# PI Regulation of a Reaction-Diffusion Equation with Delayed Boundary Control 

Hugo Lhachemi, Christophe Prieur, Emmanuel Trélat

## To cite this version:

Hugo Lhachemi, Christophe Prieur, Emmanuel Trélat. PI Regulation of a Reaction-Diffusion Equation with Delayed Boundary Control. IEEE Transactions on Automatic Control, 2021, 66 (4), pp. 1573 1587. 10.1109/TAC.2020.2996598 . hal-02294321

## HAL Id: hal-02294321

## https://hal.science/hal-02294321

Submitted on 23 Sep 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# PI Regulation of a Reaction-Diffusion Equation with Delayed Boundary Control 

Hugo Lhachemi, Christophe Prieur, Emmanuel Trélat


#### Abstract

The general context of this work is the feedback control of an infinite-dimensional system so that the closedloop system satisfies a fading-memory property and achieves the setpoint tracking of a given reference signal. More specifically, this paper is concerned with the Proportional Integral (PI) regulation control of the left Neumann trace of a onedimensional reaction-diffusion equation with a delayed right Dirichlet boundary control. In this setting, the studied reactiondiffusion equation might be either open-loop stable or unstable. The proposed control strategy goes as follows. First, a finitedimensional truncated model that captures the unstable dynamics of the original infinite-dimensional system is obtained via spectral decomposition. The truncated model is then augmented by an integral component on the tracking error of the left Neumann trace. After resorting to the Artstein transformation to handle the control input delay, the PI controller is designed by pole shifting. Stability of the resulting closed-loop infinite-dimensional system, consisting of the original reaction-diffusion equation with the PI controller, is then established thanks to an adequate Lyapunov function. In the case of a time-varying reference input and a time-varying distributed disturbance, our stability result takes the form of an exponential Input-to-State Stability (ISS) estimate with fading memory. Finally, another exponential ISS estimate with fading memory is established for the tracking performance of the reference signal by the system output. In particular, these results assess the setpoint regulation of the left Neumann trace in the presence of distributed perturbations that converge to a steady-state value and with a time-derivative that converges to zero. Numerical simulations are carried out to illustrate the efficiency of our control strategy.


Index Terms-1-D reaction-diffusion equation, PI regulation control, Neumann trace, Delay boundary control, Partial Differential Equations (PDEs).

## I. Introduction

## A. State of the art

Motivated by the efficiency of Proportional-Integral (PI) controllers for the stabilization and regulation control of finitedimensional systems, as well as its widespread adoption by industry [2], [3], the opportunity of using PI controllers in the context of infinite-dimensional systems has attracted much attention in the recent years. One of the early attempts in

[^0]this area was reported in [19], [20], then extended in [30], for bounded control operators. More recently, a number of works have been reported on the PI boundary control of linear hyperbolic systems [5], [11], [15], [31]. The use of a PI boundary controller for 1-D nonlinear transport equation has been studied first in [29] and then extended in [8]. In particular, the former tackled the regulation problem for a constant reference input and in the presence of constant perturbations. The regulation of the downside angular velocity of a drilling string with a PI controller was reported in [28]. The considered model consists of a wave equation coupled with ODEs in the presence of a constant disturbance. A related problem, the PI control of a drilling pipe under friction, was investigated in [4]. Recently, the opportunity to add an integral component to open-loop exponentially stable semigroups for the output tracking of a constant reference input and in the presence of a constant distributed perturbation was investigated in [26], [27] for unbounded control operators by using a Lyapunov functional design procedure.

In this paper, we are concerned with the PI regulation control of the left Neumann trace of a one-dimensional reactiondiffusion equation with a delayed right Dirichlet boundary control. Specifically, we aim at achieving the setpoint reference tracking of a time-varying reference signal in spite of both the presence of an arbitrarily large constant input delay and a timevarying distributed disturbance. One of the early contributions regarding stabilization of PDEs with an arbitrarily large input delay deals with a reaction-diffusion equation [14] where the controller was designed by resorting to the backstepping technique. A different approach, which is the one adopted in this paper, takes advantage of the following control design procedure initially reported in [23] and later used in [9], [10], [24] to stabilize semilinear heat, wave or fluid equations via (undelayed) boundary feedback control: 1) design of the controller on a finite-dimensional model capturing the unstable modes of the original infinite-dimensional system; 2) use of an adequate Lyapunov function to assess that the designed control law stabilizes the whole infinite-dimensional system. The extension of this design procedure to the delay feedback control of a one-dimensional linear reaction-diffusion equation was reported in [21]. The impact of the input-delay was handled in the control design by the synthesis of a predictor feedback via the classical Artstein transformation [1], [22] (see also [6]). This control strategy was replicated in [12] for the feedback stabilization of a linear Kuramoto-Sivashinsky equation with delay boundary control. This idea was then generalized to the boundary feedback stabilization of a class of diagonal infinite-dimensional systems with delay boundary
control for either a constant [16], [18] or a time-varying [17] input delay.

## B. Investigated control problem

Let $L>0$, let $c \in L^{\infty}(0, L)$ and let $D>0$ be arbitrary. We consider the one-dimensional reaction-diffusion equation over $(0, L)$ with delayed Dirichlet boundary control:

$$
\begin{array}{lr}
y_{t}=y_{x x}+c(x) y+d(t, x), & (t, x) \in \mathbb{R}_{+}^{*} \times(0, L) \\
y(t, 0)=0, & t \geqslant 0 \\
y(t, L)=u_{D}(t) \triangleq u(t-D), & t \geqslant 0 \\
y(0, x)=y_{0}(x), & x \in(0, L) \tag{1d}
\end{array}
$$

where $y(t, \cdot) \in L^{2}(0, L)$ is the state at time $t, u(t) \in \mathbb{R}$ is the control input, $D>0$ is the (constant) control input delay, and $d(t, \cdot) \in L^{2}(0, L)$ is a time-varying distributed disturbance, continuously differentiable with respect to $t$.

In this paper, our objective is to achieve the PI regulation control of the left Neumann trace $y_{x}(t, 0)$ to some prescribed reference signal, in the presence of the time-varying distributed disturbance $d$. More precisely, let $r: \mathbb{R}_{+} \rightarrow \mathbb{R}$ be an arbitrary continuous function (reference signal). We aim at achieving the setpoint tracking of the time-varying reference signal $r(t)$ by the left Neumann trace $y_{x}(t, 0)$.

Note that an exponentially stabilizing controller for (1a-1d) was designed in [21] in the disturbance-free case $(d=0)$ for a system trajectory evaluated in $H_{0}^{1}$-norm. The control strategy that we develop in the present paper elaborates on the one of [21], adequately combined with a PI procedure. First, a finite-dimensional model capturing all unstable modes of the original infinite-dimensional system is obtained by an appropriate spectral decomposition. Following the standard PI approach, the tracking error on the left Neumann trace is then added as a new component to the resulting finite-dimensional system. Before synthetizing the PI controller, the control input delay is handled thanks to the Artstein transformation. A predictor feedback control, obtained by pole shifting, is then designed to exponentially stabilize the aforementioned truncated model. The core of the proof consists of establishing that this PI feedback controller exponentially stabilizes as well the complete infinite-dimensional system. This is done by an appropriate Lyapunov-based argument. The obtained results take the form of exponential Input-to-State Stability (ISS) estimates [25] with fading memory of the reference input and the distributed perturbation. In the case where $r(t) \rightarrow r_{e}$, $d(t) \rightarrow d_{e}$ and $\dot{d}(t) \rightarrow 0$ when $t \rightarrow+\infty$, these estimates ensure the convergence of the state of the system, as well as the fulfillment of the desired setpoint regulation $y_{x}(t, 0) \rightarrow r_{e}$.

The paper is organized as follows. The proposed control strategy is introduced in Section II. The study of the equilibrium points of the closed-loop system and the associated dynamics are presented in Section III. Then, the stability analysis of the closed-loop system is presented in Section IV while the assessment of the tracking performance is reported in Section V. The obtained results are illustrated by numerical simulations in Section VI. Finally, concluding remarks are formulated in Section VII.

## II. Control design strategy

The sets of nonnegative integers, positive integers, real, nonnegative real, and positive real are denoted by $\mathbb{N}, \mathbb{N}^{*}, \mathbb{R}, \mathbb{R}_{+}$, and $\mathbb{R}_{+}^{*}$, respectively. All the finite-dimensional spaces $\mathbb{R}^{p}$ are endowed with the usual Euclidean inner product $\langle x, y\rangle=x^{\top} y$ and the associated 2-norm $\|x\|=\sqrt{\langle x, x\rangle}=\sqrt{x^{\top} x}$. For any matrix $M \in \mathbb{R}^{p \times q},\|M\|$ stands for the induced norm of $M$ associated with the above 2 -norms. For a given symmetric matrix $P \in \mathbb{R}^{p \times p}, \lambda_{m}(P)$ and $\lambda_{M}(P)$ denote its smallest and largest eigenvalues, respectively. In the sequel, the time derivative $\partial f / \partial t$ is either denoted by $f_{t}$ or $\dot{f}$ while the spatial derivative $\partial f / \partial x$ is either denoted by $f_{x}$ or $f^{\prime}$.

## A. Augmented system for PI feedback control

The control design objective is: 1) to stabilize the reactiondiffusion system (1a-1d); 2) to ensure the setpoint tracking of the reference signal $r(t)$ by the left Neumann trace $y_{x}(t, 0)$. We address this problem by designing a PI controller. Following the general PI scheme, we augment the system by introducing a new state $z(t) \in \mathbb{R}$ taking the form of the integral of the tracking error $y_{x}(t, 0)-r(t)$ (as for finite-dimensional systems, the objective of this integral component is to ensure the setpoint tracking of the reference signal in the presence of the distributed disturbance $d$ ):

$$
\begin{array}{lr}
y_{t}=y_{x x}+c(x) y+d(t, x), & (t, x) \in \mathbb{R}_{+}^{*} \times(0, L) \\
\dot{z}(t)=y_{x}(t, 0)-r(t), & t \geqslant 0 \\
y(t, 0)=0, & t \geqslant 0 \\
y(t, L)=u_{D}(t) \triangleq u(t-D), & t \geqslant 0 \\
y(0, x)=y_{0}(x), & x \in(0, L) \\
z(0)=z_{0} &
\end{array}
$$

where $z_{0} \in \mathbb{R}$ stands for the initial condition of the integral component. As we are only concerned in prescribing the future of the system, we assume that the system is uncontrolled for $t<0$, i.e., $u(t)=0$ for $t<0$. Consequently, due to the input delay $D>0$, the system is in open loop over the time range $[0, D)$ as the impact of the control strategy actually applies in the boundary condition only for $t \geqslant D$.

## B. Modal decomposition

It is convenient to rewrite ( $2 \mathrm{a}-2 \mathrm{f}$ ) as an equivalent homogeneous Dirichlet problem. Specifically, assuming ${ }^{1}$ that $u$ is continuously differentiable and setting $w(t, x)=y(t, x)-$ $\frac{x}{L} u_{D}(t)$, we have

$$
\begin{align*}
& w_{t}=w_{x x}+c(x) w+\frac{x}{L} c(x) u_{D}-\frac{x}{L} \dot{u}_{D}(t)+d(t, x)  \tag{3a}\\
& \dot{z}(t)=w_{x}(t, 0)+\frac{1}{L} u_{D}(t)-r(t)  \tag{3b}\\
& w(t, 0)=w(t, L)=0  \tag{3c}\\
& w(0, x)=y_{0}(x)-\frac{x}{L} u_{D}(0)  \tag{3d}\\
& z(0)=z_{0} \tag{3e}
\end{align*}
$$

[^1]for $t>0$ and $x \in(0,1)$. We consider the real statespace $L^{2}(0,1)$ endowed with its usual inner product $\langle f, g\rangle=$ $\int_{0}^{L} f(x) g(x) \mathrm{d} x$. Introducing the operator $\mathcal{A}=\partial_{x x}+c$ id : $D(\mathcal{A}) \subset L^{2}(0, L) \rightarrow L^{2}(0, L)$ defined on the domain $D(\mathcal{A})=H^{2}(0, L) \cap H_{0}^{1}(0, L)$, (3a-3c) can be rewritten as
\[

$$
\begin{align*}
& w_{t}(t, \cdot)=\mathcal{A} w(t, \cdot)+a(\cdot) u_{D}(t)+b(\cdot) \dot{u}_{D}(t)+d(t, \cdot)  \tag{4a}\\
& \dot{z}(t)=w_{x}(t, 0)+\frac{1}{L} u_{D}(t)-r(t) \tag{4b}
\end{align*}
$$
\]

with $a(x)=\frac{x}{L} c(x)$ and $b(x)=-\frac{x}{L}$ for every $x \in(0, L)$, with initial conditions (3d-3e). Since $\mathcal{A}$ is self-adjoint and of compact resolvent, we consider a Hilbert basis $\left(e_{j}\right)_{j \geqslant 1}$ of $L^{2}(0, L)$ consisting of eigenfunctions of $\mathcal{A}$ associated with the sequence of real eigenvalues

$$
-\infty<\cdots<\lambda_{j}<\cdots<\lambda_{1} \quad \text { with } \quad \lambda_{j} \underset{j \rightarrow+\infty}{\longrightarrow}-\infty
$$

Note that $e_{j}(\cdot) \in H_{0}^{1}(0, L) \cap C^{2}([0, L])$ for every $j \geqslant 1$ and

$$
\begin{equation*}
e_{j}^{\prime}(0) \sim \sqrt{\frac{2}{L}} \sqrt{\left|\lambda_{j}\right|}, \quad \lambda_{j} \sim-\frac{\pi^{2} j^{2}}{L^{2}} \tag{5}
\end{equation*}
$$

when $j \rightarrow+\infty$. The solution $w(t, \cdot) \in H^{2}(0, L) \cap H_{0}^{1}(0, L)$ of (4a) can be expanded as a series in the eigenfunctions $e_{j}(\cdot)$, convergent in $H_{0}^{1}(0, L)$,

$$
\begin{equation*}
w(t, \cdot)=\sum_{j=1}^{+\infty} w_{j}(t) e_{j}(\cdot) \tag{6}
\end{equation*}
$$

Therefore ( $4 a-4 b$ ) is equivalent to the infinite-dimensional control system:

$$
\begin{align*}
\dot{w}_{j}(t) & =\lambda_{j} w_{j}(t)+a_{j} u_{D}(t)+b_{j} \dot{u}_{D}(t)+d_{j}(t)  \tag{7a}\\
\dot{z}(t) & =\sum_{j \geqslant 1} w_{j}(t) e_{j}^{\prime}(0)+\frac{1}{L} u_{D}(t)-r(t) \tag{7b}
\end{align*}
$$

for $j \in \mathbb{N}^{*}$, with

$$
\begin{aligned}
w_{j}(t) & =\left\langle w(t, \cdot), e_{j}\right\rangle=\int_{0}^{L} w(t, x) e_{j}(x) \mathrm{d} x \\
a_{j} & =\left\langle a, e_{j}\right\rangle=\frac{1}{L} \int_{0}^{L} x c(x) e_{j}(x) \mathrm{d} x \\
b_{j} & =\left\langle b, e_{j}\right\rangle=-\frac{1}{L} \int_{0}^{L} x e_{j}(x) \mathrm{d} x \\
d_{j}(t) & =\left\langle d(t, \cdot), e_{j}\right\rangle=\int_{0}^{L} d(t, x) e_{j}(x) \mathrm{d} x
\end{aligned}
$$

Introducing the auxiliary control input $v \triangleq \dot{u}$, and denoting $v_{D}(t) \triangleq v(t-D),(7 a-7 b)$ can be rewritten as

$$
\begin{align*}
\dot{u}_{D}(t) & =v_{D}(t)  \tag{8a}\\
\dot{w}_{j}(t) & =\lambda_{j} w_{j}(t)+a_{j} u_{D}(t)+b_{j} v_{D}(t)+d_{j}(t)  \tag{8b}\\
\dot{z}(t) & =\sum_{j \geqslant 1} w_{j}(t) e_{j}^{\prime}(0)+\frac{1}{L} u_{D}(t)-r(t) \tag{8c}
\end{align*}
$$

for $j \in \mathbb{N}^{*}$. Since $u(t)=0$ for $t<0$, ( 8 a ) imposes that the auxiliary control input is such that $v(t)=0$ for $t<0$, and that the corresponding initial condition satisfies $u_{D}(0)=$ $u(-D)=0$. In the sequel, we design the control law $v$ in order to stabilize ( $8 \mathrm{a}-8 \mathrm{c}$ ). In this context, the actual control input $u$ associated with the original system (2a-2f) is $u(t)=$ $\int_{0}^{t} v(\tau) \mathrm{d} \tau$ for every $t \geqslant 0$.

## C. Finite-dimensional truncated model

In what follows, we fix the integer $n \in \mathbb{N}$ such that $\lambda_{n+1}<$ $0 \leqslant \lambda_{n}$. In particular, we have $\lambda_{j} \geqslant 0$ when $1 \leqslant j \leqslant n$ and $\lambda_{j} \leqslant \lambda_{n+1}<0$ when $j \geqslant n+1$.

Remark 1: In the case of an open-loop stable reactiondiffusion equation, we have $n=0$. In this particular case, as discussed in the sequel, the objective of the control design is to ensure the output regulation while preserving the stability of the closed-loop system.

Let us first show how to obtain a finite-dimensional truncated model capturing the $n$ first modes of the reaction diffusion-equation. We follow [21]. Setting

$$
\begin{gathered}
X_{1}(t)=\left(\begin{array}{c}
u_{D}(t) \\
w_{1}(t) \\
\vdots \\
w_{n}(t)
\end{array}\right), \quad A_{1}=\left(\begin{array}{cccc}
0 & 0 & \cdots & 0 \\
a_{1} & \lambda_{1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_{n} & 0 & \cdots & \lambda_{n}
\end{array}\right), \\
B_{1}=\left(\begin{array}{llll}
1 & b_{1} & \ldots & b_{n}
\end{array}\right)^{\top} \\
D_{1}(t)=\left(\begin{array}{llll}
0 & d_{1}(t) & \ldots & d_{n}(t)
\end{array}\right)^{\top}
\end{gathered}
$$

with $X_{1}(t) \in \mathbb{R}^{n+1}, A_{1} \in \mathbb{R}^{(n+1) \times(n+1)}, B_{1} \in \mathbb{R}^{n+1}$, $D_{1}(t) \in \mathbb{R}^{n+1},(8 \mathrm{a})$ and the $n$ first equations of (8b) yield

$$
\begin{equation*}
\dot{X}_{1}(t)=A_{1} X_{1}(t)+B_{1} v_{D}(t)+D_{1}(t) \tag{9}
\end{equation*}
$$

We could now augment the state-vector $X_{1}$ to include the integral component $z$ in the control design. However, the time derivative of $z$, given by ( 8 c ), involves all coefficients $w_{j}(t)$, $j \geqslant 1$. Thus, the direct augmentation of the state vector $X_{1}$ with the integral component $z$ does not allow the derivation of an ODE involving only the $n$ first modes of the reactiondiffusion equation. To overcome this issue, we set

$$
\begin{equation*}
\zeta(t) \triangleq z(t)-\sum_{j \geqslant n+1} \frac{e_{j}^{\prime}(0)}{\lambda_{j}} w_{j}(t) \tag{10}
\end{equation*}
$$

Noting that $\left|\frac{e_{j}^{\prime}(0)}{\lambda_{j}}\right|^{2} \sim \frac{2 L}{\pi^{2} j^{2}}$ when $j \rightarrow+\infty$ and thus that $\left(e_{j}^{\prime}(0) / \lambda_{j}\right)_{j}$ and $\left(w_{j}(t)\right)_{j}$ are square summable sequences, using the Cauchy-Schwarz inequality, we see that the series (10) is convergent and that

$$
\begin{aligned}
\dot{\zeta}(t) & =\dot{z}(t)-\sum_{j \geqslant n+1} \frac{e_{j}^{\prime}(0)}{\lambda_{j}} \dot{w}_{j}(t) \\
& =\alpha u_{D}(t)+\beta v_{D}(t)-\gamma(t)+\sum_{j=1}^{n} w_{j}(t) e_{j}^{\prime}(0)
\end{aligned}
$$

where we have used (8b-8c), with

$$
\begin{gather*}
\alpha=\frac{1}{L}-\sum_{j \geqslant n+1} \frac{e_{j}^{\prime}(0)}{\lambda_{j}} a_{j}, \quad \beta=-\sum_{j \geqslant n+1} \frac{e_{j}^{\prime}(0)}{\lambda_{j}} b_{j}  \tag{11a}\\
\gamma(t)=r(t)+\sum_{j \geqslant n+1} \frac{e_{j}^{\prime}(0)}{\lambda_{j}} d_{j}(t) \tag{11b}
\end{gather*}
$$

The convergence of the above series follow again by the Cauchy-Schwarz inequality. Then we have

$$
\begin{equation*}
\dot{\zeta}(t)=L_{1} X_{1}(t)+\beta v_{D}(t)-\gamma(t) \tag{12}
\end{equation*}
$$

with $L_{1}=\left(\begin{array}{llll}\alpha & e_{1}^{\prime}(0) & \ldots & e_{n}^{\prime}(0)\end{array}\right) \in \mathbb{R}^{1 \times(n+1)}$. Now, defining the augmented state-vector $X(t)=\left[\begin{array}{ll}X_{1}(t)^{\top} \quad \zeta(t)\end{array}\right]^{\top} \in$ $\mathbb{R}^{n+2}$, the exogenous input $\Gamma(t)=\left[\begin{array}{ll}D_{1}(t)^{\top} & -\gamma(t)\end{array}\right]^{\top} \in$ $\mathbb{R}^{n+2}$ and the matrices

$$
A=\left(\begin{array}{ll}
A_{1} & 0  \tag{13}\\
L_{1} & 0
\end{array}\right) \in \mathbb{R}^{(n+2) \times(n+2)}, \quad B=\binom{B_{1}}{\beta} \in \mathbb{R}^{n+2}
$$

we obtain from (9) and (12) the control system

$$
\begin{equation*}
\dot{X}(t)=A X(t)+B v_{D}(t)+\Gamma(t) \tag{14}
\end{equation*}
$$

which is the finite-dimensional truncated model capturing the unstable part of the infinite-dimensional augmented with an integral component for generating the actual control input and an integral component for setpoint reference tracking. In particular, the system (14) only involves the $n$ first modes of the reaction-diffusion equation.

Remark 2: The above developments allow the particular case $n=0$, which corresponds to the configuration where (1a-1d) is open-loop stable. In this configuration, the vectors and matrices of the truncated model (14) reduce to $X(t)=$ $\left[\begin{array}{ll}u_{D}(t) & \zeta(t)\end{array}\right]^{\top} \in \mathbb{R}^{2}, \Gamma(t)=\left[\begin{array}{cc}1 & -\gamma(t)\end{array}\right]^{\top} \in \mathbb{R}^{2}$,

$$
A=\left(\begin{array}{cc}
0 & 0 \\
\alpha & 0
\end{array}\right), \quad B=\binom{1}{\beta}
$$

In this setting, the control objective consists of ensuring the setpoint tracking of the system output $y_{x}(t, 0)$ while preserving the stability of the closed-loop system.

Putting together the finite-dimensional truncated model (14) along with (8b) for $j \geqslant n+1$ which correspond to the modes of the original infinite-dimensional system neglected by the truncated model, we get the final representation used for both control design and stability analyses:

$$
\begin{align*}
\dot{X}(t) & =A X(t)+B v_{D}(t)+\Gamma(t)  \tag{15a}\\
\dot{w}_{j}(t) & =\lambda_{j} w_{j}(t)+a_{j} u_{D}(t)+b_{j} v_{D}(t)+d_{j}(t) \tag{15b}
\end{align*}
$$

with $j \geqslant n+1$.

## D. Controllability of the finite-dimensional truncated model

As mentioned in the introduction, the control design strategy relies now on the two following steps. First, we want to design a controller for the finite-dimensional system (14). Second, we aim at assessing that the obtained PI controller successfully stabilizes the original infinite-dimensional system ( $2 \mathrm{a}-2 \mathrm{f}$ ) and provides the desired setpoint reference tracking. In order to fulfill the first objective, we first establish the controllability property for the pair $(A, B)$.

Lemma 1: The pair $(A, B)$ satisfies the Kalman condition.
Proof: Considering the structures of $A$ and $B$ defined by (13), we apply Lemma 7 reported in Appendix. More specifically, from the implication $(i) \Rightarrow(i i)$, we need to check that the pair $\left(A_{1}, B_{1}\right)$ satisfies the Kalman condition and the square matrix $\left(\begin{array}{cc}A_{1} & B_{1} \\ L_{1} & \beta\end{array}\right)$ is invertible. The first condition is indeed true as straightforward computations show that $\operatorname{det}\left(B_{1}, A_{1} B_{1}, \ldots, A_{1}^{n} B_{1}\right)=\prod_{j=1}^{n}\left(a_{j}+\right.$ $\left.\lambda_{j} b_{j}\right) \operatorname{VdM}\left(\lambda_{1}, \ldots, \lambda_{n}\right) \neq 0$, where VdM is a Vandermonde
determinant, because all eigenvalues are distinct and, using $\mathcal{A} e_{j}=\lambda_{j} e_{j}$ and an integration by parts, $a_{j}+\lambda_{j} b_{j}=$ $-e_{j}^{\prime}(L) \neq 0$ by Cauchy uniqueness (see also [21]). Thus, we focus on the invertibility condition:

$$
\begin{aligned}
\operatorname{det}\left(\begin{array}{cc}
A_{1} & B_{1} \\
L_{1} & \beta
\end{array}\right) & =\operatorname{det}\left(\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 1 \\
a_{1} & \lambda_{1} & \cdots & 0 & b_{1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n} & 0 & \cdots & \lambda_{n} & b_{n} \\
\alpha & e_{1}^{\prime}(0) & \cdots & e_{n}^{\prime}(0) & \beta
\end{array}\right) \\
& =(-1)^{n+1} \operatorname{det}\left(\begin{array}{cccc}
a_{1} & \lambda_{1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_{n} & 0 & \cdots & \lambda_{n} \\
\alpha & e_{1}^{\prime}(0) & \cdots & e_{n}^{\prime}(0)
\end{array}\right)
\end{aligned}
$$

We now consider two distinct cases depending on whether $\lambda=0$ is an eigenvalue of $\mathcal{A}$ or not.

Let us first consider the case where $\lambda=0$ is not an eigenvalue of $\mathcal{A}$. In particular, $\lambda_{1}, \ldots, \lambda_{n}$ are all non zero and thus row operations applied to the last row yield:

$$
\left.\begin{array}{l}
\operatorname{det}\left(\begin{array}{cc}
A_{1} & B_{1} \\
L_{1} & \beta
\end{array}\right) \quad\left(\begin{array}{cccc}
a_{1} & \lambda_{1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_{n} & 0 & \cdots & \lambda_{n} \\
& =(-1)^{n+1} \operatorname{det}\left(\sum_{i=1}^{n} a_{i} \frac{e_{i}^{\prime}(0)}{\lambda_{i}}\right. & 0 & \cdots
\end{array} 0\right.
\end{array}\right)
$$

Consequently, based on the definition of the constant $\alpha$ given by (11a), the above determinant is not zero if and only if

$$
\begin{equation*}
\sum_{j \geqslant 1} a_{i} \frac{e_{i}^{\prime}(0)}{\lambda_{i}} \neq \frac{1}{L} \tag{16}
\end{equation*}
$$

We note that this condition is independent of the number $n$ of modes of the infinite-dimensional system captured by the truncated model and we show in the sequel that (16) always holds true. To do so, let $y_{e}$ be the stationary solution of (1a-1d) associated with the constant boundary input $u_{e}=1$ and zero distributed disturbance, i.e., $\left(y_{e}\right)_{x x}+c y_{e}=0$ with $y_{e}(0)=0$ and $y_{e}(L)=1$. Such a function $y_{e}$ indeed exists and can be obtained as follows. By assumption, $\lambda=0$ is not an eigenvalue of $\mathcal{A}$. Thus the solution $y_{0}$ of $\left(y_{0}\right)_{x x}+c y_{0}=0$ with $y_{0}(0)=0$ and $y_{0}^{\prime}(0)=1$ satisfies $y_{0}(L) \neq 0$. Hence, one can obtain the claimed function by defining $y_{e}(x)=y_{0}(x) / y_{0}(L)$. Now, $w_{e}(x) \triangleq y_{e}(x)-\frac{x}{L}$ is a stationary solution of (3a) and (3c$3 \mathrm{~d})$ in the sense that $\left(w_{e}\right)_{x x}+c w_{e}+\frac{x}{L} c=0$ with $w_{e}(0)=$ $w_{e}(L)=0$. From (7a), $\lambda_{j} w_{e, j}+a_{j}=0$ and thus $w_{e, j}=-\frac{a_{j}}{\lambda_{j}}$. We deduce that

$$
\left(w_{e}\right)_{x}(0)=\sum_{j \geqslant 1} w_{e, j} e_{j}^{\prime}(0)=-\sum_{j \geqslant 1} \frac{a_{j}}{\lambda_{j}} e_{j}^{\prime}(0)
$$

Hence (16) holds if and only if $\left(w_{e}\right)_{x}(0) \neq-\frac{1}{L}$, which is equivalent to $\left(y_{e}\right)_{x}(0) \neq 0$. By Cauchy uniqueness, the condition $\left(y_{e}\right)_{x}(0)=0$, along with $\left(y_{e}\right)_{x x}+c y_{e}=0$ and $y_{e}(0)=0$, implies that $y_{e}=0$, which contradicts $y_{e}(L)=1$. Thus (16) holds and the system is controllable.

Let us now consider the second case, i.e., $\lambda=0$ is an eigenvalue of $\mathcal{A}$. Based on the definition of the integer $n$, we have $n \geqslant 1$ and $\lambda_{n}=0$ while $\lambda_{k}>0$ for all $1 \leqslant k \leqslant n-1$. Expanding the determinant, first, along the $(n+1)$-th column, and then, along the $n$-th row, we obtain

$$
\operatorname{det}\left(\begin{array}{cc}
A_{1} & B_{1} \\
L_{1} & \beta
\end{array}\right)=a_{n} e_{n}^{\prime}(0) \prod_{i=1}^{n-1} \lambda_{i}
$$

By Cauchy uniqueness, we have $e_{n}^{\prime}(0) \neq 0$ (otherwise $e_{n}$ would be solution of a second-order ODE with the boundary conditions $e_{n}(0)=e_{n}^{\prime}(0)=0$, yielding the contradiction $\left.e_{n}=0\right)$. Thus, the above determinant is nonzero if and only if $a_{n} \neq 0$. We proceed by contradiction. Using $e_{n}^{\prime \prime}+c e_{n}=$ $\mathcal{A} e_{n}=0$ and $a_{n}=\frac{1}{L} \int_{0}^{L} x c(x) e_{n}(x) \mathrm{d} x=0$, we obtain by integration by parts: $0=-\int_{0}^{L} x e_{n}^{\prime \prime}(x) \mathrm{d} x=-\left[x e_{n}^{\prime}(x)\right]_{x=0}^{x=L}+$ $\int_{0}^{L} e_{n}^{\prime}(x) \mathrm{d} x=-L e_{n}^{\prime}(L)$, whence $e_{n}^{\prime}(L)=0$. This result, along with $e_{n}^{\prime \prime}+c e_{n}=0$ and $e_{n}(L)=0$, yields by Cauchy uniqueness the contradiction $e_{n}=0$. Thus, $a_{n} \neq 0$ and the system is controllable.

## E. Control design strategy

Using the controllability property of the pair $(A, B)$, we propose to resort to the classical predictor feedback to stabilize the finite-dimensional truncated model (15a). Specifically, introducing the Artstein transformation

$$
\begin{equation*}
Z(t)=X(t)+\int_{t-D}^{t} e^{A(t-D-\tau)} B v(\tau) \mathrm{d} \tau \tag{17}
\end{equation*}
$$

(see [1]), straightforward computations show that

$$
\dot{Z}(t)=A Z(t)+e^{-D A} B v(t)+\Gamma(t)
$$

Since $(A, B)$ satisfies the Kalman condition, the pair $\left(A, e^{-D A} B\right)$ also satisfies the Kalman condition and we infer the existence of a feedback gain $K \in \mathbb{R}^{1 \times(n+2)}$ such that $A_{K} \triangleq A+e^{-D A} B K$ is Hurwitz. We choose the control law

$$
\begin{equation*}
v(t)=\chi_{[0,+\infty)}(t) K Z(t) \tag{18}
\end{equation*}
$$

where $\chi_{[0,+\infty)}$ denotes the characteristic function of the interval $[0,+\infty)$, which is used to capture the fact that we are only concerned by imposing a non zero control input for $t>0$. Then we obtain the stable closed-loop dynamics

$$
\dot{Z}(t)=A_{K} Z(t)+\Gamma(t)
$$

Remark 3: The first component of $Z(t)$ is $u(t)$. Indeed, denoting by $E_{1}=\left[\begin{array}{llll}1 & 0 & \ldots & 0\end{array}\right] \in \mathbb{R}^{1 \times(n+2)}$, we have

$$
\begin{aligned}
E_{1} Z(t) & =E_{1} X(t)+\int_{t-D}^{t} E_{1} e^{(t-s-D) A} B v(s) \mathrm{d} s \\
& =u_{D}(t)+\int_{t-D}^{t} v(s) \mathrm{d} s=u(t)
\end{aligned}
$$

where we have used that the first row of $A$ is null, that $v(t)=$ $\dot{u}(t)$ for $t \geqslant 0$ and that $u(t)=0$ when $t \leqslant 0$.

Remark 4: Putting together (17-18) and using the fact that $v(t)=0$ for $t \leqslant 0$, we obtain that the control input $v$ is solution of the fixed point implicit equation
$v(t)=\chi_{[0,+\infty)}(t) K X(t)+K \int_{\max (t-D, 0)}^{t} e^{A(t-D-\tau)} B v(\tau) \mathrm{d} \tau$.

Existence and uniqueness of the solution of the above equation as well as regularity properties and inversion of the Artstein transformation are reported in [6].
$\circ$
The main objective is now to establish that the feedback control (18) stabilizes as well the original infinite-dimensional system (or, in the case $n=0$, preserves the stability property of the system) while providing a setpoint tracking of the time-varying reference signal $r(t)$ by the left Neumann trace $y_{x}(t, 0)$. The former is studied in Section IV while the latter is investigated in Section V.

## III. EQUILIBRIUM CONDITION AND RELATED DYNAMICS

In the sequel, $r_{e} \in \mathbb{R}$ and $d_{e} \in L^{2}(0, L)$ stand for "nominal values" of the time-varying reference signals $r(t)$ and the distributed disturbance $d(t)$, respectively. Even if $r_{e}$ and $d_{e}$ can be selected arbitrarily, the two following (distinct) cases will be of particular interest in the sequel:

- $\left|r(t)-r_{e}\right| \leqslant \delta_{r}$ and $\left\|d(t)-d_{e}\right\| \leqslant \delta_{d}$ for some $\delta_{r}, \delta_{d}>0$;
- $r(t) \rightarrow r_{e}$ and $d(t) \rightarrow d_{e}$ when $t \rightarrow+\infty$.


## A. Characterization of equilibrium for the closed-loop system

Setting $d_{e, j}=\left\langle d_{e}, e_{j}\right\rangle=\int_{0}^{L} d_{e}(x) e_{j}(x) \mathrm{d} x$ for $j \geqslant 1$, $\Delta r=r-r_{e}, \Delta d=d-d_{e}, \Delta d_{j}=d_{j}-d_{e, j}$,
$\Gamma_{e}=\left(\begin{array}{c}0 \\ d_{e, 1} \\ \vdots \\ d_{e, n} \\ -r_{e}-\sum_{j \geqslant n+1} \frac{e_{j}^{\prime}(0)}{\lambda_{j}} d_{e, j}\end{array}\right), \Delta \Gamma=\left(\begin{array}{c}0 \\ \Delta d_{1} \\ \vdots \\ \Delta d_{n} \\ -\Delta r-\sum_{j \geqslant n+1} \frac{e_{j}^{\prime}(0)}{\lambda_{j}} \Delta d_{j}\end{array}\right)$
we obtain from (15a-15b) and (18)

$$
\begin{aligned}
\dot{Z}(t) & =A_{K} Z(t)+\Gamma_{e}+\Delta \Gamma(t) \\
\dot{w}_{j}(t) & =\lambda_{j} w_{j}(t)+a_{j} u_{D}(t)+b_{j} v_{D}(t)+d_{e, j}+\Delta d_{j}(t)
\end{aligned}
$$

for $j \geqslant n+1$. We now characterize the equilibrium condition of the above closed-loop system associated with the constant reference input $r(t)=r_{e} \in \mathbb{R}$ and the constant distributed disturbance $d(t)=d_{e} \in L^{2}(0, L)$ (i.e., $\Delta r=0$ and $\Delta d=0$ ). In the sequel, we denote by a subscript " $e$ " the equilibrium value of the different quantities. For instance, $Z_{e}$ denotes the equilibrium value of $Z$. Noting that $u_{D, e}=u_{e}$ and $v_{D, e}=v_{e}$, we obtain

$$
\begin{aligned}
& 0=A_{K} Z_{e}+\Gamma_{e} \\
& 0=\lambda_{j} w_{j, e}+a_{j} u_{e}+b_{j} v_{e}+d_{e, j}, \quad j \geqslant n+1
\end{aligned}
$$

In particular, from $v_{e}=K Z_{e}$, we have

$$
0=A_{K} Z_{e}+\Gamma_{e}=A Z_{e}+e^{-D A} B v_{e}+\Gamma_{e}
$$

Since the first rows of $A$ and $\Gamma_{e}$ are null and $E_{1} e^{-D A} B=1$, we obtain $v_{e}=0$ and

$$
\begin{align*}
Z_{e} & =-A_{K}^{-1} \Gamma_{e}  \tag{19a}\\
u_{e} & =E_{1} Z_{e}=-E_{1} A_{K}^{-1} \Gamma_{e}  \tag{19b}\\
w_{j, e} & =-\frac{a_{j}}{\lambda_{j}} u_{e}-\frac{d_{e, j}}{\lambda_{j}}, \quad j \geqslant n+1 \tag{19c}
\end{align*}
$$

We introduce $X_{e}=Z_{e}$ because $A X_{e}+B v_{D, e}+\Gamma_{e}=A_{K} Z_{e}+$ $\Gamma_{e}=0$, which is compatible with the Artstein transformation since $v_{e}=0$ implies $Z_{e}=X_{e}+\int_{t-D}^{t} e^{(t-s-D) A} B v_{e} \mathrm{~d} s$. The equilibrium condition of the integral component for reference tracking is given by

$$
\begin{equation*}
\zeta_{e}=E_{n+2} X_{e}=-E_{n+2} A_{K}^{-1} \Gamma_{e} \tag{19d}
\end{equation*}
$$

where $E_{n+2}=\left[\begin{array}{llll}0 & \ldots & 0 & 1\end{array}\right] \in \mathbb{R}^{1 \times(n+2)}$. Noting that $\lambda_{j} w_{j, e}=-a_{j} u_{e}-d_{e, j}$ for $j \geqslant n+1$ where $\left(a_{j}\right)_{j}$ and $\left(d_{e, j}\right)_{j}$ are square-summable sequences and $\lambda_{j} \rightarrow+\infty$ when $j \rightarrow+\infty$, both $\left(w_{j, e}\right)_{j}$ and $\left(\lambda_{j} w_{j, e}\right)_{j}$ are square-summable sequences. Hence we define

$$
\begin{equation*}
w_{e} \triangleq \sum_{j \geqslant 1} w_{j, e} e_{j} \in D(\mathcal{A})=H^{2}(0, L) \cap H_{0}^{1}(0, L) \tag{20}
\end{equation*}
$$

which is convergent in $H_{0}^{1}(0, L)$. In particular, we obtain from the last line of $A X_{e}+\Gamma_{e}=0$ and using (19c) that

$$
\begin{aligned}
& L_{1} X_{1, e}=r_{e}+\sum_{j \geqslant n+1} \frac{e_{j}^{\prime}(0)}{\lambda_{j}} d_{e, j} \\
\Leftrightarrow & \frac{1}{L} u_{e}-\sum_{j \geqslant n+1} \frac{e_{j}^{\prime}(0)}{\lambda_{j}} a_{j} u_{e}+\sum_{j=1}^{n} w_{j, e} e_{j}^{\prime}(0) \\
& =r_{e}+\sum_{j \geqslant n+1} \frac{e_{j}^{\prime}(0)}{\lambda_{j}} d_{e, j} \\
\Leftrightarrow & \sum_{j \geqslant 1} w_{j, e} e_{j}^{\prime}(0)+\frac{1}{L} u_{e}=r_{e} \\
\Leftrightarrow & w_{e}^{\prime}(0)+\frac{1}{L} u_{e}=r_{e} .
\end{aligned}
$$

Then, introducing $y_{e} \triangleq w_{e}+\frac{x}{L} u_{e} \in L^{2}(0, L)$, we obtain $y_{e}^{\prime}(0)=r_{e}$, which corresponds to the desired reference tracking. Finally, since

$$
\begin{aligned}
& \mathcal{A} w_{e}=\sum_{j \geqslant 1} \lambda_{j} w_{j, e} e_{j}=-\sum_{j \geqslant 1} a_{j} e_{j} u_{e}-\sum_{j \geqslant 1} d_{e, j} e_{j} \\
&=-a u_{e}-b v_{e}-d_{e}
\end{aligned}
$$

we have $\mathcal{A} w_{e}+a u_{D, e}+b v_{D, e}+d_{e}=0$.
Remark 5: The above developments show that the equilibrium point of the closed-loop infinite-dimensional system given by (19a-19d) and (20) is fully determined by the constant values of the reference signal $r_{e}$ and the distributed disturbance $d_{e}$.

## B. Dynamics of deviations

We now define the deviations of the various quantities with respect to their equilibrium value: $\Delta X=X-X_{e}, \Delta Z=$ $Z-Z_{e}, \Delta w=w-w_{e}, \Delta w_{j}=w_{j}-w_{j, e}, \Delta \zeta=\zeta-\zeta_{e}$, $\Delta u=u-u_{e}$ (first component of $\Delta Z$ ), $\Delta u_{D}=u_{D}-u_{e}$ (first component of $\Delta X), \Delta v=v-v_{e}$, and $\Delta v_{D}=v_{D}-v_{D, e}$. Then, in original coordinates:

$$
\begin{equation*}
\Delta w_{t}=\mathcal{A} \Delta w+a \Delta u_{D}+b \Delta v_{D}+\Delta d \tag{21}
\end{equation*}
$$

and

$$
\Delta \dot{X}(t)=\mathcal{A} \Delta X(t)+B \Delta v_{D}(t)+\Delta \Gamma(t)
$$

$$
\Delta \dot{w}_{j}(t)=\lambda_{j} \Delta w_{j}(t)+a_{j} \Delta u_{D}(t)+b_{j} \Delta v_{D}(t)+\Delta d_{j}(t)
$$

for $j \geqslant n+1$ with the auxiliary control input $\Delta v(t)=$ $\chi_{[0,+\infty)}(t) K \Delta Z(t)$ where

$$
\begin{equation*}
\Delta Z(t)=\Delta X(t)+\int_{t-D}^{t} e^{(t-s-D) A} B \Delta v(s) \mathrm{d} s \tag{22}
\end{equation*}
$$

In $Z$ coordinates, the closed-loop dynamics is given by

$$
\begin{align*}
\Delta \dot{Z}(t) & =A_{K} \Delta Z(t)+\Delta \Gamma(t)  \tag{23a}\\
\Delta \dot{w}_{j}(t) & =\lambda_{j} \Delta w_{j}(t)+a_{j} \Delta u_{D}(t)+b_{j} \Delta v_{D}(t)+\Delta d_{j}(t) \tag{23b}
\end{align*}
$$

for $j \geqslant n+1$.

## IV. Stability analysis

## A. Main stability result

The objective of this section is to establish the following stability result, taking the form of an Input-to-State Stability (ISS) estimate with fading memory of both the reference input $r$ and the distributed perturbation $d$.

Theorem 1: There exist $\kappa, \bar{C}_{1}>0$ such that, for every $\epsilon \in[0,1)$, there exists $\bar{C}_{2}(\epsilon)>0$ such that

$$
\begin{align*}
& \Delta u_{D}(t)^{2}+\Delta \zeta(t)^{2}+\|\Delta w(t)\|_{H_{0}^{1}(0, L)}^{2}  \tag{24}\\
& \leqslant \\
& \quad \bar{C}_{1} e^{-2 \kappa t}\left(\Delta u_{D}(0)^{2}+\Delta \zeta(0)^{2}+\|\Delta w(0)\|_{H_{0}^{1}(0, L)}^{2}\right) \\
& \quad+\bar{C}_{2}(\epsilon) \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\}
\end{align*}
$$

Moreover, the constants $\kappa, \bar{C}_{1}, \bar{C}_{2}(\epsilon)$ can be chosen independently of $r_{e}$ and $d_{e}$.

Since $\Delta w(t, x)=\Delta y(t, x)-\frac{x}{L} \Delta u_{D}(t)$, we deduce from the continuous embedding $H_{0}^{1}(0, L) \subset L^{\infty}(0, L)$ (see, e.g., [7]) the following corollary.

Corollary 1: Let $\kappa>0$ be provided by Theorem 1. There exists $\tilde{C}_{1}>0$ such that, for every $\epsilon \in[0,1)$, there exists $\tilde{C}_{2}(\epsilon)>0$ such that

$$
\begin{align*}
& \|\Delta y(t)\|_{L^{\infty}(0, L)}  \tag{25}\\
& \leqslant \tilde{C}_{1} e^{-\kappa t}\left(\left|\Delta u_{D}(0)\right|+|\Delta \zeta(0)|+\|\Delta w(0)\|_{H_{0}^{1}(0, L)}\right) \\
& \quad+\tilde{C}_{2}(\epsilon) \sup _{0 \leqslant s \leqslant t} e^{-\epsilon \kappa(t-s)}\{|\Delta r(s)|+\|\Delta d(s)\|\} .
\end{align*}
$$

We also deduce the following corollary concerning the asymptotic behavior of the closed-loop system in the case of convergent reference signal $r(t)$ and distributed disturbance $d(t)$ as $t \rightarrow+\infty$.

Corollary 2: Assume that $r(t) \rightarrow r_{e}$ and $d(t) \rightarrow d_{e}$ when $t \rightarrow+\infty$. Then $w(t) \rightarrow w_{e}$ in $H_{0}^{1}$ norm, $y(t) \rightarrow y_{e}$ in both $L^{\infty}$ and $L^{2}$ norm, $u(t) \rightarrow u_{e}$, and $\zeta(t) \rightarrow \zeta_{e}$ with exponential vanishing of the contribution of the initial conditions.

Remark 6: In the particular case $n=0$, which corresponds to an exponentially stable open-loop reaction-diffusion equation (1a-1d), the above results ensure that the stability of the closed-loop system is preserved after introduction of the two integral states $v$ and $z$.

In order to prove the claimed stability result, we resort as in [21] to the Lyapunov function

$$
\begin{equation*}
V(t)=\frac{M}{2} \Delta Z(t)^{\top} P \Delta Z(t) \tag{26}
\end{equation*}
$$

$$
\begin{aligned}
& +\frac{M}{2} \int_{\max (t-D, 0)}^{t} \Delta Z(s)^{\top} P \Delta Z(s) \mathrm{d} s \\
& -\frac{1}{2} \sum_{j \geqslant 1} \lambda_{j} \Delta w_{j}(t)^{2}
\end{aligned}
$$

where $P \in \mathbb{R}^{(n+2) \times(n+2)}$ is the solution of the Lyapunov equation $A_{K}^{\top} P+P A_{K}=-I$ and $M>0$ is chosen such that

$$
M>\max \left(\frac{\gamma_{1} \lambda_{1}}{\lambda_{m}(P)}, 4\left(\gamma_{1}\|a\|^{2}+2\|b\|^{2}\left\|e^{-D A_{K}}\right\|^{2}\|K\|^{2}\right)\right)
$$

with $\gamma_{1} \triangleq 2 \max \left(1, D e^{2 D\|A\|}\|B K\|^{2}\right)$.
Remark 7: The first term in the definition (26) of $V$ accounts for the stability of the finite-dimensional truncated model (23a), expressed in $Z$ coordinates, capturing the $n$ first modes of the reaction-diffusion equation. The motivation behind the introduction of the second (integral) term relies on the fact that it allows, in conjunction with (22), the derivation of an upper-estimate of $\|\Delta X(t)\|$ (i.e., the state of the truncated model in its original $X$ coordinates) based on $V(t)$. Finally, the last term is used to capture the countable infinite number of modes of the original reaction-diffusion equation (21), including those that where neglected in the control design. Note that $\langle\mathcal{A} \Delta w(t), \Delta w(t)\rangle=\sum_{j \geqslant 1} \lambda_{j} \Delta w_{j}(t)^{2}$.

## B. Preliminary Lemmas for the proof of Theorem 1

We derive hereafter various lemmas that will be useful in the sequel to establish the stability properties of the closed-loop system. First, we estimate $\Delta \Gamma(t)$ as follows.

Lemma 2: There exists a constant $M_{d}>0$ such that

$$
\|\Delta \Gamma(t)\|^{2} \leqslant M_{d}^{2}\left(\Delta r(t)^{2}+\|\Delta d(t)\|^{2}\right) \quad \forall t \geqslant 0
$$

Proof: By definition of $\Delta \Gamma(t)$ and using the CauchySchwarz inequality we have

$$
\begin{aligned}
& \|\Delta \Gamma(t)\|^{2} \\
& =\left\|\Delta D_{1}(t)\right\|^{2}+\left|\Delta r(t)+\sum_{j \geqslant n+1} \frac{e_{j}^{\prime}(0)}{\lambda_{j}} \Delta d_{j}(t)\right|^{2} \\
& \leqslant \sum_{j=1}^{n} \Delta d_{j}(t)^{2}+2 \Delta r(t)^{2}+2 \sum_{j \geqslant n+1}\left|\frac{e_{j}^{\prime}(0)}{\lambda_{j}}\right|^{2} \sum_{j \geqslant n+1} \Delta d_{j}(t)^{2} \\
& \leqslant M_{d}^{2}\left(\Delta r(t)^{2}+\|\Delta d(t)\|^{2}\right)
\end{aligned}
$$

with $M_{d}^{2}=2 \max \left(1, \sum_{j \geqslant n+1}\left|\frac{e_{j}^{\prime}(0)}{\lambda_{j}}\right|^{2}\right)<+\infty$, since, by (5), we have $\left|\frac{e_{j}^{\prime}(0)}{\lambda_{j}}\right|^{2} \sim \frac{2 L}{\pi^{2} j^{2}}$ when $j \rightarrow+\infty$.

Lemma 3: There exists a constant $C_{1}>0$ such that

$$
\begin{align*}
& V(t) \geqslant C_{1} \sum_{j \geqslant 1}\left(1+\left|\lambda_{j}\right|\right) \Delta w_{j}(t)^{2}  \tag{27a}\\
& V(t) \geqslant C_{1}\left(\Delta u_{D}(t)^{2}+\Delta \zeta(t)^{2}+\|\Delta w(t)\|_{H_{0}^{1}(0, L)}^{2}\right)  \tag{27b}\\
& V(t) \geqslant C_{1}\|\Delta Z(t)\|^{2} \tag{27c}
\end{align*}
$$

for every $t \geqslant 0$.
Proof: From (22) with $\Delta v=K \Delta Z$, we obtain that

$$
\|\Delta X(t)\|^{2}
$$

$$
\begin{align*}
\leqslant & 2\|\Delta Z(t)\|^{2} \\
& +2 D e^{2 D\|A\|}\|B K\|^{2} \int_{\max (t-D, 0)}^{t}\|\Delta Z(s)\|^{2} \mathrm{~d} s \\
\leqslant & \gamma_{1}\left(\|\Delta Z(t)\|^{2}+\int_{\max (t-D, 0)}^{t}\|\Delta Z(s)\|^{2} \mathrm{~d} s\right) \tag{28}
\end{align*}
$$

with $\gamma_{1}=2 \max \left(1, D e^{2 D\|A\|}\|B K\|^{2}\right)>0$. Thus, we have

$$
\begin{aligned}
\Delta Z(t)^{\top} P \Delta Z(t)+\int_{(t-D, t) \cap(0,+\infty)} & \Delta Z(s)^{\top} P \Delta Z(s) \mathrm{d} s \\
& \geqslant \frac{\lambda_{m}(P)}{\gamma_{1}}\|\Delta X(t)\|^{2}
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\sum_{j \geqslant 1} \lambda_{j} \Delta w_{j}(t)^{2} & \leqslant \sum_{j \geqslant n+1} \lambda_{j} \Delta w_{j}(t)^{2}+\lambda_{1} \sum_{j=1}^{n} \Delta w_{j}(t)^{2} \\
& \leqslant \sum_{j \geqslant n+1} \lambda_{j} \Delta w_{j}(t)^{2}+\lambda_{1}\|\Delta X(t)\|^{2}
\end{aligned}
$$

we obtain
$V(t) \geqslant\left(\frac{M \lambda_{m}(P)}{2 \gamma_{1}}-\frac{\lambda_{1}}{2}\right)\|\Delta X(t)\|^{2}-\frac{1}{2} \sum_{j \geqslant n+1} \lambda_{j} \Delta w_{j}(t)^{2}$.
Since $M>\frac{\gamma_{1} \lambda_{1}}{\lambda_{m}(P)}>0$, we obtain the existence of $\gamma_{2}=$ $\frac{1}{2} \min \left(\frac{M \lambda_{m}(P)}{\gamma_{1}}-\lambda_{1}, 1\right)>0$ such that

$$
\begin{equation*}
V(t) \geqslant \gamma_{2}\left(\|\Delta X(t)\|^{2}-\sum_{j \geqslant n+1} \lambda_{j} \Delta w_{j}(t)^{2}\right) \tag{29}
\end{equation*}
$$

from which we obtain (27a). Now, as in [21], from the series expansions (6) and (20) that are convergent in $H_{0}^{1}(0, L)$, we infer that

$$
\begin{gather*}
\|\Delta w(t)\|_{H_{0}^{1}(0, L)}^{2}=\sum_{i, j \geqslant 1} \Delta w_{i}(t) \Delta w_{j}(t) \int_{0}^{L} e_{i}^{\prime}(x) e_{j}^{\prime}(x) \mathrm{d} x \\
=\int_{0}^{L} c(x) \Delta w(t, x)^{2} \mathrm{~d} x-\sum_{j \geqslant 1} \lambda_{j} \Delta w_{j}(t)^{2} \tag{30}
\end{gather*}
$$

where the second equality follows from an integration by part and the facts that $e_{j}^{\prime \prime}+c e_{j}=\lambda_{j} e_{j}, e_{j}(0)=e_{j}(L)=0$, and $\left(e_{i}\right)_{i \geqslant 1}$ is a Hilbert basis of $L^{2}(0, L)$. Hence, using the fact that $-\sum_{1 \leqslant j \leqslant n} \lambda_{j} \Delta w_{j}(t)^{2} \leqslant 0$, the following estimates hold:

$$
\begin{aligned}
& \|\Delta w(t)\|_{H_{0}^{1}(0, L)}^{2} \\
& \leqslant\|c\|_{L^{\infty}(0, L)} \sum_{j \geqslant 1} \Delta w_{j}(t)^{2}-\sum_{j \geqslant n+1} \lambda_{j} \Delta w_{j}(t)^{2} \\
& \leqslant\|c\|_{L^{\infty}(0, L)} \sum_{j=1}^{n} \Delta w_{j}(t)^{2}-\sum_{j \geqslant n+1}\left(\lambda_{j}-\|c\|_{L^{\infty}(0, L)}\right) \Delta w_{j}(t)^{2} \\
& \leqslant \gamma_{3}\left(\sum_{j=1}^{n} \Delta w_{j}(t)^{2}-\sum_{j \geqslant n+1} \lambda_{j} \Delta w_{j}(t)^{2}\right)
\end{aligned}
$$

for some constant $\gamma_{3}>0$ because $\lambda_{j} \underset{j \rightarrow+\infty}{\longrightarrow}-\infty$ whence $-\left(\lambda_{j}-\|c\|_{L^{\infty}(0, L)}\right) \sim-\lambda_{j}$ when $j \rightarrow+\infty$ with $\lambda_{j}<0$ for all $j \geqslant n+1$. Therefore, we obtain from (29) that

$$
V(t) \geqslant \gamma_{2}\left(\Delta u_{D}(t)^{2}+\Delta \zeta(t)^{2}\right)+\frac{\gamma_{2}}{\gamma_{3}}\|\Delta w(t)\|_{H_{0}^{1}(0, L)}^{2}
$$

which provides (27b). Finally, from the definition of $V$ given by (26) and using (28), we also have

$$
\begin{aligned}
V(t) \geqslant & \frac{M \lambda_{m}(P)}{2}\left(\|\Delta Z(t)\|^{2}+\int_{\max (t-D, 0)}^{t}\|\Delta Z(s)\|^{2} \mathrm{~d} s\right) \\
& \underbrace{-\frac{1}{2} \sum_{j \geqslant n+1} \lambda_{j} \Delta w_{j}(t)^{2}}_{\geqslant 0}-\frac{\lambda_{1}}{2}\|\Delta X(t)\|^{2} \\
\geqslant & \frac{1}{2}\left(M \lambda_{m}(P)-\lambda_{1} \gamma_{1}\right) \\
& \times\left(\|\Delta Z(t)\|^{2}+\int_{\max (t-D, 0)}^{t}\|\Delta Z(s)\|^{2} \mathrm{~d} s\right) \\
\geqslant & \gamma_{1} \gamma_{2}\|\Delta Z(t)\|^{2},
\end{aligned}
$$

which gives (27c).

## C. End of proof of Theorem 1

We are now in a position to establish the stability properties of the closed-loop system and prove Theorem 1. We first study the exponential decay properties of $V$ for $t \geqslant D$.

Lemma 4: There exist $\kappa, C_{2}>0$ such that, for every $\epsilon \in$ $[0,1)$,

$$
\begin{aligned}
V(t) \leqslant & e^{-2 \kappa(t-D)} V(D) \\
& +\frac{C_{2}}{1-\epsilon} \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\}
\end{aligned}
$$

for every $t \geqslant D$.
Proof: First, we note that, for $t>D$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t} & {\left[\int_{t-D}^{t} \Delta Z(s)^{\top} P \Delta Z(s) \mathrm{d} s\right] } \\
& =\Delta Z(t)^{\top} P \Delta Z(t)-\Delta Z(t-D)^{\top} P \Delta Z(t-D) \\
& =\int_{t-D}^{t} \frac{\mathrm{~d}}{\mathrm{~d} \tau}\left[\Delta Z(\tau)^{\top} P \Delta Z(\tau)\right](s) \mathrm{d} s
\end{aligned}
$$

Since $\mathcal{A}$ is self-adjoint and since $\langle\mathcal{A} \Delta w(t), \Delta w(t)\rangle=$ $\sum_{j \geqslant 1} \lambda_{j} \Delta w_{j}(t)^{2}$, we have for every $t>D$,

$$
\begin{aligned}
& \dot{V}(t) \\
&= \frac{M}{2} \Delta Z(t)^{\top}\left(A_{K}^{\top} P+P A_{K}\right) \Delta Z(t)+M \Delta Z(t)^{\top} P \Delta \Gamma(t) \\
&+\frac{M}{2} \int_{t-D}^{t} \Delta Z(s)^{\top}\left(A_{K}^{\top} P+P A_{K}\right) \Delta Z(s) \mathrm{d} s \\
&+M \int_{t-D}^{t} \Delta Z(s)^{\top} P \Delta \Gamma(s) \mathrm{d} s-\left\langle\mathcal{A} \Delta w(t), \Delta w_{t}(t)\right\rangle \\
&=-\frac{M}{2}\|\Delta Z(t)\|^{2}+M \Delta Z(t)^{\top} P \Delta \Gamma(t) \\
&-\frac{M}{2} \int_{t-D}^{t}\|\Delta Z(s)\|^{2} \mathrm{~d} s+M \int_{t-D}^{t} \Delta Z(s)^{\top} P \Delta \Gamma(s) \mathrm{d} s \\
&-\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2}-\langle\mathcal{A} \Delta w(t), a\rangle \Delta u_{D}(t) \\
&-\langle\mathcal{A} \Delta w(t), b\rangle \Delta v_{D}(t)-\langle\mathcal{A} \Delta w(t), \Delta d(t)\rangle .
\end{aligned}
$$

Using the following estimates:

$$
\Delta Z(t)^{\top} P \Delta \Gamma(t) \leqslant\|\Delta Z(t)\|\|P\|\|\Delta \Gamma(t)\|
$$

$$
\leqslant \frac{1}{4}\|\Delta Z(t)\|^{2}+\|P\|^{2} M_{d}^{2}\left(\Delta r(t)^{2}+\|\Delta d(t)\|^{2}\right)
$$

$$
\begin{aligned}
\int_{t-D}^{t} & \Delta Z(s)^{\top} P \Delta \Gamma(s) \mathrm{d} s \\
& \leqslant \int_{t-D}^{t}\|\Delta Z(s)\|\|P\|\|\Delta \Gamma(s)\| \mathrm{d} s \\
\leqslant & \frac{1}{4} \int_{t-D}^{t}\|\Delta Z(s)\|^{2} \mathrm{~d} s \\
& +D\|P\|^{2} M_{d}^{2} \sup _{t-D \leqslant s \leqslant t}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\}
\end{aligned}
$$

$\left|\langle\mathcal{A} \Delta w(t), a\rangle \Delta u_{D}(t)\right|$

$$
\leqslant \frac{1}{4}\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2}+\|a\|^{2}\left|\Delta u_{D}(t)\right|^{2}
$$

$$
\leqslant \frac{1}{4}\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2}+\|a\|^{2}\|\Delta X(t)\|^{2}
$$

$$
\leqslant \frac{1}{4}\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2}
$$

$$
+\gamma_{1}\|a\|^{2}\left(\|\Delta Z(t)\|^{2}+\int_{t-D}^{t}\|\Delta Z(s)\|^{2} \mathrm{~d} s\right)
$$

$\left|\langle\mathcal{A} \Delta w(t), b\rangle \Delta v_{D}(t)\right|$

$$
\begin{aligned}
\leqslant & \frac{1}{4}\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2}+\|b\|^{2}\left|\Delta v_{D}(t)\right|^{2} \\
\leqslant & \frac{1}{4}\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2}+\|b\|^{2}\|K\|^{2}\|\Delta Z(t-D)\|^{2} \\
\leqslant & \frac{1}{4}\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2}+2\|b\|^{2}\left\|e^{-D A_{K}}\right\|^{2}\|K\|^{2}\|\Delta Z(t)\|^{2} \\
& +2 M_{d}^{2} D^{2} e^{2 D\left\|A_{K}\right\|}\|b\|^{2}\|K\|^{2} \sup _{t-D \leqslant s \leqslant t}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\}
\end{aligned}
$$

where we have used in the latter inequality that $\Delta \dot{Z}(t)=$ $A_{K} \Delta Z(t)+\Delta \Gamma(t)$ whence

$$
\Delta Z(t-D)=e^{-D A_{K}} \Delta Z(t)+\int_{t}^{t-D} e^{(t-D-s) A_{K}} \Delta \Gamma(s) \mathrm{d} s
$$

and

$$
|\langle\mathcal{A} \Delta w(t), \Delta d(t)\rangle| \leqslant \frac{1}{4}\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2}+\|\Delta d(t)\|^{2}
$$

we obtain that, for every $t>D$,

$$
\begin{aligned}
& \dot{V}(t) \leqslant\left(-\frac{M}{4}+\gamma_{1}\|a\|^{2}+2\|b\|^{2}\left\|e^{-D A_{K}}\right\|^{2}\|K\|^{2}\right) \\
& \times\left(\|\Delta Z(t)\|^{2}+\int_{t-D}^{t}\|\Delta Z(s)\|^{2} \mathrm{~d} s\right) \\
&-\frac{1}{4}\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2}+\gamma_{4} \sup _{t-D \leqslant s \leqslant t}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\}
\end{aligned}
$$

where
$\gamma_{4}=1+M_{d}^{2}\left\{(1+D) M\|P\|^{2}+2 D^{2} e^{2 D\left\|A_{K}\right\|}\|b\|^{2}\|K\|^{2}\right\}$.
Since $M>4\left(\gamma_{1}\|a\|^{2}+2\|b\|^{2}\left\|e^{-D A_{K}}\right\|^{2}\|K\|^{2}\right)$, setting

$$
\gamma_{5}=M / 4-\left(\gamma_{1}\|a\|^{2}+2\|b\|^{2}\left\|e^{-D A_{K}}\right\|^{2}\|K\|^{2}\right)>0
$$

we have
$\dot{V}(t)$

$$
\begin{aligned}
\leqslant & -\gamma_{5}\left(\|\Delta Z(t)\|^{2}+\int_{t-D}^{t}\|\Delta Z(s)\|^{2} \mathrm{~d} s\right) \\
& -\frac{1}{4}\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2}+\gamma_{4} \sup _{t-D \leqslant s \leqslant t}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\} \\
\leqslant & -\frac{\gamma_{5}}{\lambda_{M}(P)}\left(\Delta Z(t)^{\top} P \Delta Z(t)+\int_{t-D}^{t} \Delta Z(s)^{\top} P \Delta Z(s) \mathrm{d} s\right) \\
& -\frac{1}{4}\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2}+\gamma_{4} \sup _{t-D \leqslant s \leqslant t}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\},
\end{aligned}
$$

for every $t>D$. Now, since $\lambda_{j} \geqslant 0$ when $1 \leqslant j \leqslant n$ and $\lambda_{j} \leqslant \lambda_{n+1}<0$ when $j \geqslant n+1$, we have, for every $t \geqslant 0$,

$$
\begin{aligned}
& -\sum_{j \geqslant 1} \lambda_{j} \Delta w_{j}(t)^{2} \leqslant-\sum_{j=n+1}^{+\infty} \lambda_{j} \Delta w_{j}(t)^{2} \\
& \quad \leqslant \gamma_{6} \sum_{j=n+1}^{+\infty} \lambda_{j}^{2} \Delta w_{j}(t)^{2} \leqslant \gamma_{6}\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2}
\end{aligned}
$$

with $\gamma_{6}=1 /\left|\lambda_{n+1}\right|>0$. Setting $\Delta p(s)^{2}=\Delta r(s)^{2}+$ $\|\Delta d(s)\|^{2}$, we infer that
$\dot{V}(t)$

$$
\begin{aligned}
\leqslant & -\frac{2 \gamma_{5}}{M \lambda_{M}(P)} \frac{M}{2} \\
& \times\left(\Delta Z(t)^{\top} P \Delta Z(t)+\int_{t-D}^{t} \Delta Z(s)^{\top} P \Delta Z(s) \mathrm{d} s\right) \\
- & \frac{1}{2} \frac{1}{2 \gamma_{6}} \gamma_{6}\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2}+\gamma_{4} \sup _{t-D \leqslant s \leqslant t} \Delta p(s)^{2} \\
\leqslant & -2 \kappa \frac{M}{2}\left(\Delta Z(t)^{\top} P \Delta Z(t)+\int_{t-D}^{t} \Delta Z(s)^{\top} P \Delta Z(s) \mathrm{d} s\right) \\
& -2 \kappa \frac{1}{2} \gamma_{6}\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2}+\gamma_{4} \sup _{t-D \leqslant s \leqslant t} \Delta p(s)^{2} \\
\leqslant & -2 \kappa V(t)+\gamma_{4} \sup _{t-D \leqslant s \leqslant t}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\}
\end{aligned}
$$

for every $t>D$ where $\kappa=\frac{1}{2} \min \left(\frac{2 \gamma_{5}}{M \lambda_{M}(P)}, \frac{1}{2 \gamma_{6}}\right)>0$. Then, we obtain, for every $t \geqslant D$ and every $\epsilon \in[0,1)$,

$$
\begin{aligned}
& V(t)-e^{-2 \kappa(t-D)} V(D) \\
& \leqslant \gamma_{4} e^{-2 \kappa t} \int_{D}^{t} e^{2 \kappa \tau} \sup _{\tau-D \leqslant s \leqslant \tau} \Delta p(s)^{2} \mathrm{~d} \tau \\
& \leqslant \gamma_{4} e^{-2 \kappa t} \int_{D}^{t} e^{2(1-\epsilon) \kappa \tau} \mathrm{d} \tau \times \sup _{D \leqslant \tau \leqslant t}\left[e^{2 \epsilon \kappa \tau} \sup _{\tau-D \leqslant s \leqslant \tau} \Delta p(s)^{2}\right] \\
& \leqslant \frac{\gamma_{4}}{2(1-\epsilon) \kappa} e^{-2 \kappa t} e^{2(1-\epsilon) \kappa t} \times \sup _{D \leqslant \tau \leqslant t} \sup _{\tau-D \leqslant s \leqslant \tau} e^{2 \epsilon \kappa \tau} \Delta p(s)^{2} \\
& \leqslant \frac{\gamma_{4}}{2(1-\epsilon) \kappa} e^{-2 \epsilon \kappa t} \times \sup _{D \leqslant \tau \leqslant t} \sup _{\tau-D \leqslant s \leqslant \tau} e^{2 \epsilon \kappa(s+D)} \Delta p(s)^{2} \\
& \leqslant \frac{\gamma_{4} e^{2 \epsilon \kappa D}}{2(1-\epsilon) \kappa} e^{-2 \epsilon \kappa t} \sup _{D \leqslant \tau \leqslant t} \sup _{\tau-D \leqslant s \leqslant \tau} e^{2 \epsilon \kappa s} \Delta p(s)^{2} \\
& \leqslant \frac{\gamma_{4} e^{2 \kappa D}}{2(1-\epsilon) \kappa} e^{-2 \epsilon \kappa t} \sup _{0 \leqslant s \leqslant t} e^{2 \epsilon \kappa s}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\} \\
& \leqslant \frac{\gamma_{4} e^{2 \kappa D}}{2(1-\epsilon) \kappa} \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\}
\end{aligned}
$$

where we have used to establish the fourth inequality that, for a given $\tau \in[D, t], \tau-D \leqslant s \leqslant \tau$ implies $\tau \leqslant s+D$. The claimed estimate holds with $C_{2}=\gamma_{4} e^{2 \kappa D} /(2 \kappa)$.

Lemma 5: There exist constants $C_{3}, C_{4}>0$ such that

$$
\begin{aligned}
V(t) \leqslant & C_{3}\left(\Delta u_{D}(0)^{2}+\Delta \zeta(0)^{2}+\|\Delta w(0)\|_{H_{0}^{1}(0, L)}^{2}\right) \\
& +C_{4} \sup _{0 \leqslant s \leqslant t}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\}
\end{aligned}
$$

for every $t \in[0, D]$ with $\Delta u_{D}(0)=-u_{e}$.
Proof: For $0 \leqslant t \leqslant D$, we have

$$
\begin{aligned}
V(t)= & \frac{M}{2}\left(\Delta Z(t)^{\top} P \Delta Z(t)+\int_{0}^{t} \Delta Z(s)^{\top} P \Delta Z(s) \mathrm{d} s\right) \\
& -\frac{1}{2} \sum_{j \geqslant 1} \lambda_{j} \Delta w_{j}(t)^{2} .
\end{aligned}
$$

We note that, for $0 \leqslant t<D, \Delta u_{D}(t)=u(t-D)-u_{e}=$ $-u_{e}=\Delta u_{D}(0)$ and $\Delta v_{D}(t)=v(t-D)-v_{e}=0$, whence

$$
\dot{V}(t)
$$

$$
\begin{aligned}
= & \frac{M}{2} \Delta Z(t)^{\top}\left(A_{K}^{\top} P+P A_{K}\right) \Delta Z(t)+M \Delta Z(t)^{\top} P \Delta \Gamma(t) \\
& +\frac{M}{2} \Delta Z(t)^{\top} P \Delta Z(t)-\left\langle\mathcal{A} \Delta w(t), \Delta w_{t}(t)\right\rangle \\
= & -\frac{M}{2}\|\Delta Z(t)\|^{2}+M \Delta Z(t)^{\top} P \Delta \Gamma(t) \\
& +\frac{M}{2} \Delta Z(t)^{\top} P \Delta Z(t)-\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2} \\
& -\langle\mathcal{A} \Delta w(t), a\rangle \Delta u_{D}(t)-\langle\mathcal{A} \Delta w(t), \Delta d(t)\rangle \\
\leqslant & \frac{M\left(\|P\|+\lambda_{M}(P)-1\right)}{2}\|\Delta Z(t)\|^{2}+\frac{M\|P\|}{2}\|\Delta \Gamma(t)\|^{2} \\
& -\frac{1}{2}\|\mathcal{A} \Delta w(t)\|_{L^{2}(0, L)}^{2}+\|a\|^{2}\left|\Delta u_{D}(0)\right|^{2}+\|\Delta d(t)\|^{2} \\
\leqslant & \frac{M\left(\|P\|+\lambda_{M}(P)-1\right)}{2}\|\Delta Z(t)\|^{2}+\|a\|^{2}\|\Delta X(0)\|^{2} \\
& +\max \left(1, \frac{M M_{d}^{2}\|P\|}{2}\right)\left(\Delta r(t)^{2}+\|\Delta d(t)\|^{2}\right) .
\end{aligned}
$$

Noting that $\Delta \dot{Z}(t)=A_{K} \Delta Z(t)+\Delta \Gamma(t)$ and $\Delta Z(0)=$ $\Delta X(0)$, we have

$$
\Delta Z(t)=e^{A_{K} t} \Delta X(0)+\int_{0}^{t} e^{A_{K}(t-\tau)} \Delta \Gamma(\tau) \mathrm{d} \tau
$$

Thus, we obtain the existence of $\gamma_{7}, \gamma_{8}>0$ such that

$$
\dot{V}(t) \leqslant \gamma_{7}\|\Delta X(0)\|^{2}+\gamma_{8} \sup _{0 \leqslant s \leqslant t}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\}
$$

for $0 \leqslant t<D$. Therefore,

$$
\begin{aligned}
V(t) \leqslant V(0)+D \gamma_{7} \| & \|X(0)\|^{2} \\
& +D \gamma_{8} \sup _{0 \leqslant s \leqslant t}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\}
\end{aligned}
$$

for $0 \leqslant t \leqslant D$. To conclude, using (30), we estimate $V(0)+$ $D \gamma_{7}\|\Delta X(0)\|^{2}$ as follows:

$$
\begin{aligned}
& V(0)+D \gamma_{7}\|\Delta X(0)\|^{2} \\
& =\frac{M}{2} \Delta X(0)^{\top} P \Delta X(0)+D \gamma_{7}\|\Delta X(0)\|^{2}-\frac{1}{2} \sum_{j \geqslant 1} \lambda_{j} \Delta w_{j}(0)^{2} \\
& \leqslant \frac{M \lambda_{M}(P)+2 D \gamma_{7}}{2}\left(\Delta u_{D}(0)^{2}+\Delta \zeta(0)^{2}+\sum_{j=1}^{n} \Delta w_{j}(0)^{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{2}\left(\|\Delta w(0)\|_{H_{0}^{1}(0, L)}^{2}+\|c\|_{L^{\infty}(0, L)}\|\Delta w(0)\|_{L^{2}(0, L)}^{2}\right) \\
\leqslant & \frac{M \lambda_{M}(P)+2 D \gamma_{7}}{2}\left(\Delta u_{D}(0)^{2}+\Delta \zeta(0)^{2}\right) \\
& +\frac{1}{2}\left(1+L^{2}\left\{\|c\|_{L^{\infty}(0, L)}+M \lambda_{M}(P)+2 D \gamma_{7}\right\}\right) \\
& \times\|\Delta w(0)\|_{H_{0}^{1}(0, L)}^{2},
\end{aligned}
$$

where we have used the Poincaré inequality to derive the last estimate: $\|f\|_{L^{2}(0, L)} \leqslant L\|f\|_{H_{0}^{1}(0, L)}$ for every $f \in H_{0}^{1}(0, L)$. Combining the two latter estimates, the result follows.

Lemma 6: There exist $\kappa, C_{5}>0$ such that, for every $\epsilon \in$ $[0,1)$, there exists $C_{6}(\epsilon)>0$ such that

$$
\begin{aligned}
& V(t) \leqslant C_{5} e^{-2 \kappa t}\left(\Delta u_{D}(0)^{2}+\Delta \zeta(0)^{2}+\|\Delta w(0)\|_{H_{0}^{1}(0, L)}^{2}\right) \\
& +C_{6}(\epsilon) \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\} \quad \forall t \geqslant 0 .
\end{aligned}
$$

Proof: When $0 \leqslant t \leqslant D$, Lemma 5 yields

$$
\begin{aligned}
& V(t) \\
& \leqslant C_{3} e^{2 \kappa D} e^{-2 \kappa t}\left(\Delta u_{D}(0)^{2}+\Delta \zeta(0)^{2}+\|\Delta w(0)\|_{H_{0}^{1}(0, L)}^{2}\right) \\
& \quad+C_{4} e^{2 \epsilon \kappa D} \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\}
\end{aligned}
$$

because $D-t \geqslant 0$ and $D-t+s \geqslant 0$ for all $0 \leqslant s \leqslant t \leqslant$ $D$. When $t \geqslant D$, we infer from Lemma 4 , from the latter estimate evaluated in $t=D$, and by using again the notation $\Delta p(s)^{2}=\Delta r(s)^{2}+\|\Delta d(s)\|^{2}$, that

$$
\begin{aligned}
& V(t) \\
& \leqslant e^{-2 \kappa(t-D)} V(D)+\frac{C_{2}}{1-\epsilon} \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)} \Delta p(s)^{2} \\
& \leqslant C_{3} e^{-2 \kappa(t-D)}\left(\Delta u_{D}(0)^{2}+\Delta \zeta(0)^{2}+\|\Delta w(0)\|_{H_{0}^{1}(0, L)}^{2}\right) \\
&+C_{4} e^{-2 \epsilon \kappa(t-D)} \sup _{0 \leqslant s \leqslant D} e^{2 \epsilon \kappa s}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\} \\
&+\frac{C_{2}}{1-\epsilon} \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\} \\
& \leqslant C_{3} e^{2 \kappa D} e^{-2 \kappa t}\left(\Delta u_{D}(0)^{2}+\Delta \zeta(0)^{2}+\|\Delta w(0)\|_{H_{0}^{1}(0, L)}^{2}\right) \\
&+\left(C_{4} e^{2 \epsilon \kappa D}+\frac{C_{2}}{1-\epsilon}\right) \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)} \Delta p(s)^{2} .
\end{aligned}
$$

The claimed estimate holds with $C_{5}=C_{3} e^{2 \kappa D}$ and $C_{6}(\epsilon)=$ $C_{4} e^{2 \epsilon \kappa D}+\frac{C_{2}}{1-\epsilon}$.

We are now in a position to prove the main result of this section, namely the stability result stated in Theorem 1. Indeed, from Lemmas 3 and 6 , we infer the existence of constants $\bar{C}_{1}=C_{5} / C_{1}>0$ and $\bar{C}_{2}(\epsilon)=C_{6}(\epsilon) / C_{1}>0$ such that (24) holds. Similarly, we obtain the following estimates which will be useful in the next section concerning the tracking performance:

$$
\begin{aligned}
& \sum_{j \geqslant 1}\left(1+\left|\lambda_{j}\right|\right) \Delta w_{j}(t)^{2} \\
& \leqslant \bar{C}_{1} e^{-2 \kappa t}\left(\Delta u_{D}(0)^{2}+\Delta \zeta(0)^{2}+\|\Delta w(0)\|_{H_{0}^{1}(0, L)}^{2}\right) \\
& \quad+\bar{C}_{2}(\epsilon) \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\},
\end{aligned}
$$

for every $t \geqslant 0$, and, as $\Delta v(t)=K \Delta Z(t)$ for $t \geqslant 0$ and $\Delta v(t)=0$ for $t<0$,

$$
\begin{align*}
& \left\|\Delta v_{D}(t)\right\|^{2}  \tag{32}\\
& \leqslant \hat{C}_{1} e^{-2 \kappa t}\left(\Delta u_{D}(0)^{2}+\Delta \zeta(0)^{2}+\|\Delta w(0)\|_{H_{0}^{1}(0, L)}^{2}\right) \\
& \quad+\hat{C}_{2}(\epsilon) \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)}\left\{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}\right\},
\end{align*}
$$

for every $t \geqslant 0$ with $\hat{C}_{1}=\|K\|^{2} \bar{C}_{1} e^{2 \kappa D}$ and $\hat{C}_{2}(\epsilon)=$ $\|K\|^{2} \bar{C}_{2}(\epsilon) e^{2 \epsilon \kappa D}$. This concludes the proof of Theorem 1.

Remark 8: All the constants $C_{i}$ for $1 \leqslant i \leqslant 6$, and thus $\bar{C}_{1}, \bar{C}_{2}(\epsilon)$, defined in this section are independent of the considered equilibrium condition characterized by the quantities $r_{e}$ and $d_{e}$. Consequently, one can apply the result of Theorem 1, for the same values of the constants $\bar{C}_{1}, \bar{C}_{2}(\epsilon)$, to distinct equilibrium points associated with different constant values $r_{e}$ and $d_{e}$ successively taken by the reference signal $r(t)$ and the distributed disturbance $d(t)$, respectively. This feature will be illustrated in numerical computations in Section VI. 。

## V. Setpoint reference tracking analysis

It now remains to assess that the setpoint tracking of the reference signal $r(t)$ is achieved in the presence of the distributed disturbance $d(t)$. Specifically, we establish in this section the following tracking result.

Theorem 2: Let $\kappa>0$ be provided by Theorem 1. There exists $\bar{C}_{3}>0$ such that, for every $\epsilon \in[0,1)$, there exists $\bar{C}_{4}(\epsilon)>0$ such that

$$
\begin{align*}
& \left|y_{x}(t, 0)-r(t)\right|  \tag{33}\\
& \begin{array}{r}
\leqslant \bar{C}_{3} e^{-\kappa t}\left(\left|\Delta u_{D}(0)\right|+|\Delta \zeta(0)|+\|\Delta w(0)\|_{H_{0}^{1}(0, L)}\right. \\
+ \\
\left.+\|\mathcal{A} \Delta w(0)\|_{L^{2}(0, L)}\right) \\
+\bar{C}_{4}(\epsilon) \sup _{0 \leqslant s \leqslant t} e^{-\epsilon \kappa(t-s)}\{|\Delta r(s)|+\|\Delta d(s)\|+\|\dot{d}(s)\|\} .
\end{array}
\end{align*}
$$

Moreover, the constants $\bar{C}_{3}, \bar{C}_{4}(\epsilon)$ can be chosen independently of the parameters $r_{e}$ and $d_{e}$.

Corollary 3: Assume that $r(t) \rightarrow r_{e}, d(t) \rightarrow d_{e}$, and $\dot{d}(t) \rightarrow 0$ when $t \rightarrow+\infty$. Then $y_{x}(t, 0) \rightarrow r_{e}$ with exponential vanishing of the contribution of the initial conditions.

Remark 9: In the particular case $n=0$, which corresponds to an exponentially stable open-loop reaction-diffusion equation ( $1 \mathrm{a}-1 \mathrm{~d}$ ), the above results also ensure that the proposed control strategy achieves the setpoint reference tracking of the reference signal $r(t)$ while preserving the stability of the closed-loop system.

Proof of Theorem 2. Based on the identity $w_{e, x}(0)+\frac{1}{L} u_{e}=$ $r_{e}$, we have the estimates:

$$
\begin{align*}
& \left|y_{x}(t, 0)-r(t)\right| \quad \leqslant\left|w_{x}(t, 0)+\frac{1}{L} u_{D}(t)-r_{e}\right|+|\Delta r(t)| \\
& \quad \leqslant\left|w_{x}(t, 0)-w_{e, x}(0)\right|+\frac{1}{L}\left|\Delta u_{D}(t)\right|+|\Delta r(t)| . \tag{34}
\end{align*}
$$

From the estimate of $\Delta u_{D}(t)$ provided by (24), it is sufficient to study the term $w_{x}(t, 0)-w_{e, x}(0)=\sum_{j \geqslant 1} \Delta w_{j}(t) e_{j}^{\prime}(0)$. Since $e_{j}^{\prime}(0) \sim \sqrt{2 / L} \sqrt{\left|\lambda_{j}\right|}$, there exists a constant $\gamma_{9}>0$ such that $\left|e_{j}^{\prime}(0)\right| \leqslant \gamma_{9} \sqrt{\left|\lambda_{j}\right|}$ for all $j \geqslant n+1$. Let $m \geqslant n+1$ be such
that $\eta \triangleq-\lambda_{m}>\kappa>0$. Thus $\lambda_{j} \leqslant-\eta<-\kappa<0$ for all $j \geqslant m$. We infer from the Cauchy-Schwarz inequality that

$$
\begin{align*}
& \left|w_{x}(t, 0)-w_{e, x}(0)\right| \leqslant \sum_{j \geqslant 1}\left|\Delta w_{j}(t)\right|\left|e_{j}^{\prime}(0)\right| \\
& \leqslant \sum_{j=1}^{m-1}\left|\Delta w_{j}(t)\right|\left|e_{j}^{\prime}(0)\right|+\gamma_{9} \sum_{j \geqslant m} \sqrt{\left|\lambda_{j}\right| \mid} \Delta w_{j}(t) \mid \\
& \leqslant \sqrt{\sum_{j=1}^{m-1} e_{j}^{\prime}(0)^{2}} \sqrt{\sum_{j=1}^{m-1} \Delta w_{j}(t)^{2}} \\
& \quad+\gamma_{9} \sqrt{\sum_{j \geqslant m} \frac{1}{\left|\lambda_{j}\right|}} \sqrt{\sum_{j \geqslant m} \lambda_{j}^{2} \Delta w_{j}(t)^{2}} \tag{35}
\end{align*}
$$

where $\sum_{j \geqslant m} \frac{1}{\left|\lambda_{j}\right|}<+\infty$ because $\lambda_{j} \sim-\pi^{2} j^{2} / L^{2}$. Based on (31), it is sufficient to study the term $\sqrt{\sum_{j \geqslant m} \lambda_{j}^{2} \Delta w_{j}(t)^{2}}$. To do so, we integrate for $j \geqslant m$ the dynamics (23b) of the coefficient $\Delta w_{j}(t)$ as follows:

$$
\begin{align*}
& \lambda_{j} \Delta w_{j}(t)=e^{\lambda_{j} t} \lambda_{j} \Delta w_{j}(0)  \tag{36}\\
& \quad+\int_{0}^{t} \lambda_{j} e^{\lambda_{j}(t-\tau)}\left\{a_{j} \Delta u_{D}(\tau)+b_{j} \Delta v_{D}(\tau)+\Delta d_{j}(\tau)\right\} \mathrm{d} \tau
\end{align*}
$$

Now, integrating by parts and noting that $\Delta \dot{d}_{j}(\tau)=\dot{d}_{j}(\tau)$, we have

$$
\begin{align*}
& \int_{0}^{t} \lambda_{j} e^{\lambda_{j}(t-\tau)} \Delta d_{j}(\tau) \mathrm{d} \tau  \tag{37}\\
& \quad=-\Delta d_{j}(t)+e^{\lambda_{j} t} \Delta d_{j}(0)+\int_{0}^{t} e^{\lambda_{j}(t-\tau)} \dot{d}_{j}(\tau) \mathrm{d} \tau
\end{align*}
$$

whence,

$$
\begin{aligned}
& \left|\lambda_{j} \Delta w_{j}(t)\right| \\
& \leqslant \\
& \quad e^{\lambda_{j} t}\left|\lambda_{j} \Delta w_{j}(0)\right| \\
& \quad+\int_{0}^{t}\left(-\lambda_{j}\right) e^{\lambda_{j}(t-\tau)}\left\{\left|a_{j}\right|\left|\Delta u_{D}(\tau)\right|+\left|b_{j}\right|\left|\Delta v_{D}(\tau)\right|\right\} \mathrm{d} \tau \\
& \quad+\left|\Delta d_{j}(t)\right|+e^{\lambda_{j} t}\left|\Delta d_{j}(0)\right|+\int_{0}^{t} e^{\lambda_{j}(t-\tau)}\left|\dot{d}_{j}(\tau)\right| \mathrm{d} \tau \\
& \leqslant \\
& \quad e^{-\eta t}\left|\lambda_{j} \Delta w_{j}(0)\right|+\left|a_{j}\right| \int_{0}^{t}\left(-\lambda_{j}\right) e^{\lambda_{j}(t-\tau)}\left|\Delta u_{D}(\tau)\right| \mathrm{d} \tau \\
& \quad+\left|b_{j}\right| \int_{0}^{t}\left(-\lambda_{j}\right) e^{\lambda_{j}(t-\tau)}\left|\Delta v_{D}(\tau)\right| \mathrm{d} \tau \\
& \quad+\left|\Delta d_{j}(t)\right|+e^{-\eta t}\left|\Delta d_{j}(0)\right|+\int_{0}^{t} e^{-\eta(t-\tau)}\left|\dot{d}_{j}(\tau)\right| \mathrm{d} \tau
\end{aligned}
$$

Now, as $\lambda_{j} \leqslant-\eta<-\kappa<-\epsilon \kappa$, the use of estimate (24) and the introduction of the notations $\Delta \mathrm{CI}=$ $\sqrt{\Delta u_{D}(0)^{2}+\Delta \zeta(0)^{2}+\|\Delta w(0)\|_{H_{0}^{1}(0, L)}^{2}}$ and $\Delta p(s)=$ $\sqrt{\Delta r(s)^{2}+\|\Delta d(s)\|^{2}}$ yield

$$
\begin{aligned}
& \int_{0}^{t}\left(-\lambda_{j}\right) e^{\lambda_{j}(t-\tau)}\left|\Delta u_{D}(\tau)\right| \mathrm{d} \tau \\
& \leqslant\left(-\lambda_{j}\right) \sqrt{\bar{C}_{1}} e^{\lambda_{j} t} \int_{0}^{t} e^{-\lambda_{j} \tau} e^{-\kappa \tau} \mathrm{d} \tau \Delta \mathrm{CI} \\
& \quad+\left(-\lambda_{j}\right) \sqrt{\bar{C}_{2}(\epsilon)} e^{\lambda_{j} t} \int_{0}^{t} e^{-\lambda_{j} \tau} \sup _{0 \leqslant s \leqslant \tau} e^{-\epsilon \kappa(\tau-s)} \Delta p(s) \mathrm{d} \tau
\end{aligned}
$$

$$
\begin{aligned}
\leqslant & \left(-\lambda_{j}\right) \sqrt{\bar{C}_{1}} e^{\lambda_{j} t} \int_{0}^{t} e^{-\left(\lambda_{j}+\kappa\right) \tau} \mathrm{d} \tau \Delta \mathrm{CI} \\
& +\left(-\lambda_{j}\right) \sqrt{\bar{C}_{2}(\epsilon)} e^{\lambda_{j} t} \int_{0}^{t} e^{-\left(\lambda_{j}+\epsilon \kappa\right) \tau} \sup _{0 \leqslant s \leqslant \tau} e^{\epsilon \kappa s} \Delta p(s) \mathrm{d} \tau \\
\leqslant & \frac{\lambda_{j}}{\lambda_{j}+\kappa} \sqrt{\bar{C}_{1}} e^{\lambda_{j} t}\left(e^{-\left(\lambda_{j}+\kappa\right) t}-1\right) \Delta \mathrm{CI} \\
& +\frac{\lambda_{j}}{\lambda_{j}+\epsilon \kappa} \sqrt{\bar{C}_{2}(\epsilon)} e^{\lambda_{j} t}\left(e^{-\left(\lambda_{j}+\epsilon \kappa\right) t}-1\right) \sup _{0 \leqslant s \leqslant t} e^{\epsilon \kappa s} \Delta p(s) \\
\leqslant & \frac{\eta}{\eta-\kappa} \sqrt{\bar{C}_{1}} e^{-\kappa t} \Delta \mathrm{CI} \\
& +\frac{\eta}{\eta-\epsilon \kappa} \sqrt{\bar{C}_{2}(\epsilon)} e^{-\epsilon \kappa t} \sup _{0 \leqslant s \leqslant t} e^{\epsilon \kappa s} \Delta p(s) \\
\leqslant & \frac{\eta}{\eta-\kappa} \sqrt{\bar{C}_{1}} e^{-\kappa t} \Delta \mathrm{CI} \\
& +\frac{\eta}{\eta-\epsilon \kappa} \sqrt{\bar{C}_{2}(\epsilon)} \sup _{0 \leqslant s \leqslant t} e^{-\epsilon \kappa(t-s)} \Delta p(s) .
\end{aligned}
$$

Similarly, the use of the estimate (32) yields

$$
\begin{aligned}
& \int_{0}^{t}\left(-\lambda_{j}\right) e^{\lambda_{j}(t-\tau)}\left|\Delta v_{D}(\tau)\right| \mathrm{d} \tau \\
& \leqslant \frac{\eta}{\eta-\kappa} \sqrt{\hat{C}_{1}} e^{-\kappa t} \Delta \mathrm{CI} \\
& \quad+\frac{\eta}{\eta-\epsilon \kappa} \sqrt{\hat{C}_{2}(\epsilon)} \sup _{0 \leqslant s \leqslant t} e^{-\epsilon \kappa(t-s)} \Delta p(s)
\end{aligned}
$$

Finally, since $\eta>\kappa$, we also infer from the Cauchy-Schwarz inequality that

$$
\begin{aligned}
& \int_{0}^{t} e^{-\eta(t-\tau)}\left|\dot{d}_{j}(\tau)\right| \mathrm{d} \tau=\int_{0}^{t} e^{-(\eta-\kappa)(t-\tau)} e^{-\kappa(t-\tau)}\left|\dot{d}_{j}(\tau)\right| \mathrm{d} \tau \\
& \leqslant \sqrt{\int_{0}^{t} e^{-2(\eta-\kappa)(t-\tau)} \mathrm{d} \tau} \sqrt{\int_{0}^{t} e^{-2 \kappa(t-\tau)}\left|\dot{d}_{j}(\tau)\right|^{2} \mathrm{~d} \tau} \\
& \leqslant \sqrt{\frac{1}{2(\eta-\kappa)}} \sqrt{\int_{0}^{t} e^{-2 \kappa(t-\tau)}\left|\dot{d}_{j}(\tau)\right|^{2} \mathrm{~d} \tau}
\end{aligned}
$$

We infer from the three above estimates that

$$
\begin{aligned}
& \left|\lambda_{j} \Delta w_{j}(t)\right| \\
& \leqslant e^{-\eta t}\left|\lambda_{j} \Delta w_{j}(0)\right| \\
& \quad+\frac{\eta}{\eta-\kappa}\left(\left|a_{j}\right| \sqrt{\bar{C}_{1}}+\left|b_{j}\right| \sqrt{\hat{C}_{1}}\right) e^{-\kappa t} \Delta \mathrm{CI} \\
& \quad+\frac{\eta}{\eta-\epsilon \kappa}\left(\left|a_{j}\right| \sqrt{\bar{C}_{2}(\epsilon)}+\left|b_{j}\right| \sqrt{\hat{C}_{2}(\epsilon)}\right) \sup _{0 \leqslant s \leqslant t} e^{-\epsilon \kappa(t-s)} \Delta p(s) \\
& \quad+\left|\Delta d_{j}(t)\right|+e^{-\eta t}\left|\Delta d_{j}(0)\right| \\
& \quad+\sqrt{\frac{1}{2(\eta-\kappa)}} \sqrt{\int_{0}^{t} e^{-2 \kappa(t-\tau)}\left|\dot{d}_{j}(\tau)\right|^{2} \mathrm{~d} \tau}
\end{aligned}
$$

Consequently we have:

$$
\begin{aligned}
& \left|\lambda_{j} \Delta w_{j}(t)\right|^{2} \\
& \leqslant 6 e^{-2 \eta t}\left|\lambda_{j} \Delta w_{j}(0)\right|^{2} \\
& \quad+\frac{6 \eta^{2}}{(\eta-\kappa)^{2}}\left(\left|a_{j}\right| \sqrt{\bar{C}_{1}}+\left|b_{j}\right| \sqrt{\hat{C}_{1}}\right)^{2} e^{-2 \kappa t} \Delta \mathrm{CI}^{2} \\
& \quad+\frac{6 \eta^{2}}{(\eta-\epsilon \kappa)^{2}}\left(\left|a_{j}\right| \sqrt{\bar{C}_{2}(\epsilon)}+\left|b_{j}\right| \sqrt{\hat{C}_{2}(\epsilon)}\right)^{2}
\end{aligned}
$$

$$
\begin{aligned}
& \times \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)} \Delta p(s)^{2} \\
+ & 6\left|\Delta d_{j}(t)\right|^{2}+6 e^{-2 \eta t}\left|\Delta d_{j}(0)\right|^{2} \\
+ & \frac{3}{\eta-\kappa} \int_{0}^{t} e^{-2 \kappa(t-\tau)}\left|\dot{d}_{j}(\tau)\right|^{2} \mathrm{~d} \tau
\end{aligned}
$$

whence

$$
\begin{aligned}
& \sum_{j \geqslant m} \lambda_{j}^{2} \Delta w_{j}(t)^{2} \\
& \leqslant 6 e^{-2 \kappa t}\|\mathcal{A} \Delta w(0)\|_{L^{2}(0, L)}^{2} \\
& +\frac{12 \eta^{2}}{(\eta-\epsilon \kappa)^{2}}\left(\|a\|^{2} \bar{C}_{1}+\|b\|^{2} \hat{C}_{1}\right) e^{-2 \kappa t} \Delta \mathrm{CI}^{2} \\
& +\frac{12 \eta^{2}}{(\eta-\epsilon \kappa)^{2}}\left(\|a\|^{2} \bar{C}_{2}(\epsilon)+\|b\|^{2} \hat{C}_{2}(\epsilon)\right) \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)} \Delta p(s)^{2} \\
& +6\|\Delta d(t)\|^{2}+6 e^{-2 \kappa t}\|\Delta d(0)\|^{2} \\
& +\frac{3}{\eta-\kappa \int_{0}^{t} e^{-2 \kappa(t-\tau)}\|\dot{d}(\tau)\|^{2} \mathrm{~d} \tau} \\
& \leqslant 6 e^{-2 \kappa t}\|\mathcal{A} \Delta w(0)\|_{L^{2}(0, L)}^{2} \\
& +\frac{12 \eta^{2}}{(\eta-\epsilon \kappa)^{2}}\left(\|a\|^{2} \bar{C}_{1}+\|b\|^{2} \hat{C}_{1}\right) e^{-2 \kappa t} \Delta \mathrm{CI}^{2} \\
& +\frac{12 \eta^{2}}{(\eta-\epsilon \kappa)^{2}}\left(\|a\|^{2} \bar{C}_{2}(\epsilon)+\|b\|^{2} \hat{C}_{2}(\epsilon)\right) \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)} \Delta p(s)^{2} \\
& +12 \sup e^{-2 \kappa(t-\tau)}\|\Delta d(\tau)\|^{2} \\
& +\frac{0 \leqslant s \leqslant t}{2(1-\epsilon)(\eta-\kappa) \kappa} \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)}\|\dot{d}(s)\|^{2}
\end{aligned}
$$

where we have used that

$$
\begin{aligned}
& \int_{0}^{t} e^{-2 \kappa(t-\tau)}\|\dot{d}(\tau)\|^{2} \mathrm{~d} \tau \\
& \quad=\int_{0}^{t} e^{-2(1-\epsilon) \kappa(t-\tau)} e^{-2 \epsilon \kappa(t-\tau)}\|\dot{d}(\tau)\|^{2} \mathrm{~d} \tau \\
& \quad \leqslant \int_{0}^{t} e^{-2(1-\epsilon) \kappa(t-\tau)} \mathrm{d} \tau \times \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)}\|\dot{d}(s)\|^{2} \\
& \quad \leqslant \frac{1}{2(1-\epsilon) \kappa} \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)}\|\dot{d}(s)\|^{2}
\end{aligned}
$$

We deduce the existence of constants $C_{7}, C_{8}(\epsilon)>0$ such that

$$
\begin{aligned}
& \sum_{j \geqslant m} \lambda_{j}^{2} \Delta w_{j}(t)^{2} \\
& \leqslant C_{7} e^{-2 \kappa t}\left(\Delta u_{D}(0)^{2}+\Delta \zeta(0)^{2}+\|\Delta w(0)\|_{H_{0}^{1}(0, L)}^{2}\right. \\
& \left.+\|\mathcal{A} \Delta w(0)\|_{L^{2}(0, L)}^{2}\right) \\
& \quad+C_{8}(\epsilon) \sup _{0 \leqslant s \leqslant t} e^{-2 \epsilon \kappa(t-s)}\left(\Delta r(s)^{2}+\|\Delta d(s)\|^{2}+\|\dot{d}(s)\|^{2}\right)
\end{aligned}
$$

Using now (34) along with (35) and estimates (24), (31), and (38), we obtain the existence of the claimed constants $C_{3}, C_{4}(\epsilon)>0$ such that the estimate (33) holds.

Remark 10: At first sight, it might seem surprising that the estimate (33) on the tracking performance only involves the time derivative $\dot{d}$ of the distributed disturbance but not the time derivative $\dot{r}$ of the reference signal. Such a dissimilarity between the reference signal and the distributed disturbance is
due to the explicit occurrence of the distributed perturbation $d$ in the dynamics (23b) of the coefficient of projection $\Delta w_{j}$. Indeed, in order to estimate the term $\left|\lambda_{j} \Delta w_{j}(t)\right|$ from (36), one needs to estimate the term $\int_{0}^{t} \lambda_{j} e^{\lambda_{j}(t-\tau)} \Delta d_{j}(\tau) \mathrm{d} \tau$. To do so, one first needs to eliminate the multiplicative factor $\lambda_{j}$ using, e.g., either an integration or an integration by parts. Simultaneously, we need to use Parseval identity in order to gather all coefficients $\Delta d_{j}(t)$. However, contrarily to the constant coefficients $a_{j}, b_{j}$, each coefficient $\Delta d_{j}(t)$ is a function of time and thus cannot be pulled out of the integral. This remark motivates the integration by parts carried out in (37). This way, the multiplicative factor $\lambda_{j}$ is eliminated and the subsequent estimates can be obtained. However, this is ${ }_{2}$ at the price of the emergence of the term $\dot{d}$ in the resulting tracking estimate.

## VI. Numerical illustration

We take $c=1.25, L=2 \pi$, and $D=1 \mathrm{~s}$. The three first eigenvalues of the open-loop system are $\lambda_{1}=1, \lambda_{2}=0.25$, and $\lambda_{2}=-1$. Only the two first modes need to be stabilized. Thus we have $n=2$ and we compute the feedback gain $K \in$ $\mathbb{R}^{1 \times 4}$ such that the poles of the closed-loop truncated model 2 (capturing the two unstable modes of the infinite-dimensional system plus two integral components, one for the control input and one for the reference tracking) are given by $-0.5,-0.6$, -0.7 , and -0.8 . The adopted numerical scheme is the modal approximation of the infinite-dimensional system using its first 10 modes. The initial condition is $y_{0}(x)=\frac{x}{L}\left(1-\frac{x}{L}\right)$.

The simulation results for a time-varying reference $r(t)$ evolving within the range $[0,50]$ and the constant distributed disturbance $d(t, x)=x$ are depicted in Fig. 1. Applying first the obtained stability results for $t<30 \mathrm{~s}$ with $r_{e}=0$ and $d_{e}(x)=x$, we obtain that $y \rightarrow 0$ in $L^{\infty}(0, L)$ norm, $u(t) \rightarrow 0$, and $y_{x}(t, 0) \rightarrow 0$, when $t \rightarrow+\infty$. This is compliant with the simulation result observed for increasing values of $t$ approaching $t=30 \mathrm{~s}$. Consequently, the numerical simulation confirms that the proposed control strategy achieves the exponential stabilization of the closed-loop system while ensuring a zero steady-state left Neumann trace. Then, for $30 \mathrm{~s}<t<60 \mathrm{~s}$, the tracking error remains bounded in the presence of an oscillatory reference signal. Finally, for $t>60 \mathrm{~s}$, we apply again the obtained stability results but for $r_{e}=50$ and $d_{e}(x)=x$. This time, we obtain that $y \rightarrow y_{e} \neq 0$ in $L^{\infty}(0, L)$ norm and $u(t) \rightarrow u_{e} \neq 0$ as $y_{x}(t, 0) \rightarrow r_{e}=50$ when $t \rightarrow+\infty$. In particular, conforming to the obtained ISS estimates with fading memory (24-25) and (33), the impact of the variations of the reference signal around its nominal value $r_{e}$, i.e., configuration for which $\Delta r(t) \neq 0$, are eliminated as $t$ increases due to the action of the PI controller. This result provides a numerical confirmation of the efficiency of the proposed PI control strategy for the regulation control of the left Neumann trace of the system.

The simulation results for a constant reference $r(t)=50$ and the time-varying distributed disturbance $d(t, x)=d_{0}(t) x$ with $d_{0}$ given by Fig. 2(d) are depicted in Fig. 2. First, for time $t<30 \mathrm{~s}$, the left Neumann trace successfully tracks the reference signal $r(t)=r_{e}=50$ under the constant


Fig. 1. Time evolution of the closed-loop system for a time-varying reference signal $r(t)$ and a constant distributed perturbation $d(t, x)=x$
distributed perturbation $d(t, x)=d_{e}(x)=x$. Then, around $t=30 \mathrm{~s}$, the magnitude of the perturbation increases from $d_{0}(t)=1$ to $d_{0}(t)=2$ with an overshoot around the value of 4.5. This time-varying perturbation induces a perturbation on the setpoint reference tracking of the the left Neumann trace over the time-range $[30,40] \mathrm{s}$. However, once the perturbation reaches its steady-state value $d(t, x)=d_{e}(x)=2 x$, the impact of off-equilibrium perturbations are eliminated providing $y_{x}(t, 0) \rightarrow r_{e}=50$. This is compliant with the obtained ISS estimates with fading memory (24-25) and (33) as the contribution of the variations of the perturbation around its nominal value $d_{e}$, i.e., configuration for which $\Delta d(t) \neq 0$, are eliminated as $t$ increases due to the action of the PI controller.

(d) Time varying component $d_{0}(t)$ of the distributed disturbance $d(t)=d_{0}(t) x$

Fig. 2. Time evolution of the closed-loop system for a constant reference signal $r(t)=50$ and a time-varying distributed disturbance $d(t, x)=d_{0}(t) x$

## VII. CONCLUSION

We have achieved the PI regulation control of the left Neumann trace of a one dimensional linear reaction-diffusion equation with delayed right Dirichlet boundary control. The proposed control design approach extends to PI control a recently proposed approach for the delay boundary feedback control of infinite-dimensional systems via spectral reduction. Specifically, a finite-dimensional model capturing the unstable modes of the open-loop system has been obtained by spectral decomposition. Based on the classical Artstein transformation (used to handle the delay in the control input) and the pole schifting theorem, a PI controller has been derived. Then, the stability of the full infinite-dimensional closed-loop system has been assessed by using an adequate Lyapunov function, yielding an exponential Input-to-State Stability (ISS) estimate with fading memory of the time-varying reference signal and the time-varying distributed disturbance. Finally, a similar exponential ISS estimate with fading memory has been derived for the setpoint regulation control of the reference signal by the left Neumann trace.

As a conclusion, we indicate here potential directions for the extension of the work reported in this paper.

First, it would be of interest to investigate whether the proposed PI control strategy can be used for the delay boundary regulation control of analogous PDEs. Good candidates in this direction are the linear Kuramoto-Sivashinsky equation [12] and the wave equation as studied in [10].

Second, the work presented here was devoted to the control of a 1-D reaction diffusion. A natural research direction relies in the investigation of whether the proposed PI boundary control strategy could be applied to a multi-dimensional reactiondiffusion equation. This is not a straightforward extension of the developments presented in this paper since, in particular, we instrumentally used the fact that, in 1-D, $\sum 1 /\left|\lambda_{j}\right|<+\infty$. Such a condition fails in multi-D.

Finally, we assumed in this work that the measure of the full state is available. Future developments may be concerned with the development of an observer and the study of the stability of the resulting closed-loop system.

## Appendix

## Technical lemma

The following lemma generalizes the result of [13, Chap. 12.4] to the case $D \neq 0$.

Lemma 7: Let $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$ be given matrices. The two following properties are equivalent:
(i) The pair $(A, B)$ satisfies the Kalman condition and $\operatorname{rank}\left(\begin{array}{ll}A & B \\ C & D\end{array}\right)=n+p$.
(ii) The pair $\left(\left(\begin{array}{ll}A & 0_{n \times p} \\ C & 0_{p \times p}\end{array}\right),\binom{B}{D}\right)$ satisfies the Kalman
condition.

In order to prove Lemma 7, we will use the following result.
Lemma 8: Let $M \in \mathbb{R}^{q \times q}$ and $N \in \mathbb{R}^{q \times r}$ be given matrices. Assume that $(M, N)$ satisfies the Kalman condition.

Then $\operatorname{Ran}\left(\begin{array}{ll}M & N\end{array}\right)=\mathbb{R}^{q}$, i.e., the matrix $\left(\begin{array}{ll}M & N\end{array}\right)$ is surjective.

Proof of Lemma 8: Noting that the surjectivity of $\left(\begin{array}{ll}M & N\end{array}\right)$ is equivalent to the condition $\operatorname{ker} M^{\top} \cap \operatorname{ker} N^{\top}=\{0\}$, let $\psi \in$ $\mathbb{R}^{n}$ be such that $\psi^{\top}(M \quad N)=0$. We have then $\psi^{\top} N=0$ and $\psi^{\top} M=0$, hence $\psi^{\top} M^{k}=0$ for every $k \in \mathbb{N}^{*}$ and thus $\psi M^{k} N=0$ for every $k \in \mathbb{N}$. Since $(M, N)$ satisfies the Kalman condition, we infer that $\psi=0$.

Proof of Lemma 7: $\left((i) \Rightarrow\right.$ (ii)) Let us prove that, if $\psi_{1} \in$ $\mathbb{R}^{n}$ et $\psi_{2} \in \mathbb{R}^{p}$ are such that

$$
\left(\begin{array}{ll}
\psi_{1}^{\top} & \psi_{2}^{\top}
\end{array}\right)\left(\begin{array}{ccccc}
B & A B & A^{2} B & \cdots & A^{n+p-1} B \\
D & C B & C A B & \cdots & C A^{n+p-2} B
\end{array}\right)=0
$$

then $\psi_{1}=\psi_{2}=0$. Indeed, we have then $\psi_{1}^{\top} B+\psi_{2}^{\top} D=0$ and $\left(\psi_{1}^{\top} A+\psi_{2}^{\top} C\right) A^{i} B=0$ for all $0 \leqslant i \leqslant n+p-2$. Noting that $n+p-2 \geqslant n-1$ and since $(A, B)$ satisfies the Kalman condition, we obtain that $\psi_{1}^{\top} A+\psi_{2}^{\top} C=0$. Consequently

$$
\left(\begin{array}{ll}
\psi_{1}^{\top} & \psi_{2}^{\top}
\end{array}\right)\left(\begin{array}{ll}
A & B \\
C & D
\end{array}\right)=0
$$

and hence $\psi_{1}=\psi_{2}=0$ since $\left(\begin{array}{ll}A & B \\ C & D\end{array}\right)$ is surjective.
$((i i) \Rightarrow(i))$ Let us prove that if $\psi \in \mathbb{R}^{n}$ is such that $\psi^{\top}\left(\begin{array}{llll}B & A B & \cdots & \left.A^{n-1} B\right)=0 \text { then } \psi=0 \text {. Indeed, we }\end{array}\right.$ first infer from the Hamilton-Cayley theorem that $\psi^{\top} A^{k} B=0$ for every $k \in \mathbb{N}$, and then

$$
\left(\begin{array}{ll}
\psi^{\top} & 0
\end{array}\right)\left(\begin{array}{ccccc}
B & A B & A^{2} B & \cdots & A^{n+p-1} B \\
D & C B & C A B & \cdots & C A^{n+p-2} B
\end{array}\right)=0
$$

Since the pair $\left(\left(\begin{array}{ll}A & 0 \\ C & 0\end{array}\right),\binom{B}{D}\right)$ satisfies the Kalman condition, we obtain $\psi=0$. This shows that $(A, B)$ satisfies the Kalman condition. Besides, by Lemma 8, since $\left(\left(\begin{array}{ll}A & 0 \\ C & 0\end{array}\right),\binom{B}{D}\right)$ satisfies the Kalman condition, $\left(\begin{array}{lll}A & 0 & B \\ C & 0 & D\end{array}\right)$ is surjective. The lemma is proved.

## References

[1] Z. Artstein, "Linear systems with delayed controls: a reduction," IEEE Transactions on Automatic Control, vol. 27, no. 4, pp. 869-879, 1982.
[2] K. J. Åström and T. Hägglund, PID controllers: theory, design, and tuning. Instrument society of America Research Triangle Park, NC, 1995, vol. 2.
[3] K. J. Astrom and R. M. Murray, "Feedback systems," in An introduction for Scientists and Engineers. Princeton Unvirsity Press, 2008.
[4] M. Barreau, F. Gouaisbaut, and A. Seuret, "Practical stabilization of a drilling pipe under friction with a PI-controller," arXiv preprint arXiv:1904.10658, 2019.
[5] G. Bastin, J.-M. Coron, and S. O. Tamasoiu, "Stability of linear density-flow hyperbolic systems under PI boundary control," Automatica, vol. 53, pp. 37-42, 2015.
[6] D. Bresch-Pietri, C. Prieur, and E. Trélat, "New formulation of predictors for finite-dimensional linear control systems with input delay," Systems \& Control Letters, vol. 113, pp. 9-16, 2018.
[7] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations. Springer Science \& Business Media, 2010.
[8] J.-M. Coron and A. Hayat, "PI controllers for 1-D nonlinear transport equation," IEEE Transactions on Automatic Control, 2019, in press.
[9] J.-M. Coron and E. Trélat, "Global steady-state controllability of onedimensional semilinear heat equations," SIAM Journal on Control and Optimization, vol. 43, no. 2, pp. 549-569, 2004.
[10] _-, "Global steady-state stabilization and controllability of 1D semilinear wave equations," Communications in Contemporary Mathematics, vol. 8, no. 04, pp. 535-567, 2006.
[11] V. Dos Santos, G. Bastin, J.-M. Coron, and B. d'Andréa Novel, "Boundary control with integral action for hyperbolic systems of conservation laws: Stability and experiments," Automatica, vol. 44, no. 5, pp. 13101318, 2008.
[12] P. Guzmán, S. Marx, and E. Cerpa, "Stabilization of the linear Kuramoto-Sivashinsky equation with a delayed boundary control," 2019.
[13] H. K. Khalil and J. W. Grizzle, Nonlinear systems. Prentice Hall Upper Saddle River, NJ, 2002, vol. 3.
[14] M. Krstic, "Control of an unstable reaction-diffusion PDE with long input delay," Systems \& Control Letters, vol. 58, no. 10-11, pp. 773782, 2009.
[15] P.-O. Lamare and N. Bekiaris-Liberis, "Control of $2 \times 2$ linear hyperbolic systems: Backstepping-based trajectory generation and PI-based tracking," Systems \& Control Letters, vol. 86, pp. 24-33, 2015.
[16] H. Lhachemi and C. Prieur, "Feedback stabilization of a class of diagonal infinite-dimensional systems with delay boundary control," arXiv preprint arXiv:1902.05086, 2019.
[17] H. Lhachemi, C. Prieur, and R. Shorten, "An LMI condition for the robustness of constant-delay linear predictor feedback with respect to uncertain time-varying input delays," Automatica, vol. 109, no. 108551, 2019.
[18] H. Lhachemi, R. Shorten, and C. Prieur, "Control law realification for the feedback stabilization of a class of diagonal infinite-dimensional systems with delay boundary control," IEEE Control Systems Letters, vol. 3, no. 4, pp. 930-935, 2019.
[19] S. Pohjolainen, "Robust multivariable PI-controller for infinite dimensional systems," IEEE Transactions on Automatic Control, vol. 27, no. 1, pp. 17-30, 1982.
[20] ——, "Robust controller for systems with exponentially stable strongly continuous semigroups," Journal of mathematical analysis and applications, vol. 111, no. 2, pp. 622-636, 1985.
[21] C. Prieur and E. Trélat, "Feedback stabilization of a 1D linear reactiondiffusion equation with delay boundary control," IEEE Transactions on Automatic Control, vol. 64, no. 4, pp. 1415-1425, 2019.
[22] J.-P. Richard, "Time-delay systems: an overview of some recent advances and open problems," Automatica, vol. 39, no. 10, pp. 1667-1694, 2003.
[23] D. L. Russell, "Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions," SIAM Review, vol. 20, no. 4, pp. 639-739, 1978.
[24] M. Schmidt and E. Trélat, "Controllability of Couette flows," Commun. Pure Appl. Anal., vol. 5, no. 1, pp. 201-211, 2006.
[25] E. D. Sontag, "Smooth stabilization implies coprime factorization," IEEE Trans. Autom. Control, vol. 34, no. 4, pp. 435-443, Apr. 1989.
[26] A. Terrand-Jeanne, V. Andrieu, V. D. S. Martins, and C.-Z. Xu, "Lyapunov functionals for output regulation of exponentially stable semigroups via integral action and application to hyperbolic systems," in 2018 IEEE Conference on Decision and Control (CDC), Miami Beach, FL, USA, 2018, pp. 4631-4636.
[27] ——, "Adding integral action for open-loop exponentially stable semigroups and application to boundary control of PDE systems," arXiv preprint arXiv:1901.02208, 2019.
[28] A. Terrand-Jeanne, V. D. S. Martins, and V. Andrieu, "Regulation of the downside angular velocity of a drilling string with a PI controller," in 2018 European Control Conference (ECC), Limassol, Cyprus, 2018, pp. 2647-2652.
[29] N.-T. Trinh, V. Andrieu, and C.-Z. Xu, "Design of integral controllers for nonlinear systems governed by scalar hyperbolic partial differential equations," IEEE Transactions on Automatic Control, vol. 62, no. 9, pp. 4527-4536, 2017.
[30] C.-Z. Xu and H. Jerbi, "A robust PI-controller for infinite-dimensional systems," International Journal of Control, vol. 61, no. 1, pp. 33-45, 1995.
[31] C.-Z. Xu and G. Sallet, "Multivariable boundary PI control and regulation of a fluid flow system," Mathematical Control and Related Fields, vol. 4, no. 4, pp. 501-520, 2014.


[^0]:    This publication has emanated from research supported in part by a research grant from Science Foundation Ireland (SFI) under grant number 16/RC/3872 and is co-funded under the European Regional Development Fund and by I-Form industry partners.

    Hugo Lhachemi is with the School of Electrical and Electronic Engineering, University College Dublin, Dublin, Ireland (e-mail: hugo.lhachemi@ucd.ie). Christophe Prieur is with Université Grenoble Alpes, CNRS, GrenobleINP, GIPSA-lab, F-38000, Grenoble, France (e-mail: christophe.prieur@gipsalab.fr).

    Emmanuel Trélat is with Sorbonne Université, CNRS, Université de Paris, Inria, Laboratoire Jacques-Louis Lions (LJLL), F-75005 Paris, France (email: emmanuel.trelat@sorbonne-universite.fr).

[^1]:    ${ }^{1}$ This property will be ensured by the construction carried out in the sequel.

