Inc., Boston, MA, 1993.

[DMM86] Gianni Dal Maso and Luciano Modica. Nonlinear stochastic homogenization and ergodic theory. *J. Reine Angew. Math.*, 368:28–42, 1986.

[DZ15] Anthony H. Dooley and Guohua Zhang. Local entropy theory of a random dynamical system. *Mem. Amer. Math. Soc.*, 233(1099):vi+106, 2015.

[Fed69] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[FHK99] B. Franchi, P. Hajłasz, and P. Koskela. Definitions of Sobolev classes on metric spaces. *Ann. Inst. Fourier (Grenoble)*, 49(6):1903–1924, 1999.

[GH13] Jasun Gong and Piotr Hajłasz. Differentiability of p-harmonic functions on metric measure spaces. *Potential Anal.*, 38(1):79–93, 2013.

[Gre69] Frederick P. Greenleaf. *Invariant means on topological groups and their applications*. Van Nostrand Mathematical Studies, No. 16. Van Nostrand Reinhold Co., New York-Toronto, Ont.-London, 1969.

[GT01] Vladimir Gol'dshtein and Marc Troyanov. Axiomatic theory of Sobolev spaces. *Expo. Math.*, 19(4):289–336, 2001.

[Haj03] Piotr Hajłasz. Sobolev spaces on metric-measure spaces. In *Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)*, volume 338 of *Contemp. Math.*, pages 173–218. Amer. Math. Soc., Providence, RI, 2003.

[Hei07] Juha Heinonen. Nonsmooth calculus. *Bull. Amer. Math. Soc. (N.S.)*, 44(2):163–232, 2007.

[HK98] Juha Heinonen and Pekka Koskela. Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.*, 181(1):1–61, 1998.

[HKST15] Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam, and Jeremy T. Tyson. *Sobolev spaces on metric measure spaces*, volume 27 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2015. An approach based on upper gradients.

[JKO94] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik. *Homogenization of differential operators and integral functionals*. Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian.

[Kei04] Stephen Keith. A differentiable structure for metric measure spaces. *Adv. Math.*, 183(2):271–315, 2004.

[Kre85] Ulrich Krengel. *Ergodic theorems*, volume 6 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1985. With a supplement by Antoine Brunel.

[Lin01] Elon Lindenstrauss. Pointwise theorems for amenable groups. *Invent. Math.*, 146(2):259–295, 2001.

[Man13] Jean-Philippe Mandallena. Localization principle and relaxation. *Adv. Calc. Var.*, 6(2):217–246, 2013.

[Mar78] Paolo Marcellini. Periodic solutions and homogenization of nonlinear variational problems. *Ann. Mat. Pura Appl. (4)*, 117:139–152, 1978.

[MH94] Jerrold E. Marsden and Thomas J. R. Hughes. *Mathematical foundations of elasticity*. Dover Publications, Inc., New York, 1994. Corrected reprint of the 1983 original.

[MM94] K. Messaoudi and G. Michaille. Stochastic homogenization of nonconvex integral functionals. *RAIRO Modél. Math. Anal. Numér.*, 28(3):329–356, 1994.

[Mül87] Stefan Müller. Homogenization of nonconvex integral functionals and cellular elastic materials. *Arch. Rational Mech. Anal.*, 99(3):189–212, 1987.

[OW87] Donald S. Ornstein and Benjamin Weiss. Entropy and isomorphism theorems for actions of amenable groups. *J. Analyse Math.*, 48:1–141, 1987.

[Pat88] Alan L. T. Paterson. *Amenability*, volume 29 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.

[Sha00] Nageswari Shanmugalingam. Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. *Rev. Mat. Iberoamericana*, 16(2):243–279, 2000.

[Tem92] Arkady Tempelman. *Ergodic theorems for group actions*, volume 78 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1992. Informational and thermodynamical aspects, Translated and revised from the 1986 Russian original.

[Zhi02] V. V. Zhikov. Averaging of problems in the theory of elasticity on singular structures. *Izv. Ross. Akad. Nauk Ser. Mat.*, 66(2):81–148, 2002.

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