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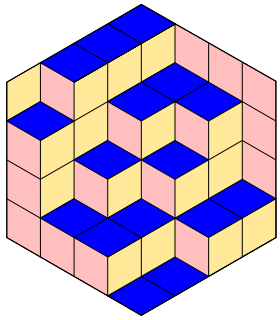

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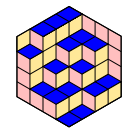

On structure constants of Iwahori–Hecke algebras for Kac–Moody groups

Nicole Bardy-Panse & Guy Rousseau

Abstract We consider the Iwahori–Hecke algebra ${}^I\mathscr{H}$ associated to an almost split Kac–Moody group G (affine or not) over a nonarchimedean local field $\mathcal{K}$. It has a canonical double-coset basis $(T_{\mathbf{w}})_{\mathbf{w}\in W^+}$ indexed by a sub-semigroup W^+ of the affine Weyl group W. The multiplication is given by structure constants $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} \in \mathbb{N} = \mathbb{Z}_{\geqslant 0}$: $T_{\mathbf{w}} * T_{\mathbf{v}} = \sum_{\mathbf{u}\in P_{\mathbf{w},\mathbf{v}}} a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} T_{\mathbf{u}}$. A conjecture, by Braverman, Kazhdan, Patnaik, Gaussent and the authors, tells that $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is a polynomial, with coefficients in $\mathbb{N}$, in the parameters $q_i - 1, q'_i - 1$ of G over $\mathcal{K}$. We prove this conjecture when $\mathbf{w}$ and $\mathbf{v}$ are spherical or, more generally, when they are said to be generic: this includes all cases of $\mathbf{w}, \mathbf{v} \in W^+$ if G is of affine or strictly hyperbolic type. In the split affine case (where $q_i = q'_i = q$, $\forall i$) we get a universal Iwahori–Hecke algebra with the same basis $(T_{\mathbf{w}})_{\mathbf{w}\in W^+}$ over a polynomial ring $\mathbb{Z}[Q]$; it specializes to ${}^I\mathscr{H}$ when one sets $Q = q$.

Introduction

Let G be a split, semi-simple, simply connected algebraic group over a non archimedean local field $\mathcal{K}$. So $\mathcal{K}$ is complete for a discrete, non trivial valuation with a finite residue field κ. We write $\mathcal{O} \subset \mathcal{K}$ for the ring of integers and q for the cardinality of κ. Then G is locally compact. In this situation, Nagayoshi Iwahori and Hideya Matsumoto in [22], introduced an open compact subgroup K_I of G, now known as an Iwahori subgroup. If N is the normalizer of a suitable split maximal torus $T \simeq (\mathcal{K}^*)^n$, then (K_I, N) is a BN pair. The Iwahori–Hecke algebra of G is the algebra ${}^I\mathscr{H}_R = {}^I\mathscr{H}_R(G, K_I)$ of locally constant, compactly supported functions on G, with values in a ring R, that are bi-invariant by the left and right actions of K_I. The multiplication is given by the convolution product.

If $H \simeq (\mathcal{O}^*)^n$ is the maximal compact subgroup of T, then $H \subset K_I$ and $W = N/H$ is the affine Weyl group. One has the Bruhat decomposition $G = K_I \cdot W \cdot K_I = \sqcup_{\mathbf{w}\in W} K_I \cdot \mathbf{w} \cdot K_I$. If one considers the characteristic function $T_{\mathbf{w}}$ of $K_I \cdot \mathbf{w} \cdot K_I$, we get a basis of ${}^I\mathscr{H}_R$: ${}^I\mathscr{H}_R = \oplus_{\mathbf{w}\in W} R \cdot T_{\mathbf{w}}$. The convolution product is given by $T_{\mathbf{w}} * T_{\mathbf{v}} = \sum_{\mathbf{u}\in P_{\mathbf{w},\mathbf{v}}} a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} T_{\mathbf{u}}$, with $P_{\mathbf{w},\mathbf{v}}$ a finite subset of W. The numbers $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} \in R$ are the structure constants of ${}^I\mathscr{H}_R$. The unit is $1 = T_e$.

Iwahori and Matsumoto gave a precise (and now classical) definition of ${}^I\mathscr{H}_R$ by generators and relations. The group W is an infinite Coxeter group generated by $\{r_0, \dots, r_n\}$. Then ${}^I\mathscr{H}_R$ is generated by $\{T_{r_0}, \dots, T_{r_n}\}$ with relations $T^2_{r_i} = q \cdot 1 + (q-1) \cdot T_{r_i}$ and $T_{r_i} * T_{r_j} * T_{r_i} * \cdots = T_{r_j} * T_{r_i} * T_{r_j} * \cdots$ (with $m_{i,j}$ factors on each side) for $i \neq j$, if $m_{i,j}$ is the finite order of $r_i r_j$. For $\mathbf{w} = r_{i_1} \cdot \ldots \cdot r_{i_s}$ a reduced expression in W,

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one has $T_{\mathbf{w}} = T_{r_{i_1}} * \cdots * T_{r_{i_s}}$. In a Coxeter group one knows the rules to get (using the Coxeter relations between the r_i) a reduced expression from a non reduced expression (e.g. the product of two reduced expressions $\mathbf{w} = r_{i_1} \cdot \ldots \cdot r_{i_s}$ and $\mathbf{v} = r_{j_1} \cdot \ldots \cdot r_{j_t}$). So one deduces easily (using the above relations between the T_{r_i}) that each structure constant $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ (for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in W$) is in $\mathbb{Z}[q]$. More precisely it is a polynomial in $q-1$ with coefficients in $\mathbb{N} = \mathbb{Z}_{\geqslant 0}$. This polynomial depends only on $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and W.

So one has a universal description of ${}^I\mathscr{H}_{\mathbb{Z}}$ as a $\mathbb{Z}[q]-$algebra, depending only on W.

There are various generalizations of the above situation. First one may replace G by a general reductive group over $\mathcal{K}$, isotropic but potentially non split. Then one has to consider the relative affine Weyl group W, which is a Coxeter group. One may still define a compact, open Iwahori subgroup K_I and there is a Bruhat decomposition $G = K_I \cdot W \cdot K_I$. Now the description of ${}^I\mathscr{H}_R$ involves parameters q_i (satisfying $T^2_{r_i} = q_i \cdot 1 + (q_i - 1) \cdot T_{r_i}$) which are potentially different from q. This gives the Iwahori–Hecke algebra with unequal parameters. There is a pleasant description of ${}^I\mathscr{H}_R$ using the Bruhat–Tits building associated to the BN pair (K_I, N), see e.g. [29].

For now more than twenty years, there is an increasing interest in the study of Kac–Moody groups over local fields, see the works of Braverman, Garland, Kapranov, Kazhdan, Patnaik, Gaussent and the authors: e.g. [3, 4, 5, 6, 7, 8, 16, 17, 19, 24]. It has been possible to define and study for Kac–Moody groups (supposed at first affine) the spherical Hecke algebra, the Iwahori–Hecke algebra, the Satake isomorphism, ... This is also closely related to more abstract works on Hecke algebras by Cherednik and Macdonald, e.g. [13, 14, 25].

We are mainly interested in Iwahori–Hecke algebras for Kac–Moody groups over local fields. They were introduced and described by Braverman, Kazhdan and Patnaik in the affine case [8] and then in general by Gaussent and the authors [3]. So let us consider a Kac–Moody group G (affine or not) over the local field $\mathcal{K}$. We suppose it split (as defined by Tits [34]) or more generally almost split [30]. Let us choose also a maximal split subtorus. To this situation are associated an affine (relative) Weyl group W and an Iwahori subgroup K_I (defined up to conjugacy by W), see 1.4.5 and 1.4.7 below. This group W is not a Coxeter group but may be described as a semi-direct product $W = W^v \ltimes Y$, where W^v is a Coxeter group, the relative Weyl group, and Y is (essentially) the cocharacter group of the torus.

Unfortunately the Bruhat decomposition "$G = K_I \cdot W \cdot K_I$" fails to be true (even in the untwisted affine case, i.e. for loop groups). One has to consider the sub-semigroup $W^+ = W^v \ltimes Y^+$ (resp. $W^{+g} = W^v \ltimes Y^{+g}$) of W, where Y^+ (resp. Y^{+g}) is the intersection of Y with the Tits cone $\mathcal{T}$ (resp. with a cone $\mathcal{T}^\circ \cup V_0 \subset \mathcal{T}$, where $\mathcal{T}^\circ$ is the open Tits cone) in $V = Y \otimes_{\mathbb{Z}} \mathbb{R}$ (see 1.2, 1.5, and 1.8 below). Then $G^+ = K_I \cdot W^+ \cdot K_I$ (resp. $G^{+g} = K_I \cdot W^{+g} \cdot K_I \subset G^+$) is a sub-semigroup of G: the Kac–Moody–Tits semigroup (resp. the generic Kac–Moody–Tits semigroup). We may consider the characteristic functions $T_{\mathbf{w}}$ of the double cosets $K_I \cdot \mathbf{w} \cdot K_I$ and one proves in [3] that:

The space ${}^I\mathscr{H}_R$ (resp. ${}^I\mathscr{H}^g_R$) of $R-$valued functions with finite support on $K_I \backslash G^+ / K_I$ (resp. $K_I \backslash G^{+g} / K_I$) is naturally endowed with a structure of algebra (see 1.11). We get thus the Iwahori–Hecke algebra ${}^I\mathscr{H}_R = \oplus_{\mathbf{w} \in W^+} R \cdot T_{\mathbf{w}}$ (resp. the generic Iwahori–Hecke algebra ${}^I\mathscr{H}^g_R = \oplus_{\mathbf{w} \in W^{+g}} R \cdot T_{\mathbf{w}}$). The product is given by structure constants $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} \in \mathbb{N} = \mathbb{Z}_{\geqslant 0}$: $T_{\mathbf{w}} * T_{\mathbf{v}} = \sum_{\mathbf{u} \in P_{\mathbf{w},\mathbf{v}}} a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} T_{\mathbf{u}}$.

Conjecture 1 ([3, 2.5]). *Each $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is a polynomial, with coefficients in $\mathbb{N} = \mathbb{Z}_{\geqslant 0}$, in the parameters $q_i - 1, q'_i - 1$ of the situation, see 1.4.6 below. This polynomial depends only on the affine Weyl group W acting on the apartment $\mathbb{A}$ and on $\mathbf{w}, \mathbf{v}, \mathbf{u} \in W^+$.*

One may consider that this is a translation of the following question of Braverman, Kazhdan and Patnaik:

Question ([8, end of 1.2.4]). *Has the algebra ${}^I\mathscr{H}_{\mathbb{C}}$ a purely algebraic or combinatorial description with respect to the coset basis $(T_{\mathbf{w}})_{\mathbf{w}\in W^+}$?*

But a more precise formulation of this question is as follows:

Conjecture 2. *The algebra ${}^I\mathscr{H}_{\mathbb{Z}}$ (or ${}^I\mathscr{H}^g_{\mathbb{Z}}$) is the specialization of an algebra ${}^I\mathscr{H}_{\mathbb{Z}[\mathscr{Q}]}$ (or ${}^I\mathscr{H}^g_{\mathbb{Z}[\mathscr{Q}]}$) with the same basis $(T_{\mathbf{w}})_{\mathbf{w}\in W^+}$ (or $(T_{\mathbf{w}})_{\mathbf{w}\in W^{+g}}$) over $\mathbb{Z}[\mathscr{Q}]$. Here $\mathscr{Q}$ is a set of indeterminates Q_i, Q'_i (with some equalities between them, see* 1.4.6 *below) and the specialization is given by $Q_i \mapsto q_i, Q'_i \mapsto q'_i, \forall i \in I$. The algebra ${}^I\mathscr{H}_{\mathbb{Z}[\mathscr{Q}]}$ (or ${}^I\mathscr{H}^g_{\mathbb{Z}[\mathscr{Q}]}$) depends only on the affine Weyl group W acting on the apartment $\mathbb{A}$.*

Let us consider the split case: G is a split Kac–Moody group, all parameters q_i, q'_i are equal to $q = |\kappa|$ and all indeterminates Q_i, Q'_i are equal to a single indeterminate Q. Then the conjecture 1 has already been proved by Gaussent and the authors [3, 6.7] and independently by Muthiah [28] if, moreover, G is untwisted affine. Actually the same proof gives also conjecture 2, see 1.4.7 below.

In the general (non split) case, weakened versions were obtained in [3]: the $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ are Laurent polynomials in the q_i, q'_i [l.c. 6.7]; they are true polynomials if $\mathbf{w}, \mathbf{v} \in W^v \ltimes (Y \cap \mathcal{T}^\circ)$ and $\mathbf{v}$ is "regular" [l.c. 3.8].

In this article, we prove the conjecture 1 when $\mathbf{w}$ and $\mathbf{v}$ are in W^{+g} (see 3.4). We remark also that $W^+ = W^{+g}$ in the affine case (twisted or not) or the strictly hyperbolic case, even if G is not split. This is a first step towards the description of an abstract algebra ${}^I\mathscr{H}_{\mathbb{Z}[\mathscr{Q}]}$ (resp. ${}^I\mathscr{H}^g_{\mathbb{Z}[\mathscr{Q}]}$) over $\mathbb{Z}[\mathscr{Q}]$ in the affine (or strictly hyperbolic) case (resp. in the general case).

One should mention here that one may give a more precise description of the Iwahori–Hecke algebra using a Bernstein–Lusztig presentation (see [17], [8] and [3]). But this description is given in a new basis and the coefficients of the change of basis matrix are Laurent polynomials in the parameters q_i, q'_i. So this description is not sufficient to prove the conjecture.

Actually this article is written in a more general framework explained in Section 1: as in [3], we work with an abstract masure $\mathscr{I}$ and we take G to be a strongly transitive group of vectorially-Weyl automorphisms of $\mathscr{I}$. In Section 2 we gather the additional technical tools (e.g. decorated Hecke paths) needed to improve the results of [3, Section 3]. We get our main results about $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ in Section 3: we deal with the cases $\mathbf{w}, \mathbf{v}$ spherical. In Section 4 we deal with the remaining cases where $\mathbf{w}, \mathbf{v}$ are in W^{+g}, i.e. when $\mathbf{w}, \mathbf{v}$ are said generic.

1. General framework

1.1. Vectorial data. We consider a quadruple $(V, W^v, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ where V is a finite dimensional real vector space, W^v a subgroup of $\mathrm{GL}(V)$ (the vectorial Weyl group), I a finite set, $(\alpha_i^\vee)_{i\in I}$ a free family in V and $(\alpha_i)_{i\in I}$ a free family in the dual V^*. We ask these data to satisfy the conditions of [31, 1.1]. In particular, the formula $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$ defines a linear involution in V which is an element in W^v and $(W^v, \{r_i \mid i \in I\})$ is a Coxeter system.

To be more concrete, we consider the Kac–Moody case of [l.c. ; 1.2]: the matrix $\mathbb{M} = (\alpha_j(\alpha_i^\vee))_{i,j\in I}$ is a generalized Cartan matrix. Then W^v is the Weyl group of the corresponding Kac–Moody Lie algebra $\mathfrak{g}_{\mathbb{M}}$ and the associated real root system is

$$\Phi = \{w(\alpha_i) \mid w \in W^v, i \in I\} \subset Q = \bigoplus_{i\in I} \mathbb{Z}\cdot\alpha_i.$$

We set $\Phi^\pm = \Phi \cap Q^\pm$ where $Q^\pm = \pm(\bigoplus_{i\in I}(\mathbb{Z}_{\geqslant 0})\cdot\alpha_i)$ and $Q^\vee = (\bigoplus_{i\in I}\mathbb{Z}\cdot\alpha_i^\vee)$, $Q^\vee_\pm = \pm(\bigoplus_{i\in I}(\mathbb{Z}_{\geqslant 0})\cdot\alpha_i^\vee)$. We have $\Phi = \Phi^+ \cup \Phi^-$ and, for $\alpha = w(\alpha_i) \in \Phi$, $r_\alpha = w\cdot r_i \cdot w^{-1}$ and $r_\alpha(v) = v - \alpha(v)\alpha^\vee$, where the coroot $\alpha^\vee = w(\alpha_i^\vee)$ depends only on α.

The set Φ is an (abstract, reduced) real root system in the sense of [26], [27] or [1]. We shall sometimes also use the set $\Delta = \Phi \cup \Delta_{im}^+ \cup \Delta_{im}^-$ of all roots (with $-\Delta_{im}^- = \Delta_{im}^+ \subset Q^+$, W^v-stable) defined in [23]. It is an (abstract, reduced) root system in the sense of [1].

The *fundamental positive chamber* is $C_f^v = \{v \in V \mid \alpha_i(v) > 0, \forall i \in I\}$. Its closure $\overline{C_f^v}$ is the disjoint union of the vectorial faces $F^v(J) = \{v \in V \mid \alpha_i(v) = 0, \forall i \in J, \alpha_i(v) > 0, \forall i \in I \smallsetminus J\}$ for $J \subset I$. We set $V_0 = F^v(I)$. The positive (resp. negative) vectorial faces are the sets $w \cdot F^v(J)$ (resp. $-w \cdot F^v(J)$) for $w \in W^v$ and $J \subset I$. The support of such a face is the vector space it generates. The set J or the face $w \cdot F^v(J)$ or an element of this face is called *spherical* if the group $W^v(J)$ generated by $\{r_i \mid i \in J\}$ (which is the fixator or stabilizer in W^v of $F^v(J)$) is finite. An element of a vectorial chamber $\pm w \cdot C_f^v$ is called *regular*.

The *Tits cone* $\mathcal{T}$ (resp. its interior $\mathcal{T}^\circ$) is the (disjoint) union of the positive (resp. and spherical) vectorial faces. It is a W^v-stable convex cone in V. One has $\mathcal{T} = \mathcal{T}^\circ = V$ (resp. $V_0 \subset \mathcal{T} \smallsetminus \mathcal{T}^\circ$) in the classical (resp. non classical) case, i.e. when W^v is finite (resp. infinite). By the above characterization of spherical faces, $\mathcal{T}^\circ$ is the set of $x \in \mathcal{T}$ whose fixator in W^v is finite.

We say that $\mathbb{A}^v = (V, W^v)$ is a *vectorial apartment*.

1.2. The model apartment. As in [31, 1.4] the model apartment $\mathbb{A}$ is V considered as an affine space and endowed with a family $\mathcal{M}$ of walls. These walls are affine hyperplanes directed by $\ker(\alpha)$ for $\alpha \in \Phi$. More precisely, they may be written $M(\alpha, k) = \{v \in V \mid \alpha(v) + k = 0\}$, for $\alpha \in \Phi$ and $k \in \mathbb{R}$.

We ask this apartment to be *semi-discrete* and the origin 0 to be *special*. This means that these walls are the hyperplanes $M(\alpha, k) = \{v \in V \mid \alpha(v) + k = 0\}$ for $\alpha \in \Phi$ and $k \in \Lambda_\alpha$, with $\Lambda_\alpha = k_\alpha \cdot \mathbb{Z}$ a non trivial discrete subgroup of $\mathbb{R}$. Using [19, Lemma 1.3] (i.e. replacing Φ by another system Φ_1) we may (and shall) assume that $\Lambda_\alpha = \mathbb{Z}, \forall \alpha \in \Phi$.

For $\alpha = w(\alpha_i) \in \Phi$, $k \in \mathbb{Z}$ and $M = M(\alpha, k)$, the reflection $r_{\alpha,k} = r_M$ with respect to M is the affine involution of $\mathbb{A}$ with fixed points the wall M and associated linear involution r_α. The affine Weyl group W^a is the group generated by the reflections r_M for $M \in \mathcal{M}$; we assume that W^a stabilizes $\mathcal{M}$. We know that $W^a = W^v \ltimes Q^\vee$ and we write $W_{\mathbb{R}}^a = W^v \ltimes V$; here $Q^\vee$ and V have to be understood as groups of translations.

An automorphism of $\mathbb{A}$ is an affine bijection $\varphi : \mathbb{A} \to \mathbb{A}$ stabilizing the set of pairs $(M, \alpha^\vee)$ of a wall M and the coroot associated with $\alpha \in \Phi$ such that $M = M(\alpha, k)$, $k \in \mathbb{Z}$. The group $\mathrm{Aut}(\mathbb{A})$ of these automorphisms contains W^a and normalizes it. We consider also the group $\mathrm{Aut}_{\mathbb{R}}^W(\mathbb{A}) = \{\varphi \in \mathrm{Aut}(\mathbb{A}) \mid \overrightarrow{\varphi} \in W^v\} = \mathrm{Aut}(\mathbb{A}) \cap W_{\mathbb{R}}^a$.

For $\alpha \in \Phi$ and $k \in \mathbb{R}$, $D(\alpha, k) = \{v \in V \mid \alpha(v) + k \geqslant 0\}$ is a half-space, it is called a *half-apartment* if $k \in \mathbb{Z}$. We write $D(\alpha, \infty) = \mathbb{A}$.

The Tits cone $\mathcal{T}$ and its interior $\mathcal{T}^o$ are convex and W^v-stable cones, therefore, we can define three W^v-invariant preorder relations on $\mathbb{A}$:

$$x \leqslant y \;\Leftrightarrow\; y - x \in \mathcal{T}; \quad x \overset{o}{<} y \;\Leftrightarrow\; y - x \in \mathcal{T}^o; \quad x \overset{o}{\leqslant} y \;\Leftrightarrow\; y - x \in \mathcal{T}^o \cup V_0.$$

If W^v has no fixed point in $V \smallsetminus \{0\}$ (i.e. $V_0 = \{0\}$) and no finite factor, then they are orders; but, in general, they are not.

1.3. Faces, sectors. The faces in $\mathbb{A}$ are associated to the above systems of walls and half-apartments. As in [9], they are no longer subsets of $\mathbb{A}$, but filters of subsets of $\mathbb{A}$. For the definition of that notion and its properties, we refer to [9] or [18].

If F is a subset of $\mathbb{A}$ containing an element x in its closure, the germ of F in x is the filter $\mathrm{germ}_x(F)$ consisting of all subsets of $\mathbb{A}$ which contain intersections of F

and neighbourhoods of x. In particular, if $x \neq y \in \mathbb{A}$, we denote the germ in x of the segment $[x, y]$ (resp. of the interval $]x, y]$) by $[x, y)$ (resp. $]x, y)$).

For $y \neq x$, the segment germ $[x, y)$ is called of sign $\pm$ if $y - x \in \pm\mathcal{T}$. The segment $[x, y]$ (or the segment germ $[x, y)$ or the ray with origin x containing y) is called *preordered* if $x \leqslant y$ or $y \leqslant x$ and *generic* if $x \overset{o}{<} y$ or $y \overset{o}{<} x$.

Given F a filter of subsets of $\mathbb{A}$, its *strict enclosure* $\mathrm{cl}_{\mathbb{A}}(F)$ (resp. *closure* $\overline{F}$) is the filter made of the subsets of $\mathbb{A}$ containing an element of F of the shape $\cap_{\alpha\in\Delta} D(\alpha, k_\alpha)$, where $k_\alpha \in \mathbb{Z} \cup \{\infty\}$ (resp. containing the closure $\overline{S}$ of some $S \in F$). One considers also the (larger) *enclosure* $\mathrm{cl}^{\#}_{\mathbb{A}}(F)$ of [33, 3.6.1] (introduced in [10, 11, 12] and well studied in [21], see also [20]). It is the filter made of the subsets of $\mathbb{A}$ containing an element of F of the shape $\cap_{\alpha\in\Psi} D(\alpha, k_\alpha)$, with $\Psi \subset \Phi$ finite and $k_\alpha \in \mathbb{Z}$ (i.e. a finite intersection of half apartments).

A *local face* F in the apartment $\mathbb{A}$ is associated to a point $x \in \mathbb{A}$, its vertex, and a vectorial face F^v in V, its direction. It is defined as $F = \mathrm{germ}_x(x + F^v)$ and we denote it by $F = F^\ell(x, F^v)$. Its closure is $\overline{F^\ell}(x, F^v) = \mathrm{germ}_x(x + \overline{F^v})$. There is an order on the local faces: the assertions "F is a face of F'", "F' covers F" and "$F \leqslant F'$" are by definition equivalent to $F \subset \overline{F'}$. The dimension of a local face F is the smallest dimension of an affine space generated by some $S \in F$. The (unique) such affine space E of minimal dimension is the support of F; if $F = F^\ell(x, F^v)$, $\mathrm{supp}(F) = x + \mathrm{supp}(F^v)$. A local face $F = F^\ell(x, F^v)$ is spherical if the direction of its support meets the open Tits cone (i.e. when F^v is spherical), then its pointwise stabilizer W_F in W^a or $W^a_{\mathbb{R}}$ is finite and fixes x.

We shall actually here speak only of local faces, and sometimes forget the word local or write $F = F(x, F^v)$.

A *local chamber* is a maximal local face, i.e. a local face $F^\ell(x, \pm w\cdot C^v_f)$ for $x \in \mathbb{A}$ and $w \in W^v$. The *fundamental local positive (resp. negative)* chamber is $C^+_0 = \mathrm{germ}_0(C^v_f)$ (resp. $C^-_0 = \mathrm{germ}_0(-C^v_f)$).

A *(local) panel* is a spherical local face maximal among local faces which are not chambers, or, equivalently, a spherical face of dimension $n - 1$. Its support is a hyperplane parallel to a wall.

A *sector* in $\mathbb{A}$ is a $V-$translate $\mathfrak{s} = x + C^v$ of a vectorial chamber $C^v = \pm w \cdot C^v_f$, $w \in W^v$. The point x is its *base point* and C^v its *direction*. Two sectors have the same direction if, and only if, they are conjugate by $V-$translation, and if, and only if, their intersection contains another sector.

The *sector-germ* of a sector $\mathfrak{s} = x+C^v$ in $\mathbb{A}$ is the filter $\mathfrak{S}$ of subsets of $\mathbb{A}$ consisting of the sets containing a $V-$translate of $\mathfrak{s}$, it is well determined by the direction C^v. So, the set of translation classes of sectors in $\mathbb{A}$, the set of vectorial chambers in V and the set of sector-germs in $\mathbb{A}$ are in canonical bijection.

A *sector-face* in $\mathbb{A}$ is a $V-$translate $\mathfrak{f} = x + F^v$ of a vectorial face $F^v = \pm w \cdot F^v(J)$. The sector-face-germ of $\mathfrak{f}$ is the filter $\mathfrak{F}$ of subsets containing a translate $\mathfrak{f}'$ of $\mathfrak{f}$ by an element of F^v (i.e. $\mathfrak{f}' \subset \mathfrak{f}$). If F^v is spherical, then $\mathfrak{f}$ and $\mathfrak{F}$ are also called spherical. The sign of $\mathfrak{f}$ and $\mathfrak{F}$ is the sign of F^v.

1.4. The Masure. In this section, we recall the definition and some properties of a masure given by Guy Rousseau in [31] and simplified by Auguste Hébert [21].

1.4.1. An apartment of type $\mathbb{A}$ is a set A endowed with a set $\mathrm{Isom}^W(\mathbb{A}, A)$ of bijections (called Weyl-isomorphisms) such that, if $f_0 \in \mathrm{Isom}^W(\mathbb{A}, A)$, then $f \in \mathrm{Isom}^W(\mathbb{A}, A)$ if, and only if, there exists $w \in W^a$ satisfying $f = f_0 \circ w$. An isomorphism (resp. a Weyl-isomorphism, a vectorially-Weyl isomorphism) between two apartments $\varphi : A \to A'$

is a bijection such that, for any $f \in \mathrm{Isom}^W(\mathbb{A}, A)$, $f' \in \mathrm{Isom}^W(\mathbb{A}, A')$, $f'^{-1} \circ \varphi \circ f \in \mathrm{Aut}(\mathbb{A})$ (resp. $\in W^a$, $\in \mathrm{Aut}^W_{\mathbb{R}}(\mathbb{A})$); the group of these isomorphisms is written $\mathrm{Isom}(A, A')$ (resp. $\mathrm{Isom}^W(A, A')$, $\mathrm{Isom}^W_{\mathbb{R}}(A, A')$). As the filters in $\mathbb{A}$ defined in 1.3 above (e.g. local faces, sectors, walls,...) are permuted by $\mathrm{Aut}(\mathbb{A})$, they are well defined in any apartment of type $\mathbb{A}$ and exchanged by any isomorphism.

A *masure* (formerly called an *ordered affine hovel*) of type $\mathbb{A}$ is a set $\mathscr{I}$ endowed with a covering $\mathcal{A}$ of subsets called apartments, each endowed with some structure of an apartment of type $\mathbb{A}$. We recall here the simplification and improvement of the original definition given by Auguste Hébert in [21]: these data have to satisfy the following two axioms:

(MA ii) If two apartments A, A' are such that $A \cap A'$ contains a generic ray, then $A \cap A'$ is a finite intersection of half-apartments (i.e. $A \cap A' = \mathrm{cl}^{\#}_A(A \cap A')$) and there exists a Weyl isomorphism $\varphi : A \to A'$ fixing $A \cap A'$.

(MA iii) If $\mathfrak{R}$ is the germ of a splayed chimney and if F is a local face or a germ of a chimney, then there exists an apartment containing $\mathfrak{R}$ and F.

Actually a filter or subset in $\mathscr{I}$ is called a preordered (or generic) segment (or segment germ), a local face, a spherical sector face or a spherical sector face germ if it is included in some apartment A and is called like that in A. We do not recall here what is (a germ of) a (splayed) chimney; it contains (the germ of) a (spherical) sector face. We shall actually use (MA iii) uniquely through its consequence (b) below.

In the affine case the hypothesis "$A \cap A'$ contains a generic ray" may be omitted in (MA ii).

We list now some of the properties of masures we shall use.

(a) If F is a point, a preordered segment, a local face or a spherical sector face in an apartment A and if A' is another apartment containing F, then $A \cap A'$ contains the enclosure $\mathrm{cl}^{\#}_A(F)$ of F and there exists a Weyl-isomorphism from A onto A' fixing $\mathrm{cl}^{\#}_A(F)$, see [21, 5.11] or [20, 4.4.10]. Hence any isomorphism from A onto A' fixing F fixes $\overline{F}$ (and even $\mathrm{cl}^{\#}_A(F) \cap \mathrm{supp}(F)$).

More generally the intersection of two apartments A, A' is always closed (in A and A'), see [21, 3.9] or [20, 4.2.17].

(b) If $\mathfrak{F}$ is the germ of a spherical sector face and if F is a local face or a germ of a sector face, then there exists an apartment that contains $\mathfrak{F}$ and F.

(c) If two apartments A, A' contain $\mathfrak{F}$ and F as in (b), then their intersection contains $\mathrm{cl}^{\#}_A(\mathfrak{F} \cup F)$ and there exists a Weyl-isomorphism from A onto A' fixing $\mathrm{cl}^{\#}_A(\mathfrak{F} \cup F)$.

(d) We consider the relations, $\overset{o}{<}$ and $\overset{o}{\leqslant}$ on $\mathscr{I}$ defined as follows:

$$x \leqslant y \text{ (resp. } x \overset{o}{<} y, x \overset{o}{\leqslant} y)$$
$$\iff \exists A \in \mathcal{A} \text{ such that } x, y \in A \text{ and } x \leqslant_A y \text{ (resp. } x \overset{o}{<}_A y, x \overset{o}{\leqslant}_A y).$$

Then $\leqslant$ (resp. $\overset{o}{<}$, $\overset{o}{\leqslant}$) is a well defined preorder relation, in particular transitive; it is called the *Tits preorder* (resp. *Tits open preorder, large Tits open preorder*), see [21].

(e) We ask here $\mathscr{I}$ to be thick of *finite thickness*: the number of local chambers covering a given (local) panel in a wall has to be finite $\geqslant 3$. This number is the same for any panel F in a given wall M [31, 2.9]; we denote it by $1 + q_M = 1 + q_F$.

(f) An automorphism (resp. a Weyl-automorphism, a vectorially-Weyl automorphism) of $\mathscr{I}$ is a bijection $\varphi : \mathscr{I} \to \mathscr{I}$ such that $A \in \mathcal{A} \iff \varphi(A) \in \mathcal{A}$ and

then $\varphi|_A : A \to \varphi(A)$ is an isomorphism (resp. a Weyl-isomorphism, a vectorially-Weyl isomorphism). We write $\mathrm{Aut}(\mathscr{I})$ (resp. $\mathrm{Aut}^W(\mathscr{I})$, $\mathrm{Aut}^W_{\mathbb{R}}(\mathscr{I})$) the group of these automorphisms.

1.4.2. For $x \in \mathscr{I}$, the set $\mathcal{T}_x^+\mathscr{I}$ (resp. $\mathcal{T}_x^-\mathscr{I}$) of segment germs $[x, y)$ for $y > x$ (resp. $y < x$) may be considered as a building, the *positive* (resp. *negative*) *tangent building*. The corresponding faces are the local faces of positive (resp. negative) direction and vertex x. For such a local face F, we write sometimes $[x, y) \in F$ if $]x, y) \subset F$. The associated Weyl group is W^v. If the $W-$distance (calculated in $\mathcal{T}_x^{\pm}\mathscr{I}$) of two local chambers is $d^W(C_x, C'_x) = w \in W^v$, to any reduced decomposition $w = r_{i_1} \cdots r_{i_n}$ corresponds a unique minimal gallery from C_x to C'_x of type $(i_1, \cdots, i_n)$.

The buildings $\mathcal{T}_x^+\mathscr{I}$ and $\mathcal{T}_x^-\mathscr{I}$ are actually twinned. The codistance $d^{*W}(C_x, C'_x)$ of two opposite sign chambers C_x and C'_x is the $W-$distance $d^W(C_x, \mathrm{op}\, C'_x)$, where $\mathrm{op}\, C'_x$ denotes the opposite chamber to C'_x in an apartment containing C_x and C'_x. Similarly two segment germs $\eta \in \mathcal{T}_x^+\mathscr{I}$ and $\zeta \in \mathcal{T}_x^-\mathscr{I}$ are said opposite if they are in a same apartment A and opposite in this apartment (i.e. in the same line, with opposite directions).

1.4.3. *Lemma.* ([31, 2.9]) Let D be a half-apartment in $\mathscr{I}$ and $M = \partial D$ its wall (i.e. its boundary). One considers a panel F in M and a local chamber C in $\mathscr{I}$ covering F. Then there is an apartment containing D and C.

1.4.4. We assume that $\mathscr{I}$ has a strongly transitive group of automorphisms G, i.e. 1.4.1(a) and (c) above (after replacing $\mathrm{cl}_A^{\#}$ by cl_A) are satisfied by isomorphisms induced by elements of G, cf. [33, 4.10] and [15, 4.7].

We choose in $\mathscr{I}$ a fundamental apartment which we identify with $\mathbb{A}$. As G is strongly transitive, the apartments of $\mathscr{I}$ are the sets $g \cdot \mathbb{A}$ for $g \in G$. The stabilizer N of $\mathbb{A}$ in G induces a group $W = \nu(N) \subset \mathrm{Aut}(\mathbb{A})$ of affine automorphisms of $\mathbb{A}$ which permutes the walls, local faces, sectors, sector-faces... and contains the affine Weyl group $W^a = W^v \ltimes Q^\vee$ [33, 4.13.1].

We denote the stabilizer of $0 \in \mathbb{A}$ in G by K and the pointwise stabilizer (or fixator) of C_0^+ (resp. C_0^-) by $K_I = K_I^+$ (resp. K_I^-). This group K_I is called the *Iwahori subgroup*.

1.4.5. We ask $W = \nu(N)$ to be *vectorially-Weyl* for its action on the vectorial faces. This means that the associated linear map $\overrightarrow{w}$ of any $w \in \nu(N)$ is in W^v. As $\nu(N)$ contains W^a and stabilizes $\mathcal{M}$, we have $W = \nu(N) = W^v \ltimes Y$, where W^v fixes the origin 0 of $\mathbb{A}$ and Y is a group of translations such that: $Q^\vee \subset Y \subset P^\vee = \{v \in V \mid \alpha(v) \in \mathbb{Z}, \forall \alpha \in \Phi\}$. An element $\mathbf{w} \in W$ will often be written $\mathbf{w} = \lambda \cdot w$, with $\lambda \in Y$ and $w \in W^v$.

We ask Y to be *discrete* in V. This is clearly satisfied if Φ generates V^* i.e. $(\alpha_i)_{i \in I}$ is a basis of V^*.

1.4.6. Note that there is only a finite number of constants q_M as in the definition of thickness. Indeed, we must have $q_{wM} = q_M$, $\forall w \in \nu(N)$ and $w \cdot M(\alpha, k) = M(w(\alpha), k), \forall w \in W^v$. So now, fix $i \in I$, as $\alpha_i(\alpha_i^\vee) = 2$ the translation by $\alpha_i^\vee$ permutes the walls $M = M(\alpha_i, k)$ (for $k \in \mathbb{Z}$) with two orbits. So, $Q^\vee \subset W^a$ has at most two orbits in the set of the constants $q_{M(\alpha_i,k)}$: one containing the $q_i = q_{M(\alpha_i,0)}$ and the other containing the $q'_i = q_{M(\alpha_i,\pm 1)}$. Hence, the number of (possibly) different q_M is at most $2 \cdot |I|$. We denote this set of parameters by $\mathcal{Q} = \{q_i, q'_i \mid i \in I\}$.

In [3, 1.4.5] one proves the following further equalities: $q_i = q'_i$ if $\alpha_i(Y) = \mathbb{Z}$ and $q_i = q'_i = q_j = q'_j$ if $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$.

We consider also the polynomial algebra $\mathbb{Z}[\mathscr{Q}]$, where $\mathscr{Q}$ is the set $\mathscr{Q} = \{Q_i, Q'_i \mid i \in I\}$ of indeterminates, satisfying the same equalities: $Q_i = Q'_i$ if $\alpha_i(Y) = \mathbb{Z}$ and $Q_i = Q'_i = Q_j = Q'_j$ if $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$. See [3, 6.1] where $Q_i = \sigma_i^2, Q'_i = (\sigma'_i)^2$.

1.4.7. *Examples.* The main examples of all the above situation are provided by the Kac–Moody theory, as already indicated in the introduction. More precisely let G be an almost split Kac–Moody group over a non archimedean complete field $\mathcal{K}$. We suppose moreover the valuation of $\mathcal{K}$ discrete and its residue field κ perfect. Then there is a masure $\mathscr{I}$ on which G acts strongly transitively by vectorially Weyl automorphisms. If $\mathcal{K}$ is a local field (i.e. κ is finite), then we are in the situation described above. This is the main result of [10], [11], [12] and [33].

When G is actually split, this result was known previously by [19] and [32]. And in this case all the constants q_M, q_i, q'_i are equal to the cardinality q of the residue field κ.

We gave in [3, 6.7] a proof of conjecture 1 for this split case; see also [28]. Actually these proofs are proofs of conjecture 2, as the polynomials $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ are Laurent polynomials inherited from the description of ${}^I\mathscr{H}$ as a specialization of the associative Bernstein–Lusztig algebra over $\mathbb{Z}[\mathscr{Q}]$: the algebra ${}^I\mathscr{H}_{\mathbb{Z}[\mathscr{Q}]}$ over $\mathbb{Z}[\mathscr{Q}]$ defined by these structure constants on the basis $(T_{\mathbf{w}})_{\mathbf{w}\in W^+}$ is associative.

1.4.8. *Remark.* All isomorphisms in [31] are Weyl-isomorphisms, and, when G is strongly transitive, all isomorphisms constructed in l.c. are induced by an element of G.

1.5. Type 0 vertices. The elements of Y, through the identification $Y = N\cdot 0 \subset \mathbb{A}$, are called *vertices of type* 0 in $\mathbb{A}$; they are special vertices. We note $Y^+ = Y \cap \mathcal{T}$, $Y^{+g} = Y \cap (\mathcal{T}^\circ \cup V_0)$, $Y^{+0} = Y \cap V_0$ and $Y^{++} = Y \cap \overline{C_f^v}$. The type 0 vertices in $\mathscr{I}$ are the points on the orbit $\mathscr{I}_0$ of 0 by G. This set $\mathscr{I}_0$ is often called the affine Grassmannian as it is equal to G/K, where $K = \mathrm{Stab}_G(\{0\})$. But in general, G is not equal to $KYK = KNK$ [18, 6.10] i.e. $\mathscr{I}_0 \neq K\cdot Y$.

We know that $\mathscr{I}$ is endowed with a $G-$invariant preorder $\leqslant$ which induces the known one on $\mathbb{A}$. Moreover, if $x \leqslant y$, then x and y are in the same apartment.

We set $\mathscr{I}^+ = \{x \in \mathscr{I} \mid 0 \leqslant x\}$, $\mathscr{I}_0^+ = \mathscr{I}_0 \cap \mathscr{I}^+$, $G^+ = \{g \in G \mid 0 \leqslant g\cdot 0\}$ and $G^{+g} = \{g \in G \mid 0 \overset{o}{\leqslant} g\cdot 0\}$; so $\mathscr{I}_0^+ = G^+\cdot 0 = G^+/K$. As $\leqslant$ (resp. $\overset{o}{\leqslant}$) is a $G-$invariant preorder, G^+ (resp. G^{+g}) is a semigroup, called the *Kac–Moody–Tits semigroup* (resp. the *generic Kac–Moody–Tits semigroup*).

One has $G^+ = K(N \cap G^+)K$; more precisely the map $Y^{++} \to K\backslash G^+/K$ is a bijection, if we identify $\lambda \in Y^{++} \subset W^v \ltimes Y = W = N/\ker\nu$ with its class in N modulo $\ker\nu \subset K$. Clearly $G^{+g} = K(Y^{++} \cap Y^{+g})K$.

1.6. Vectorial distance. For x in the Tits cone $\mathcal{T}$, we denote by x^{++} the unique element in $\overline{C_f^v}$ conjugated by W^v to x.

Let $\mathscr{I} \times_{\leqslant} \mathscr{I} = \{(x,y) \in \mathscr{I}\times\mathscr{I} \mid x \leqslant y\}$ be the set of increasing pairs in $\mathscr{I}$. Such a pair (x,y) is always in a same apartment $g\cdot\mathbb{A}$; so $(g^{-1})\cdot y - (g^{-1})\cdot x \in \mathcal{T}$ and we define the *vectorial distance* $d^v(x,y) \in \overline{C_f^v}$ by $d^v(x,y) = ((g^{-1})\cdot y - (g^{-1})\cdot x)^{++}$. It does not depend on the choices we made (by 1.8(b) below).

For $(x,y) \in \mathscr{I}_0 \times_{\leqslant} \mathscr{I}_0 = \{(x,y) \in \mathscr{I}_0\times\mathscr{I}_0 \mid x \leqslant y\}$, the vectorial distance $d^v(x,y)$ takes values in Y^{++}. Actually, as $\mathscr{I}_0 = G\cdot 0$, K is the stabilizer of 0 and $\mathscr{I}_0^+ = K\cdot Y^{++}$ (with uniqueness of the element in Y^{++}), the map d^v induces a bijection between the set $(\mathscr{I}_0 \times_{\leqslant} \mathscr{I}_0)/G$ of $G-$orbits in $\mathscr{I}_0 \times_{\leqslant} \mathscr{I}_0$ and Y^{++}.

Further, d^v gives the inverse of the map $Y^{++} \to K\backslash G^+/K$, as any $g \in G^+$ is in $K\cdot d^v(0, g\cdot 0)\cdot K$.

1.7. Paths and retractions. We consider piecewise linear continuous paths $\pi : [0,1] \to \mathbb{A}$ such that each (existing) tangent vector $\pi'(t)$ belongs to an orbit $W^v \cdot \lambda$ for some $\lambda \in \overline{C_f^v}$. Such a path is called a $\lambda-$*path*; it is increasing with respect to the preorder relation $\leqslant$ on $\mathbb{A}$. If $\lambda \in \overline{C}_f^v \cap (\mathcal{T}^\circ \cup V_0)$, then it is increasing for $\overset{o}{\leqslant}$.

For any $t \neq 0$ (resp. $t \neq 1$), we let $\pi'_-(t)$ (resp. $\pi'_+(t)$) denote the derivative of π at t from the left (resp. from the right). Further, we define $w_\pm(t) \in W^v$ to be the smallest element in its $(W^v)_\lambda-$class such that $\pi'_\pm(t) = w_\pm(t) \cdot \lambda$ (where $(W^v)_\lambda$ is the stabilizer in W^v of λ).

Moreover, we denote by $\pi_-(t) = \pi(t) - [0,1)\pi'_-(t) = [\pi(t), \pi(t-\varepsilon)\,)$ (resp. $\pi_+(t) = \pi(t) + [0,1)\pi'_+(t) = [\pi(t), \pi(t+\varepsilon)\,)$ (for $\varepsilon > 0$ small) the negative (resp. positive) segment-germ of π at t, for $0 < t \leqslant 1$ (resp. $0 \leqslant t < 1$).

Let C_z (resp. $\mathfrak{S}$) be a local chamber with vertex z (resp. a sector germ) in an apartment A of $\mathscr{I}$. For all $x \in \mathscr{I}_{\geqslant z} = \{y \in \mathscr{I} \mid y \geqslant z\}$ (resp. $x \in \mathscr{I}$) there is an apartment A' containing x and C_z (resp. $\mathfrak{S}$). And this apartment is conjugated to A by an element of G fixing C_z (resp. $\mathfrak{S}$) (cf. 1.4.1(a) and 1.4.4). So, by the usual arguments we can define the retraction $\rho = \rho_{A,C_z}$ from $\mathscr{I}_{\geqslant z}$ (resp. $\rho = \rho_{A,\mathfrak{S}}$ from $\mathscr{I}$) onto $A_{\geqslant z} = A \cap \mathscr{I}_{\geqslant z}$ (resp. onto the apartment A) with center C_z (resp. $\mathfrak{S}$).

For any such retraction ρ, the image of any segment $[x,y]$ with $(x,y) \in \mathscr{I} \times_\leqslant \mathscr{I}$ and $d^v(x,y) = \lambda \in \overline{C_f^v}$ (with moreover $x, y \in \mathscr{I}_{\geqslant z}$ if $\rho = \rho_{A,C_z}$) is a $\lambda-$path [18, 4.4]. In particular, $\rho(x) \leqslant \rho(y)$. By definition, if A' is another apartment containing $\mathfrak{S}$ (resp. C_z), then ρ induces an isomorphism from A' onto A. As we assume the existence of the strongly transitive group G, this isomorphism is the restriction of an automorphism of $\mathscr{I}$.

1.8. Preordered convexity. Let $\mathscr{C}^\pm$ (resp. $\mathscr{C}_0^\pm$) be the set of all local chambers of direction $\pm$ (resp. with moreover vertices of type 0). A positive (resp. negative) local chamber of vertex $x \in \mathscr{I}$ will often be written C_x (resp. C_x^-) and its direction $C_x^v = \overrightarrow{C_x}$ (resp. $C_x^{-\,v} = \overrightarrow{C_x^-}$). We consider the set $\mathscr{C}^+ \times_\leqslant \mathscr{C}^+ = \{(C_x, C_y) \in \mathscr{C}^+ \times \mathscr{C}^+ \mid x \leqslant y\}$ (resp. $\mathscr{C}^+ \times^\circ_\leqslant \mathscr{C}^+ = \{(C_x, C_y) \in \mathscr{C}^+ \times \mathscr{C}^+ \mid x \overset{o}{\leqslant} y\}$). We sometimes write $C_x \leqslant C_y$ (resp. $C_x \overset{o}{\leqslant} C_y$) when $x \leqslant y$ (resp. $x \overset{o}{\leqslant} y$).

Proposition. *Let $x, y \in \mathscr{I}$ with $x \leqslant y$. We consider two local faces F_x, F_y with respective vertices x, y. Then*

(a) *F_x and F_y are contained in a common apartment.*
(b) *If A, B are two apartments containing $\{x,y\}$ (resp. $F_x \cup F_y$), then there is a Weyl-isomorphism from A onto B, fixing the enclosure $\mathrm{cl}_A^\#(\{x,y\}) = \mathrm{cl}_B^\#(\{x,y\}) \supset [x,y]$ (resp. the closed convex hull $\overline{\mathrm{conv}}_A(F_x \cup F_y) = \overline{\mathrm{conv}}_B(F_x \cup F_y)$).*

This improvement of results in [31, 5.4, 5.1] and [3, 1.10] is proved by Auguste Hébert: [21, 5.17, 5.18], see also [20, 4.4.16, 4.4.17]. In (b) the case of $\{x,y\}$ is proved in [31, 5.4] as, by [21, 5.1] or [20, 4.4.1], one may replace cl by $\mathrm{cl}^\#$. This property is called the *preordered convexity* of intersections of apartments.

Consequence. We define $W^+ = W^v \ltimes Y^+$ (resp. $W^{+g} = W^v \ltimes Y^{+g}$) which is a subsemigroup of W, and call it the *Tits–Weyl* (resp. *generic Tits–Weyl*) *semigroup*. An element $\mathbf{w} \in W^{+g}$ is called *generic* (in a large sense) and *spherical* if, moreover, $\lambda \in \mathcal{T}^\circ \cap Y^+$.

Let $\varepsilon, \eta \in \{+,-\}$. If $C_x^\varepsilon \in \mathscr{C}_0^\varepsilon$ and $0 \leqslant x$, we know by (b) above, that there is an apartment A containing C_0^η and C_x^ε. But all apartments containing C_0^η are conjugated

to $\mathbb{A}$ by K_I^η (by 1.4.1(a)), so there is $k \in K_I^\eta$ with $k^{-1}\cdot C_x^\varepsilon \subset \mathbb{A}$. Now the vertex $k^{-1}\cdot x \in \mathscr{I}_0$ of $k^{-1}\cdot C_x^\varepsilon$ satisfies $k^{-1}\cdot x \geqslant 0$, so there is $\mathbf{w} \in W^+$ such that $k^{-1}\cdot C_x^\varepsilon = \mathbf{w}\cdot C_0^\varepsilon$.

When $g \in G^+$, $g\cdot C_0^\varepsilon$ is in $\mathscr{C}_0^\varepsilon$ and there are $k \in K_I^\eta$, $\mathbf{w} \in W^+$ with $g\cdot C_0^\varepsilon = k\cdot\mathbf{w}\cdot C_0^\varepsilon$, i.e. $g \in K_I^\eta\cdot W^+\cdot K_I^\varepsilon$. We have proved the *Bruhat decompositions* $G^+ = K_I^\pm\cdot W^+\cdot K_I^\pm$ and the *Birkhoff decompositions* $G^+ = K_I^\mp\cdot W^+\cdot K_I^\pm$. For uniqueness, see 1.10 below.

Similarly we also have $G^{+g} = K_I^\pm\cdot W^{+g}\cdot K_I^\pm$ and $G^{+g} = K_I^\mp\cdot W^{+g}\cdot K_I^\pm$.

1.9. Remark. If the generalized Cartan matrix $\mathbb{M}$ is of affine or strictly hyperbolic type (in the sense of [23, 4.3 or Ex. 4.1]), then any non spherical vectorial face is $w\cdot F^v(I) = F^v(I) = V_0 = \{v \in V \mid \alpha_i(v) = 0, \forall i \in I\}$. So the Tits cones satisfy $\mathcal{T} = \mathcal{T}^\circ \sqcup V_0$ and $Y^+ = Y^{+g}$, $W^+ = W^{+g}$.

1.10. W-distance. Let $(C_x, C_y) \in \mathscr{C}_0^+ \times_\leqslant \mathscr{C}_0^+$, there is an apartment A containing C_x and C_y. We identify $(\mathbb{A}, C_0^+)$ with (A, C_x) i.e. we consider the unique $f \in \mathrm{Isom}_{\mathbb{R}}^W(\mathbb{A}, A)$ such that $f(C_0^+) = C_x$. Then $f^{-1}(y) \geqslant 0$ and there is $\mathbf{w} \in W^+$ such that $f^{-1}(C_y) = \mathbf{w}\cdot C_0^+$. By 1.8(b), $\mathbf{w}$ does not depend on the choice of A.

We define the $W-$*distance* between the two local chambers C_x and C_y to be this unique element: $d^W(C_x, C_y) = \mathbf{w} \in W^+ = Y^+ \rtimes W^v$. If $\mathbf{w} = \lambda\cdot w$, with $\lambda \in Y^+$ and $w \in W^v$, we write also $d^W(C_x, y) = \lambda$; it implies $d^v(x, y) = \lambda^{++}$. As $\leqslant$ is $G-$invariant, the $W-$distance is also $G-$invariant. When $\mathbf{w} = w \in W^v$ and $w = r_{i_1}\cdot\dots\cdot r_{i_r}$ is a reduced decomposition, we have $d^W(C_x, C_y) = w$ if and only if there is a minimal gallery (of local chambers in $\mathcal{T}_x^+\mathscr{I}$) from C_x to C_y of type $(i_1, \dots, i_r)$, in particular $x = y$. When $x = y$, this definition coincides with the one in 1.4.2.

Let us consider an apartment A and local chambers $C_x, C_y, C_z \in \mathscr{C}_0^+$ included in A. If $d^W(C_x, C_y) = \mathbf{w}$, we write $C_y = C_x * \mathbf{w}$. Conversely, for any $\mathbf{w} \in W^+$, there is a unique local chamber $C_z = C_x * \mathbf{w}$ in A such that $d^W(C_x, C_z) = \mathbf{w}$; actually $C_x * \mathbf{w}$ depends on A, but not on an identification of A with $\mathbb{A}$. For $x \leqslant y \leqslant z$, we have (in A) the Chasles relation: $d^W(C_x, C_z) = d^W(C_x, C_y)\cdot d^W(C_y, C_z)$; i.e. $(C_x, \mathbf{w}) \mapsto C_x * \mathbf{w}$ is a right action of the semi-group W^+. When (A, C_x) is identified with $(\mathbb{A}, C_0^+)$, one has $C_x * \mathbf{w} = \mathbf{w}C_x$.

When $C_x = C_0^+$ and $C_y = g\cdot C_0^+$ (with $g \in G^+$), $d^W(C_x, C_y)$ is the only $\mathbf{w} \in W^+$ such that $g \in K_I\cdot\mathbf{w}\cdot K_I$. This is the uniqueness result in Bruhat decomposition: $G^+ = \coprod_{\mathbf{w}\in W^+} K_I\cdot\mathbf{w}\cdot K_I$. Similarly we have $G^{+g} = \coprod_{\mathbf{w}\in W^{+g}} K_I\cdot\mathbf{w}\cdot K_I$.

The $W-$distance classifies the orbits of K_I on $\{C_y \in \mathscr{C}_0^+ \mid y \geqslant 0\}$, hence also the orbits of G on $\mathscr{C}_0^+ \times_\leqslant \mathscr{C}_0^+$.

1.11. Iwahori–Hecke Algebras. We consider any commutative ring with unity R. The *Iwahori–Hecke algebra* ${}^I\mathcal{H}_R$ associated to $\mathscr{I}$ with coefficients in R introduced in [3] is as follows:

To each $\mathbf{w} \in W^+$, we associate a function $T_\mathbf{w}$ from $\mathscr{C}_0^+ \times_\leqslant \mathscr{C}_0^+$ to R defined by

$$T_\mathbf{w}(C, C') = \begin{cases} 1 & \text{if } d^W(C, C') = \mathbf{w}, \\ 0 & \text{otherwise.} \end{cases}$$

The Iwahori–Hecke algebra ${}^I\mathcal{H}_R$ is the free $R-$module

$$\left\{ \sum_{\mathbf{w}\in W^+} a_\mathbf{w} T_\mathbf{w} \mid a_\mathbf{w} \in R,\ a_\mathbf{w} = 0 \text{ except for a finite number of } \mathbf{w} \right\},$$

endowed with the convolution product:

$$(\varphi * \psi)(C_x, C_y) = \sum_{C_z} \varphi(C_x, C_z)\psi(C_z, C_y),$$

where $C_z \in \mathscr{C}_0^+$ is such that $x \leqslant z \leqslant y$.

Actually, ${}^I\mathcal{H}_R$ can be identified with the natural convolution algebra of the functions $G^+ \to R$, bi-invariant under K_I and with finite support (in $K_I\backslash G^+/K_I$); this is the definition given in the introduction.

More precisely ${}^I\mathcal{H}_R$ is the space of functions $\varphi : \mathscr{C}_0^+ \times_{\leqslant} \mathscr{C}_0^+ \to R$, that are left $G-$invariant and with support a finite union of orbits (see the last two lines of 1.10). To a $\varphi \in {}^I\mathcal{H}_R$ is associated $\varphi^G : K_I\backslash G^+/K_I \to R$ such that $\varphi^G(g) = \varphi(C_0^+, g \cdot C_0^+)$. So, for $\varphi, \psi \in {}^I\mathcal{H}_R$,

$$\begin{aligned}(\varphi * \psi)^G(g) = (\varphi * \psi)(C_0^+, g \cdot C_0^+) &= \sum_{C_z} \varphi(C_0^+, C_z)\psi(C_z, g \cdot C_0^+) \\ &= \sum_{h \in G^+/K_I} \varphi(C_0^+, h \cdot C_0^+)\psi(h \cdot C_0^+, g \cdot C_0^+) \\ &= \sum_{h \in G^+/K_I} \varphi(C_0^+, h \cdot C_0^+)\psi(C_0^+, h^{-1}g \cdot C_0^+) = \sum_{h \in G^+/K_I} \varphi^G(h)\psi^G(h^{-1}g),\end{aligned}$$

we get the convolution product (in the classical case, we take a Haar measure on G with K_I of measure 1).

One also considers the subspace ${}^I\mathcal{H}_R^g = \sum_{\mathbf{w} \in W^{+g}} R \cdot T_{\mathbf{w}}$. From 4.2 and Remark 3.3(2) one sees that it is a subalgebra of ${}^I\mathcal{H}_R$. We call it the *generic Iwahori–Hecke algebra* associated to $\mathscr{I}$ with coefficients in R. From 1.9 one has ${}^I\mathcal{H}_R = {}^I\mathcal{H}_R^g$ in the affine or strictly hyperbolic cases.

We now recall some useful results of [3] in order to introduce the structure constants and a way to compute them.

Proposition 1.1 ([3, 2.3]). *Let us fix two local chambers C_x and C_y in $\mathscr{C}_0^+$ with $x \leqslant y$ and $d^W(C_x, C_y) = \mathbf{u} \in W^+$. We consider $\mathbf{w}$ and $\mathbf{v}$ in W^+. Then the number $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ of $C_z \in \mathscr{C}_0^+$ with $x \leqslant z \leqslant y$, $d^W(C_x, C_z) = \mathbf{w}$ and $d^W(C_z, C_y) = \mathbf{v}$ is finite (i.e. in $\mathbb{N}$).*

Theorem 1.2 ([3, 2.4]). *For any ring R, ${}^I\mathcal{H}_R$ is an algebra with identity element $Id = T_1$ such that*

$$T_{\mathbf{w}} * T_{\mathbf{v}} = \sum_{\mathbf{u} \in P_{\mathbf{w},\mathbf{v}}} a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} T_{\mathbf{u}}$$

where $P_{\mathbf{w},\mathbf{v}}$ is a finite subset of W^+, such that $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} = 0$ for $\mathbf{u} \notin P_{\mathbf{w},\mathbf{v}}$.

2. Projections and retractions

In this section we introduce the new tools that we shall use in the next section to compute the structure constants of the Iwahori–Hecke algebra.

2.1. Projections of chambers.

2.1.1. *Projection of a chamber C_y on a point x.* Let $x \in \mathscr{I}$, $C_y \in \mathscr{C}^+$ with $x \leqslant y$, $x \neq y$. We consider an apartment A containing x and C_y (by 1.8(a) above) and write $C_y = F(y, C_y^v)$ in A. For $y' \in y + C_y^v$ sufficiently near to y, $\alpha(y' - x) \neq 0$ for any root α and $y' - x \in \mathcal{T}^\circ$. So $]x, y')$ is in a unique positive local chamber $\mathrm{pr}_x(C_y)$ of vertex x; this chamber satisfies $[x, y) \subset \overline{\mathrm{pr}_x(C_y)} \subset \mathrm{cl}_A(\{x, y'\})$ and does not depend on the choice of y'. Moreover, if A' is another apartment containing x and C_y, we may suppose $y' \in A \cap A'$ and $]x, y')$, $\mathrm{cl}_A(\{x, y'\})$, $\mathrm{pr}_x(C_y)$ are the same in A'. The local chamber $\mathrm{pr}_x(C_y)$ is well determined by x and C_y, it is the *projection* of C_y in $\mathcal{T}_x^+\mathscr{I}$.

The same things may be done changing $+$ to $-$ or $\leqslant$ to $\geqslant$. But, in the above situation, if $C_y \in \mathscr{C}^-$, we have to assume $x \overset{o}{<} y$ to define $\mathrm{pr}_x(C_y) \in \mathscr{C}^+$: otherwise $]x, y')$ might be outside $x + \mathcal{T}$.

When $x = y$, we write $\mathrm{pr}_x(C_y) = C_y$.

2.1.2. *Projection of a chamber C_y on a generic segment germ.* Let $x \in \mathscr{I}$, $\delta = [x, x')$ a generic segment-germ and $C_y \in \mathscr{C}$ with $x \leqslant y$. By 2.1.1 we can consider $\mathrm{pr}_x(C_y) \in \mathscr{C}^+$ (with the hypothesis $x \overset{o}{<} y$ if $C_y \in \mathscr{C}^-$). We consider now an apartment A containing $[x, x')$ and $\mathrm{pr}_x(C_y)$ (by 1.8(a) above).

We consider inside A the prism denoted by $\mathrm{prism}_\delta(C_y)$ obtained as the intersection of all half-spaces $D(\alpha, k)$ (for $\alpha \in \Phi$ and $k \in \mathbb{R}$) that contain $\mathrm{pr}_x(C_y)$ and such that $\delta \subset M(\alpha, k)$. We can see that if $\delta = [x, x')$ is regular, $\mathrm{prism}_\delta(C_y) = A$. If the apartment A contains δ and C_y (hence also $\mathrm{pr}_x(C_y)$) we may replace $\mathrm{pr}_x(C_y)$ by C_y in the above definition of $\mathrm{prism}_\delta(C_y)$.

Lemma 2.1. *In $\mathrm{prism}_\delta(C_y)$, there is a unique local chamber of vertex x that contains δ in its closure. This chamber is independent of the choice of A.*

N.B. *This local chamber is, by definition, the* projection $\mathrm{pr}_\delta(C_y)$ *of the chamber C_y on the segment-germ δ. It is the local chamber containing δ in its closure which is the nearest from $\mathrm{pr}_x(C_y)$: either $d^W(\mathrm{pr}_x(C_y), \mathrm{pr}_\delta(C_y))$ is minimum or $d^{*W}(\mathrm{pr}_x(C_y), \mathrm{pr}_\delta(C_y))$ is maximum.*

The same things may be done when one supposes $y \leqslant x$ and $C_y \in \mathscr{C}^-$ or $y \overset{o}{<} x$ and $C_y \in \mathscr{C}^+$.

Proof. In the apartment A, we consider δ_+ the segment-germ δ if δ is in $\mathcal{T}_x^+\mathscr{I}$ and $\mathrm{op}_A(\delta)$ if $\delta \in \mathcal{T}_x^-\mathscr{I}$ (where $\mathrm{op}_A(\delta)$ denotes the opposite segment-germ in A). By 1.4.2, we can consider in the building $\mathcal{T}_x^+\mathscr{I}$ the minimal galleries from $\mathrm{pr}_x(C_y)$ to δ_+ (more exactly to a chamber C such that $\delta_+ \in \bar{C}$). The last chamber of each of these galleries is the same (as it has to be on the same side as $\mathrm{pr}_x(C_y)$ of any hyperplane of A, containing δ_+ and parallel to a wall); we denote it C_x^{++}. This chamber is associated to a positive system of roots Φ^+ and a root basis $(\alpha_1, \dots, \alpha_\ell)$, satisfying $\alpha_i(\delta) = 0 \iff i \leqslant r$, where $0 \leqslant r < \ell$ (we identify x and 0). Then, we have the characterization of the prism : $p \in \mathrm{prism}_\delta(C_y) \iff (\alpha_i(p) \geqslant 0$ for $1 \leqslant i \leqslant r)$. We consider w_r the element of highest length in the finite Weyl group $\langle (r_{\alpha_i})_{i \leqslant r} \rangle$.

The local chamber C_x^{++} if $\delta \in \mathcal{T}_x^+\mathscr{I}$ (resp. $\mathrm{op}_A(w_r(C_x^{++})$ if not) is the unique chamber with vertex x of $\mathrm{prism}_\delta(C_y)$ that contains δ in its closure. Indeed, if C is such a chamber, then if $]x, p) \subset C$, we have $\alpha_i(p) > 0$ for all $i \leqslant r$ (because $C \subset \mathrm{prism}_\delta(C_y)$) and $\alpha_i(p)$ of the same sign as $\alpha_i(\delta)$ if $i > r$ (because $\delta \subset \bar{C}$). So $C = C_x^{++}$ if $\delta \in \mathcal{T}_x^+\mathscr{I}$ (resp. $C = \mathrm{op}_A(w_r(C_x^{++}))$ if $\delta \in \mathcal{T}_x^-\mathscr{I}$).

In the case $\delta \in \mathcal{T}_x^+\mathscr{I}$, the characterization of C_x^{++} in the building $\mathcal{T}_x^+\mathscr{I}$ proves that it does not depend on the choice of A.

The chamber $\mathrm{op}_A(w_r(C_x^{++}))$ also only depends on δ and C_y if $\delta \in \mathcal{T}_x^-\mathscr{I}$. It is sufficient to prove that it intersects $\mathrm{conv}_A(\delta \cup \mathrm{pr}_x(C_y))$. Indeed, let us choose ξ and y such that $[x, \xi) = \delta$ and $]x, y) \subset \mathrm{pr}_x(C_y)$. We have $\alpha_i(\xi) = 0$ for $i \leqslant r$, $\alpha_i(\xi) < 0$ for $i > r$ and $\alpha_i(y) > 0$ for $i \leqslant r$. So for t near 1 enough, $\alpha_i(t\xi + (1-t)y) > 0$ for $i \leqslant r$ and < 0 for $i > r$, so $]x, t\xi + (1-t)y) \subset \mathrm{op}_A(w_r(C_x^{++})$. By Proposition 1.8, the local chamber $\mathrm{op}_A(w_r(C_x^{++}))$ is included in all apartments containing δ and $\mathrm{pr}_x(C_y)$, so is independent of the choice of A. □

2.2. Centrifugally folded galleries of chambers. Let z be a point in the standard apartment $\mathbb{A}$. We have twinned buildings $\mathcal{T}_z^+\mathscr{I}$ (resp. $\mathcal{T}_z^-\mathscr{I}$). As in 1.4.2, we consider their unrestricted structure, so the associated Weyl group is W^v and the chambers (resp. closed chambers) are the local chambers $C = \mathrm{germ}_z(z + C^v)$ (resp. local closed chambers $\overline{C} = \mathrm{germ}_z(z + \overline{C^v})$), where C^v is a vectorial chamber, cf. [18, 4.5] or [31, § 5]. The distances (resp. codistances) between these chambers

are written d^W (resp. d^{*W}). To $\mathbb{A}$ is associated a twin system of apartments $\mathbb{A}_z = (\mathbb{A}_z^-, \mathbb{A}_z^+)$.

Let $\mathbf{i} = (i_1, \dots, i_r)$ be the type of a minimal gallery. We choose in $\mathbb{A}_z^-$ a negative (local) chamber C_z^- and denote by C_z^+ its opposite in $\mathbb{A}_z^+$. We consider now galleries of (local) chambers $\mathbf{c} = (C_z^-, C_1, \dots, C_r)$ in the apartment $\mathbb{A}_z^-$ starting at C_z^- and of type $\mathbf{i}$. Their set is written $\Gamma(C_z^-, \mathbf{i})$. We consider the root β_j corresponding to the common limit hyperplane $M_j = M(\beta_j, -\beta_j(z))$ of type i_j of C_{j-1} and C_j satisfying moreover $\beta_j(C_j) \geqslant \beta_j(z)$.

We consider the system of positive roots Φ^+ associated to C_z^+. Actually, $\Phi^+ = w \cdot \Phi_f^+$, if Φ_f^+ is the system Φ^+ defined in 1.1 and $C_z^+ = \mathrm{germ}_z(z + w \cdot C_f^v)$. We denote by $(\alpha_i)_{i\in I}$ the corresponding basis of Φ and by $(r_i)_{i\in I}$ the corresponding generators of W^v. Note that this change of notation for Φ^+ and r_i is limited to subsection 2.2.

The set $\Gamma(C_z^-, \mathbf{i})$ of galleries is in bijection with the set $\Gamma(\mathbf{i}) = \{1, r_{i_1}\} \times \cdots \times \{1, r_{i_r}\}$ via the map $(c_1, \dots, c_r) \mapsto (C_z^-, c_1 C_z^-, \dots, c_1 \cdots c_r C_z^-)$. Moreover $\beta_j = -c_1 \cdots c_j(\alpha_{i_j})$.

Definition. *Let $\mathfrak{Q}$ be a chamber in $\mathbb{A}_z$. A gallery $\mathbf{c} = (C_z^-, C_1, \dots, C_r) \in \Gamma(C_z^-, \mathbf{i})$ is said to be* centrifugally folded *with respect to $\mathfrak{Q}$ if $C_j = C_{j-1}$ implies that M_j is a wall and separates $\mathfrak{Q}$ from $C_j = C_{j-1}$. We denote this set of centrifugally folded galleries by $\Gamma_{\mathfrak{Q}}^+(C_z^-, \mathbf{i})$. We write $\Gamma_{\mathfrak{Q}}^+(C_z^-, \mathbf{i}, C)$ the subset of galleries in $\Gamma_{\mathfrak{Q}}(C_z^-, \mathbf{i})$ such that C_r is a given chamber C.*

2.3. Liftings of galleries. Next, let $\rho_{\mathfrak{Q}} : \mathcal{T}_z\mathscr{I} \to \mathbb{A}_z$ be the retraction centered at $\mathfrak{Q}$. To a gallery of chambers $\mathbf{c} = (C_z^-, C_1, \dots, C_r)$ in $\Gamma(C_z^-, \mathbf{i})$, one can associate the set of all galleries of type $\mathbf{i}$ starting at C_z^- in $\mathcal{T}_z^-\mathscr{I}$ that retract onto $\mathbf{c}$, we denote this set by $\mathcal{C}_{\mathfrak{Q}}(C_z^-, \mathbf{c})$. We denote the set of galleries $\mathbf{c}' = (C_z^-, C_1', \dots, C_r')$ in $\mathcal{C}_{\mathfrak{Q}}(C_z^-, \mathbf{c})$ that are minimal (i.e. satisfy $C_{j-1}' \neq C_j'$ for any j) by $\mathcal{C}_{\mathfrak{Q}}^m(C_z^-, \mathbf{c})$. Recall from [19, Proposition 4.4], that the set $\mathcal{C}_{\mathfrak{Q}}^m(C_z^-, \mathbf{c})$ is nonempty if, and only if, the gallery $\mathbf{c}$ is centrifugally folded with respect to $\mathfrak{Q}$. Recall also from loc. cit., Corollary 4.5, that if $\mathbf{c} \in \Gamma_{\mathfrak{Q}}^+(C_z^-, \mathbf{i})$, then the number of elements in $\mathcal{C}_{\mathfrak{Q}}^m(C_z^-, \mathbf{c})$ is:

$$\sharp\mathcal{C}_{\mathfrak{Q}}^m(C_z^-, \mathbf{c}) = \prod_{j\in J_1} (q_j - 1) \times \prod_{j\in J_2} q_j$$

where $q_j = q_{M_j} \in \mathcal{Q}$,

$$J_1 = \{j \in \{1, \cdots, r\} \mid c_j = 1\} = \{j \in \{1, \cdots, r\} \mid C_{j-1} = C_j\}$$

and

$$J_2 = \{j \in \{1, \cdots, r\} \mid C_{j-1} \neq C_j \text{ and } M_j \text{ is a wall separating } \mathfrak{Q} \text{ (and } C_{j-1}) \text{ from } C_j\}.$$

One may remark that $\{1, \cdots, r\}$ contains the disjoint union $J_1 \sqcup J_2$, but may be different from it. The missing j are precisely those j such that M_j is not a wall (hence q_{M_j} is not defined) or that $\mathfrak{Q}$ (and C_j) are separated from C_{j-1} by M_j.

More generally let $\mathbf{c}^m = (C_z^-, C_1^m, \dots, C_r^m)$ be the minimal gallery in $\mathbb{A}_z^-$ of type $\mathbf{i}$. We write $\mathcal{C}^m(C_z^-, \mathbf{i})$ the set of all minimal galleries in $\mathscr{I}$ of type $\mathbf{i}$ starting from C_z^-. Its cardinality is $\prod_{j\in J_2} q_j$, where J_2 is the set of $1 \leqslant j \leqslant r$ such that the hyperplane M_j separating C_{j-1}^m from C_j^m is a wall.

N.B. *The $q_j = q_{M_j}$ in the above formulas are in the set $\mathcal{Q}$ of parameters. More precisely, by* 1.4.6*, if $M_j = M(\beta_j, k_j)$ with $\beta_j = w \cdot \alpha_i$ (for some $w \in W^v$, $i \in I$ and $k_j \in \mathbb{Z}$), then one has $q_j = q_i$ if k_j is even and $q_j = q_i'$ if k_j is odd.*

2.4. Hecke paths. The Hecke paths we consider here are slight modifications of those used in [19]. They were defined in [3], or in [2] (for the classical case).

Let us fix a local chamber $C_x \in \mathscr{C}_0 \cap \mathbb{A}$.

Definition. *A Hecke path of shape $\lambda \in Y^{++}$ with respect to C_x in $\mathbb{A}$ is a $\lambda-$path in $\mathbb{A}$ that satisfies the following assumptions. For all $p = \pi(t)$, we ask $x \overset{o}{<} p$, so we can consider the local negative chamber $C_p^- = \mathrm{pr}_p(C_x)$ by 2.1.1. Then we assume moreover that for all $t \in [0,1] \smallsetminus \{0,1\}$, there exist finite sequences $(\xi_0 = \pi'_-(t), \xi_1, \dots, \xi_s = \pi'_+(t))$ of vectors in V and $(\beta_1, \dots, \beta_s)$ of real roots such that, for all $j = 1, \dots, s$:*

(i) $r_{\beta_j}(\xi_{j-1}) = \xi_j$,
(ii) $\beta_j(\xi_{j-1}) < 0$,
(iii) $\beta_j(\pi(t)) \in \mathbb{Z}$, *i.e.* $\pi(t)$ *is in a wall of direction* $\ker \beta_j$,
(iv) $\beta_j(C^-_{\pi(t)}) < \beta_j(\pi(t))$.

One says then that these two sequences are a $(W^v_{\pi(t)}, C^-_{\pi(t)})-$chain from $\pi'_-(t)$ to $\pi'_+(t)$. Actually $W^v_{\pi(t)}$ is the subgroup of W^v generated by the r_β such that $M(\beta, -\beta(\pi(t)))$ is a wall.

When $t \in]0,1[$ is such that $s \neq 0$, one has $\pi'_-(t) \neq \pi'_+(t)$, the path is centrifugally folded with respect to C_x at $\pi(t)$.

Lemma 2.2. *Let $\pi \subset \mathbb{A}$ be a Hecke path with respect to C_x as above. Then,*

(a) *For t varying in $[0,1]$ and $p = \pi(t)$, the set of vectorial rays $\mathbb{R}_+(x - \pi(t))$ is contained in a finite set of closures of (negative) vectorial chambers.*
(b) *There is only a finite number of pairs (M, t) with a wall M containing a point $p = \pi(t)$ for $t > 0$, such that $\pi_-(t)$ is not in M and x is not in the same side of M as $\pi_-(t)$ (but may be $x \in M$).*
(c) *One writes $p_0 = \pi(t_0), p_1 = \pi(t_1), \dots, p_{\ell_\pi} = \pi(t_{\ell_\pi})$ with $0 = t_0 < t_1 < \cdots < t_{\ell_\pi - 1} < 1 = t_{\ell_\pi}$ the points $p = \pi(t)$ satisfying to (b) above (or $t = 0, t = 1$). Then any point t where the path is (centrifugally) folded with respect to C_x at $\pi(t)$ appears in the set $\{t_k \mid 1 \leqslant k \leqslant \ell_\pi - 1\}$.*

Proof. (a) The $\lambda-$path π is a union of line segments $[p'_0, p'_1] \cup [p'_1, p'_2] \cup \cdots \cup [p'_{n-1}, p'_n]$. By hypothesis on Hecke paths, for each point $p = \pi(t)$, $x - p$ is in the open negative Tits cone $-\mathcal{T}^\circ$ (in particular only in a finite number of closures of negative vectorial chambers). Let $p \in [p'_i, p'_{i+1}]$, then $x - p = x - p'_i - (p - p'_i)$ and $\mathbb{R}_+(x-p) \subset \mathrm{conv}(\mathbb{R}_+(x - p'_i), -\mathbb{R}_+(p - p'_i))$ and this convex hull is independent of p and only in a finite number of closures of (negative) vectorial chambers (as $(x - p'_i) \in -\mathcal{T}^\circ$ and $(p - p'_i) \in \mathbb{R}_+(p'_{i+1} - p'_i) \subset \mathcal{T}$). So (a) is proved.

(b) There is only a finite number of vectorial walls separating (strictly) a chamber in the set of (a) above and a vector $p'_i - p'_{i+1}$. And, for each such vectorial wall, there is only a finite number of walls with this direction meeting the compact set $\pi([0,1])$. Moreover such a wall meets a segment $]p'_i, p'_{i+1}]$ at most once or contains $[p'_i, p'_{i+1}]$ (hence $\pi_-(t) \subset M$ for $\pi(t) \in]p'_i, p'_{i+1}]$).

(c) The folding points are among $\{p_1, \dots, p_{\ell_\pi - 1}\}$ by (iv) and (ii) above for $j = 1$. □

2.5. Retractions and liftings of line segments.

2.5.1. *Local study.* In tangent buildings, the centrifugally folded galleries are related with retractions of opposite segment germs, by the following lemma proved in [19, Lemma 4.6].

We consider a point $z \in \mathbb{A}$ and a negative local chamber C_z^- in $\mathbb{A}_z^-$. Let ξ and η be two segment germs in $\mathbb{A}_z^+ = \mathbb{A} \cap \mathcal{T}_z^+ \mathscr{I}$. Let $-\eta$ and $-\xi$ opposite respectively η and ξ

in $\mathbb{A}_z^-$. Let $\mathbf{i}$ be the type of a minimal gallery between C_z^- and $C_{-\xi}$, where $C_{-\xi}$ is the negative (local) chamber containing $-\xi$ such that $d^W(C_z^-, C_{-\xi})$ is of minimal length. Let $\mathfrak{Q}$ be a chamber of $\mathbb{A}_z^+$ containing η. We suppose ξ and η conjugated by W_z^v.

LEMMA. *The following conditions are equivalent:*

(a) *There exists an opposite ζ to η in $\mathcal{T}_z^-\mathscr{I}$ such that $\rho_{\mathbb{A}_z, C_z^-}(\zeta) = -\xi$.*
(b) *There exists a gallery $\mathbf{c} \in \Gamma_{\mathfrak{Q}}^+(C_z^-, \mathbf{i})$ ending in $-\eta$.*
(c) *There exists a $(W_z^v, C_z^-)-$chain from ξ to η.*

Moreover the possible ζ are in one-to-one correspondence with the disjoint union of the sets $\mathcal{C}_{\mathfrak{Q}}^m(C_z^-, \mathbf{c})$ for $\mathbf{c}$ in the set $\Gamma_{\mathfrak{Q}}^+(C_z^-, \mathbf{i}, -\eta)$ of galleries in $\Gamma_{\mathfrak{Q}}^+(C_z^-, \mathbf{i})$ ending in $-\eta$.

2.5.2. *Consequence.* Let C_x be a positive local chamber in $\mathbb{A}$ and $z \in \mathbb{A}$ a point such that $x \overset{o}{<} z$. We consider $C_z^- = \mathrm{pr}_z(C_x)$. Then one knows that the restriction of the retraction $\rho = \rho_{\mathbb{A}, C_x}$ to the tangent twin building $\mathcal{T}_z\mathscr{I}$ is the retraction $\rho_{\mathbb{A}_z, C_z^-}$.

We consider two points y, z_0 in $\mathscr{I}$ such that $x \overset{o}{<} z_0 \leqslant y$, with $d^v(z_0, y) = \lambda \in Y^{++}$. By 1.7, the image $\rho([z_0, y])$ is a $\lambda-$path π from $\rho(z_0)$ to $\rho(y)$. For $z \in [z_0, y[$, we consider an apartment A containing $[z, y)$ and C_x, hence also C_z^-. We write $p = \rho(z)$. The restriction $\rho|_A$ is the restriction to A of an automorphism φ of $\mathscr{I}$ fixing C_x (and an isomorphism from A to $\mathbb{A}$); φ induces an isomorphism $\varphi|_{\mathcal{T}_z\mathscr{I}}$ from $\mathcal{T}_z\mathscr{I}$ onto $\mathcal{T}_z\mathscr{I}$. One has $\rho|_{\mathcal{T}_z\mathscr{I}} = \rho_{\mathbb{A}_p, C_p^-} \circ \varphi|_{\mathcal{T}_z\mathscr{I}} = \varphi|_{A_z} \circ \rho_{A_z, C_z^-}$. So one may use the above Lemma, more precisely the implication (a) $\Longrightarrow$ (c): we get a $(W_p^v, C_p^-)-$chain from $\pi'_-(t)$ to $\pi'_+(t)$ (if $p = \pi(t)$).

We have proved that $\pi = \rho([z_0, y])$ is a Hecke path of shape λ with respect to C_x in $\mathbb{A}$. This result is a part of [3, Theorem 3.4]. It is also a consequence of the proof of [2, Th. 3.8] which deals with the classical case of buildings.

2.5.3. *Liftings of Hecke paths.* One considers in $\mathbb{A}$ a positive local chamber C_x, a Hecke path π of shape $\lambda \in Y^{++}$ with respect to C_x and the retraction $\rho = \rho_{\mathbb{A}, C_x}$. Given a point $y \in \mathscr{I}$ with $\rho(y) = \pi(1)$, we consider the set $S_{C_x}(\pi, y)$ of all segment germs $[z, y]$ in $\mathscr{I}$ such that $\rho([z, y]) = \pi$. The above Lemma (essentially (b)) is used in [3] to compute the cardinality of $S_{C_x}(\pi, y)$.

We consider the notation of 1.7 and the numbers t_k of Lemma 2.2. Then $p_k = \pi(t_k)$, $\xi_k = -\pi_-(t_k)$, $\eta_k = \pi_+(t_k)$ and $\mathbf{i}_k$ is the type of a minimal gallery between $C_{p_k}^-$ and $C_{-\xi_k}$, where $C_{-\xi_k}$ is the negative (local) chamber such that $-\xi_k \subset \overline{C_{-\xi_k}}$ and $d^W(C_{p_k}^-, C_{-\xi_k})$ is of minimal length. Let $\mathfrak{Q}_k$ be a fixed chamber in $\mathbb{A}_{z_k}^+$ containing η_k in its closure and $\Gamma_{\mathfrak{Q}_k}^+(C_{p_k}^-, \mathbf{i}_k, -\eta_k)$ be the set of all the galleries $(C_{z_k}^-, C_1, \ldots, C_r)$ of type $\mathbf{i}_k$ in $\mathbb{A}_{z_k}^-$, centrifugally folded with respect to $\mathfrak{Q}_k$ and with $-\eta_k \in \overline{C_r}$.

The following result is Theorem 3.4 in [3]. One uses the notation of 2.2 and 2.3. One considers paths π more general than Hecke paths. The idea is to lift the path π step by step starting from its end by using the above Lemma. We shall generalize it in Theorem 3.3 by lifting decorated Hecke paths (see just below).

THEOREM 2.3. *The set $S_{C_x}(\pi, y)$ is non empty if, and only if, π is a Hecke path with respect to C_x. Then, we have a bijection*

$$S_{C_x}(\pi, y) \simeq \left(\prod_{k=1}^{\ell_\pi - 1} \coprod_{\mathbf{c} \in \Gamma_{\mathfrak{Q}_k}^+(C_{p_k}^-, \mathbf{i}_k, -\eta_k)} \mathcal{C}_{\mathfrak{Q}_k}^m(C_{p_k}^-, \mathbf{c}) \right) . \mathcal{C}^m(C_y^-, \mathbf{i}_{\ell_\pi}).$$

In particular, the number of elements in this set is a polynomial in the numbers $q \in \mathcal{Q}$ with coefficients in $\mathbb{Z}$ depending only on $\mathbb{A}$.

2.6. Decorated segments and paths. Let us consider z_0 and y in $\mathscr{I}$ such that $z_0 \overset{o}{<} y$.

Definition 2.4. *A* decorated segment $\underline{[z_0, y]}$ *is the datum of a segment* $[z_0, y]$ *as above and, for any* $z \in [z_0, y[$ *(resp.* $z \in]z_0, y]$*) of a positive (resp. negative) chamber* C_z^+ *(resp.* C_z''*) with vertex* z *and containing the segment germ* $[z, y)$ *(resp.* $[z, z_0)$*) in its closure. One asks moreover that* $C_z^+ = \mathrm{pr}_{[z,y)}(C)$ *(resp.* $C_z'' = \mathrm{pr}_{[z,z_0)}(C)$*) for any local chamber* $C = C_{z'}^+$ *or* $C = C_{z'}''$ *as above. One may remark that, then,* $C_z^+ = \mathrm{pr}_z(C)$ *(resp.* $C_z'' = \mathrm{pr}_z(C)$*) if* $z' \in [z, y]$ *(resp.* $z' \in [z_0, z]$*).*

Clearly the decorated segment $\underline{[z_0, y]}$ is entirely determined by the segment $[z_0, y]$ and any of the local chambers $C_{z'}^+$ or $C_{z'}''$. It is entirely contained in any apartment containing $[z_0, y]$ and one local chamber $C_{z'}^+$ or $C_{z'}''$ (by 2.1).

For points $z_0' \neq y'$ in $[z_0, y]$ in the order z_0, z_0', y', y (i.e. $z_0' \overset{o}{<} y'$) the datum $\underline{[z_0', y']} = ([z_0', y'], (C_z^+)_{z \in [z_0', y'[}, (C_z'')_{z \in]z_0', y']})$ is a decorated segment.

Lemma 2.5. *Let* $[z_0, y]$ *be a segment as above,* $z_1 \in [z_0, y]$ *and* C_{z_1} *a local chamber with vertex* z_1 *contained in a same apartment* A *as* $[z_0, y]$*. Let us define* $C_z^+ = \mathrm{pr}_{[z,y)}(C_{z_1})$ *and* $C_z'' = \mathrm{pr}_{[z,z_0)}(C_{z_1})$*. Then* $\underline{[z_0, y]} = ([z_0, y], (C_z^+)_{z \in [z_0, y[}, (C_z'')_{z \in]z_0, y]})$ *is a decorated segment. Moreover in* A *all chambers* C_z^+ *(resp.* C_z''*) are deduced from each other by a translation.*

N.B. *If* z_1 *is* z_0 *or* y *then any local chamber* C_{z_1} *with vertex* z_1 *is contained in a same apartment as* $[z_0, y]$*.*

Proof. We have to prove that $C_z^+ = \mathrm{pr}_{[z,y)}(C)$ (resp. $C_z'' = \mathrm{pr}_{[z,z_0)}(C)$) for any local chamber $C = C_{z'}^+$ or $C = C_{z'}''$. Let us recall that the chamber C_z^+ (resp. C_z'') is the unique chamber, that contains $\delta = [z, y)$ (resp. $\delta = [z, z_0)$) in its closure, of the prism $\mathrm{prism}_\delta(C_{z_1})$ defined in A as the intersection of all half-spaces $D(\alpha, k)$ (for $\alpha \in \Phi$ and $k \in \mathbb{R}$) that contain C_{z_1} and such that $\delta \subset M(\alpha, k)$. In fact each prism considered to define all these chambers in these definitions is the same prism $\mathrm{prism}_{[z_0,y]}(C_{z_1})$, as $\delta \subset M(\alpha, k) \iff [z_0, y] \subset M(\alpha, k)$. Moreover, as already partially remarked in 2.1.2, $\mathrm{prism}_{[z_0,y]}(C_{z_1}) = \mathrm{prism}_{[z_0,y]}(C)$ for $C = C_{z'}^+$ or $C = C_{z'}''$. Indeed, such a C is in $\mathrm{prism}_{[z_0,y]}(C_{z_1})$ and any $M(\alpha, k)$ containing $[z_0, y]$ cannot cut C, so $\mathrm{prism}_{[z_0,y]}(C_{z_1}) = \mathrm{prism}_{[z_0,y]}(C)$.

It is now clear that $C_z^+ = \mathrm{pr}_{[z,y)}(C)$ (resp. $C_z'' = \mathrm{pr}_{[z,z_0)}(C)$) for any local chamber $C = C_{z'}^+$ or $C = C_{z'}''$. Moreover the translations of a vector in the direction of the line of A containing δ stabilize the prism and exchange the segment germs. So the last assertion of the lemma is clear. □

Definitions 2.6. *A* decorated $\lambda-$path $\underline{\pi}$ *is the datum of:*

- *a* $\lambda-$*path* $\{\pi(t) \mid 0 \leqslant t \leqslant 1\}$,
- *a positive (resp. a negative) local chamber* $C_{\pi(t)}^+$ *(resp.* $C_{\pi(t)}''$*) of vertex* $\pi(t)$ *for* $0 \leqslant t < 1$ *(resp.* $0 < t \leqslant 1$*).*

Such that there are numbers $0 = t_0' < t_1' < \cdots < t_r' = 1$ *satisfying, for any* $1 \leqslant i \leqslant r$*,*

- $\{\pi(t) \mid t_{i-1}' \leqslant t \leqslant t_i'\}$ *is a segment* $[\pi(t_{i-1}'), \pi(t_i')]$,
- $\underline{[\pi(t_{i-1}'), \pi(t_i')]} = ([\pi(t_{i-1}'), \pi(t_i')], (C_{\pi(t)}^+)_{t \in [t_{i-1}', t_i'[}, (C_{\pi(t)}'')_{t \in]t_{i-1}', t_i']})$ *is a decorated segment (in particular* $\pi(t_{i-1}') \overset{o}{<} \pi(t_i')$*), hence* λ *is spherical).*

A *decorated Hecke path* of shape λ with respect to C_x in $\mathbb{A}$ is a decorated $\lambda-$path $\underline{\pi}$ such that the underlying path π is a Hecke path of shape λ with respect to C_x in

$\mathbb{A}$. One assumes moreover that the numbers $0 < t'_1 < \cdots < t'_r = 1$ are equal to the numbers $0 < t_1 < t_2 < \cdots < t_{\ell_\pi} = 1$ of Lemma 2.2 above.

PROPOSITION 2.7. *Let $\underline{[z_0, y]}$ be a decorated segment (with $d^v(z_0, y) = \lambda \in Y^{++}$ spherical), C_x a chamber of vertex x in $\mathbb{A}$ with $x \overset{o}{<} z_0$ (hence $x \overset{o}{<} z$ for any $z \in [z_0, y]$) and $\rho = \rho_{\mathbb{A},C_x}$ the associated retraction. We parametrize $[z_0, y]$ by $z(t) = z_0 + t(y - z_0)$ in any apartment containing $[z_0, y]$. Then:*

$$\rho(\underline{[z_0, y]}) = (\pi = \rho \circ z, (C^+_{\rho z(t)} = \rho C^+_{z(t)})_{t\in[0,1[}, (C^*_{\rho z(t)} = \rho C''_{z(t)})_{t\in]0,1]})$$

is a decorated Hecke path of shape λ with respect to C_x in $\mathbb{A}$.

N.B. *Conversely a decorated Hecke path is not always the image by ρ of a decorated segment. But the calculations of the number of such liftings (as in Theorem* 2.3*) is the main ingredient of our main theorem (*3.3 *below) generalizing the Theorem 3.7 in* [3].

Proof. For any $z \in [z_0, y[$ (resp. $z \in]z_0, y]$), we consider an apartment A^+_z (resp. A''_z) containing C_x and C^+_z (resp. C''_z). Then $A^+_z \cup A''_z$ (or $A^+_{z_0}$, A''_y) contains a neighbourhood of z (or z_0, y) in the segment $[z_0, y]$. By compactness of this segment we get numbers $0 = t'_0 < t'_1 < \cdots t'_r = 1$ and apartments A_i such that A_i contains C_x, $z([t'_{i-1}, t'_i])$ and either $C^+_{z(t'_{i-1})}$ or $C''_{z(t'_i)}$. By the projection properties of decorated segments, it contains all other $C^+_{z(t)}$ (resp. $C''_{z(t)}$) for $t \in [t'_{i-1}, t'_i[$ (resp. $t \in]t'_{i-1}, t'_i]$). As ρ sends isomorphically A_i onto $\mathbb{A}$, we get that $\rho(\underline{[z_0, y]})$ is a decorated $\lambda-$path, with underlying path a Hecke path of shape λ with respect to C_x in $\mathbb{A}$.

To get that $\rho(\underline{[z_0, y]})$ is a decorated Hecke path, we have now to prove that the t'_i may be replaced by the t_i associated to this Hecke path by Lemma 2.2. We may apply the following Lemma to $[\pi(t_{i-1}), \pi(t_i)]$. Any apartment A containing C_x and $C''_{z(t_i)}$ contains $[z(t_{i-1}), z(t_i)]$, hence also $C''_{z(t)}$ for $t_{i-1} < t \leqslant t_i$ and $C^+_{z(t)}$ for $t_{i-1} \leqslant t < t_i$, by the projection properties of decorated segments. But ρ induces an isomorphism from A onto $\mathbb{A}$. So $([\pi(t_{i-1}), \pi(t_i)], (\rho C^+_{z(t)})_{t_{i-1}\leqslant t<t_i}, (\rho C''_{z(t)})_{t_{i-1}<t\leqslant t_i})$ is a decorated segment, as expected. □

LEMMA 2.8. *In an apartment $\mathbb{A}$ of a masure $\mathscr{I}$, we consider a local chamber C_x and a line segment $[p_0, p_1]$ with $x \overset{o}{<} p_0 \leqslant p_1$. We suppose that, for any $p \in]p_0, p_1[$ and any wall M containing p, then $[p, p_0]$ is in the half-apartment containing C_x delimited by M. We consider the retraction $\rho = \rho_{\mathbb{A},C_x}$. Then, for any segment germ $[z_1, z)$ in $\mathscr{I}$ such that $\rho([z_1, z)) = [p_1, p_0)$ (hence $\rho(z_1) = p_1$), there is a unique line segment $[z_1, z_0]$ such that $[z_1, z_0) = [z_1, z)$ and $\rho([z_1, z_0]) = [p_1, p_0]$. More precisely any apartment A containing C_x and $[z_1, z)$ contains $[z_1, z_0]$.*

Proof. Let A be an apartment containing C_x and $[z_1, z)$. Up to the isomorphism ρ from A onto $\mathbb{A}$, one may suppose $A = \mathbb{A}$. Then $z_1 = p_1$ and $[p_1, p_0]$ satisfies $[p_1, p_0) = [p_1, z)$, $\rho([p_1, p_0]) = [p_1, p_0]$ as expected for $[p_1, z_0]$. Let us consider another solution $[p_1, z_0]$, so $[p_1, z_0) = [p_1, p_0)$ and $\rho([p_1, z_0]) = [p_1, p_0]$. Let z' be the point satisfying $[p_1, z'] \subset [p_1, p_0] \cap [p_1, z_0]$ that is the nearest from p_0. One has $z' \neq p_1$ and one wants to prove that $z' = p_0$. If $z' \neq p_0$, one may consider a minimal gallery $\mathbf{c}'$ in $\mathcal{T}^-_{z'}\mathscr{I}$ from $C^-_{z'} = \mathrm{pr}_{z'}(C_x)$ to the segment germ $[z', z_0)$. Clearly $\mathbf{c} = \rho(\mathbf{c}')$ is a minimal gallery in $\mathbb{A}^-_{z'}$ from $C^-_{z'}$ to the segment germ $[z', p_0)$. If we write $\mathfrak{Q} = C^-_{z'}$, we have $\mathbf{c}' \in \mathcal{C}^m_{\mathfrak{Q}}(C^-_{z'}, \mathbf{c})$, with the notation of 2.3. But by the hypotheses, no wall M containing z' separates strictly C_x (i.e. $C^-_{z'}$) from $[z', p_0)$. Hence the formula in 2.3 tells that $\mathcal{C}^m_{\mathfrak{Q}}(C^-_{z'}, \mathbf{c})$ is reduced to one element: we have $\mathbf{c}' = \mathbf{c}$, $[z', z_0) = [z', p_0)$, contrary to the hypothesis on z'. □

Remark 2.9. The definitions and results in 2.6, 2.7, 2.8 above are also true if we replace C_x by a negative sector germ $\mathfrak{S}$ in $\mathbb{A}$ and ρ by $\rho_{\mathbb{A},\mathfrak{S}}$. The corresponding results of the lemma are more or less implicit in [3], see the last paragraph of proof of Lemma 2.1 or of Proposition 2.3 in l.c.

3. Structure constants in spherical cases

In this section, we compute the structure constants $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ of the Iwahori–Hecke algebra ${}^I\mathcal{H}^{\mathscr{J}}_R$, assuming that $\mathbf{v} = \mu \cdot v$ and $\mathbf{w} = \lambda \cdot w$ are spherical, i.e. μ and λ are spherical (see 1.1 for the definitions). As in [3], we will adapt some results obtained in the spherical case in [19] to our situation.

These structure constants depend on the shape of the standard apartment $\mathbb{A}$ and on the numbers q_M of 1.4.6. Recall that the number of (possibly) different parameters is at most $2 \cdot |I|$. We denoted by $\mathcal{Q} = \{q_1, \cdots, q_l, q'_1 = q_{l+1}, \cdots, q'_l = q_{2l}\}$ this set of parameters.

For $\lambda \in Y^+$ spherical, we denote w_λ (resp. w^+_λ) the smallest (resp. longest) element $w \in W^v$ such that $w \cdot \lambda \in \overline{C^v_f}$. We start by several lemmas.

Lemma 3.1 ([3, 3.6]). *Let $C_x, C_z \in \mathscr{C}^+_0$ with $x \leqslant z$ and $\lambda \in Y^+$ spherical, $w \in W^v$. We write $C^-_z = \mathrm{pr}_z(C_x)$. Then*

$$d^W(C_x, C_z) = \lambda \cdot w \Longleftrightarrow \begin{cases} d^W(C_x, z) = \lambda \\ d^{*W}(C^-_z, C_z) = w^+_\lambda w. \end{cases}$$

Lemma 3.2. *Let $C_z, C_y \in \mathscr{C}^+_0$ with $z \overset{o}{<} y$ and $\mu \in Y^+$ spherical, $v \in W^v$. We write $C^+_z = \mathrm{pr}_z(C_y)$ and $C''_y = \mathrm{pr}_{[y,z)}(C^+_z) = \mathrm{pr}_y(C^+_z)$. Then*

(a) $d^W(C_z, C_y) = \mu v \Longleftrightarrow \begin{cases} d^W(C_z, C^+_z) = v(w_{v^{-1}\cdot\mu})^{-1} \\ d^W(C^+_z, C_y) = \mu^{++} w_{v^{-1}\cdot\mu}. \end{cases}$

(b) $d^W(C^+_z, C_y) = \mu^{++} w_{v^{-1}\cdot\mu} \iff \begin{cases} d^W(C^+_z, y) = \mu^{++} \\ d^{*W}(C''_y, C_y) = w^+_{\mu^{++}} w_{v^{-1}\cdot\mu} \end{cases}$

Proof. (a) Let us fix an apartment A' containing C_z, C_y and so C^+_z and identify (A', C_z) with $(\mathbb{A}, C^+_0)$.

Let us suppose that $d^W(C_z, C_y) = \mu v$ and denote $C^+_y := C^+_z + \mu$. Clearly $d^W(C_z, C_z + \mu) = \mu$ and, by Chasles in A', $\mu \cdot v = d^W(C_z, C_y) = d^W(C_z, C_z + \mu) d^W(C_z + \mu, C_y)$, hence $d^W(C_z + \mu, C_y) = v$ i.e. $C_y = (C_z + \mu) * v$ (cf. 1.10). By $G-$invariance of d^W and Chasles, we have $d^W(C_z, C^+_z) = d^W(C_z + \mu, C^+_y) = d^W(C_z + \mu, C_y) d^W(C_y, C^+_y) = v d^W(C_y, C^+_y)$. Among the walls containing $[z, y]$, no one separates C^+_y from C_y, so the local chamber C^+_y is the closest chamber to C_y among those containing the segment-germ $]y, y + \mu)$ in their closure, i.e. $C^+_y = \mathrm{pr}_{[y,y+\mu)}(C_y)$ and $d^W(C_y, C^+_y) = w'$ where w' is the smallest $w \in W^v \subset W^+$ (for the Bruhat order of W^v) such that $]y, y + \mu) \subset \overline{C_y * w} = \overline{C_{z+\mu} * vw} = \overline{C_z * \mu v w} = \mu v w \overline{C_z}$, as we identified C_z with C^+_0. As $\mu = y - z$, we can see w' as the smallest $w \in W^v \subset W^+$ (for the Bruhat order of W^v) such that $]z, z + \mu) \subset vw\overline{C_z}$ i.e. $v^{-1}\mu \in w\overline{C^v_f}$ (as we identified C_z with C^+_0), so $w' = (w_{v^{-1}\cdot\mu})^{-1}$. Finally, we get $d^W(C_z, C^+_z) = v(w_{v^{-1}\cdot\mu})^{-1}$ and so

$$\begin{aligned} d^W(C^+_z, C_y) &= (d^W(C_z, C^+_z))^{-1} d^W(C_z, C_y) = w_{v^{-1}\cdot\mu} v^{-1} \mu v (w_{v^{-1}\cdot\mu})^{-1} w_{v^{-1}\cdot\mu} \\ &= \mu^{++} w_{v^{-1}\cdot\mu}. \end{aligned}$$

In the same way, if we suppose that $d^W(C_z, C^+_z) = v(w_{v^{-1}\cdot\mu})^{-1}$ and $d^W(C^+_z, C_y) = \mu^{++} w_{v^{-1}\cdot\mu}$, by Chasles we obtain $d^W(C_z, C_y) = \mu v$.

(b) We consider now the opposite local chamber at y of C_y^+ (resp. C_y) in A' which is denoted by $-C_y^+$ (resp. $-C_y$). If $d^W(C_z^+, C_y) = \mu^{++} w_{v^{-1}\cdot\mu}$, we have $d^W(C_z^+, y) = \mu^{++} = d^W(C_z^+, C_y^+)$ and $d^W(C_y^+, C_y) = w_{v^{-1}\cdot\mu}$, so $d^{*W}(-C_y^+, C_y) = w_{v^{-1}\cdot\mu}$. By the proof of 2.1, we see that C_y'' and $-C_y^+$ are such that $d^W(-C_y^+, C_y'') = d^W(C_y'', -C_y^+) = w^+_{\mu^{++}}$ (the longest element of $W^v_{\mu^{++}}$ the fixator of μ^{++} in W^v). By Chasles in A', we have

$$d^{*W}(C_y'', C_y) = d^W(C_y'', -C_y) = d^W(C_y'', -C_y^+) d^W(-C_y^+, -C_y) = w^+_{\mu^{++}} \cdot w_{v^{-1}\cdot\mu}.$$

The converse result is clear by Chasles. □

3.1. Local study. We shall need a partial generalization of Lemma 2.5.1 dealing with decorations.

We consider a point $z \in \mathbb{A}$, a negative local chamber C_z^- in $\mathbb{A}_z^-$ and the retraction $\rho = \rho_{\mathbb{A}_z, C_z^-}$ in $\mathcal{T}_z\mathscr{I}$. Let C_z^+ (resp. C_z^*) be a positive (resp. negative) local chamber in $\mathbb{A}_z$, we also introduce the retraction $\rho' = \rho_{\mathbb{A}_z, C_z^+}$ in $\mathcal{T}_z\mathscr{I}$. Let ξ and η be two segment germs in $\mathbb{A}_z^+ = \mathbb{A} \cap \mathcal{T}_z^+\mathscr{I}$ of the same "type" (i.e. $\eta = [z, z + w\cdot\lambda)$, $\xi = [z, z + w'\cdot\lambda)$ for some $\lambda \in Y^{++}$ and $w, w' \in W^v$). We suppose that $\overline{C_z^+}$ contains η and $\overline{C_z^*}$ contains the opposite $-\xi = [z, z - w'\lambda)$ of ξ in $\mathbb{A}_z$. We denote $-\eta = [z, z - w\cdot\lambda)$ the opposite of η in $\mathbb{A}_z$ and $\widetilde{C}_z = \mathrm{pr}_{-\eta}(C_z^+)$. Let $\mathbf{i}$ be the type of a minimal gallery from C_z^- to C_z^*.

Lemma. *The following conditions are equivalent:*

(a) *There exists a segment germ ζ opposite η in $\mathcal{T}_z^-\mathscr{I}$ and a negative local chamber C_z'' containing ζ in its closure such that $\rho(\zeta) = -\xi$, $\rho(C_z'') = C_z^*$ and $C_z'' = \mathrm{pr}_\zeta(C_z^+)$.*

(b) *There exists a gallery $\mathbf{c} \in \Gamma^+_{C_z^+}(C_z^-, \mathbf{i})$ ending in the local chamber $\widetilde{C}_z$.*

Moreover the possible (ζ, C_z'') are in one-to-one correspondence with the disjoint union of the sets $\mathcal{C}^m_{C_z^+}(C_z^-, \mathbf{c})$ for $\mathbf{c}$ in the set $\Gamma^+_{C_z^+}(C_z^-, \mathbf{i}, \widetilde{C}_z)$.

Proof. If ζ, a segment germ opposite η in $\mathcal{T}_z^-\mathscr{I}$, and C_z'', a negative local chamber containing ζ in its closure, are such that $\rho(\zeta) = -\xi$, $\rho(C_z'') = C_z^*$ and $C_z'' = \mathrm{pr}_\zeta(C_z^+)$, there is a unique minimal gallery $\mathbf{c}'$ from C_z^- to C_z'' of type $\mathbf{i}$ (as ρ induces a bijection between the minimal galleries from C_z^- to C_z'' and the minimal galleries from C_z^- to C_z^*). The gallery $\mathbf{c} = \rho'(\mathbf{c}')$ is in $\Gamma^+_{C_z^+}(C_z^-, \mathbf{i}, \widetilde{C}_z)$. Indeed, ζ is opposite η so $\rho'(\zeta) = -\eta$, hence the image of $C_z'' = \mathrm{pr}_\zeta(C_z^+)$ by ρ' is $\widetilde{C}_z = \mathrm{pr}_{-\eta}(C_z^+)$.

Reciprocally, let $\mathbf{c} \in \Gamma^+_{C_z^+}(C_z^-, \mathbf{i})$ be a gallery ending in the local chamber $\widetilde{C}_z$. We can lift this gallery with respect to ρ' while preserving the first chamber C_z^- to obtain a minimal gallery $\mathbf{c}'$ of type $\mathbf{i}$. Let us call C_z'' the last chamber of the lifted gallery. The isomorphism associated to ρ' (see 1.7) between an apartment A_z containing C_z^+ and C_z'' and $\mathbb{A}_z$ enables us to say that the lifting of $-\eta$ is a segment germ ζ opposite η in A_z and $C_z'' = \mathrm{pr}_\zeta(C_z^+)$. As the gallery $\mathbf{c}$ is of type $\mathbf{i}$, ρ sends C_z'' onto the end of the minimal gallery of same type beginning at C_z^-, so $\rho(C_z'') = C_z^*$. Moreover, ζ is of the same type as $-\eta$ (and $-\xi$), so $\rho(\zeta) = -\xi$.

From the first paragraph above, we get an injective map $(\zeta, C_z'') \mapsto \mathbf{c}'$ from the set of pairs (ζ, C_z'') as in (a) and the disjoint union of the sets $\mathcal{C}^m_{C_z^+}(C_z^-, \mathbf{c})$ for $\mathbf{c}$ in the set $\Gamma^+_{C_z^+}(C_z^-, \mathbf{i}, \widetilde{C}_z)$: indeed, ζ is fully determined by C_z'' (and λ). The second paragraph proves that this map is surjective. □

3.2. Opposite line segments. The following lemma will be useful in Theorem 3.3.

Lemma. *Let us consider in a masure $\mathscr{I}$ two preordered line segments or rays δ_1, δ_2 in apartments A_1, A_2, sharing the same origin x. One supposes the segments germs* $\mathrm{germ}_x(\delta_1)$ *and* $\mathrm{germ}_x(\delta_2)$ *opposite (in any apartment containing them both). Then there is a line in an apartment A of $\mathscr{I}$ containing δ_1 and δ_2. In particular, if δ_1, δ_2 are line segments (resp. rays), then $\delta_1 \cup \delta_2$ is also a line segment (resp. a line).*

Proof. The case of line segments is Lemma 4.9 in [19]. The case of rays may be deduced from the fact stated in part 2 of the proof of [31, Prop. 5.4]. As we shall not use it, we omit the details. □

3.3. The main formula. Let us fix two local chambers C_x and C_y in $\mathscr{C}_0^+$ with $x \leqslant y$ and $d^W(C_x, C_y) = \mathbf{u} = \nu \cdot u \in W^+$. We consider $\mathbf{w} = \lambda \cdot w$ and $\mathbf{v} = \mu \cdot v$ in W^+. Then we know that the structure constant $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is the number of $C_{z_0} \in \mathscr{C}_0^+$ with $x \leqslant z_0 \leqslant y$, $d^W(C_x, C_{z_0}) = \mathbf{w}$ and $d^W(C_{z_0}, C_y) = \mathbf{v}$; moreover this number is finite, see Proposition 1.1. In Lemmas 3.1 and 3.2 we gave conditions equivalent to these $W-$distance conditions.

We choose the standard apartment $\mathbb{A}$ containing C_x and C_y, and we identify C_x with the fundamental local chamber C_0^+. The datum of z_0 is equivalent to the datum of the segment $[z_0, y]$ or of the decorated segment $\underline{[z_0, y]}$ associated, as in 2.5, to $[z_0, y]$ and C_y. We consider then the decorated Hecke path $\underline{\pi}$ image of $\underline{[z_0, y]}$ by the retraction $\rho_{\mathbb{A}, C_x}$.

To the Hecke path π underlying a decorated Hecke path $\underline{\pi}$ are associated $\ell_\pi \in \mathbb{N}$ and numbers $t_0 = 0 < t_1 < t_2 < \cdots < t_{\ell_\pi} = 1$ as in Lemma 2.2 and Definition 2.6. We write $p_k = \pi(t_k)$. We write C_p^+ (resp. C_p^* instead of C_p'') the decorations of $\underline{\pi}$ at a point p of π. We write C_z^+ (resp. C_z'') the decorations of a decorated segment at one of its points z.

We use freely the notations from 2.1, 2.2 and 2.3.

Theorem. *Assume μ and λ spherical. Then the structure constant $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is given by:*

$$a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} = \sum_{\underline{\pi}} \prod_{k=0}^{\ell_\pi} a_{\underline{\pi}}(k)$$

where $\underline{\pi}$ runs over the decorated Hecke paths in $\mathbb{A}$ of shape μ^{++} with respect to C_x from $p_0 = x + \lambda = \lambda$ to $y = x + \nu = \nu$, and the integers $a_{\underline{\pi}}(k)$ are given by:

(a) $a_{\underline{\pi}}(\ell_\pi) = \sum_{\mathbf{d} \in \Gamma^+_{C_y}(C_y^-, \mathbf{i}_\ell, \tilde{C}_y)} \sharp \mathcal{C}^m_{C_y}(C_y^-, \mathbf{d})$, *where $\mathbf{i}_\ell$ is the type of a fixed minimal gallery from C_y^- to C_y^* and $\tilde{C}_y$ is the unique local chamber at y in $\mathbb{A}$ such that $d^{*W}(\tilde{C}_y, C_y) = w^+_{\mu^{++}} w_{v^{-1}\cdot\mu}$.*

(b) *For $1 \leqslant k \leqslant \ell_\pi - 1$, $a_{\underline{\pi}}(k) = \sum_{\mathbf{c} \in \Gamma^+_{C^+_{p_k}}(C^-_{p_k}, \mathbf{i}_k, \tilde{C}_{p_k})} \sharp \mathcal{C}^m_{C^+_{p_k}}(C^-_{p_k}, \mathbf{c})$, where $\mathbf{i}_k$ is the type of a fixed minimal gallery from $C^-_{p_k}$ to $C^*_{p_k}$ and $\tilde{C}_{p_k} = \mathrm{pr}_{-\eta_k}(C^+_{p_k})$ with $-\eta_k$ the segment germ of origin p_k in $\mathbb{A}$ opposite $\eta_k = \pi_+(t_k)$.*

(c) $a_{\underline{\pi}}(0) = \sum_{\mathbf{e} \in \Gamma^+_{C^-_{p_0}}(C^+_{p_0}, \mathbf{i}, C'_{p_0})} \sharp \mathcal{C}^m_{C^-_{p_0}}(C^+_{p_0}, \mathbf{e})$, *where $\mathbf{i}$ is the type of a fixed reduced decomposition of $w_{v^{-1}\cdot\mu} \cdot v^{-1}$ and C'_{p_0} is the unique local chamber at $p_0 = \pi(0)$ in $\mathbb{A}$ such that $d^{*W}(C^-_{p_0}, C'_{p_0}) = w^+_\lambda w$.*

Remarks.

(1) Actually $\prod_{k=1}^{\ell_\pi - 1} a_{\underline{\pi}}(k)$ is the number of decorated segments $\underline{[z_0, y]}$ such that $\rho(\underline{[z_0, y]}) = \underline{\pi}$ and $C_y^* = C_y''$. It may be zero.

(2) If $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} \neq 0$, then necessarily ν is spherical (in particular $\mathbf{u} \in W^{+g}$), as then any Hecke path of shape μ^{++} is increasing for $\overset{o}{<}$ (see 1.7). The arguments of [3] are sufficient for this result.

(3) From this theorem we deduce that $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} \neq 0$ is equivalent to the following:
- there exists a Hecke path in $\mathbb{A}$ of shape μ^{++} with respect to C_x from $p_0 = x + \lambda = \lambda$ to $y = x + \nu = \nu$,
- there exists a decoration $\underline{\pi}$ of π (always true),
- for this decorated Hecke path each of the sets $\Gamma^+_{C_y}(C_y^-, \mathbf{i}_\ell, \tilde{C}_y)$, $\Gamma^+_{C^+_{p_k}}(C^-_{p_k}, \mathbf{i}_k, \tilde{C}_{p_k})$ and $\Gamma^+_{C^-_{p_0}}(C^+_{p_0}, \mathbf{i}, C'_{p_0})$ is non empty.

(4) The number of decorated Hecke paths $\underline{\pi}$ as above is finite: we know that the number of paths π is finite (it is a consequence of Theorem 3.5 in [3]) and, as μ is spherical, the number of decorations of π is finite.

Proof. $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is the number of local chambers $C_{z_0} \in \mathscr{C}_0^+$ with $x \leqslant z_0 \leqslant y$, $d^W(C_x, C_{z_0}) = \mathbf{w}$ and $d^W(C_{z_0}, C_y) = \mathbf{v}$ (we chose C_x, C_y in $\mathbb{A}$ such that $d^W(C_x, C_y) = \mathbf{u}$). We know that this number is finite, see Proposition 1.1. The datum of z_0 is equivalent to the datum of the segment $[z_0, y]$ or of the decorated segment $\underline{[z_0, y]}$ associated, as in 2.5, to $[z_0, y]$ and C_y. We use now the retraction $\rho = \rho_{\mathbb{A},C_x} : \mathscr{I}_{\geqslant x} \to \mathbb{A}$. We have $y = \rho(y) = x + \nu$ and the condition $d^W(C_x, z_0) = \lambda$ is equivalent to $\rho(z_0) = x + \lambda = p_0$. So $\rho(\underline{[z_0, y]})$ has to be a decorated Hecke path $\underline{\pi}$ as asked in the theorem. And we get the formula:

$$a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} = \sum_{\underline{\pi}} (\text{number of liftings of } \underline{\pi}) \times (\text{number of } C_{z_0} \text{ for } z_0 \text{ given}),$$

It is possible to calculate like that for $\rho(C^+_{z_0}) = C^+_{p_0}$ is well determined by the decorated path $\underline{\pi}$. Hence (as we shall see in (b) or (c) below), the number of C_{z_0} only depends on $\underline{\pi}$ and not on the lifting of $\underline{\pi}$. In [3, Theorem 3.7] we argued the same way, but with Hecke paths (without decoration) so we had to suppose μ^{++} regular to get that $\rho(C^+_{z_0})$ was well determined by the path π.

For short, we write $\ell = \ell_\pi$. We compute the number of liftings of $\underline{\pi}$ by looking successively at the number of liftings of $\underline{[p_{\ell-1}, p_\ell]}$, $\underline{[p_{\ell-2}, p_{\ell-1}]}$, $\dots$, $\underline{[p_0, p_1]}$.

(a) The number $a_{\underline{\pi}}(\ell)$ of liftings of $\underline{[p_{\ell-1}, p_\ell = y]}$ is the number of liftings $[z_{\ell-1}, z_\ell = y]$ of $[p_{\ell-1}, p_\ell = y]$ and C''_y of C^*_y such that $[y, z_{\ell-1}) \subset \overline{C''_y}$ and $d^{*W}(C''_y, C_y) = w^+_{\mu^{++}} w_{v^{-1}\cdot\mu}$ (by Lemma 3.2(b)). But $[y, z_{\ell-1}]$ is determined by $[y, z_{\ell-1})$ (cf. Lemma 2.8) and $[y, z_{\ell-1})$ is determined by C''_y and μ^{++}. So we just have to count the liftings C''_y of C^*_y. By the same way as in the proof of Lemma 3.1, we are going to prove that the possible C''_y are in one-to-one correspondence with the disjoint union of the sets $\mathcal{C}^m_{C_y}(C_y^-, \mathbf{c})$ for $\mathbf{c}$ in $\Gamma^+_{C_y}(C_y^-, \mathbf{i}_\ell, \tilde{C}_y)$. In this case, the tools are $\rho = \rho_{\mathbb{A},C_x}$, that on $\mathcal{T}_y\mathscr{I}$, coincides with $\rho = \rho_{\mathbb{A},C_y^-}$ 2.5.2 and $\rho' = \rho_{\mathbb{A},C_y}$.

If C''_y is given, there is a unique minimal gallery $\mathbf{c}'$ from C_y^- to C''_y of type $\mathbf{i}_\ell$ (as ρ induces a bijection between the minimal galleries from C_y^- to $C''_y = \mathrm{pr}_{[y,z_{\ell-1})}(C_y)$ and those from C_y^- to $C^*_y = \mathrm{pr}_{[y,p_{\ell-1})}(C_y)$). By Lemma 3.2(b) we know that $d^{*W}(C''_y, C_y) = w^+_{\mu^{++}} w_{v^{-1}\cdot\mu}$, so $\rho'(C''_y) = \tilde{C}_y$, and the gallery $\mathbf{c} = \rho'(\mathbf{c}')$ is in $\Gamma^+_{C_y}(C_y^-, \mathbf{i}_\ell, \tilde{C}_y)$, while $\mathbf{c}'$ is in $\mathcal{C}^m_{C_y}(C_y^-, \mathbf{c})$.

Reciprocally, if $\mathbf{c}$ is in the set $\Gamma^+_{C_y}(C_y^-, \mathbf{i}_\ell, \tilde{C}_y)$, let us consider C''_y the last chamber of $\mathbf{c}'$ a lifted gallery of $\mathbf{c}$ with respect to ρ'. The condition on $\tilde{C}_y$ enables to say that $d^{*W}(C''_y, C_y) = w^+_{\mu^{++}} w_{v^{-1}\cdot\mu}$ and so, by Lemma 3.2 the decoration C''_y of $\underline{[z_{\ell-1}, y]}$ at y satisfies the expected codistance condition.

(b) For $1 \leqslant k \leqslant \ell - 1$, we suppose given the lifting $\underline{[z_k, y]}$ of $\underline{\pi}|_{[t_k,1]}$. The number $a_{\underline{\pi}}(k)$ of suitable liftings $\underline{[z_{k-1}, z_k]}$ of $\underline{[p_{k-1}, p_k]}$ is the number of pairs $([z_{k-1}, z_k], C''_{z_k})$ of liftings $[z_{k-1}, z_k]$ of $[p_{k-1}, p_k]$ and $\overline{C''_{z_k}}$ of $C^*_{p_k}$ such that $[z_k, z_{k-1})$ is opposite to $[z_k, z_{k+1})$ (see Lemma 3.2), $[z_k, z_{k-1}) \in \overline{C''_{z_k}}$ and C''_{z_k} is the decoration of $[z_k, z_{k-1}]$ associated to C_y. Let us consider an apartment A containing C_x and $C''_{z_{k+1}}$ hence also $[z_k, z_{k+1}]$ and $C^+_{z_k}$ (see Lemma 2.8). The restriction $\rho|_A$ is the restriction to A of an automorphism φ of $\mathscr{I}$ fixing C_x that induces an isomorphism $\varphi|_{\mathcal{T}_{z_k}\mathscr{I}}$ from $\mathcal{T}_{z_k}\mathscr{I}$ onto $\mathcal{T}_{p_k}\mathscr{I}$ and sends $C^+_{z_k} \subset A$ to $C^+_{p_k} = \rho(C^+_{z_k})$. So the map φ induces a bijection from the set of suitable liftings $([z_{k-1}, z_k], C''_{z_k})$ of $([p_{k-1}, p_k], C^*_{z_k})$ onto the set of pairs $([z'_{k-1}, p_k], C''_{p_k})$ such that $[p_k, z'_{k-1}) \in \overline{C''_{p_k}}$ is opposite to $[p_k, p_{k+1}]$ $(= \rho([z_k, z_{k+1}]) = \varphi([z_k, z_{k+1}]))$, $C''_{p_k} = \mathrm{pr}_{[p_k, z'_{k-1})}(C^+_{p_k})$ and $\rho_{\mathbb{A}, C^-_{p_k}}(C''_{p_k}) = C^*_{p_k}$ (as $\rho_{\mathbb{A}, C^-_{p_k}} \circ \varphi|_{\mathcal{T}_{z_k}\mathscr{I}}(C''_{z_k}) = \rho(C''_{z_k})$).

By Lemma 3.1 the possible $([p_k, z'_{k-1}), C''_{p_k})$ (and so the possible $([p_k, z'_{k-1}], C''_{p_k})$ by Lemma 2.8) are in one-to-one correspondence with the union of the sets $\mathcal{C}^m_{C^+_{p_k}}(C^-_{p_k}, \mathbf{c})$ for $\mathbf{c}$ in the set $\Gamma^+_{C^+_{p_k}}(C^-_{p_k}, \mathbf{i}_\ell, \tilde{C}_{p_k})$, with $\tilde{C}_{p_k} = \mathrm{pr}_{-\eta_k}(C^+_{p_k})$.

(c) For the last step of the lifting, by the same way as before, we suppose given the lifting $\underline{[z_0, y]}$ and we suppose $z_0 = p_0$. So we know that $C^+_{p_0} = C^+_{z_0}$. The Lemma 3.1 says that $d^{*W}(C^-_{p_0}, C_{z_0}) = w^+_\lambda w$, and Lemma 3.2 that $d^W((C^+_{p_0}, C_{z_0}) = w_{v^{-1}\mu} v^{-1}$. So, as before, the number of C_{z_0} is the number of elements of the different sets $\mathcal{C}^m_{C^-_{p_0}}(C^+_{p_0}, \mathbf{e})$ where $\mathbf{e}$ is a gallery of $\Gamma^+_{C^-_{p_0}}(C^+_{p_0}, \mathbf{i}, C'_{p_0})$ as $\mathbf{i}$ is the type of a minimal gallery from $C^+_{p_0}$ to C_{z_0} that retracts by $\rho_{\mathbb{A}, C^-_{p_0}}$ to a gallery from $C^+_{p_0}$ to C'_{p_0}. □

3.4. Consequence. The above explicit formula, together with the formula for $\sharp\mathcal{C}^m_{\mathfrak{Q}}(C^-_z, \mathbf{c})$ in 2.3, tell us that the structure constant $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is a polynomial in the parameters $q_i - 1, q'_i - 1$ for $q_i, q'_i \in \mathcal{Q}$ with coefficients in $\mathbb{N} = \mathbb{Z}_{\geqslant 0}$ and that this polynomial depends only on $\mathbb{A}$, W, $\mathbf{w}$, $\mathbf{v}$ and $\mathbf{u}$. So we have proved the conjecture 1 of the introduction in this generic case: when λ and μ are spherical.

Note that we did not obtain all the structure constants $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ for the generic Iwahori–Hecke algebra ${}^I\mathcal{H}^g_{\mathbb{Z}}$. The cases $\mathbf{w} \in W^v \ltimes V_0$ or $\mathbf{v} \in W^v \ltimes V_0$ (i.e. $\lambda \in V_0$ or $\mu \in V_0$ in the above notation) are missing. We deal with them in the following section.

4. Structure constants in remaining generic cases

4.1. The problem. Let us choose $C_x, C_y \in \mathscr{C}^+_0$ with $x \leqslant y$ and $d^W(C_x, C_y) = \mathbf{u} = \nu \cdot u \in W^+ = W^v \ltimes Y^+$. Then the structure constant $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ (for $\mathbf{w} = \lambda \cdot w$ and $\mathbf{v} = \mu \cdot v$ in W^+) is the number of $C_{z_0} \in \mathscr{C}^+_0$ with $x \leqslant z_0 \leqslant y$, $d^W(C_x, C_{z_0}) = \mathbf{w}$ and $d^W(C_{z_0}, C_y) = \mathbf{v}$, see Proposition 1.1.

In Theorem 3.3, we computed $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ when $\mathbf{w}, \mathbf{v}$ are spherical (i.e. $\lambda, \mu \in Y \cap \mathcal{T}^\circ$). We shall compute it below in the remaining cases where $\mathbf{w}, \mathbf{v} \in W^{+g} = W^v \ltimes (Y \cap (\mathcal{T}^\circ \cup V_0))$. So, in the affine or strictly hyperbolic cases, we shall get $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ for any $\mathbf{w}, \mathbf{v} \in W^+$. But we get, in general, these structure constants for $\mathbf{w}, \mathbf{v} \in W^{+g} = W^v \ltimes Y^{+g}$, i.e. we get the structure constants of ${}^I\mathcal{H}^g$, see 3.4 and 4.5.

We start with a lemma analogous to lemmas 3.1 and 3.2.

Lemma 4.1. *Let $C_x, C_z \in \mathscr{C}^+_0$ with $x \leqslant z$ and $\lambda \in Y^{+0}$, $w \in W^v$. We write $C^+_x = \mathrm{pr}_x(C_z)$, then*

$$d^W(C_x, C_z) = \lambda \cdot w \Longleftrightarrow \begin{cases} d^W(C_x, z) = \lambda \\ d^W(C^-_z, C_z) = w \end{cases} \Longleftrightarrow \begin{cases} d^W(C_x, z) = \lambda \\ d^W(C_x, C^+_x) = w. \end{cases}$$

Actually $d^W(C_x, z) = \lambda \in V_0$ implies $x \leqslant z$ and $z \leqslant x$. So $C_z^- := \mathrm{pr}_z(C_x)$ is well defined, by 2.1.1*, and is a positive local chamber.*

Proof. By definition $d^W(C_x, C_z) = \lambda \cdot w$ implies $d^W(C_x, z) = \lambda$ (1.10). Suppose now $d^W(C_x, z) = \lambda$. Then $d^v(x, z) = \lambda \in V_0$, so any apartment A containing x or z contains z or x and, in A, one has $z = x + \lambda \leqslant x$; this is a consequence of 1.4.1(a), as any enclosure is stable under V_0. Hence $C_z^- = \mathrm{pr}_z(C_x) \in A$ is well defined, by 2.1.1, and is a positive local chamber. Actually $C_z^- = C_x + \lambda$ (calculation in A). We have also $C_x^+ = C_z - \lambda$. It is now clear that $d^W(C_x, C_z) = \lambda \cdot w \iff d^W(C_z^-, C_z) = w \iff d^W(C_x, C_x^+) = w$. □

4.2. First reduction. We consider $\mathbf{u}, \mathbf{v}, \mathbf{w} \in W^+$ and write $\mathbf{u} = \nu \cdot u, \mathbf{v} = \mu \cdot v, \mathbf{w} = \lambda \cdot w$ with $\lambda, \mu, \nu \in Y^+$ and $u, v, w \in W^v$. We choose $C_x, C_y \in \mathscr{C}_0^+$ with $x \leqslant y$ and $d^W(C_x, C_y) = \mathbf{u}$; we may suppose $C_x, C_y \subset \mathbb{A}$. We choose $C_{z_0} \in \mathscr{C}_0^+$ with $x \leqslant z_0 \leqslant y$, $d^W(C_x, C_{z_0}) = \mathbf{w}$ and $d^W(C_{z_0}, C_y) = \mathbf{v}$.

If $\lambda \in Y^{+0} = Y \cap V_0$, one has $d^W(C_x, z_0) = \lambda$ (Lemma 4.1) and $z_0 \in \mathbb{A}$, more precisely $z_0 = x + \lambda$ (as we saw in the proof of Lemma 4.1).

If $\mu \in Y^{+0}$, then we get $z_0 \in \mathbb{A}$, more precisely $z_0 = y - \mu$, by Lemma 4.1 applied to C_{z_0}, C_y instead of C_x, C_z.

In both cases z_0 has to be a well determined point in $\mathbb{A}$ and $\nu = d^v(x, y) \in W^v\lambda + W^v\mu$. In particular, if $\mathbf{w}, \mathbf{v} \in W^{+g}$ i.e. $\lambda, \mu \in Y^{+g}$, one has also $\nu \in Y^{+g}$ i.e. $\mathbf{u} \in W^{+g}$.

We now want to compute the number $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ of $C_{z_0} \in \mathscr{C}_0^+$ with $x \leqslant z_0 \leqslant y$, $d^W(C_x, C_{z_0}) = \mathbf{w}$ and $d^W(C_{z_0}, C_y) = \mathbf{v}$. For this we separate below the cases $\lambda \in Y^{+0}$ and $\mu \in Y^{+0}$.

4.3. The case $\mu \in Y^{+0}$. We suppose $\lambda \in Y \cap \mathcal{T}^\circ$ (resp. $\lambda \in Y^{+0}$). By Lemma 4.1 above and Lemma 3.1, we have to find the number $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ of $C_{z_0} \in \mathscr{C}_0^+$ satisfying (with $C_{z_0}^+ = \mathrm{pr}_{[z_0,y)}(C_y) = \mathrm{pr}_{z_0}(C_y)$):

(a) $d^W(C_x, z_0) = \lambda$,
(b) $d^W(C_{z_0}, y) = \mu$,
(c) $d^W(C_{z_0}, C_{z_0}^+) = v$
(d) $d^{*W}(C_{z_0}^-, C_{z_0}) = w_\lambda^+ \cdot w$ (resp. (d) $d^W(C_{z_0}^-, C_{z_0}) = w$).

Actually $\mu \in V_0$ is fixed by W^v and $y, C_{z_0}, C_{z_0}^+$ are in a same apartment (containing C_y and C_{z_0}), so $d^W(C_{z_0}, y) = \mu \iff d^W(C_{z_0}^+, y) = \mu$. Then $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is the number of $C_{z_0} \in \mathscr{C}_0^+$ satisfying (a), (b′) $d^W(C_{z_0}^+, y) = \mu$, (c) and (d). The first two conditions involve only $z_0, C_x, C_y \in \mathbb{A}$.

Proposition. *The number $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is either* 0 *(if the conditions* (a), (b′) *above are incompatible) or*

$$\sum_{\mathbf{e} \in \Gamma^+_{C_{z_0}^-}(C_{z_0}^+, \mathbf{i}, C'_{z_0})} \sharp \mathcal{C}^m_{C_{z_0}^-}(C_{z_0}^+, \mathbf{e})$$

where $\mathbf{i}$ is the type of a fixed reduced decomposition of v^{-1} and C'_{z_0} is the unique local chamber at z_0 in $\mathbb{A}$ such that $d^{*W}(C_{z_0}^-, C'_{z_0}) = w_\lambda^+ \cdot w$ (resp. $d^W(C_{z_0}^-, C'_{z_0}) = w$).

Remark. The coefficient $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is zero when (a) and (b′) are incompatible, i.e. when $\nu \neq \lambda + \mu$: if in $\mathbb{A}$ we identify C_x to the fundamental chamber C_0^+, (a) is equivalent to $z_0 = x + \lambda$, (b′) to $y = z_0 + \mu$ and $d^W(C_x, C_y) = \nu \cdot u$ implies $y = x + \nu$.

But the other case where $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} = 0$ is when $\Gamma^+_{C_{z_0}^-}(C_{z_0}^+, \mathbf{i}, C'_{z_0})$ is empty.

Proof. We have to translate the conditions (c) and (d). We consider the retraction $\rho = \rho_{\mathbb{A},C^-_{z_0}}$. The condition (c) is equivalent to the existence of a minimal gallery $\mathbf{c}$ starting from $C^+_{z_0}$, of type $\mathbf{i}$ (i.e. $\mathbf{c} \in \mathcal{C}^m(C^+_{z_0}, \mathbf{i})$) ending in C_{z_0}; and there is a bijection between these $\mathbf{c}$ and the C_{z_0} satisfying (c). Now the condition (d) is equivalent to $\rho(C_{z_0}) = C'_{z_0}$ (as ρ preserves the $W-$distances to $C^-_{z_0}$). Considering $\mathbf{e} = \rho(\mathbf{c})$, the proposition is now clear. □

4.4. The case $\lambda \in Y^{+0}$ (and $\mu \in Y \cap \mathcal{T}^\circ$). By Lemma 4.1 above and Lemma 3.2, we have to find the number $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ of $C_{z_0} \in \mathscr{C}^+_0$ satisfying:

(a) $d^W(C_x, z_0) = \lambda$,
(b) $d^W(C^+_{z_0}, y) = \mu^{++}$,
(c) $d^{*W}(C'_y, C_y) = w^+_{\mu^{++}} w_{v^{-1}\cdot\mu}$
(d) $d^W(C^-_{z_0}, C_{z_0}) = w$
(e) $d^W(C^+_{z_0}, C_{z_0}) = w_{v^{-1}\cdot\mu} \cdot v^{-1}$.

But $C^+_{z_0} = \mathrm{pr}_{z_0}(C_y)$, $C'_y = \mathrm{pr}_y(C^+_{z_0})$ and $C_x, C_y, z_0 = x + \lambda$ are in $\mathbb{A}$. So the conditions (a), (b), (c) involve only C_x, C_y and z_0.

Proposition. *The number $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is either* 0 *(if the conditions* (a), (b), (c) *above are incompatible) or*

$$\sum_{\mathbf{e}\in\Gamma^+_{C^-_{z_0}}(C^+_{z_0},\mathbf{i},C'_{z_0})} \sharp\mathcal{C}^m_{C^-_{z_0}}(C^+_{z_0}, \mathbf{e})$$

where $\mathbf{i}$ *is the type of a fixed reduced decomposition of* $w_{v^{-1}\cdot\mu} \cdot v^{-1}$ *and* C'_{z_0} *is the unique local chamber at* z_0 *in* $\mathbb{A}$ *such that* $d^W(C^-_{z_0}, C'_{z_0}) = w$.

Remark. The coefficient $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is zero when (a), (b) and (c) are incompatible, i.e. when z_0, determined by (b) does not satisfy (a) and (c). But it is more difficult than in 4.3 to translate it simply. It is also zero when $\Gamma^+_{C^-_{z_0}}(C^+_{z_0}, \mathbf{i}, C'_{z_0})$ is empty.

Proof. We have to translate conditions (d) and (e). It goes the same way as in 4.3. □

4.5. Conclusion. In all cases where $\lambda, \mu \in Y^{+g} = Y \cap (\mathcal{T}^\circ \cup V_0)$, we may use the formula for $\mathcal{C}^m_\Omega(C'_z, \mathbf{c})$ in 2.3, the Theorem 3.3 and/or the Propositions 4.3, 4.4. We get the expected result: the structure constant $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is a polynomial in the parameters $q_i - 1, q'_i - 1$ for $q_i, q'_i \in \mathcal{Q}$ with coefficients in $\mathbb{N} = \mathbb{Z}_{\geqslant 0}$ and this polynomial depends only on $\mathbb{A}$, W, $\mathbf{w}$, $\mathbf{v}$ and $\mathbf{u}$. We have proved Conjecture 1 in these cases, in particular in the affine or strictly hyperbolic cases.

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Nicole Bardy-Panse, Université de Lorraine, CNRS, IECL, Nancy, F-54000, France
E-mail : Nicole.Panse@univ-lorraine.fr

Guy Rousseau, Université de Lorraine, CNRS, IECL, Nancy, F-54000, France
E-mail : Guy.Rousseau@univ-lorraine.fr