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High-gain observers for a class of 2×2 quasilinear hyperbolic systems with 2 different velocities

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Abstract: In this work we extend recently proposed observer designs based on high-gain to a more general first order quasilinear hyperbolic system of balance laws. This class of systems is written in an observable form with two states, two different characteristic velocities and distributed measurement. The exponential stability of the related observation error is fully established by means of Lyapunov-based analysis.

Keywords: high-gain observers, quasilinear hyperbolic systems of balance laws, Lyapunov analysis, H^1 exponential stability.

1. INTRODUCTION

The classical high-gain observer design for finite-dimensional nonlinear systems has gained great academic interest through the last decades. They apply to a large class of cases corresponding to *uniformly observable* systems Gauthier and Bornard [1981], Gauthier et al [1992]. It has been extensively studied in the literature and remains widely considered, see Khalil [2017] and references therein. In the recent paper Kitsos et al [2018], we extended this approach to a class of hyperbolic systems, for which first results on high-gain observer design have been proposed for a particular case of uniformly observable systems, written as an $n \times n$ *quasi-linear hyperbolic system of balance laws* and considering distributed measurements. In a more recent work Kitsos et al [2019], we considered a more general case, where a 2×2 system is written in an observable form, differentiating from Kitsos et al [2018] where we needed to consider identical characteristic velocities. There exist some results on observer design for hyperbolic systems in the literature, mainly considering the full state vector on the boundaries as measurement. Amongst others, one can refer to Di Meglio et al [2013] and Hasan et al [2016] for the backstepping design, to Besançon et al [2006] for a discretization approach, to Castillo et al [2013] for direct infinite-dimension-based Lyapunov techniques (see also Besançon et al [2013]) or to Nguyen et al [2016] for optimization methods. For semigroup-based methods see Curtain [1982], Christophides and Daoutidis [1996] and Schaum et al [2015].

The present paper aims at providing sufficient conditions for high-gain observer design for a class of quasilinear hyperbolic systems of balance laws. The difficulty of the present approach comes from the fact that the Lyapunov stability analysis that we employ requires the existence of a positive definite symmetric Lyapunov matrix involved in the chosen Lyapunov functional, which has to additionally

commute with the matrix of the characteristic velocities. For this reason, considering stability problems in Bastin and Coron [2016], Coron and Bastin [2015] and in Prieur et al [2014] (Proposition 2.1) and other approaches of these authors, either such a commutativity property is assumed to be satisfied, or a diagonal Lyapunov matrix is chosen. In our case, the Lyapunov matrix cannot commute with the matrix of the characteristic velocities, since it is required to satisfy simultaneously a matrix Lyapunov equation and the involved stabilizable matrix, which describes the linear part of the balance laws, does not allow diagonal stability to hold. Due to this technical limitation, in the first approach Kitsos et al [2018], the characteristic velocities were considered identical. In Kitsos et al [2019] we considered a more general problem for a 2×2 system, where space derivatives of the output were injected in the observer's equations as correction terms, in order to confront this difficulty. The contribution of the present paper is twofold: First, by employing a triangular linear coordinates transformation technique, we avoid the restrictive injection of output's space derivatives, in order to confront the problem of distinct characteristic velocities. Second, we prove a stronger result of H^1 exponential stability of the observer error, contrary to the exponential stability result in the supremum norm (Kitsos et al [2018]).

The problem is illustrated in details in Section 2. The theoretical analysis and the full proofs for the observer design that we develop are presented in Section 3, where Theorem 2 constitutes the main result. In addition, an example illustrates the nature of our methodology. Some conclusions and perspectives are discussed in Section 4.

Notation: For a given $x \in \mathbb{R}^n$, $|x|$ denotes its usual Euclidean norm. For a given constant matrix $A \in \mathbb{R}^{m \times n}$, A^T denotes its transpose, $|A| := \sup\{|Ax|, |x| = 1\}$ is its induced norm and $\text{Sym}(A) = \frac{A+A^T}{2}$ stands for its symmetric part. By $\underline{\text{eig}}(A)$ we denote the minimum eigenvalue

of a matrix A . For a function $f(\cdot)$, we use the difference operator given by $\Delta_{\hat{\xi}}[f](\xi) := f(\hat{\xi}) - f(\xi)$, parametrized by $\hat{\xi}$. For $f \in C^1$ by Df we denote its Jacobian. For a continuous (C^0) map $[0, 1] \ni x \mapsto \xi(x) \in \mathbb{R}^n$ we adopt the notation $\|\xi\|_0 := \max\{|\xi(x)|, x \in [0, 1]\}$. For a continuously differentiable (C^1) map $[0, 1] \ni x \mapsto \xi(x) \in \mathbb{R}^n$ we adopt the notation $\|\xi\|_1 := \|\xi\|_0 + \|\xi_x\|_0$. For a function $\xi \in H^1([0, 1]; \mathbb{R}^n)$ the definition of the H^1 -norm is $\|\xi\|_{H^1} := \left(\int_0^1 (|\xi|^2 + |\xi_x|^2) dx\right)^{1/2}$. By $\mathcal{B}(\delta)$ we denote the set $\mathcal{B}(\delta) := \{\xi \in C^1([0, +\infty) \times [0, 1]; \mathbb{R}^2) : \|\xi(t, \cdot)\|_1 \leq \delta, \forall t \geq 0\}$.

2. CLASS OF SYSTEMS AND MAIN OBSERVER RESULT

Let us consider the first-order quasilinear hyperbolic system described by the following equations on a time and space domain $\Pi := [0, +\infty) \times [0, 1]$:

$$\xi_t(t, x) + \Lambda(\xi_1(t, x))\xi_x(t, x) = A\xi(t, x) + f(\xi(t, x)) \quad (1a)$$

where $[\xi_1 \ \xi_2]^T = \xi : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}^2$ is the state. Consider also distributed measurement $y : [0, +\infty) \times [0, 1]$ that is available at the output, given by

$$y(t, x) = C\xi(t, x) \quad (1b)$$

We assume that

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, C = [1 \ 0],$$

$$\Lambda(\xi_1) := \text{diag}\{\lambda_1(\xi_1), \lambda_2(\xi_2)\}, f(\xi) = \begin{bmatrix} f_1(\xi_1) \\ f_2(\xi_1, \xi_2) \end{bmatrix}$$

with $\lambda_1(\xi_1), \lambda_2(\xi_1) > 0, \forall \xi_1 \in \mathbb{R}$. We observe that the system satisfies some triangular structure (as in Kitsos et al [2018]), which illustrates an analogy to the finite-dimensional case.

We consider initial and boundary conditions of the form

$$\xi(0, x) =: \xi^0(x), x \in [0, 1] \quad (2a)$$

$$\xi(t, 0) = \mathcal{H}(\xi_1(t, 1)), t \in [0, +\infty) \quad (2b)$$

Considering system's dynamics, we assume the following regularity.

A0. The involved mappings $\lambda_i \in C^1(\mathbb{R}; \mathbb{R}^2), i = 1, 2, f \in C^1(\mathbb{R}^2; \mathbb{R}^2), \mathcal{H} \in C^1(\mathbb{R}; \mathbb{R}^2), \xi^0 \in C^1([0, 1]; \mathbb{R}^2)$.

The following assumption is essential to assert the well-posedness of our system, along with the minimal observer design requirement of "forward completeness" and, furthermore, it imposes boundedness of the classical solutions in the C^1 -norm, which is essential in the design of our observer. For more detailed presentation, the reader can refer to Bastin and Coron [2016], Li [1985] and references therein, where sufficient conditions for the well-posedness of quasilinear hyperbolic systems of balance laws are given.

A1. There exists nonempty compact set

$\mathcal{M} \subset C^1([0, 1]; \mathbb{R}^2)$, such that, for any initial conditions $\xi^0 \in \mathcal{M}$, satisfying zero-order and one-order compatibility conditions, problem (1a), (2) admits a unique classical solution in $C^1([0, +\infty) \times [0, 1]; \mathbb{R}^2)$. Moreover, for any ξ^0 in the above-mentioned class, there exists $\delta > 0$, such that $\|\xi(t, \cdot)\|_1 \leq \delta, \forall t \in [0, +\infty)$. Also, $\lambda_1(C\xi) \geq \lambda_2(C\xi), \forall \xi \in \mathcal{B}(\delta)$.

The following global Lipschitzness assumption is crucial for the stability analysis of the observer error equation.

A2. There exists Lipschitz constant $L_f > 0$, such that for all $\xi \in \mathbb{R}^2, |Df(\xi)| \leq L_f$. There exists Lipschitz constant $L_{f'} > 0$, such that for all $\xi, \hat{\xi} \in \mathbb{R}^2, |\Delta_{\hat{\xi}}[Df](\xi)| \leq L_{f'}|\hat{\xi} - \xi|$.

In order to be able to adopt a high-gain observer methodology, we need to first perform the following invertible linear transformation, in order to obtain a more appropriate form:

$$\zeta = T\xi \quad (3a)$$

$$T := \begin{bmatrix} 1 & 0 \\ a_2 & 1 \end{bmatrix} \quad (3b)$$

with a_2 to be defined later.

Now system (1a), (1b), (2b) is rewritten as follows:

$$\zeta_t(t, x) + \bar{\Lambda}(\zeta_1(t, x))\zeta_x(t, x) = A\zeta(t, x) + M\zeta(t, x) + Tf(T^{-1}\zeta(t, x)) \quad (4a)$$

$$y(t, x) = C\zeta(t, x) \quad (4b)$$

$$\zeta(t, 0) = T\mathcal{H}(\zeta_1(t, 1)) \quad (4c)$$

defined on the domain Π , where M is given by

$$M := \begin{bmatrix} -a_2 & 0 \\ -a_2^2 & a_2 \end{bmatrix} \quad (5)$$

and

$$\bar{\Lambda}(\zeta_1) := TC^T C(\lambda_1(\zeta_1) - \lambda_2(\zeta_1)) + \lambda_2(\zeta_1) I_{2 \times 2} \quad (6)$$

Consider now $K \in \mathbb{R}^2$ and P a symmetric and positive definite matrix satisfying

$$2\text{Sym}(P(A + KC)) = -I_{2 \times 2} \quad (7)$$

which is always feasible, due to the observability of the pair (A, C) . Define also diagonal matrix Θ , given by

$$\Theta := \text{diag}\{\theta, \theta^2\} \quad (8)$$

where $\theta > 1$ is the candidate high-gain constant of the observer, which will be selected later. Then, set

$$a_2 := a_{2,0}/a_{1,0} \quad (9a)$$

$$\begin{bmatrix} a_{1,0} \\ a_{2,0} \end{bmatrix} := \theta^{-1}\Theta P^{-1}C^T \quad (9b)$$

Notice that a_2 is written in the form

$$a_2 = \theta \bar{a}_2 \quad (10)$$

where \bar{a}_2 is independent of θ and depends only on components of P .

Let us now introduce our candidate observer dynamics defined on the domain Π and its boundary conditions for system (4), as follows:

$$\begin{aligned} \hat{\zeta}_t(t, x) + \bar{\Lambda}(y(t, x))\hat{\zeta}_x &= A\hat{\zeta}(t, x) + M\hat{\zeta}(t, x) \\ - \left(\Theta K - \begin{bmatrix} 2a_2 \\ a_2^2 \end{bmatrix} \right) \left(y(t, x) - C\hat{\zeta}(t, x) \right) &+ Tf(T^{-1}\hat{\zeta}(t, x)) \end{aligned} \quad (11a)$$

$$\hat{\zeta}(t, 0) = T\mathcal{H}(y(t, 1)), t \geq 0 \quad (11b)$$

The following lemma guarantees the existence of unique global classical solutions for our candidate observer. We invoke Kmit [2008], where an analogous result is proven under Lipschitzness properties of the dynamics. It is easy

to check that our candidate observer under the transformation $\zeta := T^{-1}\hat{\zeta}$ is written in a well-posed characteristic form and satisfies semilinear hyperbolic laws. Assumptions A0 - A2 in conjunction with the previously mentioned comments (details are left to the reader) are compatible with the sufficient conditions of Theorem 2.1 in Kmit [2008] and, thereby similar global existence result is established for our observer system, as given in the following result.

Lemma 1. Under Assumptions A0, A2 and considering $y \in C^1([0, +\infty) \times [0, 1]; \mathbb{R})$, with y globally bounded in the C^1 -norm, the problem described by (11) on domain Π and initial conditions $\hat{\zeta}^0 := \hat{\zeta}(0, x), \forall x \in [0, 1]$, satisfying zero-order and one-order compatibility conditions (see Bastin and Coron [2016] for details on compatibility conditions) admits a unique classical solution on Π , i.e., there exists a unique solution $\hat{\zeta} \in C^1([0, +\infty) \times [0, 1]; \mathbb{R}^2)$.

We are now in a position to present our main result on the observer design.

Theorem 2. Consider system (1a) - (2), defined on Π with output (1b) and suppose that Assumptions A0, A1 and A2 hold. Let also $P \in \mathbb{R}^{2 \times 2}$ be a positive definite symmetric matrix and K a vector both satisfying (7). Then, for $\theta > 1$, system (11) with initial condition $\hat{\zeta}^0 \in C^1([0, 1]; \mathbb{R}^2)$, with $\hat{\zeta}(0, x) = \hat{\zeta}^0(x)$, satisfying zero-order and one-order compatibility conditions, is a well-posed high-gain observer for $\zeta = T\xi$, in the sense that it admits a unique classical solution in Π on the one hand, providing an estimate for the state of system (1a) - (2) for choice of θ large enough on the other hand. More precisely, for any $\kappa > 0$, there exist $\theta > 1$, such that the following estimate is satisfied:

$$\|\xi(t, \cdot) - T^{-1}\hat{\zeta}(t, \cdot)\|_{H^1} \leq l e^{-\kappa t} \|\xi^0 - T^{-1}\hat{\zeta}^0\|_{H^1}, t \geq 0 \quad (12)$$

for some $l > 0$ depending on θ .

This theorem states that for system (1a) - (2) we have a systematic high-gain observer design providing an estimate of its full state, with a convergence rate adjustable via the high gain θ .

3. OBSERVER CONVERGENCE PROOF

This section is dedicated to the proof of Theorem 2.

We define the linearly transformed error by

$$\epsilon := \Theta^{-1} \left(\hat{\zeta} - \zeta \right)$$

and we obtain the following equations defined on Π :

$$\begin{aligned} \epsilon_t + \Lambda_1(y)\epsilon_x &= \theta(A + KC)\epsilon \\ + \theta\bar{a}_2\epsilon + \Theta^{-1}T\Delta_{T^{-1}\hat{\zeta}}[f](T^{-1}\zeta) & \end{aligned} \quad (13)$$

$$\epsilon(t, 0) = 0 \quad (14)$$

where

$$\Lambda_1(y) := \frac{1}{a_{1,0}} P^{-1} C^T C (\lambda_1(y) - \lambda_2(y)) + \lambda_2(y) I_{2 \times 2} \quad (15)$$

Observe from (15) that $P\Lambda_1(y)$ is symmetric for any symmetric matrix P , a fact that will allow an integration by parts that we will perform later.

At this point, let us introduce the operator

$$\mathcal{K} : C^1([0, +\infty) \times [0, 1]; \mathbb{R}^2) \rightarrow C^0([0, +\infty) \times [0, 1]; \mathbb{R}^{2 \times 2})$$

defined by

$$\mathcal{K}[\xi] := \Lambda_1'(C\xi)\Lambda_1^{-1}(C\xi)C \frac{\partial}{\partial t} \xi \quad (16)$$

Define also

$$\begin{aligned} \mathcal{K}_1^{\hat{\zeta}} : \\ C^1([0, +\infty) \times [0, 1]; \mathbb{R}^2) \rightarrow C^0([0, +\infty) \times [0, 1]; \mathbb{R}^{2 \times 2}) \end{aligned}$$

$$\begin{aligned} \mathcal{K}_2^{\hat{\zeta}} : \\ C^1([0, +\infty) \times [0, 1]; \mathbb{R}^2) \rightarrow C^0([0, +\infty) \times [0, 1]; \mathbb{R}^2) \end{aligned}$$

(parametrized by $\hat{\zeta}$ and acting on $\xi = T^{-1}\zeta$) by

$$\mathcal{K}_1^{\hat{\zeta}}[\xi] := \mathcal{K}[\xi] + \Theta^{-1}TDf(T^{-1}\hat{\zeta})T^{-1}\Theta \quad (17a)$$

$$\begin{aligned} \mathcal{K}_2^{\hat{\zeta}}[\xi] := -\mathcal{K}[\xi]\Theta^{-1}T\Delta_{T^{-1}\hat{\zeta}}[f](T^{-1}\zeta) \\ + \Theta^{-1}T\Delta_{T^{-1}\hat{\zeta}}[Df](T^{-1}\zeta) \frac{\partial}{\partial t} \xi \end{aligned} \quad (17b)$$

Next by temporarily assuming that ϵ is of class C^2 , we derive the following hyperbolic equations for ϵ_t :

$$\begin{aligned} \epsilon_{tt} + \Lambda_1(y)\epsilon_{tx} &= \mathcal{K}_1^{\hat{\zeta}}[\xi]\epsilon_t + \theta\bar{a}_2\epsilon_t \\ + \theta(A + KC)\epsilon_t - \theta\mathcal{K}[\xi](A + KC)\epsilon + \mathcal{K}_2^{\hat{\zeta}}[\xi] & \end{aligned} \quad (18)$$

$$\epsilon_t(t, 0) = 0 \quad (19)$$

Remark 3. Notice that whenever $\xi \in \mathcal{B}(\delta)$, due to continuity and positiveness of $\lambda_i, i = 1, 2$ and furthermore the fact that $\|\xi(t, \cdot)\|_0 \leq \delta, \forall t \geq 0, 0 < \sup_{\xi \in \mathcal{B}(\delta)} (\lambda_i(C\xi)), \inf_{\xi \in \mathcal{B}(\delta)} (\lambda_i(C\xi)) < +\infty, i = 1, 2$. In addition, whenever $\xi \in \mathcal{B}(\delta)$, as a result of the hyperbolic dynamics (1a), we easily calculate constant $\delta_1 > 0$, such that

$$\|\xi_t(t, \cdot)\|_0 = \|T^{-1}\zeta_t(t, \cdot)\|_0 \leq \delta_1, \forall t \geq 0 \quad (20)$$

By virtue of the above arguments, the fact that $\theta > 1$, (20), continuity and global Lipschitzness (Assumptions A0, A2) of the involved mappings, we can easily calculate positive constants $\gamma_i, i = 1, \dots, 6$, such that whenever $\xi \in \mathcal{B}(\delta)$, the following inequalities are satisfied for all $\hat{\zeta} \in C^0([0, +\infty) \times [0, 1]; \mathbb{R}^2), t \geq 0, x \in [0, 1]$:

$$|\Theta^{-1}TDf(T^{-1}\hat{\zeta}(t, x))T^{-1}\Theta| \leq \gamma_1 \quad (21a)$$

$$|\Theta^{-1}T\Delta_{T^{-1}\hat{\zeta}}[f](T^{-1}\zeta(t, x))| \leq \gamma_2|\epsilon(t, x)| \quad (21b)$$

$$|\Theta^{-1}T\Delta_{T^{-1}\hat{\zeta}}[Df](T^{-1}\zeta(t, x))| \leq \gamma_3|\epsilon(t, x)| \quad (21c)$$

$$|\Lambda_1'(\xi_1(t, x))| \leq \gamma_4 \quad (21d)$$

$$\gamma_5 \leq |\Lambda_1(\xi_1(t, x))| \leq \gamma_6 \quad (21e)$$

Note that all the above constants $\gamma_1, \dots, \gamma_6$ depend only on the Lipschitz constants of Assumption A2, constants δ, δ_1 , elements of matrices P and Λ and are independent of the high gain θ , due to the triangular structures of the nonlinear mappings. We can similarly obtain constants $\gamma_7, \gamma_8 > 0$, such that for all $(t, x) \in \Pi$, we get

$$|\mathcal{K}_1^{\hat{\zeta}}[\xi](t, x)| \leq \gamma_7 := \gamma_1 + \delta_1\gamma_4/\gamma_5,$$

$$|\mathcal{K}_2^{\hat{\zeta}}[\xi](t, x)| \leq \gamma_8|\epsilon(t, x)|; \gamma_8 := \delta_1\gamma_4\gamma_2/\gamma_5 + \gamma_3\delta_1 \quad (22)$$

To prove the error exponential stability, let us define functional $\mathcal{W} : C^1([0, 1]; \mathbb{R}^2) \rightarrow \mathbb{R}$ by

$$\mathcal{W}[\epsilon] := \int_0^1 e^{-\mu\theta\bar{a}_2x} (\epsilon^T P \epsilon + \rho_0 \epsilon_t^T P \epsilon_t) dx \quad (23)$$

where $\rho_0 \in (0, 1]$ is a constant (to be chosen appropriately), $P \in \mathbb{R}^{2 \times 2}$ is a positive definite symmetric matrix satisfying (7) for appropriate K and $\mu > 0$ will be chosen appropriately later.

By invoking Lemma 1 and Assumption A1, which establish global unique classical solutions for observer system (11) and system (1a), (2) respectively, we are now in a position to define $W : [0, +\infty) \rightarrow \mathbb{R}$ by $W(t) := \mathcal{W}[\epsilon](t), t \geq 0$ (where we use the notation $\epsilon(t) := \epsilon(t, x), \forall x \in [0, 1]$). Calculating the time-derivative \dot{W} along the classical solutions of (13) - (14), (18) - (19), we get

$$\begin{aligned} \dot{W} &= \int_0^1 e^{-\mu\theta\bar{a}_2x} \\ &\times (\epsilon_t^T P \epsilon + \epsilon^T P \epsilon_t + \rho_0 \epsilon_{tt}^T P \epsilon_t + \rho_0 \epsilon_t^T P \epsilon_{tt}) dx \end{aligned} \quad (24)$$

In the following, we omit the arguments (t, x) of the mappings inside the integrals.

After substituting the dynamical equations (13) and (18) into the above equation and applying integration by parts, \dot{W} can be written in the following form:

$$\dot{W} = T_1 + T_2 + T_3 + T_4 \quad (25)$$

where

$$\begin{aligned} T_1 &:= -e^{-\mu\theta\bar{a}_2} (\epsilon(1)^T P \Lambda_1(y(1)) \epsilon(1) \\ &+ \rho_0 \epsilon_t(1)^T P \Lambda_1(y(1)) \epsilon_t(1)) \end{aligned} \quad (26a)$$

$$\begin{aligned} T_2 &:= \int_0^1 e^{-\mu\theta\bar{a}_2x} (-\mu\theta\bar{a}_2 \epsilon^T P \Lambda_1(y) \epsilon - \mu\theta\bar{a}_2 \rho_0 \epsilon_t^T P \Lambda_1(y) \epsilon_t \\ &+ \epsilon^T P \Lambda_1'(y) y_x \epsilon + \rho_0 \epsilon_t^T P \Lambda_1'(y) y_x \epsilon_t) dx \end{aligned} \quad (26b)$$

$$\begin{aligned} T_3 &:= 2 \int_0^1 e^{-\mu\theta\bar{a}_2x} (\epsilon^T P \Theta^{-1} T \Delta_{T^{-1}\zeta} [f] (T^{-1}\zeta) \\ &+ \rho_0 \epsilon_t^T P \mathcal{K}_2^{\hat{\zeta}}[\xi] + \rho_0 \epsilon_t^T \text{Sym}(P \mathcal{K}_1^{\hat{\zeta}}[\xi]) \epsilon_t \\ &+ \theta \bar{a}_2 (\epsilon^T P \epsilon + \rho_0 \epsilon_t^T P \epsilon_t)) dx \end{aligned} \quad (26c)$$

$$\begin{aligned} T_4 &:= \theta \int_0^1 e^{-\mu\theta\bar{a}_2x} \\ &\times (2\epsilon^T \text{Sym}(P(A + KC)) \epsilon + 2\rho_0 \epsilon_t^T \text{Sym}(P(A + KC)) \epsilon_t \\ &- \rho_0 \epsilon_t^T P \mathcal{K}[\xi](A + KC) \epsilon - \rho_0 \epsilon^T (A + KC)^T \mathcal{K}^T[\xi] P \epsilon_t) dx \end{aligned} \quad (26d)$$

It turns out from the above equations that

$$T_1 \leq 0 \quad (27)$$

Next, observe that the term T_2 can be bounded as follows:

$$\begin{aligned} T_2 &\leq (\alpha - \mu\theta\bar{a}_2 \inf_{\xi \in \mathcal{B}(\delta)} \lambda_2(C\xi)) W \\ &- \frac{\mu\theta\bar{a}_2}{a_{1,0}} \int_0^1 e^{-\mu\theta\bar{a}_2x} (\lambda_1(y) - \lambda_2(y)) (\epsilon_1^2 + \rho_0 (\partial_t \epsilon_1)^2) dx \end{aligned} \quad (28)$$

where $\alpha := \frac{|P|\delta\gamma_4}{\text{eig}(P)}$ and by virtue of Assumption A1, we finally obtain

$$T_2 \leq (\alpha - \mu\theta\bar{a}_2 \inf_{\xi \in \mathcal{B}(\delta)} \lambda_2(C\xi)) W \quad (29)$$

By exploiting (21e), (22), T_3 can be bounded as follows:

$$\begin{aligned} T_3 &\leq \int_0^1 e^{-\mu\theta\bar{a}_2x} \\ &\times (2|P|(\gamma_2|\epsilon|^2 + \rho_0\gamma_8|\epsilon||\epsilon_t| + \rho_0\gamma_7|\epsilon_t|^2)) dx + 2\theta\bar{a}_2 W \\ &\leq \left((\gamma_8 + 2 \max(\gamma_2, \gamma_7)) \frac{|P|}{\text{eig}(P)} + 2\theta\bar{a}_2 \right) W \end{aligned} \quad (30)$$

The term T_4 can be rewritten in the following form:

$$T_4 := -\theta \int_0^1 e^{-\mu\theta\bar{a}_2x} \left(\begin{bmatrix} \epsilon^T & \epsilon_t^T \end{bmatrix} \Sigma[\xi] \begin{bmatrix} \epsilon \\ \epsilon_t \end{bmatrix} \right) dx \quad (31)$$

where, after utilizing (7), $\Sigma : \mathcal{B}(\delta) \rightarrow C^0([0, +\infty) \times [0, 1]; \mathbb{R}^{4 \times 4})$ is given by

$$\Sigma[\xi] := \begin{bmatrix} I_{2 \times 2} & -\rho_0(A + KC)^T \mathcal{K}^T[\xi] P \\ -\rho_0 P \mathcal{K}[\xi](A + KC) & \rho_0 I_{2 \times 2} \end{bmatrix} \quad (32)$$

We can easily verify that $\Sigma[\xi] \succ 0$ (by solving a simple LMI) for every $\xi \in \mathcal{B}(\delta)$ if

$$\rho_0 < \frac{\gamma_5^2}{\gamma_4^2 \delta^2 |P|^2 |A + KC|^2} \quad (33)$$

It turns out that for every choice of matrices P and K satisfying equation (7), there always exists a ρ_0 (sufficiently small), such that the above inequality is satisfied and this fact renders $\Sigma[\xi]$ positive. Consequently, for appropriate choice of ρ_0 , there exists $\sigma > 0$, such that

$$T_4 \leq -\theta \frac{\sigma}{|P|} W \quad (34)$$

Now, choose μ , such that

$$\mu > \frac{2}{\inf_{\xi \in \mathcal{B}(\delta)} \lambda_2(C\xi)} \quad (35)$$

with the right-hand side being well-defined by Remark 3.1. Combining equations (27), (29), (30), (34), (35) with (25), we deduce

$$\dot{W} \leq (-\theta\omega_1 + \omega_2) W \quad (36)$$

where $\omega_1 := \frac{\sigma}{|P|}$, $\omega_2 := \alpha + (\gamma_8 + 2 \max(\gamma_2, \gamma_7)) \frac{|P|}{\text{eig}(P)}$. We obtained the estimate (36) of \dot{W}_p for ϵ of class C^2 , but the proof so far implies that the result does not depend on the C^2 -norms. Therefore, by invoking density arguments, the results remain valid with ϵ only of class C^1 .

Applying the comparison lemma to (36), we get

$$W(t) \leq e^{-(\theta\omega_1 - \omega_2)t} W(0), \forall t \geq 0 \quad (37)$$

Now, one can select the high-gain θ , such that

$$\theta > \max(1, \theta_0); \theta_0 := \omega_2 / \omega_1 \quad (38)$$

and, therefore, for sufficiently large θ we achieve to obtain $\theta\omega_1 - \omega_2 > 0$. In order to derive an estimation of the H^1 -norm, we can observe that a relationship of the following form can be deduced from the dynamics (13):

$$\begin{aligned} \gamma_5 \|\epsilon_x(t, \cdot)\|_{L^2} &\leq \|\epsilon(t, \cdot)\|_{L^2} \\ &\leq \gamma_6 \|\epsilon_x(t, \cdot)\|_{L^2} + (\theta|A + KC| + \theta|\bar{a}_2| + \gamma_2) \|\epsilon(t, \cdot)\|_{L^2} \end{aligned} \quad (39)$$

Performing trivial inequalities and using the above-mentioned relation, we can derive a constant $c > 0$ dependent on θ , such that for all $t \geq 0$,

$$\|\epsilon(t, \cdot)\|_{H^1} \leq c \rho_0^{-1/2} \sqrt{\frac{|P|}{\text{eig}(P)}} e^{-\frac{\omega_1}{2}(\theta - \theta_0)t} \|\epsilon^0\|_{H^1} \quad (40)$$

where $\epsilon^0(x) := \epsilon(0, x)$ is the initial condition of the error.

In the above derivations we have used the inequality

$$\rho_0 \text{eig}(P) \|\epsilon(t, \cdot)\|_{H^1}^2 \leq W(t) \leq e^{\mu\bar{a}_2\theta} |P| \|\epsilon(t, \cdot)\|_{H^1}^2, t \geq 0 \quad (41)$$

Now, it is clear that for any $\kappa > 0$, one can choose $\theta = \frac{2}{\omega_1} \kappa + \theta_0$, so as to get an estimation error as in (12), with l being dependent on θ .

Hence, we designed an exponential in the H^1 -norm high-gain observer which convergences to zero after an initial time t_0 and with tunable convergence rate κ , dependent

on the selection of θ . The higher the values θ attains, the faster the observation error converges to zero.

Remark 4. Let us remark here, that in (12), constant l depends exponentially on θ , contrary to the classical high-gain observer design results, where it should only have polynomial dependence. In different formulation, we can rewrite (12) as

$$\|\xi(t, \cdot) - T^{-1}\hat{\zeta}(t, \cdot)\|_{H^1} \leq \bar{l}e^{-\kappa(t-t_0)}\|\xi^0 - T^{-1}\hat{\zeta}^0\|_{H^1}, t \geq 0 \quad (42)$$

where, as we showed in the proof, \bar{l} depends polynomially on θ and

$$t_0 := \frac{\mu\bar{a}_2\theta}{\omega_1(\theta - \theta_0)} \quad (43)$$

denotes the time from which the exponential converges starts and, by definition, it depends only on the minimum value of $\lambda_2(\cdot)$. It is worthwhile to remark that, the previous limitation leads to a slightly weaker result, compared to our previous approaches Kitsos et al [2018], Kitsos et al [2019], although here, we confronted the problem with distinct characteristic velocities by avoiding the restrictive limitation of injecting output's spatial derivatives, as in Kitsos et al [2019]. Future studies will be dedicated to this.

To better illustrate the nature of the high-gain observer design, we use an example.

Example 1. Consider system

$$\partial_t \xi_1 + 0.1(2 + \cos(\xi_1))\partial_x \xi_1 = \xi_2 + \sin(\xi_1), \quad (44a)$$

$$\partial_t \xi_2 + 0.1(2 + \sin(\xi_1))\partial_x \xi_2 = \sin(\xi_2 - \xi_1), \quad (44b)$$

$\forall(t, x) \in \Pi$, with distributed measurement

$$y = \xi_1 \quad (44c)$$

and boundary conditions of the form

$$\xi(t, 0) = 0, t \geq 0 \quad (44d)$$

Consider initial conditions $\xi_1^0(x) = x^2$, $\xi_2^0(x) = -x^2/2$, $x \in [0, 1]$. System (44) is of the form (1a) with boundary conditions described by (2). More particularly, $\Lambda(\xi_1) = \text{diag}(2 + \cos(\xi_1), 2 + \sin(\xi_1))$, $f(\xi) = \begin{bmatrix} \sin(\xi_1) \\ \sin(\xi_2 - \xi_1) \end{bmatrix}$,

$\mathcal{H}(\cdot) = 0$. All Assumptions A0 - A2 that we have assumed for system (1a) are satisfied for the choice of these initial conditions. We choose vector gain $K = (-2, -1)^T$ and after calculating all the essential constants that are used in Theorem 2, we can proceed to the observer design. We appropriately choose the high gain constant θ being equal to 50. We also calculate a_2 by (9) in order to perform the transformation T as in (3) into the system ζ and we can, therefore, obtain the high-gain observer dynamics for ζ as in (11). Finally, we choose arbitrary observer initial conditions (in accordance with the compatibility conditions) $\hat{\zeta}_1^0(x) = \hat{\zeta}_2^0(x) = 0$, $x \in [0, 1]$.

In Figure 1 the solution ξ_1 is shown and in Figures 2 and 3 we illustrate the estimation error functions for both states, which exhibit exponential convergence to zero, as predicted by Theorem 2.

4. CONCLUSION

In this paper we designed a high-gain observer for a class of observable hyperbolic systems with distributed measurement. This result constitutes an extension of the high-gain

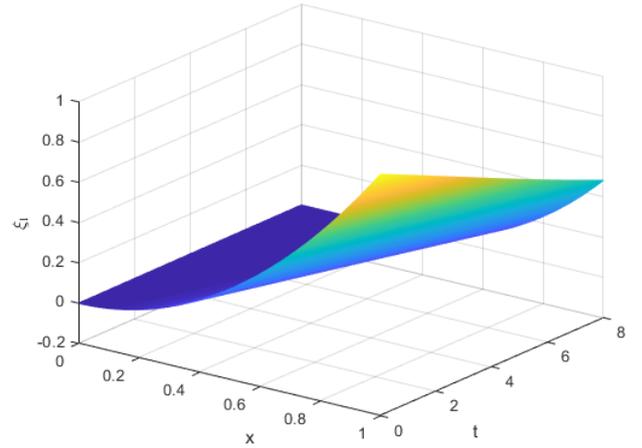


Fig. 1. Time and Space Evolution of Solution ξ_1 (system's output)

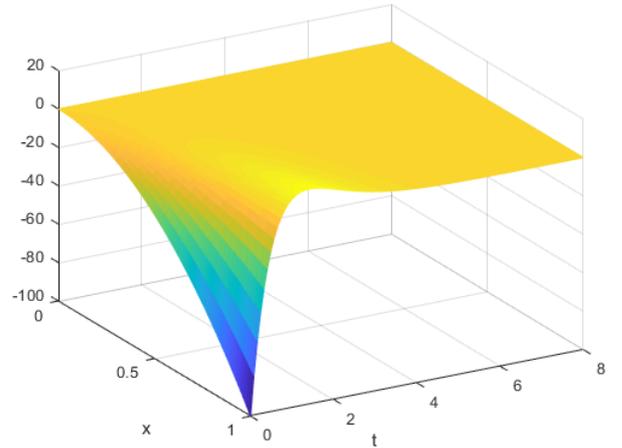


Fig. 2. Time and Space Evolution of Observer Error $T^{-1}\hat{\zeta}_1 - \xi_1$

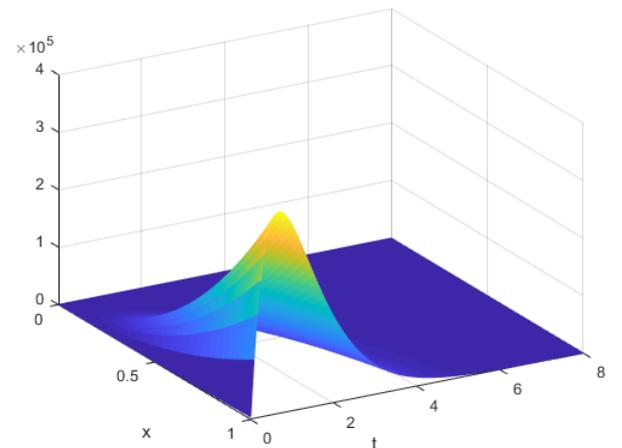


Fig. 3. Time and Space Evolution of Observer Error $T^{-1}\hat{\zeta}_2 - \xi_2$

observer design for finite-dimensional systems to a class of hyperbolic systems and, also, an extension of our previous works towards this direction, as we considered here distinct characteristic velocities. We proved the exponential decay of the observer error in the H^1 - norm step by step by first choosing an appropriate Lyapunov functional. The extension of this methodology to more general cases of hyperbolic systems and weakening some of our assumptions, in order this methodology to apply to real systems, like chemical reactors, is subject to our future approaches.

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