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L^p theory for the interaction between the incompressible Navier-Stokes system and a damped beam

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Abstract

We consider a viscous incompressible fluid governed by the Navier-Stokes system written in a domain where a part of the boundary is moving as a damped beam under the action of the fluid. We prove the existence and uniqueness of global strong solutions for the corresponding fluid-structure interaction system in an L^p - L^q setting. The main point in the proof consists in the study of a linear parabolic system coupling the non stationary Stokes system and a damped beam. We show that this linear system possesses the maximal regularity property by proving the \mathcal{R} -sectoriality of the corresponding operator.

Key words. Incompressible Navier-Stokes System, Fluid-structure interaction, Strong solutions, Maximal L^p regularity.

AMS subject classifications. 35Q35, 76D03, 76D05, 74F10.

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1 Introduction

In this work, we study the interaction between a viscous incompressible Newtonian fluid and an elastic structure located on one part of the fluid domain boundary. More precisely, if there is no displacement of the structure, the fluid occupies a smooth bounded domain \mathcal{F} such that its boundary $\partial\mathcal{F}$ contains a flat part Γ_S . We can assume $\Gamma_S = (0, 1) \times \{0\}$ and we set $\Gamma_0 := \partial\mathcal{F} \setminus \overline{\Gamma_S}$.

On Γ_S , we assume that there is a beam that can deform through the force exerted by the fluid onto the structure, whereas Γ_0 remains unchanged. We denote by $\eta(t, s)$ the displacement of the beam at the position $s \in (0, 1)$ and at time t . Thus Γ_S transforms into

$$\Gamma_S(\eta(t)) := \{(s, \eta(t, s)) ; s \in (0, 1)\} \quad (1.1)$$

and the fluid domain $\mathcal{F}(\eta(t))$ is the interior of $\Gamma_0 \cup \overline{\Gamma_S(\eta(t))}$. We assume that

$$\eta(t, 0) = \eta(t, 1) = \partial_s \eta(t, 0) = \partial_s \eta(t, 1) = 0, \quad (1.2)$$

and that

$$\Gamma_0 \cap \Gamma_S(\eta(t)) = \emptyset, \quad (1.3)$$

so that $\Gamma_0 \cup \overline{\Gamma_S(\eta(t))}$ is a closed, simple and regular curve (if $\eta(t)$ is regular).

The fluid-structure system that we consider reads as follows

$$\begin{cases} \partial_t \tilde{v} + (\tilde{v} \cdot \nabla) \tilde{v} - \operatorname{div} \mathbb{T}(\tilde{v}, \tilde{\pi}) = 0, & t > 0, x \in \mathcal{F}(\eta(t)), \\ \operatorname{div} \tilde{v} = 0 & t > 0, x \in \mathcal{F}(\eta(t)), \\ \tilde{v}(t, s, \eta(t, s)) = \partial_t \eta(t, s) e_2 & t > 0, s \in (0, 1), \\ \tilde{v} = 0 & t > 0, x \in \Gamma_0, \\ \partial_{tt} \eta + \Delta_s^2 \eta - \Delta_s \partial_t \eta = \mathbb{H}(\tilde{v}, \tilde{\pi}, \eta) & t > 0, s \in (0, 1), \\ \eta = \partial_s \eta = 0 & t > 0, s \in \{0, 1\}, \end{cases} \quad (1.4)$$

where $e_2 = (0, 1)^\top$ is the unit normal to Γ_S , exterior to \mathcal{F} . The fluid stress tensor $\mathbb{T}(\tilde{v}, \tilde{\pi})$ is given by

$$\mathbb{T}(\tilde{v}, \tilde{\pi}) = 2\nu D(\tilde{v}) - \tilde{\pi} I_2, \quad D(\tilde{v}) = \frac{1}{2} (\nabla \tilde{v} + \nabla \tilde{v}^\top). \quad (1.5)$$

We have set

$$\Delta_s = \partial_{ss} \quad \text{and} \quad \Delta_s^2 = \partial_{ssss}.$$

The function \mathbb{H} is defined by

$$\mathbb{H}(\tilde{v}, \tilde{\pi}, \eta) = -\sqrt{1 + (\partial_s \eta)^2} (\mathbb{T}(\tilde{v}, \tilde{\pi}) \tilde{n})|_{\Gamma_S(\eta(t))} \cdot e_2, \quad (1.6)$$

where

$$\tilde{n} = \frac{1}{\sqrt{1 + (\partial_s \eta)^2}} (-\partial_s \eta, 1)^\top,$$

is the unit normal to $\Gamma_S(\eta(t))$ outward $\mathcal{F}(\eta(t))$. The above system is completed by the following initial data

$$\eta(0, \cdot) = \eta_1^0, \quad \partial_t \eta(0, \cdot) = \eta_2^0, \quad \text{in } (0, 1), \quad \tilde{v}(0) = \tilde{v}^0 \quad \text{in } \mathcal{F}(\eta_1^0). \quad (1.7)$$

Due to the divergence free condition, the solution of the system (1.4)-(1.7) satisfies

$$0 = \int_{\mathcal{F}(\eta(t))} \operatorname{div} \tilde{v} \, dx = \int_{\Gamma_S(\eta(t))} \tilde{v} \cdot \tilde{n} \, d\Gamma = \frac{d}{dt} \int_0^1 \eta \, ds.$$

Consequently, we look for solutions with constant mean value for the displacement η . For simplicity, we assume that

$$\int_0^1 \eta(t, s) \, ds = 0, \text{ for all } t \geq 0, \quad (1.8)$$

and we thus consider the space

$$L_m^q(0, 1) = \left\{ f \in L^q(0, 1) ; \int_0^1 f \, ds = 0 \right\}, \quad (1.9)$$

and the projection $P_m : L^q(0, 1) \rightarrow L_m^q(0, 1)$ defined by

$$P_m f = f - \int_0^1 f \, ds \quad (f \in L^q(0, 1)). \quad (1.10)$$

We project the beam equation in (1.4) on $L_m^q(0, 1)$ and on $L_m^q(0, 1)^\perp$: this gives

$$\partial_{tt}\eta + P_m \Delta_s^2 \eta - \Delta_s \partial_t \eta = P_m (\mathbb{H}(\tilde{v}, \tilde{\pi}, \eta)) \quad t > 0, s \in (0, 1), \quad (1.11)$$

and

$$\int_0^1 \tilde{\pi}(t, s, \eta(t, s)) \, ds = \int_0^1 \Delta_s^2 \eta + 2\nu \int_0^1 \sqrt{1 + (\partial_s \eta)^2} [(D\tilde{v})\tilde{n}] (t, s, \eta(t, s)) \cdot e_2 \, ds. \quad (1.12)$$

This means that, at the contrary to the Navier-Stokes system without structure, the pressure is not determined up to a constant. In what follows, we only consider the first equation and solve the system up to constant for the pressure, and equation (1.12) will fix the constant at the end. We thus consider the following system

$$\begin{cases} \partial_t \tilde{v} + (\tilde{v} \cdot \nabla) \tilde{v} - \operatorname{div} \mathbb{T}(\tilde{v}, \tilde{\pi}) = 0 & t > 0, x \in \mathcal{F}(\eta(t)), \\ \operatorname{div} \tilde{v} = 0 & t > 0, x \in \mathcal{F}(\eta(t)), \\ \tilde{v}(t, s, \eta(t, s)) = \partial_t \eta(t, s) e_2 & t > 0, s \in (0, 1), \\ \tilde{v} = 0 & t > 0, x \in \Gamma_0, \\ \partial_{tt}\eta + P_m \Delta_s^2 \eta - \Delta_s \partial_t \eta = P_m \mathbb{H}(\tilde{v}, \tilde{\pi}, \eta) & t > 0, s \in (0, 1), \\ \eta = \partial_s \eta = 0 & t > 0, s \in \{0, 1\}, \\ \eta(0, \cdot) = \eta_1^0, \quad \partial_t \eta(0, \cdot) = \eta_2^0, \text{ in } (0, 1), \quad \tilde{v}(0) = \tilde{v}^0 \text{ in } \mathcal{F}(\eta_1^0). \end{cases} \quad (1.13)$$

To state our main result, we need to introduce some notations. Firstly $W^{s,q}(\Omega)$, with $s \geq 0$ and $q \geq 1$, denotes the usual Sobolev space. Let $k, m \in \mathbb{N}$, $k < m$. For $1 \leq p < \infty$, $1 \leq q < \infty$, we consider the standard definition of the Besov spaces by real interpolation of Sobolev spaces

$$B_{q,p}^s(\mathcal{F}) = \left(W^{k,q}(\mathcal{F}), W^{m,q}(\mathcal{F}) \right)_{\theta,p} \text{ where } s = (1 - \theta)k + \theta m, \quad \theta \in (0, 1).$$

We refer to [1] and [36] for a detailed presentation of the Besov spaces. We also need to introduce function spaces for the fluid velocity and pressure depending on the displacement η of the structure. Let $1 < p, q < \infty$ and $\eta \in L^p(0, \infty; W^{4,q}(0, 1)) \cap W^{2,p}(0, \infty; L^q(0, 1))$ satisfying (1.2) and (1.3). We show in Section 2 that there exists a mapping $X = X_\eta$ such that $X(t, \cdot)$ is a C^1 -diffeomorphism from \mathcal{F} onto $\mathcal{F}(\eta(t))$ and such that $X \in L^p(0, \infty; W^{2,q}(\mathcal{F})) \cap W^{2,p}(0, \infty; L^q(\mathcal{F}))$. Then for $T \in (0, \infty]$, we define

$$\begin{aligned} L^p(0, T; L^q(\mathcal{F}(\eta(\cdot)))) &:= \{v \circ X^{-1} ; v \in L^p(0, T; L^q(\mathcal{F}))\}, \\ L^p(0, T; W^{2,q}(\mathcal{F}(\eta(\cdot)))) &:= \{v \circ X^{-1} ; v \in L^p(0, T; W^{2,q}(\mathcal{F}))\}, \\ W^{1,p}(0, T; L^q(\mathcal{F}(\eta(\cdot)))) &:= \{v \circ X^{-1} ; v \in W^{1,p}(0, T; L^q(\mathcal{F}))\}, \\ C([0, T]; W^{1,q}(\mathcal{F}(\eta(\cdot)))) &:= \{v \circ X^{-1} ; v \in C([0, T]; W^{1,q}(\mathcal{F}))\}, \\ C([0, T]; B_{q,p}^{2(1-1/p)}(\mathcal{F}(\eta(\cdot)))) &:= \{v \circ X^{-1} ; v \in C([0, T]; B_{q,p}^{2(1-1/p)}(\mathcal{F}))\}, \end{aligned}$$

where we have set $(v \circ X^{-1})(t, x) := v(t, (X(t, \cdot))^{-1}(x))$ for simplicity.

Finally, let us give the conditions we need on the initial conditions for the system (1.13): we assume

$$\eta_1^0 \in B_{q,p}^{2(2-1/p)}(0, 1), \quad \eta_2^0 \in B_{q,p}^{2(1-1/p)}(0, 1), \quad \tilde{v}^0 \in B_{q,p}^{2(1-1/p)}(\mathcal{F}(\eta_1^0)) \quad (1.14)$$

with the compatibility conditions

$$\eta_1^0 = \partial_s \eta_1^0 = 0 \quad \text{at } \{0, 1\}, \quad \Gamma_0 \cap \Gamma_S(\eta_1^0) = \emptyset, \quad \int_0^1 \eta_1^0 ds = 0, \quad \int_0^1 \eta_2^0 ds = 0, \quad \operatorname{div} \tilde{v}^0 = 0 \quad \text{in } \mathcal{F}(\eta_1^0), \quad (1.15)$$

and

$$\begin{cases} \tilde{v}^0(s, \eta_1^0(s)) \cdot \tilde{n}^0 = \eta_2^0(s) e_2 \cdot \tilde{n}^0 & s \in (0, 1), \quad \tilde{v}^0 \cdot \tilde{n}^0 = 0 \quad \text{on } \Gamma_0 & \text{if } \frac{1}{p} + \frac{1}{2q} > 1, \\ \tilde{v}^0(s, \eta_1^0(s)) = \eta_2^0(s) e_2 & s \in (0, 1), \quad \tilde{v}^0 = 0 \quad \text{on } \Gamma_0, \quad \eta_2^0 = 0 \quad \text{at } \{0, 1\} & \text{if } \frac{1}{p} + \frac{1}{2q} < 1. \end{cases} \quad (1.16)$$

Here \tilde{n}^0 is the unit normal to $\Gamma_S(\eta_1^0)$ outward $\mathcal{F}(\eta_1^0)$.

We are now in a position to state our main results. The first one is the local in time existence and uniqueness of strong solutions for (1.13).

Theorem 1.1. *Let $p, q \in (1, \infty)$ such that*

$$\frac{1}{p} + \frac{1}{2q} \neq 1 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} < \frac{3}{2}. \quad (1.17)$$

Let us assume that $\eta_1^0 = 0$ and (η_2^0, \tilde{v}^0) satisfies (1.14), (1.15), (1.16). Then there exists $T > 0$, depending only on (η_2^0, \tilde{v}^0) , such that the system (1.13) admits a unique strong solution $(\tilde{v}, \tilde{\pi}, \eta)$ in the class of functions satisfying

$$\begin{aligned} \tilde{v} &\in L^p(0, T; W^{2,q}(\mathcal{F}(\eta(\cdot)))) \cap W^{1,p}(0, T; L^q(\mathcal{F}(\eta(\cdot)))) \cap L^\infty(0, T; B_{q,p}^{2(1-1/p)}(\mathcal{F}(\eta(\cdot))))), \\ \tilde{\pi} &\in L^p(0, T; W_m^{1,q}(\mathcal{F}(\eta(\cdot))))), \\ \eta &\in L^p(0, \infty; W^{4,q}(0, 1)) \cap W^{2,p}(0, \infty; L^q(0, 1)). \end{aligned}$$

Moreover, $\Gamma_0 \cap \Gamma_S(\eta(t)) = \emptyset$ for all $t \in [0, T]$.

Our second main result asserts the global existence and uniqueness of strong solution for (1.13) under a smallness condition on the initial data.

Theorem 1.2. *Let $p, q \in (1, \infty)$ such that*

$$\frac{1}{p} + \frac{1}{2q} \neq 1 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} \leq \frac{3}{2}. \quad (1.18)$$

Then there exists $\beta_0 > 0$ such that, for all $\beta \in (0, \beta_0)$ there exist ε_0 and $C > 0$, such that for any $(\eta_1^0, \eta_2^0, \tilde{v}^0)$ satisfying (1.14), (1.15), (1.16) and

$$\|\tilde{v}^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F}(\eta_1^0))} + \|\eta_1^0\|_{B_{q,p}^{2(2-1/p)}(0,1)} + \|\eta_2^0\|_{B_{q,p}^{2(1-1/p)}(0,1)} < \varepsilon_0, \quad (1.19)$$

the system (1.13) admits a unique strong solution $(\tilde{v}, \tilde{\pi}, \eta)$ in the class of functions satisfying

$$\begin{aligned} & \|e^{\beta(\cdot)}\tilde{v}\|_{L^p(0,\infty;W^{2,q}(\mathcal{F}(\eta(\cdot))))} + \|e^{\beta(\cdot)}\partial_t\tilde{v}\|_{L^p(0,\infty;L^q(\mathcal{F}(\eta(\cdot))))} + \|e^{\beta(\cdot)}\tilde{v}\|_{L^\infty(0,\infty;B_{q,p}^{2(1-1/p)}(\mathcal{F}(\eta(\cdot))))} \\ & + \|e^{\beta(\cdot)}\tilde{\pi}\|_{L^p(0,\infty;W_m^{1,q}(\mathcal{F}(\eta(\cdot))))} + \|e^{\beta(\cdot)}\eta\|_{L^p(0,\infty;W^{4,q}(0,1))} + \|e^{\beta(\cdot)}\eta\|_{W^{2,p}(0,\infty;L^q(0,1))} \\ & \leq C \left(\|\tilde{v}^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F}(\eta_1^0))} + \|\eta_1^0\|_{B_{q,p}^{2(2-1/p)}(0,1)} + \|\eta_2^0\|_{B_{q,p}^{2(1-1/p)}(0,1)} \right). \end{aligned} \quad (1.20)$$

Moreover, $\Gamma_0 \cap \Gamma_S(\eta(t)) = \emptyset$ for all $t \in [0, \infty)$.

In the above statement, we have used a similar notation as in (1.9):

$$L_m^q(\mathcal{F}) := \left\{ f \in L^q(\mathcal{F}) ; \int_{\mathcal{F}} f = 0 \, dx \right\}, \quad W_m^{s,q}(\mathcal{F}) := W^{s,q}(\mathcal{F}) \cap L_m^q(\mathcal{F}).$$

We also set

$$W_m^{s,q}(0,1) = W^{s,q}(0,1) \cap L_m^q(0,1).$$

We denote by $W_0^{s,q}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{s,q}(\Omega)$ and we set

$$W_{0,m}^{s,q}(\Omega) = W_0^{s,q}(\Omega) \cap L_m^q(\Omega).$$

Finally, we also need the following notation in what follows:

$$\begin{aligned} W_{p,q}^{1,2}((0, \infty) \times \mathcal{F}) &= L^p(0, \infty; W^{2,q}(\mathcal{F})) \cap W^{1,p}(0, \infty; L^q(\mathcal{F})) \\ W_{p,q}^{2,4}((0, \infty) \times (0, 1)) &= L^p(0, \infty; W^{4,q}(0, 1)) \cap W^{2,p}(0, \infty; L^q(0, 1)) \\ W_{p,q}^{1,2}((0, \infty) \times (0, 1)) &= L^p(0, \infty; W^{2,q}(0, 1)) \cap W^{1,p}(0, \infty; L^q(0, 1)). \end{aligned}$$

Let us give some remarks on Theorem 1.1 and Theorem 1.2. First let us point out that the system (1.13) has already been studied by several authors: existence of weak solutions ([6], [20], [30]), uniqueness of weak solutions ([19]), existence of strong solutions ([5], [25], [27]), feedback stabilization ([32], [4]), global existence of strong solutions and study of the contacts ([16]). Some works consider also the case of a beam/plate without damping (that is without the term $-\Delta_s \partial_t \eta$): [15], [17]. We refer, for instance, to [18] and references therein for a concise description of recent progress in this field. It is important to notice that all the above works correspond to a ‘‘Hilbert’’ framework whereas our results are done in a ‘‘ L^p - L^q ’’ framework.

For this approach, several recent results have been obtained for fluid systems, with or without structure. For instance, one can quote [14] (viscous incompressible fluid), [11], (viscous compressible fluid), [22], [21] (viscous compressible fluid with rigid bodies), [13], [28] (incompressible viscous fluid and rigid bodies). Here we consider an incompressible viscous fluid coupled with a structure satisfying an infinite-dimensional system and we thus need to go beyond the theory developed for instance in [28].

Our approach to prove Theorem 1.1 and Theorem 1.2 is quite classical. Since the fluid domain $\mathcal{F}(\eta(t))$ depends on the structure displacement η , we first reformulate the problem in a fixed domain. This is achieved by “geometric” change of variables. Next we associate the original nonlinear problem to a linear one. The linear system preserves the fluid-structure coupling. A crucial step here is to establish the L^p - L^q regularity property in the infinite time horizon. This is done by showing the associate linear operator \mathcal{R} -sectorial and generates an exponentially stable semigroup. We then use the Banach fixed point theorem to prove existence and uniqueness results.

Let us remark that this work could also be done in the corresponding 3D/2D model, that is \mathcal{F} a regular bounded domain in \mathbb{R}^3 such that $\partial\mathcal{F}$ contains a flat part $\Gamma_S = \omega \times \{0\}$, where ω is a smooth bounded domain of \mathbb{R}^2 . In that case, we would obtain the same result as in Theorem 1.1 and in Theorem 1.2 but with the following condition on p, q (instead of (1.18)):

$$\frac{1}{p} + \frac{1}{2q} \neq \frac{1}{2} \quad \text{and} \quad \frac{1}{p} + \frac{3}{2q} \leq \frac{3}{2}.$$

or (instead of (1.17))

$$\frac{1}{p} + \frac{1}{2q} \neq \frac{1}{2} \quad \text{and} \quad \frac{1}{p} + \frac{3}{2q} < \frac{3}{2}.$$

The plan of the paper is as follows. In the next section, we use a change of variables to rewrite the governing equations in a cylindrical domain and we also restate our result after change of variables. Then, in Section 3, we recall several important results about maximal L^p regularity for Cauchy problems and in particular how to use the \mathcal{R} -sectoriality property. We use these results to study in Section 4 the linearized homogeneous system. We complete the study of the linearized system in Section 5. Finally in Section 6 and in Section 7, we estimate the nonlinear terms which allows us to prove the main results with a fixed point argument.

2 Change of Variables

In order to prove Theorem 1.2, we first rewrite the system (1.13) in the cylindrical domain $(0, \infty) \times \mathcal{F}$ by constructing an invertible mapping $X(t, \cdot)$ from the reference configuration \mathcal{F} onto $\mathcal{F}(\eta(t))$. We follow the approach of [4]: we consider the set

$$\mathcal{V}_\alpha = (0, 1) \times (-\alpha, 0) \tag{2.1}$$

with $\alpha > 0$ small enough so that $\mathcal{V}_\alpha \subset \mathcal{F}$. Notice that, $\partial\mathcal{V}_\alpha \cap \partial\mathcal{F} = \Gamma_S$. We consider $\psi \in C_c^\infty(\mathbb{R})$ such that

$$\psi = 1 \text{ in } (-\alpha/2, \alpha/2), \quad \psi = 0 \text{ in } \mathbb{R} \setminus (-\alpha, \alpha), \quad 0 \leq \psi \leq 1. \tag{2.2}$$

Assume $\eta \in W_0^{2,q}(0, 1)$. We can extend η by 0 in $\mathbb{R} \setminus (0, 1)$ so that $\eta \in W^{2,q}(\mathbb{R})$ and we can define X_η by

$$X_\eta(y) = y + \psi(y_2)\eta(y_1)e_2. \tag{2.3}$$

Assume that

$$\|\eta\|_{L^\infty(0,1)} \leq c_0 := \frac{1}{2\|\psi'\|_{L^\infty(\mathbb{R})}}. \quad (2.4)$$

In particular, X_η is a C^1 -diffeomorphism from \mathcal{F} onto $\mathcal{F}(\eta)$ with $X_\eta(\Gamma_S) = \Gamma_S(\eta)$. This leads us to set

$$X(t, y) := X_{\eta(t)}(y) \quad (2.5)$$

so that if

$$\|\eta\|_{L^\infty((0,\infty)\times(0,1))} \leq c_0, \quad (2.6)$$

then X is a C^1 -diffeomorphism from \mathcal{F} onto $\mathcal{F}(\eta(t))$. For each $t \geq 0$, we denote by $Y(t, \cdot) = X(t, \cdot)^{-1}$, the inverse of $X(t, \cdot)$.

Note that we have the following properties: for all $t \in (0, \infty)$, $y \in \mathcal{V}_{\alpha/2}$,

$$\det \nabla X(t, y) = 1, \quad \text{Cof}(\nabla X)(t, y) = \begin{bmatrix} 1 & -\partial_s \eta(t, y_1) \\ 0 & 1 \end{bmatrix}. \quad (2.7)$$

Let us now assume $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{2q} \neq 1$. Using that (see [1, Theorem 7.34]),

$$W_{p,q}^{2,4}((0, \infty) \times (0, 1)) \hookrightarrow C_b([0, \infty); B_{q,p}^{2(2-1/p)}(0, 1)) \hookrightarrow C_b([0, \infty); C^1([0, 1])),$$

we deduce the existence of \tilde{c}_0 such that if

$$\eta \in W_{p,q}^{2,4}((0, \infty) \times (0, 1)), \quad \|\eta\|_{W_{p,q}^{2,4}((0,\infty)\times(0,1))} \leq \tilde{c}_0, \quad (2.8)$$

then X is well-defined and $X \in C_b([0, \infty); C^1(\overline{\mathcal{F}}))$.

We consider the following change of unknowns

$$v(t, y) = \tilde{v}(t, X(t, y)), \quad \pi(t, y) = \tilde{\pi}(t, X(t, y)), \quad (t, y) \in (0, \infty) \times \mathcal{F}. \quad (2.9)$$

The system (1.13) can be rewritten in the form

$$\left\{ \begin{array}{ll} \partial_t v - \text{div } \mathbb{T}(v, \pi) = F(v, \pi, \eta) & t > 0, y \in \mathcal{F}, \\ \text{div } v = \text{div } G(v, \pi, \eta) & t > 0, y \in \mathcal{F}, \\ v(t, s, 0) = \partial_t \eta(t, s) e_2 & t > 0, s \in (0, 1), \\ v = 0 & t > 0, y \in \Gamma_0, \\ \partial_{tt} \eta + P_m(\Delta_s^2 \eta) - \Delta_s \partial_t \eta & \\ \quad = -P_m(\mathbb{T}(v, \pi)|_{\Gamma_S} e_2 \cdot e_2) + P_m(H(v, \pi, \eta)) & t > 0, s \in (0, 1), \\ \eta = \partial_s \eta = 0 & t > 0, s \in \{0, 1\}, \\ \eta(0, \cdot) = \eta_1^0, \quad \partial_t \eta(0, \cdot) = \eta_2^0, \quad \text{in } (0, 1), \quad v(0) = v^0 \text{ in } \mathcal{F}, & \end{array} \right. \quad (2.10)$$

where

$$v^0 := \tilde{v}^0(X(0, \cdot)). \quad (2.11)$$

Using the notation

$$Z(t, y) = (Z_{ij})_{i \leq i, j \leq 3} = [\nabla X]^{-1}(t, y), \quad t \geq 0, y \in \mathcal{F}, \quad (2.12)$$

the nonlinear terms in (2.10) can be defined as

$$F_i(v, \pi, \eta) = -(v - \partial_t X) \cdot Z^\top \nabla v_i + \nu \sum_{j,k,l} \frac{\partial^2 v_i}{\partial y_l \partial y_k} (Z_{kj} Z_{lj} - \delta_{kj} \delta_{lj}) \\ + \nu \sum_{j,k,l} \frac{\partial v_i}{\partial y_k} \frac{\partial}{\partial y_l} (Z_{kj}) Z_{lj} + ((I_2 - Z^\top) \nabla \pi)_i, \quad (2.13)$$

$$G(v, \pi, \eta) = (I_2 - \text{Cof}(\nabla X)^\top) v, \quad (2.14)$$

$$H(v, \pi, \eta) = \frac{\nu}{2} \left[(\partial_s \eta) \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) (\cdot, 0) - (\partial_s \eta)^2 \frac{\partial v_2}{\partial y_2} (\cdot, 0) \right]. \quad (2.15)$$

In the above calculation we have used the fact that

$$\text{div } \tilde{v} = 0 \iff \text{div} \left(\text{Cof } \nabla X^\top v \right) = 0.$$

The hypotheses (1.14), (1.15), (1.16) on the initial conditions are transformed into the following conditions:

$$\eta_1^0 \in B_{q,p}^{2(2-1/p)}(0, 1), \quad \eta_2^0 \in B_{q,p}^{2(1-1/p)}(0, 1), \quad v^0 \in B_{q,p}^{2(1-1/p)}(\mathcal{F}), \quad (2.16)$$

$$\eta_1^0 = \partial_s \eta_1^0 = 0 \quad \text{at } \{0, 1\}, \quad \Gamma_0 \cap \Gamma_S(\eta_1^0) = \emptyset, \quad \int_0^1 \eta_1^0 ds = 0, \quad \int_0^1 \eta_2^0 ds = 0, \\ \text{div}(\text{Cof } \nabla X(0, \cdot)^\top v^0) = 0 \quad \text{in } \mathcal{F}, \quad (2.17)$$

$$\begin{cases} -\partial_s \eta_1^0(s) v_1^0(s, 0) + v_2^0(s, 0) = \eta_2^0(s) & s \in (0, 1), \quad v^0 \cdot n = 0 \quad \text{on } \Gamma_0 & \text{if } \frac{1}{p} + \frac{1}{2q} > 1, \\ v^0(s, 0) = \eta_2^0(s) e_2 & s \in (0, 1), \quad v^0 = 0 \quad \text{on } \Gamma_0, \quad \eta_2^0 = 0 \quad \text{at } \{0, 1\} & \text{if } \frac{1}{p} + \frac{1}{2q} < 1. \end{cases} \quad (2.18)$$

Here n is the unit normal to $\partial \mathcal{F}$ outward \mathcal{F} .

Using the above change of variables Theorem 1.1 and Theorem 1.2 can be rephrased as

Theorem 2.1. *Let $p, q \in (1, \infty)$ satisfying the condition (1.17). Let us assume that $\eta_1^0 = 0$ and (η_2^0, v^0) satisfies (2.16), (2.17), (2.18). Then there exists $T > 0$, depending only on (η_2^0, v^0) , such that the system (2.10) admits a unique strong solution (v, π, η) in the class of functions satisfying*

$$v \in L^p(0, T; W^{2,q}(\mathcal{F})) \cap W^{1,p}(0, T; L^q(\mathcal{F})) \cap L^\infty(0, T; B_{q,p}^{2(1-1/p)}(\mathcal{F})), \\ \pi \in L^p(0, T; W_m^{1,q}(\mathcal{F})), \\ \eta \in L^p(0, \infty; W^{4,q}(0, 1)) \cap W^{2,p}(0, \infty; L^q(0, 1)).$$

Moreover, $X(t, \cdot) : \mathcal{F} \rightarrow \mathcal{F}(\eta(t))$ is a C^1 -diffeomorphism for all $t \in [0, T]$.

Theorem 2.2. *Let $p, q \in (1, \infty)$ satisfying the condition (1.18). Then there exists $\beta_0 > 0$ such that, for all $\beta \in (0, \beta_0)$ there exist ε_0 and $C > 0$, such that for any $(\eta_1^0, \eta_2^0, v^0)$ satisfying (2.16), (2.17), (2.18) and*

$$\|\eta_1^0\|_{B_{q,p}^{2(2-1/p)}(0,1)} + \|\eta_2^0\|_{B_{q,p}^{2(1-1/p)}(0,1)} + \|v^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})} < \varepsilon_0, \quad (2.19)$$

the system (2.10) admits a unique strong solution (v, π, η) in the class of functions satisfying

$$\begin{aligned} & \|e^{\beta(\cdot)}v\|_{L^p(0,\infty;W^{2,q}(\mathcal{F}))} + \|e^{\beta(\cdot)}\partial_t v\|_{L^p(0,\infty;L^q(\mathcal{F}))} + \|e^{\beta(\cdot)}v\|_{L^\infty(0,\infty;B_{q,p}^{2(1-1/p)}(\mathcal{F}))} \\ & + \|e^{\beta(\cdot)}\pi\|_{L^p(0,\infty;W_m^{1,q}(\mathcal{F}))} + \|e^{\beta(\cdot)}\eta\|_{L^p(0,\infty;W^{4,q}(0,1))} + \|e^{\beta(\cdot)}\eta\|_{W^{2,p}(0,\infty;L^q(0,1))} \\ & \leq C \left(\|v^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})} + \|\eta_1^0\|_{B_{q,p}^{2(2-1/p)}(0,1)} + \|\eta_2^0\|_{B_{q,p}^{2(1-1/p)}(0,1)} \right). \end{aligned}$$

Moreover, $X(t, \cdot) : \mathcal{F} \rightarrow \mathcal{F}(\eta(t))$ is a C^1 -diffeomorphism for all $t \in [0, \infty)$.

3 Some Background on \mathcal{R} -sectorial Operators

In this section, we recall some important facts on \mathcal{R} -sectorial operators. This notion is associated with the property of \mathcal{R} -boundedness (\mathcal{R} for Randomized) for a family of operators. One can find the definition of \mathcal{R} -boundedness in [37].

For any $\beta \in (0, \pi)$, we write

$$\Sigma_\beta = \{\lambda \in \mathbb{C} \setminus \{0\} ; |\arg(\lambda)| < \beta\}.$$

We recall the following definition.

Definition 3.1 (sectorial and \mathcal{R} -sectorial operators). *Let A be a densely defined closed linear operator on a Banach space \mathcal{X} with domain $\mathcal{D}(A)$. We say that A is a (\mathcal{R})-sectorial operator of angle $\beta \in (0, \pi)$ if*

$$\Sigma_\beta \subset \rho(A)$$

and if the set

$$R_\beta = \{\lambda(\lambda - A)^{-1} ; \lambda \in \Sigma_\beta\}$$

is (\mathcal{R})-bounded in $\mathcal{L}(\mathcal{X})$.

We denote by $M_\beta(A)$ (respectively $\mathcal{R}_\beta(A)$) the bound (respectively the \mathcal{R} -bound) of R_β . One can replace in the above definitions R_β by the set

$$\widetilde{R}_\beta = \{A(\lambda - A)^{-1} ; \lambda \in \Sigma_\beta\}.$$

In that case, we denote the uniform bound and the \mathcal{R} -bound by $\widetilde{M}_\beta(A)$ and $\widetilde{\mathcal{R}}_\beta(A)$.

This notion of \mathcal{R} -sectorial operators is related to the maximal regularity of type L^p by the following result due to [37] (see also [7, p.45]).

Theorem 3.2. *Let \mathcal{X} be a UMD Banach space and A a densely defined, closed linear operator on \mathcal{X} . Then the following assertions are equivalent:*

1. For any $T \in (0, \infty]$ and for any $f \in L^p(0, T; \mathcal{X})$, the Cauchy problem

$$u' = Au + f \quad \text{in } (0, T), \quad u(0) = 0 \quad (3.1)$$

admits a unique solution u with $u', Au \in L^p(0, T; \mathcal{X})$ and there exists a constant $C > 0$ such that

$$\|u'\|_{L^p(0, T; \mathcal{X})} + \|Au\|_{L^p(0, T; \mathcal{X})} \leq C\|f\|_{L^p(0, T; \mathcal{X})}.$$

2. A is \mathcal{R} -sectorial of angle $> \frac{\pi}{2}$.

We recall that \mathcal{X} is a UMD Banach space if the Hilbert transform is bounded in $L^p(\mathbb{R}; \mathcal{X})$ for $p \in (1, \infty)$. In particular, the closed subspaces of $L^q(\Omega)$ for $q \in (1, \infty)$ are UMD Banach spaces. We refer the reader to [2, pp.141–147] for more information on UMD spaces.

Combining the above theorem with [9, Theorem 2.4] and [35, Theorem 1.8.2], we can deduce the following result on the system

$$u' = Au + f \quad \text{in } (0, \infty), \quad u(0) = u_0. \quad (3.2)$$

Corollary 3.3. *Let \mathcal{X} be a UMD Banach space, $1 < p < \infty$ and let A be a closed, densely defined operator in \mathcal{X} with domain $\mathcal{D}(A)$. Let us assume that A is a \mathcal{R} -sectorial operator of angle $> \frac{\pi}{2}$ and that the semigroup generated by A has negative exponential type. Then for every $u_0 \in (\mathcal{X}, \mathcal{D}(A))_{1-1/p, p}$ and for every $f \in L^p(0, \infty; \mathcal{X})$, Eq. (3.2) admits a unique solution in $L^p(0, \infty; \mathcal{D}(A)) \cap W^{1,p}(0, \infty; \mathcal{X})$.*

Let us also mention, the following useful result on the perturbation theory of \mathcal{R} -sectoriality, obtained in [24, Corollary 2].

Proposition 3.4. *Let A be a \mathcal{R} -sectorial operator of angle β on a Banach space \mathcal{X} . Let $B : \mathcal{D}(B) \rightarrow \mathcal{X}$ be a linear operator such that $\mathcal{D}(A) \subset \mathcal{D}(B)$ and such that there exist $a, b \geq 0$ satisfying*

$$\|Bx\|_{\mathcal{X}} \leq a\|Ax\|_{\mathcal{X}} + b\|x\|_{\mathcal{X}} \quad (x \in \mathcal{D}(A)). \quad (3.3)$$

If

$$a < \frac{1}{\widetilde{M}_\beta(A)\widetilde{\mathcal{R}}_\beta(A)} \quad \text{and} \quad \lambda > \frac{bM_\beta(A)\widetilde{\mathcal{R}}_\beta(A)}{1 - a\widetilde{M}_\beta(A)\widetilde{\mathcal{R}}_\beta(A)},$$

then $A + B - \lambda$ is \mathcal{R} -sectorial of angle β .

4 Linearized System

In order to study the system (2.10), we linearized it and use the theory of the previous section. We thus consider the following linear system

$$\begin{cases} \partial_t v - \operatorname{div} \mathbb{T}(v, \pi) = 0 & \text{in } (0, \infty) \times \mathcal{F}, \\ \operatorname{div} v = 0 & \text{in } (0, \infty) \times \mathcal{F}, \\ v(t, s, 0) = \eta_2(t, s)e_2 & \text{in } (0, \infty) \times (0, 1), \\ v = 0 & \text{on } (0, \infty) \times \Gamma_0, \\ \partial_t \eta_1 = \eta_2 & \text{in } (0, \infty) \times (0, 1), \\ \partial_t \eta_2 + P_m(\Delta_s^2 \eta_1) - \Delta_s \eta_2 = -P_m(\mathbb{T}(v, \pi)|_{\Gamma_S} e_2 \cdot e_2) & \text{in } (0, \infty) \times (0, 1), \\ \eta_1 = \partial_s \eta_1 = 0 & \text{at } (0, \infty) \times \{0, 1\}, \\ \eta_1(0, \cdot) = \eta_1^0, \quad \eta_2(0, \cdot) = \eta_2^0, \quad \text{in } (0, 1), \quad v(0) = v^0 \text{ in } \mathcal{F}, \end{cases} \quad (4.1)$$

We introduce the operator $\mathcal{T} : L^2(0, 1) \rightarrow \mathbf{L}^2(\partial\mathcal{F})$ defined by

$$\begin{aligned} (\mathcal{T}\eta)(y) &= (P_m\eta(s)) e_2 \quad \text{if } y = (s, 0) \in \Gamma_S, \\ (\mathcal{T}\eta)(y) &= 0 \quad \text{if } y \in \Gamma_0. \end{aligned} \quad (4.2)$$

One can simplify the system (4.1): using that $\operatorname{div} v = 0$ in \mathcal{F} and $v_1(\cdot, 0) = 0$ on $(0, 1)$ we deduce that $(Dv)|_{\Gamma_S} e_2 \cdot e_2 = 0$. Thus

$$-P_m\left(\mathbb{T}(v, \pi)|_{\Gamma_S} e_2 \cdot e_2\right) = \gamma_m \pi,$$

where γ_m is the following modified trace operator:

$$\gamma_m f := P_m(f|_{\Gamma_S}) = f(s, 0) - \int_0^1 f(s', 0) ds' \quad (f \in W^{r,q}(\mathcal{F}) \text{ with } r > 1/q). \quad (4.3)$$

This cancelation plays no role in our result and is only used to simplify the calculation.

4.1 The fluid operator

Here we recall some results on the Stokes operator in the L^q framework. Let us introduce the Banach space

$$W_{\operatorname{div}}^q(\mathcal{F}) = \{\varphi \in L^q(\mathcal{F}) ; \operatorname{div} \varphi \in L^q(\mathcal{F})\},$$

equipped with the norm

$$\|\varphi\|_{W_{\operatorname{div}}^q(\mathcal{F})} := \|\varphi\|_{L^q(\mathcal{F})} + \|\operatorname{div} \varphi\|_{L^q(\mathcal{F})}.$$

We recall (see, for instance, [12, Lemma 1]) that the normal trace can be extended as a continuous and surjective map

$$\begin{aligned} \gamma_n : W_{\operatorname{div}}^q(\mathcal{F}) &\rightarrow W^{-1/q,q}(\partial\mathcal{F}), \\ \varphi &\mapsto \varphi \cdot n. \end{aligned}$$

In particular, we can define

$$L_\sigma^q(\mathcal{F}) = \{\varphi \in L^q(\mathcal{F}) ; \operatorname{div} \varphi = 0 \text{ in } \mathcal{F}, \varphi \cdot n = 0 \text{ on } \partial\mathcal{F}\}.$$

We have the following Helmholtz-Weyl decomposition (see, for instance Section 3 and Theorem 2 of [12]):

$$L^q(\mathcal{F}) = L_\sigma^q(\mathcal{F}) \oplus G^q(\mathcal{F}), \quad \text{where } G^q(\mathcal{F}) = \{\nabla\varphi ; \varphi \in W^{1,q}(\mathcal{F})\}.$$

The corresponding projection operator \mathcal{P} from $L^q(\mathcal{F})^2$ onto $L_\sigma^q(\mathcal{F})$ can be obtained as

$$\mathcal{P}f = f - \nabla\varphi, \quad (4.4)$$

where $\varphi \in W^{1,q}(\mathcal{F})$ is a solution of the following Neumann problem

$$\Delta\varphi = \operatorname{div} f \quad \text{in } \mathcal{F}, \quad \frac{\partial\varphi}{\partial n} = f \cdot n \quad \text{on } \partial\mathcal{F}. \quad (4.5)$$

Let us denote by $A_F = \mathcal{P}\Delta$, the Stokes operator in $L_\sigma^q(\mathcal{F})$ with domain

$$\mathcal{D}(A_F) = W^{2,q}(\mathcal{F}) \cap W_0^{1,q}(\mathcal{F}) \cap L_\sigma^q(\mathcal{F}).$$

Theorem 4.1. *Assume $1 < q < \infty$. Then the Stokes operator A_F generates a C^0 -semigroup of negative type. Moreover A_F is an \mathcal{R} -sectorial operator in $L_\sigma^q(\mathcal{F})$ of angle β for any $\beta \in (0, \pi)$.*

For the proof, we refer to Corollary 1.2 and Theorem 1.4 in [14].

4.2 The structure operator

Let us set

$$\mathcal{X}_S = W_{0,m}^{2,q}(0,1) \times L_m^q(0,1)$$

and let us consider the operator $A_S : \mathcal{D}(A_S) \rightarrow \mathcal{X}_S$ defined by

$$\mathcal{D}(A_S) = \left(W^{4,q}(0,1) \cap W_{0,m}^{2,q}(0,1) \right) \times W_{0,m}^{2,q}(0,1), \quad A_S = \begin{pmatrix} 0 & \text{Id} \\ -P_m \Delta^2 & \Delta \end{pmatrix},$$

where P_m is defined by (1.10).

Theorem 4.2. *Let us assume that $1 < q < \infty$. Then there exists $\gamma_1 > 0$ such that $A_S - \gamma_1$ is an \mathcal{R} -sectorial operator on \mathcal{X}_S of angle $\beta_1 > \pi/2$.*

Proof. We first consider

$$\mathcal{X}_S^0 := W_0^{2,q}(0,1) \times L^q(0,1)$$

and the operator A_S^0 defined by

$$\mathcal{D}(A_S^0) = \left(W^{4,q}(0,1) \cap W_0^{2,q}(0,1) \right) \times W_0^{2,q}(0,1), \quad A_S^0 = \begin{pmatrix} 0 & \text{Id} \\ -\Delta^2 & \Delta \end{pmatrix}.$$

Applying Theorem 5.1 in [8], we have that A_S^0 is \mathcal{R} -sectorial in \mathcal{X}_S^0 of angle $\beta_0 > \pi/2$.

Now we can extend A_S on $\mathcal{D}(A_S^0)$ by $\tilde{A}_S = A_S^0 + B_S$ where

$$B_S = \begin{pmatrix} 0 & 0 \\ (\text{Id} - P_m)\Delta^2 & 0 \end{pmatrix}, \quad (\text{Id} - P_m)\Delta^2 \eta_1 = \partial_{sss} \eta_1(1) - \partial_{sss} \eta_1(0).$$

Using standard result on the trace operator, we see that B_S satisfies the hypotheses of Proposition 3.4 and in particular for any $a > 0$ there exists $b > 0$ such that (3.3) holds. Therefore, there exists $\gamma_1 > 0$ such that $\tilde{A}_S - \gamma_1$ is an \mathcal{R} -sectorial operator on \mathcal{X}_S^0 of angle β_0 .

Let $\lambda \neq 0$, $(g_1, g_2) \in \mathcal{X}_S$ and $(\eta_1, \eta_2) \in \mathcal{D}(A_S^0)$ such that

$$(\lambda - \tilde{A}_S) \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

We can write this equation as

$$\begin{aligned} \lambda \eta_1 - \eta_2 &= g_1 && \text{in } (0,1), \\ \lambda \eta_2 + P_m \Delta^2 \eta_1 - \Delta \eta_2 &= g_2, && \text{in } (0,1), \\ \eta_1 = \partial_s \eta_1 = \eta_2 = \partial_s \eta_2 &= 0 && \text{at } \{0,1\}. \end{aligned}$$

Integrating the first two equations over $(0,1)$ we find that $(\eta_1, \eta_2) \in \mathcal{D}(A_S)$. Thus

$$\left[(\lambda - \tilde{A}_S)^{-1} \right]_{|\mathcal{X}_S} = (\lambda - A_S)^{-1}.$$

Using basic properties on \mathcal{R} -boundedness, we deduce the result. \square

4.3 The fluid-structure operator

In this subsection we rewrite (4.1) in a suitable operator form. The idea is to eliminate the pressure from both the fluid and the structure equations. To eliminate the pressure from the fluid equation we use the Leray projector \mathcal{P} defined in Eq. (4.4). Following [31], first, we decompose the fluid velocity into two parts $\mathcal{P}u$ and $(\text{Id} - \mathcal{P})u$. Next, we split the pressure into two parts, one which depends on $\mathcal{P}u$ and another part which depends on η_2 . This will lead us an equation of evolution for $(\mathcal{P}u, \eta_1, \eta_2)$ and an algebraic equation for $(\text{Id} - \mathcal{P})u$.

The advantage of this formulation is that the \mathcal{R} -boundedness of the fluid-structure operator can be obtained just by using the fact that the operators A_F and A_S are \mathcal{R} -sectorial and a perturbation argument. This idea has been used in several fluid-solid interaction problems in the Hilbert space setting as well as in L^q -setting (see, for instance, [33, 21, 29, 27] and the references therein).

First, let us consider the following problem :

$$\begin{cases} -\operatorname{div} \mathbb{T}(w, \psi) = f & \text{in } \mathcal{F}, \\ \operatorname{div} w = 0 & \text{in } \mathcal{F}, \\ w = \mathcal{T}g & \text{on } \partial\mathcal{F}, \\ \int_{\mathcal{F}} \psi \, dx = 0. \end{cases} \quad (4.6)$$

From [34, Proposition 2.3, p. 35], we have the following result:

Lemma 4.3. *Assume $1 < q < \infty$. For any $f \in L^q(\mathcal{F})$ and $g \in W_{0,m}^{2,q}(0,1)$, the system (4.6) admits a unique solution $(w, \psi) \in W^{2,q}(\mathcal{F}) \times W_m^{1,q}(\mathcal{F})$.*

This allows us to introduce the following operators: we consider

$$D_v \in \mathcal{L}(W_{0,m}^{2,q}(0,1), W^{2,q}(\mathcal{F})) \quad \text{and} \quad D_p \in \mathcal{L}(W_{0,m}^{2,q}(0,1), W_m^{1,q}(\mathcal{F}))$$

defined by

$$D_v g = w, \quad D_p g = \psi, \quad (4.7)$$

where (w, ψ) is the solution to the problem (4.6) associated with g and in the case $f = 0$.

Second, we consider the Neumann problem

$$\Delta \varphi = 0 \text{ in } \mathcal{F}, \quad \frac{\partial \varphi}{\partial n} = h \text{ on } \partial\mathcal{F}, \quad \int_{\mathcal{F}} \varphi \, dx = 0. \quad (4.8)$$

Let us denote by N the operator defined by

$$Nh = \varphi. \quad (4.9)$$

It is all known that, the above system is well-posed (see for instance [26]) and

$$N \in \mathcal{L}(W_m^{1-1/q,q}(\partial\mathcal{F}), W_m^{2,q}(\mathcal{F})), \quad N \in \mathcal{L}(W_m^{-1/q,q}(\partial\mathcal{F}), W_m^{1,q}(\mathcal{F})), \quad (4.10)$$

and thus by interpolation

$$N \in \mathcal{L}(L_m^q(\partial\mathcal{F}), W_m^{1+1/q,q}(\mathcal{F})).$$

Recall that $W_m^{-1/q,q}(\partial\mathcal{F})$ is defined by (4.11).

$$W_m^{-1/q,q}(\partial\mathcal{F}) = \left\{ h \in W^{-1/q,q}(\partial\mathcal{F}) ; \langle h, 1 \rangle_{W^{-1/q,q}, W^{1-1/q',q'}} = 0 \right\}, \quad (4.11)$$

where q' the conjugate of q i.e. $\frac{1}{q} + \frac{1}{q'} = 1$.

We also define

$$N_S \in \mathcal{L}(L^q(0, 1), W_m^{1+1/q, q}(\mathcal{F})), \quad N_S g = N h \quad \text{with} \quad h(y) = \begin{cases} g(s, 0) & \text{if } y = (s, 0) \in \Gamma_S, \\ 0 & \text{if } y \in \Gamma_0. \end{cases} \quad (4.12)$$

Finally, we introduce the operator $N_{HW} \in \mathcal{L}(L^q(\mathcal{F}), W_m^{1, q}(\mathcal{F}))$ defined by

$$N_{HW} f = \varphi, \quad (4.13)$$

where φ solves (4.5).

Using the above operators, we can obtain the two following proposition. Their proof are similar to the proof of [29, Proposition 3.7] and we thus omit them.

Proposition 4.4. *Let $1 < q < \infty$ and let us assume that $f \in L^q(\mathcal{F})$ and $g \in W_{0, m}^{2, q}(0, 1)$. A pair $(w, \psi) \in W^{2, q}(\mathcal{F}) \times W_m^{1, q}(\mathcal{F})$ is a solution to (4.6) if and only if*

$$\begin{cases} -A_F \mathcal{P} w + A_F \mathcal{P} D_\nu g = \mathcal{P} f, \\ (\text{Id} - \mathcal{P}) w = (\text{Id} - \mathcal{P}) D_\nu g, \\ \psi = N(\nu \Delta \mathcal{P} w \cdot n) + N_{HW} f. \end{cases} \quad (4.14)$$

Proposition 4.5. *Let $1 < p, q < \infty$. Assume*

$$\begin{aligned} v &\in W_{p, q}^{1, 2}((0, \infty) \times \mathcal{F}), \quad \pi \in L^p(0, \infty; W_m^{1, q}(\mathcal{F})), \\ \eta_1 &\in W_{p, q}^{2, 4}((0, \infty) \times (0, 1)), \quad \eta_2 \in W_{p, q}^{1, 2}((0, \infty) \times (0, 1)). \end{aligned}$$

Then (v, π, η_1, η_2) is a solution of (4.1) if and only if

$$\begin{cases} \mathcal{P} v' = A_F \mathcal{P} v - A_F \mathcal{P} D_\nu \eta_2 & \text{in } (0, \infty), \\ \mathcal{P} v(0) = \mathcal{P} v^0 \\ (\text{Id} - \mathcal{P}) v = (\text{Id} - \mathcal{P}) D_\nu \eta_2 & \text{in } (0, \infty), \\ \pi = N(\nu \Delta \mathcal{P} v \cdot n) - N_S \partial_t \eta_2 & \text{in } (0, \infty), \\ \partial_t \eta_1 = \eta_2 & \text{in } (0, \infty), \\ (\text{Id} + \gamma_m N_S) \partial_t \eta_2 + P_m \Delta^2 \eta_1 - \Delta \eta_2 = \gamma_m N(\nu \Delta \mathcal{P} v \cdot n) & \text{in } (0, \infty), \\ \eta_1(0) = \eta_1^0, \quad \eta_2(0) = \eta_2^0. \end{cases} \quad (4.15)$$

In the literature, the operator $\text{Id} + \gamma_m N_S$ is known as added mass operator. We are going to show that it is invertible.

Lemma 4.6. *The operator $M_S = \text{Id} + \gamma_m N_S \in \mathcal{L}(L_m^q(\Gamma_S))$ is an automorphism in $W_m^{s, q}(\Gamma_S)$ for any $s \in [0, 1]$. Moreover, $M_S^{-1} - \text{Id}$ is a compact operator on $L_m^q(0, 1)$. In particular, $M_S^{-1} - \text{Id} \in \mathcal{L}(L_m^q(0, 1), W_m^{1, q}(0, 1))$.*

Proof. At first, we show that M_S is an invertible operator on $L_m^q(0, 1)$. Since

$$\gamma_m N_S \in \mathcal{L}(L_m^q(0, 1), W_m^{1, q}(0, 1)),$$

it is sufficient to show that the kernel of M_S is reduced to $\{0\}$: assume

$$(\text{Id} + \gamma_m N_S)f = 0. \quad (4.16)$$

Then $f \in W_m^{1,q}(0,1) \subset L_m^2(0,1)$. In particular (see (4.12)), $\vartheta = N_S f \in H^1(\mathcal{F})$ is the weak solution of

$$\Delta \vartheta = 0 \text{ in } \mathcal{F}, \quad \frac{\partial \vartheta}{\partial n} = f \text{ on } \Gamma_S, \quad \frac{\partial \vartheta}{\partial n} = 0 \text{ on } \Gamma_0.$$

Multiplying (4.16) by f and using the above system, we deduce after integration by parts,

$$\int_0^1 [(\text{Id} + \gamma_m N_S)f] f \, ds = \int_0^1 f^2 \, ds + \int_{\mathcal{F}} |\nabla \vartheta|^2 \, dy = 0.$$

Thus $f = 0$ and M_S is an invertible operator on $L_m^q(0,1)$. Let $s \in [0,1]$ and $f_0 \in W_m^{s,q}(0,1)$. By the above argument, there exists a unique $f \in L_m^q(0,1)$ such that

$$(\text{Id} + \gamma_m N_S)f = f_0.$$

As $\gamma_m N_S f \in W_m^{1,q}(\Gamma_S)$ we conclude that $f \in W_m^{s,q}(\Gamma_S)$. Thus M_S is an invertible operator on $W_m^{s,q}(\Gamma_S)$. Finally, the compactness of the operator $M_S^{-1} - \text{Id}$ follows from the following identity

$$M_S^{-1} - \text{Id} = M_S^{-1} - M_S^{-1} M_S = -M_S^{-1} \gamma_S N_S.$$

□

We are now in a position to rewrite the system (4.1) in a suitable operator form. Let us set

$$\mathcal{X} = L_\sigma^q(\mathcal{F}) \times \mathcal{X}_S, \quad (4.17)$$

and consider the operator $\mathcal{A}_{FS} : \mathcal{D}(\mathcal{A}_{FS}) \rightarrow \mathcal{X}$ defined by

$$\mathcal{D}(\mathcal{A}_{FS}) = \left\{ (w, \eta_1, \eta_2) \in L_\sigma^q(\mathcal{F}) \cap W^{2,q}(\mathcal{F}) \times \mathcal{D}(A_S) ; w - \mathcal{P} D_\nu \eta_2 \in \mathcal{D}(A_F) \right\},$$

and $\mathcal{A}_{FS} = \mathcal{A}_{FS}^0 + \mathcal{B}_{FS}$, with

$$\mathcal{A}_{FS}^0 := \begin{bmatrix} A_F & 0 & -A_F \mathcal{P} D_\nu \\ 0 & 0 & \text{Id} \\ 0 & -P_m \Delta^2 & \Delta \end{bmatrix} \quad (4.18)$$

and

$$\mathcal{B}_{FS} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ M_S^{-1} \gamma_m N(\nu \Delta(\cdot) \cdot n) & -(M_S^{-1} - \text{Id}) P_m \Delta^2 & (M_S^{-1} - \text{Id}) \Delta \end{bmatrix}. \quad (4.19)$$

Combining Proposition 4.5, Lemma 4.6 and the operators introduced above, we have the following result:

Theorem 4.7. *Let us assume the hypotheses of Proposition 4.5. Then (v, π, η_2, η_2) is a solution of (4.1) if and only if*

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} \mathcal{P}v \\ \eta_1 \\ \eta_2 \end{bmatrix} &= \mathcal{A}_{FS} \begin{bmatrix} \mathcal{P}v \\ \eta_1 \\ \eta_2 \end{bmatrix}, & \begin{bmatrix} \mathcal{P}v \\ \eta_1 \\ \eta_2 \end{bmatrix} (0) &= \begin{bmatrix} \mathcal{P}v^0 \\ \eta_1^0 \\ \eta_2^0 \end{bmatrix}, \\ (\text{Id} - \mathcal{P})v &= (\text{Id} - \mathcal{P}) D_\nu \eta_2, \\ \pi &= N(\nu \Delta \mathcal{P}v \cdot n) - N_S \partial_t \eta_2. \end{aligned} \quad (4.20)$$

4.4 \mathcal{R} -sectoriality of the operator \mathcal{A}_{FS} .

In this subsection we prove the following theorem

Theorem 4.8. *Let $1 < q < \infty$. There exists $\gamma_2 > 0$ such that $\mathcal{A}_{FS} - \gamma_2$ is an \mathcal{R} -sectorial operator in \mathcal{X} of angle $> \pi/2$.*

Proof. Observe that

$$\lambda(\lambda - \mathcal{A}_{FS}^0)^{-1} = \begin{bmatrix} \lambda(\lambda - A_F)^{-1} & A_F(\lambda - A_F)^{-1} \tilde{\mathcal{P}} \tilde{D}_v \lambda(\lambda - A_S)^{-1} \\ 0 & \lambda(\lambda - A_S)^{-1} \end{bmatrix},$$

where $\tilde{D}_v [f_1, f_2]^\top = D_v f_2$. Using a standard transposition method and Lemma 4.3, we see that

$$D_v \in \mathcal{L}(L_m^q(0, 1), L^q(\mathcal{F})). \quad (4.21)$$

Therefore by Theorem 4.1 and Theorem 4.2, there exists $\gamma > 0$ such that $\mathcal{A}_{FS}^0 - \gamma$ is \mathcal{R} -sectorial operator in \mathcal{X} of angle $> \pi/2$.

Next, we want to show $\mathcal{B}_{FS} \in \mathcal{L}(\mathcal{D}(\mathcal{A}_{FS}), \mathcal{X})$ is a compact operator. Assume $(w, \eta_1, \eta_2) \in \mathcal{D}(\mathcal{A}_{FS})$. Then $\Delta w \in L^q(\mathcal{F})$ and $\operatorname{div} \Delta w = 0$ and thus from the trace result recalled in Section 4.1,

$$(\Delta w) \cdot n \in W_m^{-1/q, q}(\partial \mathcal{F}).$$

This yields $N((\Delta w) \cdot n) \in W_m^{1, q}(\mathcal{F})$, $\gamma_m N((\Delta w) \cdot n) \in W_m^{1-1/q, q}(0, 1)$ and, using Lemma 4.6,

$$M_S^{-1} \gamma_m N((\Delta w) \cdot n) \in W_m^{1-1/q, q}(0, 1).$$

On the other hand, using again Lemma 4.6, we deduce

$$(M_S^{-1} - \operatorname{Id}) P_m \Delta^2 \in \mathcal{L}(W^{4, q}(0, 1), W_m^{1, q}(0, 1)), \quad (M_S^{-1} - \operatorname{Id}) \Delta \in \mathcal{L}(W_m^{2, q}(0, 1), W_m^{1, q}(0, 1)).$$

Therefore, $\mathcal{B}_{FS} \in \mathcal{L}(\mathcal{D}(\mathcal{A}_{FS}), \mathcal{X})$ is a compact operator and by [10, Chapter III, Lemma 2.16], \mathcal{B}_{FS} is a \mathcal{A}_{FS}^0 -bounded operator with relative bound 0. Finally, using Proposition 3.4 we conclude the proof of the theorem. \square

4.5 Exponential Stability of the operator \mathcal{A}_{FS} .

In this subsection we show that \mathcal{A}_{FS} generates an exponential stable semigroup. More precisely, we prove the following theorem

Theorem 4.9. *Let $1 < q < \infty$. The operator \mathcal{A}_{FS} generates an exponentially stable semigroup on \mathcal{X} . In other words, there exist constants $C > 0$ and $\beta_0 > 0$ such that*

$$\left\| e^{t\mathcal{A}_{FS}} (v^0, \eta_1^0, \eta_2^0)^\top \right\|_{\mathcal{X}} \leq C e^{-\beta_0 t} \left\| (v^0, \eta_1^0, \eta_2^0)^\top \right\|_{\mathcal{X}}. \quad (4.22)$$

Proof. Since \mathcal{A}_{FS} generates an analytic semigroup it is sufficient to show that

$$\mathbb{C}^+ = \{\lambda \in \mathbb{C} ; \operatorname{Re} \lambda \geq 0\} \subset \rho(\mathcal{A}_{FS}).$$

Moreover, using that \mathcal{A}_{FS} has a compact resolvent and the Fredholm alternative theorem, we can show the above relation by proving that $\ker(\lambda - \mathcal{A}_{FS}) = \{0\}$ for $\lambda \in \mathbb{C}^+$. Assume $\lambda \in \mathbb{C}^+$ and

$$(v, \pi, \eta_1, \eta_2) \in W^{2,q}(\mathcal{F}) \times W_m^{1,q}(\mathcal{F}) \times W_m^{4,q}(0,1) \times W_m^{2,q}(0,1)$$

satisfy

$$\begin{cases} \lambda v - \operatorname{div} \mathbb{T}(v, \pi) = 0 & \text{in } \mathcal{F}, \\ \operatorname{div} v = 0 & \text{in } \mathcal{F}, \\ v = \mathcal{T}\eta_2 & \text{on } \partial\mathcal{F}, \\ \lambda\eta_1 - \eta_2 = 0 & \text{in } (0,1), \\ \lambda\eta_2 + P_m \Delta^2 \eta_1 - \Delta \eta_2 = \gamma_m \pi & \text{in } (0,1), \\ \eta_1 = \partial_s \eta_1 = 0 & \text{at } \{0,1\}. \end{cases} \quad (4.23)$$

First we notice that

$$(v, \pi, \eta_1, \eta_2) \in W^{2,2}(\mathcal{F}) \times W_m^{1,2}(\mathcal{F}) \times W_m^{4,2}(0,1) \times W_m^{2,2}(0,1). \quad (4.24)$$

If $q \geq 2$ then it is a consequence of Hölder's inequality. Let us assume that $1 < q < 2$ and let us take $\tilde{\lambda} \in \rho(\mathcal{A}_{FS})$ (see Theorem 4.8). We have

$$(\tilde{\lambda} - \mathcal{A}_{FS})(v, \eta_1, \eta_2)^\top = (\tilde{\lambda} - \lambda)(v, \eta_1, \eta_2)^\top$$

By following the calculation done in Section 4.3, we see that the system (4.23) can be written as

$$\begin{cases} (\tilde{\lambda} - \mathcal{A}_{FS}) \begin{bmatrix} \mathcal{P}v \\ \eta_1 \\ \eta_2 \end{bmatrix} = (\tilde{\lambda} - \lambda) \begin{bmatrix} \mathcal{P}v \\ \eta_1 \\ \eta_2 \end{bmatrix}, \\ (\operatorname{Id} - \mathcal{P})v = (\operatorname{Id} - \mathcal{P})D_\nu \eta_2, \\ \pi = N(\nu \Delta \mathcal{P}v \cdot n) - \lambda N_S \eta_2. \end{cases}$$

Since $W^{2,q}(\mathcal{F}) \subset L^2(\mathcal{F})$, $W^{2,q}(0,1) \subset L^2(0,1)$ and $(\tilde{\lambda} - \mathcal{A}_{FS})$ is invertible, we deduce (4.24).

Using (4.24), we can multiply (4.23)₁ by \bar{v} and (4.23)₅ by $\bar{\eta}_2$, and we obtain after integration by parts:

$$\lambda \int_{\mathcal{F}} |v|^2 dy + 2\nu \int_{\mathcal{F}} |D(v)|^2 dy + \lambda \int_0^1 |\eta_2|^2 ds + \bar{\lambda} \int_0^1 |\Delta_s \eta_1|^2 ds + \int_0^1 |\nabla_s \eta_2|^2 ds = 0.$$

Since $\operatorname{Re} \lambda \geq 0$, from the above equality and using the boundary conditions we obtain that $v = \pi = \eta_1 = \eta_2 = 0$. This completes the proof of the theorem. \square

5 Maximal L^p - L^q regularity for the linearized system

Using the results obtain in the previous section, we can now consider the nonhomogeneous problem associated with the linear system (4.1). More precisely, we consider the following problem:

$$\begin{cases}
\partial_t v - \operatorname{div} \mathbb{T}(v, \pi) = f & \text{in } (0, \infty) \times \mathcal{F}, \\
\operatorname{div} v = \operatorname{div} g & \text{in } (0, \infty) \times \mathcal{F}, \\
v = \mathcal{T}\eta_2 & \text{on } (0, \infty) \times \partial\mathcal{F} \\
\partial_t \eta_1 = \eta_2 & \text{in } (0, \infty) \times (0, 1), \\
\partial_t \eta_2 + P_m (\Delta_s^2 \eta_1) - \Delta_s \eta_2 = -P_m (\mathbb{T}(v, \pi)|_{\Gamma_S} e_2 \cdot e_2) + P_m h & \text{in } (0, \infty) \times (0, 1), \\
\eta_1 = \partial_s \eta_1 = 0 & \text{at } (0, \infty) \times \{0, 1\}, \\
\eta_1(0, \cdot) = \eta_1^0, \quad \eta_2(0, \cdot) = \eta_2^0, \quad \text{in } (0, 1), \quad v(0) = v^0 \text{ in } \mathcal{F},
\end{cases} \quad (5.1)$$

Remark 5.1. Since $\operatorname{div} v \neq 0$, we can not drop the term $2\nu(Dv)e_2 \cdot e_2$ from the right hand side of (5.1)₅.

Let us define

$$B_{q,p,cc}^{2(1-1/p)}(0, 1) = \left\{ \eta \in B_{q,p}^{2(1-1/p)}(0, 1) \cap L_m^q(0, 1) ; \eta = 0 \text{ at } \{0, 1\} \quad \text{if } \frac{1}{p} + \frac{1}{2q} < 1 \right\}.$$

By combining the definition (4.7) of D_v and by using (4.21), we have

$$D_v \in \mathcal{L}(B_{q,p,cc}^{2(1-1/p)}(0, 1), B_{q,p}^{2(1-1/p)}(\mathcal{F})).$$

In order to obtain a result of well-posedness on the system (5.1), we need to impose some compatibility conditions on the data:

$$\begin{aligned}
\eta_1^0 = \partial_s \eta_1^0 = 0 \quad \text{at } \{0, 1\}, \quad \int_0^1 \eta_1^0 ds = 0, \quad \int_0^1 \eta_2^0 ds = 0, \\
\operatorname{div}(v^0 - g(0, \cdot)) = 0 \quad \text{in } \mathcal{F}, \quad g = 0 \text{ on } (0, \infty) \times \partial\mathcal{F}
\end{aligned} \quad (5.2)$$

and

$$\begin{cases}
v^0 \cdot e_2 = \mathcal{T}\eta_2 \cdot e_2 \quad \text{on } \partial\mathcal{F} & \text{if } \frac{1}{p} + \frac{1}{2q} > 1, \\
v^0 = \mathcal{T}\eta_2 \quad \text{on } \partial\mathcal{F}, \quad \eta_2^0 = 0 \quad \text{at } \{0, 1\} & \text{if } \frac{1}{p} + \frac{1}{2q} < 1.
\end{cases} \quad (5.3)$$

The main theorem of this section is the following

Theorem 5.2. Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{2q} \neq 1$ and let $\beta \in [0, \beta_0)$, where β_0 is the constant in Theorem 4.9. Assume

$$v^0 \in B_{q,p}^{2(1-1/p)}(\mathcal{F}), \quad \eta_1^0 \in B_{q,p}^{2(2-1/p)}(0, 1), \quad \eta_2^0 \in B_{q,p}^{2(1-1/p)}(0, 1), \quad (5.4)$$

$$e^{\beta(\cdot)} f \in L^p(0, \infty; L^q(\mathcal{F})), \quad e^{\beta(\cdot)} g \in W_{p,q}^{1,2}((0, \infty) \times \mathcal{F}), \quad e^{\beta(\cdot)} h \in L^p(0, \infty; L^q(0, 1))$$

satisfy the compatibility conditions (5.2) and (5.3). Then the system (5.1) admits a unique strong solution

$$\begin{aligned}
e^{\beta(\cdot)} v &\in W_{p,q}^{1,2}((0, \infty) \times \mathcal{F}), \quad e^{\beta(\cdot)} \pi \in L^p(0, \infty; W_m^{1,q}(\mathcal{F})), \\
e^{\beta(\cdot)} \eta_1 &\in W_{p,q}^{2,4}((0, \infty) \times (0, 1)) \cap L^p(0, \infty; L_m^q(0, 1)), \\
e^{\beta(\cdot)} \eta_2 &\in W_{p,q}^{1,2}((0, \infty) \times (0, 1)) \cap L^p(0, \infty; L_m^q(0, 1)).
\end{aligned}$$

Moreover, there exists a constant C_L depending on p, q and the geometry such that

$$\begin{aligned}
& \left\| e^{\beta(\cdot)} v \right\|_{W_{p,q}^{1,2}((0,\infty) \times \mathcal{F})} + \left\| e^{\beta(\cdot)} v \right\|_{L^\infty(0,\infty; B_{q,p}^{2(1-1/p)}(\mathcal{F}))} + \left\| e^{\beta(\cdot)} \pi \right\|_{L^p(0,\infty; W_m^{1,q}(\mathcal{F}))} \\
& + \left\| e^{\beta(\cdot)} \eta_1 \right\|_{W_{p,q}^{2,4}((0,\infty) \times (0,1))} + \left\| e^{\beta(\cdot)} \eta_1 \right\|_{L^\infty(0,\infty; B_{q,p}^{2(2-1/p)}(0,1))} + \left\| e^{\beta(\cdot)} \eta_2 \right\|_{W_{p,q}^{1,2}((0,\infty) \times (0,1))} + \\
& + \left\| e^{\beta(\cdot)} \eta_2 \right\|_{L^\infty(0,\infty; B_{q,p}^{2(1-1/p)}(0,1))} \leq C_L \left(\|v^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})} + \|\eta_1^0\|_{B_{q,p}^{2(2-1/p)}(0,1)} + \|\eta_2^0\|_{B_{q,p}^{2(1-1/p)}(0,1)} \right. \\
& \left. + \left\| e^{\beta(\cdot)} f \right\|_{L^p(0,\infty; L^q(\mathcal{F}))} + \left\| e^{\beta(\cdot)} g \right\|_{W_{p,q}^{1,2}((0,\infty) \times \mathcal{F})} + \left\| e^{\beta(\cdot)} h \right\|_{L^p(0,\infty; L^q(0,1))} \right). \quad (5.5)
\end{aligned}$$

Proof. Let us first consider the case $\beta = 0$. We set,

$$w = v - g.$$

Then (w, π, η_1, η_2) satisfies the following system

$$\begin{cases}
\partial_t w - \operatorname{div} \mathbb{T}(w, \pi) = \tilde{f} & \text{in } (0, \infty) \times \mathcal{F}, \\
\operatorname{div} w = 0 & \text{in } (0, \infty) \times \mathcal{F}, \\
w = \mathcal{T} \eta_2 & \text{on } (0, \infty) \times \partial \mathcal{F} \\
\partial_t \eta_1 = \eta_2 & \text{in } (0, \infty) \times (0, 1), \\
\partial_t \eta_2 + P_m (\Delta_s^2 \eta_1) - \Delta_s \eta_2 = -P_m (\mathbb{T}(w, \pi)|_{\Gamma_S} e_2 \cdot e_2) + P_m \tilde{h} & \text{in } (0, \infty) \times (0, 1), \\
\eta_1 = \partial_s \eta_1 = 0 & \text{at } (0, \infty) \times \{0, 1\}, \\
\eta_1(0, \cdot) = \eta_1^0, \quad \eta_2(0, \cdot) = \eta_2^0, \quad \text{in } (0, 1), \quad w(0) = w^0 \text{ in } \mathcal{F},
\end{cases} \quad (5.6)$$

where

$$w^0 = v^0 - g(0), \quad \tilde{f} = f - \partial_t g + \nu \Delta g + \nu \nabla(\operatorname{div} g), \quad \tilde{h} = h - 2\nu Dg|_{\Gamma_S} e_2 \cdot e_2.$$

Using the regularity assumptions of v^0, f, g and the embedding

$$W_{p,q}^{1,2}((0, \infty) \times \mathcal{F}) \hookrightarrow C_b([0, \infty); B_{q,p}^{2(1-1/p)}(\mathcal{F}))$$

we see that

$$w^0 \in B_{q,p}^{2(1-1/p)}(\mathcal{F}), \quad \tilde{f} \in L^p(0, \infty; L^q(\mathcal{F})), \quad \tilde{h} \in L^p(0, \infty; L^q(0, 1)).$$

Moreover, the system (5.6) can be written as follows

$$\begin{aligned}
\frac{d}{dt} \begin{bmatrix} \mathcal{P}w \\ \eta_1 \\ \eta_2 \end{bmatrix} &= \mathcal{A}_{FS} \begin{bmatrix} \mathcal{P}w \\ \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} \mathcal{P}\tilde{f} \\ 0 \\ \bar{h} \end{bmatrix}, \quad \begin{bmatrix} \mathcal{P}w \\ \eta_1 \\ \eta_2 \end{bmatrix} (0) = \begin{bmatrix} \mathcal{P}w^0 \\ \eta_1^0 \\ \eta_2^0 \end{bmatrix}, \\
(\operatorname{Id} - \mathcal{P})w &= (\operatorname{Id} - \mathcal{P})D_\nu \eta_2, \\
\pi &= N(\nu \Delta \mathcal{P}w \cdot n) - N_S \partial_t \eta_2 + N_{HW} \tilde{f},
\end{aligned}$$

where

$$\bar{h} = M_S^{-1} P_m \tilde{h} + M_S^{-1} \gamma_m N_{HW} \tilde{f}.$$

Using (4.13) and Lemma 4.6 we can easily verify that $\bar{h} \in L^p(0, \infty; L_m^q(0, 1))$. From [3, Theorem 2.4] we also have that $(\mathcal{P}w^0, \eta_1^0, \eta_2^0) \in (\mathcal{X}, \mathcal{D}(\mathcal{A}_{FS}))_{1-1/p, p}$. From Theorem 4.8 and Theorem 4.9 we know that \mathcal{A}_{FS} generates an analytic exponentially stable semigroup on \mathcal{X} and is a \mathcal{R} -sectorial operator on \mathcal{X} . Therefore by Corollary 3.3

$$(\mathcal{P}w, \eta_1, \eta_2) \in L^p(0, \infty; \mathcal{D}(\mathcal{A}_{FS})) \cap W^{1,p}(0, \infty; \mathcal{X}).$$

From the expression of $(\text{Id} - \mathcal{P})w$ we recover that $w \in W_{p,q}^{1,2}((0, \infty) \times \mathcal{F})$ and next using relations (4.10), (4.12) and (4.13), we obtain $\pi \in L^p(0, \infty; W_m^{1,q}(\mathcal{F}))$.

The case $\beta > 0$ can be reduced to the previous case by multiplying all the functions by $e^{\beta t}$ and using the fact that $\mathcal{A}_{FS} + \beta$ is a \mathcal{R} -sectorial operator and generates an exponentially stable semigroup. \square

6 Local in time existence

The aim of this section is to prove Theorem 1.1 and Theorem 2.1. Throughout this section we assume that

$$\eta_1^0 = 0, \quad (p, q) \in (1, \infty) \text{ such that } \frac{1}{p} + \frac{1}{2q} \neq 1, \quad \frac{1}{p} + \frac{1}{q} < \frac{3}{2}.$$

Let us also assume that (η_2^0, v^0) satisfies (2.16), (2.17), (2.18). We set

$$\|v^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})} + \|\eta_2^0\|_{B_{q,p}^{2(1-1/p)}(0,1)} := M. \quad (6.1)$$

For $T < \infty$ we define \mathcal{S}_T as follows

$$\mathcal{S}_T = \left\{ (f, g, h) \mid f \in L^p(0, T; L^q(\mathcal{F})), g \in W_{p,q}^{1,2}((0, T) \times \mathcal{F}), h \in L^p(0, T; L^q(0, 1)), \right. \\ \left. g(0, \cdot) = 0 \text{ in } \mathcal{F}, \quad g = 0 \text{ on } (0, T) \times \partial\mathcal{F} \right\}, \quad (6.2)$$

with

$$\|(f, g, h)\|_{\mathcal{S}_T} = \|f\|_{L^p(0, T; L^q(\mathcal{F}))} + \|g\|_{W_{p,q}^{1,2}((0, T) \times \mathcal{F})} + \|h\|_{L^p(0, T; L^q(0, 1))}.$$

The following lemma plays an essential role in the remaining part of this section.

Lemma 6.1. *Let Ω be a smooth bounded domain in \mathbb{R}^n . Let us assume that $q \in (1, \infty)$, $s_1, s_2 \in [0, 1]$, $f_1 \in W^{s_1, q}(\Omega)$ and $f_2 \in W^{s_2, q}(\Omega)$. Then*

$$\|f_1 f_2\|_{L^q(\Omega)} \leq C(\Omega, s_1, s_2, q) \|f_1\|_{W^{s_1, q}(\Omega)} \|f_2\|_{W^{s_2, q}(\Omega)}, \quad (6.3)$$

if $s_1 + s_2 > \frac{n}{q}$ or if $s_1 > 0$ and $s_2 > 0$ and $s_1 + s_2 = \frac{n}{q}$.

Proof. We only consider the case when $0 < s_1, s_2 < n/q$. The other cases follows easily from the embedding $W^{s, q}(\mathcal{F}) \hookrightarrow L^\infty(\mathcal{F})$, if $s > 2/q$. By Hölder's inequality we have

$$\|f_1 f_2\|_{L^q(\Omega)} \leq C \|f_1\|_{L^{r'q}(\Omega)} \|f_2\|_{L^{r'q}(\Omega)}, \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

Now $W^{s_1, q}(\Omega) \hookrightarrow L^{r'q}(\Omega)$ if $r'q \leq \frac{nq}{n-s_1q}$. Similarly, $W^{s_2, q}(\Omega) \hookrightarrow L^{r'q}(\Omega)$ if $r'q \leq \frac{nq}{n-s_2q}$. Eliminating r and r' for the above relations we obtain $s_1 + s_2 \geq n/q$. \square

Our aim is to estimate the non linear terms in (2.13)-(2.15):

Proposition 6.2. *Let $p, q \in (1, \infty)$ satisfying the condition (1.17). Let us assume that $\eta_1^0 = 0$ and (η_2^0, v^0) satisfies (2.16), (2.17), (2.18). Let M be defined in (6.1). There exist $\tilde{T} < 1$, a constant $\delta > 0$ depending only on p and q , and a constant $C > 0$ depending only on p, q, M, \tilde{T} such that for $T \in (0, \tilde{T}]$ and for $(f, g, h) \in \mathcal{S}_T$ satisfying*

$$\|(f, g, h)\|_{\mathcal{S}_T} \leq 1,$$

the solution (v, π, η) of (5.1) in $[0, T]$ verifies

$$\|(F(v, \pi, \eta), G(v, \pi, \eta), H(v, \pi, \eta))\|_{\mathcal{S}_T} \leq CT^\delta,$$

where F, G, H are defined by (2.13), (2.14), (2.15).

Proof. We consider $\tilde{T} < 1$ and we assume that $T \in (0, \tilde{T}]$. The constants C appearing in this proof depend only on M . From (5.5) in Theorem 5.2, we first obtain

$$\begin{aligned} & \|v\|_{W_{p,q}^{1,2}((0,T) \times \mathcal{F})} + \|v\|_{L^\infty(0,T;B_{q,p}^{2(1-1/p)}(\mathcal{F}))} + \|\pi\|_{L^p(0,T;W_m^{1,q}(\mathcal{F}))} + \|\eta\|_{W_{p,q}^{2,4}((0,T) \times (0,1))} \\ & + \|\eta\|_{L^\infty(0,T;B_{q,p}^{2(2-1/p)}(0,1))} + \|\partial_t \eta\|_{W_{p,q}^{1,2}((0,T) \times (0,1))} + \|\partial_t \eta\|_{L^\infty(0,T;B_{q,p}^{2(1-1/p)}(0,1))} \leq C. \end{aligned} \quad (6.4)$$

Therefore by [36, Theorem (i), p.196], we have that

$$\|\partial_t \eta\|_{L^\infty(0,T;W^{s_1,q}(0,1))} + \|\partial_{ss} \eta\|_{L^\infty(0,T;W^{s_1,q}(0,1))} + \|v\|_{L^\infty(0,T;W^{s_1,q}(\mathcal{F}))} \leq C, \quad (6.5)$$

for $s_1 < \min\{2(1-1/p), 1\}$.

For all $s_2 \in (0, 1)$ we have by complex interpolation

$$\|v(t, \cdot)\|_{W^{1+s_2,q}(\mathcal{F})} \leq C \|v(t, \cdot)\|_{W^{2,q}(\mathcal{F})}^{(1+s_2)/2} \|v(t, \cdot)\|_{L^q(\mathcal{F})}^{(1-s_2)/2},$$

and thus

$$\|v\|_{L^p(0,T;W^{1+s_2,q}(\mathcal{F}))} \leq CT^{(1-s_2)/2p} \|v\|_{L^\infty(0,T;L^q(\mathcal{F}))}^{\frac{1-s_2}{2}} \|v\|_{L^p(0,T;W^{2,q}(\mathcal{F}))}^{\frac{1+s_2}{2}} \leq CT^{(1-s_2)/2p}. \quad (6.6)$$

Since $\eta(0) = 0$, we have

$$\|\eta\|_{L^\infty(0,T;W^{2,q}(0,1))} \leq CT^{1/p'} \|\partial_t \eta\|_{L^p(0,T;W^{2,q}(0,1))} \leq CT^{1/p'}.$$

In particular, there exists $\tilde{T} < 1$ such that

$$\|\eta\|_{L^\infty((0,T) \times (0,1))} \leq c_0, \quad T \leq \tilde{T},$$

where c_0 is defined in (2.4). Let X be defined as in (2.5). With the similar calculation as above, we can easily verify that

$$\begin{aligned} & \|\text{Cof } \nabla X\|_{W^{1,p}(0,T;W^{1,q}(\mathcal{F}))} + \|Z\|_{W^{1,p}(0,T;W^{1,q}(\mathcal{F}))} + \|Z\|_{L^\infty((0,T) \times \mathcal{F})} \leq C, \\ & \|\nabla X - I_2\|_{L^\infty((0,T) \times \mathcal{F})} + \|\text{Cof } \nabla X - I_2\|_{L^\infty((0,T) \times \mathcal{F})} + \|Z - I_2\|_{L^\infty((0,T) \times \mathcal{F})} \leq CT^{1/p'}. \end{aligned} \quad (6.7)$$

We are now in position to estimate the non linear terms in (2.13)-(2.15). We choose

$$0 < s_1 < \min\{2(1-1/p), 1\} \text{ and } 0 < s_2 < 1 \text{ such that } s_1 + s_2 \geq 2/q. \quad (6.8)$$

Note that, such a choice is always possible since $1/p + 1/q < 3/2$. To estimate the first term of F , we combine Lemma 6.1, (6.5) and (6.7) to obtain

$$\begin{aligned} & \left\| (v - \partial_t X) \cdot Z^\top \nabla v_i \right\|_{L^p(0,T;L^q(\mathcal{F}))} \\ & \leq C \left(\|v\|_{L^\infty(0,T;W^{s_1,q}(\mathcal{F}))} + \|\partial_t \eta\|_{L^\infty(0,T;W^{s_1,q}(0,1))} \right) \|\nabla v_i\|_{L^p(0,T;W^{s_2,q}(\mathcal{F}))} \leq CT^{(1-s_2)/2p}. \end{aligned}$$

Estimates of the other terms of F and G are similar. Finally, the estimate of H can be done as follows:

$$\left\| \nu \frac{\partial v_1}{\partial y_2}(\cdot, 0)(\partial_s \eta) \right\|_{L^p(0,T;L^q(0,1))} \leq C \|\partial_s \eta\|_{L^\infty((0,T) \times (0,1))} \|v\|_{L^p(0,T;W^{2,q}(\mathcal{F}))} \leq CT^{1/p'}.$$

□

Proposition 6.3. *Let $p, q \in (1, \infty)$ satisfying the condition (1.17). Let us assume that $\eta_1^0 = 0$ and (η_2^0, v^0) satisfies (2.16), (2.17), (2.18). Let M be defined in (6.1). There exist $\tilde{T} < 1$, a constant $\delta > 0$ depending only on p and q , and a constant $C > 0$ depending only on p, q, M, \tilde{T} such that for $T \in (0, \tilde{T}]$ we have the following property: for $(f^j, g^j, h^j) \in \mathcal{S}_T$ satisfying*

$$\|(f^j, g^j, h^j)\|_{\mathcal{S}_T} \leq 1,$$

for $j = 1, 2$, let (v^j, π^j, η^j) be the solution of (5.1) in $[0, T] \times \mathcal{F}$ corresponding to the source term (f^j, g^j, h^j) . Let us set (see (2.13), (2.14), (2.15))

$$F^j = F(v^j, \pi^j, \eta^j), \quad G^j = G(v^j, \pi^j, \eta^j), \quad H^j = H(v^j, \pi^j, \eta^j).$$

Then

$$\|(F^1, G^1, H^1) - (F^2, G^2, H^2)\|_{\mathcal{S}_T} \leq CT^\delta.$$

We are now in a position to prove Theorem 2.1 and Theorem 1.1.

Proof of Theorem 2.1. Let \tilde{T} be the constant in Proposition 6.2. For $T \in (0, \tilde{T}]$, we consider the map

$$\mathcal{N} : \mathcal{B}_T \longrightarrow \mathcal{B}_T, \quad (f, g, h) \longmapsto (F(v, \pi, \eta), G(v, \pi, \eta), H(v, \pi, \eta)),$$

where

$$\mathcal{B}_T = \{(f, g, h) \in \mathcal{S}_T \mid \|(f, g, h)\|_{\mathcal{S}_T} \leq 1\},$$

(v, π, η) is the solution to the system (5.1) in $[0, T] \times \mathcal{F}$ and F, G and H are given by (2.13)-(2.15). With Proposition 6.2 and Proposition 6.3 we have that, $\mathcal{N}(\mathcal{B}_T) \subset \mathcal{B}_T$ and $\mathcal{N}|_{\mathcal{B}_T}$ is a strict contraction for T small enough. This completes the proof of the theorem. □

From Theorem 2.1 we can now deduce Theorem 1.1.

Proof of Theorem 1.1. Let (v, π, η) be the solution of the system (2.10)- (2.15) constructed in Theorem 2.2. Since X is a C^1 -diffeomorphism from \mathcal{F} to $\mathcal{F}(\eta(t))$, we set $Y(t, \cdot) = X^{-1}(t, \cdot)$ and for $x \in \mathcal{F}(\eta(t))$, $t \geq 0$

$$\tilde{v}(t, x) = v(t, Y(t, x)), \quad \tilde{\pi}(t, x) = \pi(t, Y(t, x)), \quad \tilde{v}^0(x) = v^0(Y(0, x)).$$

One can easily verify that $(\tilde{v}^0, 0, \eta_2^0)$ satisfies the compatibility conditions (1.16). Moreover, $(\tilde{v}, \tilde{\pi}, \eta)$ satisfies the original system (1.13). □

7 Global in time existence

The aim of this section is to prove the global in time existence result. More precisely, we prove here Theorem 1.2 and Theorem 2.2. At first, we estimate the nonlinear terms F, G and H defined in (2.13) - (2.15). Throughout this section we assume that $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{2q} \neq 1$. Let us fix $\beta \in (0, \beta_0)$, where β_0 is introduced in Theorem 4.9. Let us consider the Banach space

$$\mathcal{S} = W_{p,q}^{1,2}((0, \infty) \times \mathcal{F}) \times L^p(0, \infty; W_m^{1,q}(\mathcal{F})) \times W_{p,q}^{2,4}((0, \infty) \times (0, 1)),$$

and its closed ball centered in 0 and of radius $\varepsilon > 0$:

$$\mathbf{B}_{\mathcal{S}}(\varepsilon) = \left\{ (v, \pi, \eta) \in \mathcal{S} ; \left\| (v, \pi, \eta) \right\|_{\mathcal{S}} \leq \varepsilon \right\}. \quad (7.1)$$

Assume

$$e^{\beta(\cdot)}(v, \pi, \eta) \in \mathbf{B}_{\mathcal{S}}(\varepsilon).$$

Then, we already notice that there exists a constant $C > 0$ such that

$$\left\| e^{\beta(\cdot)} v \right\|_{L^\infty(0, \infty; B_{q,p}^{2(1-1/p)}(\mathcal{F}))} + \left\| e^{\beta(\cdot)} \eta \right\|_{L^\infty(0, \infty; B_{q,p}^{2(2-1/p)}(0,1))} + \left\| e^{\beta(\cdot)} \partial_t \eta \right\|_{L^\infty(0, \infty; B_{q,p}^{2(1-1/p)}(0,1))} \leq C\varepsilon. \quad (7.2)$$

Using the embedding

$$B_{q,p}^{2(1-1/p)}(0, 1) \hookrightarrow C([0, 1]) \quad (p, q > 1), \quad (7.3)$$

there exists a constant $C > 0$ such that for all $t \geq 0$,

$$\|\eta\|_{L^\infty(0, \infty; L^\infty(0,1))} \leq C\varepsilon.$$

In particular, there exists $\varepsilon_0 \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, η satisfies (2.6). This allows us to construct X, Y and Z as in Section 2. We first give some estimates on this change of variables.

Lemma 7.1. *Let X be defined by (2.5) and Z be defined by (2.12). Then there exists a constant $C > 0$ depending only in p, q and \mathcal{F} such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for every $(v, \pi, \eta) \in \mathbf{B}_{\mathcal{S}}(\varepsilon)$, we have*

$$\|\nabla X - I_2\|_{L^\infty((0, \infty) \times \mathcal{F})} + \|\text{Cof } \nabla X - I_2\|_{L^\infty((0, \infty) \times \mathcal{F})} + \|Z - I_2\|_{L^\infty((0, \infty) \times \mathcal{F})} \leq C\varepsilon, \quad (7.4)$$

and

$$\|\nabla X\|_{L^\infty((0, \infty) \times \mathcal{F})} + \|\text{Cof } \nabla X\|_{L^\infty((0, \infty) \times \mathcal{F})} + \|Z\|_{L^\infty((0, \infty) \times \mathcal{F})} \leq C. \quad (7.5)$$

Proof. Due to the definition of X , (7.2) and (7.3), we have

$$\|\nabla X - I_2\|_{L^\infty((0, \infty) \times \mathcal{F})} \leq C\|\eta\|_{L^\infty(0, \infty; W^{1, \infty}(0,1))} \leq C\varepsilon.$$

All the other estimates are similar. □

In the following proposition we estimate the nonlinear terms F, G and H defined by (2.13)-(2.15).

Proposition 7.2. *There exists a constant $C_N > 0$ depending only in p, q, β and \mathcal{F} such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for every $(v, \pi, \eta) \in \mathbf{B}_{\mathcal{S}}(\varepsilon)$, we have*

$$\|e^{\beta(\cdot)} F\|_{L^p(0, \infty; L^q(\mathcal{F}))} + \|e^{\beta(\cdot)} G\|_{W_{p,q}^{1,2}((0, \infty) \times \mathcal{F})} + \|e^{\beta(\cdot)} H\|_{L^p(0, \infty; L^q(0,1))} \leq C_N \varepsilon^2. \quad (7.6)$$

Proof. Since $\frac{1}{p} + \frac{1}{q} \leq \frac{3}{2}$, we have the following embeddings (see for instance [23, p.58])

$$\begin{aligned} W_{p,q}^{1,2}((0, \infty) \times \mathcal{F}) &\hookrightarrow L^{3p}(0, \infty; L^{3q}(\mathcal{F})), & W_{p,q}^{1,2}((0, \infty) \times \mathcal{F}) &\hookrightarrow L^{3p/2}(0, \infty; W^{1,3q/2}(\mathcal{F})), \\ W_{p,q}^{1,2}((0, \infty) \times (0, 1)) &\hookrightarrow L^{3p}(0, \infty; L^{3q}(0, 1)), & W_{p,q}^{1,2}((0, \infty) \times (0, 1)) &\hookrightarrow L^{3p/2}(0, \infty; W^{1,3q/2}(0, 1)). \end{aligned}$$

Therefore

$$\begin{aligned} &\|e^{\beta(\cdot)}v\|_{L^{3p}(0,\infty;L^{3q}(\mathcal{F}))} + \|e^{\beta(\cdot)}\nabla v\|_{L^{3p/2}(0,\infty;W^{1,3q/2}(\mathcal{F}))} \\ &\quad + \|e^{\beta(\cdot)}\partial_t\eta\|_{L^{3p}(0,\infty;L^{3q}(0,1))} + \|e^{\beta(\cdot)}\partial_{ts}\eta\|_{L^{3p/2}(0,\infty;L^{3q/2}(0,1))} \\ &\quad + \|e^{\beta(\cdot)}\partial_{ss}\eta\|_{L^{3p}(0,\infty;L^{3q}(0,1))} + \|e^{\beta(\cdot)}\partial_{sss}\eta\|_{L^{3p/2}(0,\infty;L^{3q/2}(0,1))} \leq C\varepsilon. \end{aligned} \quad (7.7)$$

To estimate of first term of F , we combine (7.7), Lemma 7.1 and Hölder's inequality to obtain

$$\begin{aligned} &\left\| -e^{\beta(\cdot)}(v - \partial_t X) \cdot Z^\top \nabla v_i \right\|_{L^p(0,\infty;L^q(\mathcal{F}))} \\ &\leq C \left(\|e^{\beta(\cdot)}v\|_{L^{3p}(0,\infty;L^{3q}(\mathcal{F}))} + \|e^{\beta(\cdot)}\partial_t\eta\|_{L^{3p}(0,\infty;L^{3q}(\Gamma_S))} \right) \|\nabla v_i\|_{L^{3p/2}(0,\infty;L^{3q/2}(\mathcal{F}))} \leq C\varepsilon^2. \end{aligned}$$

The estimates of second and fourth terms of F follow from using Lemma 7.1. In order to estimate of third term of F , we use the expression of Z and (7.7) to find

$$\left\| \frac{\partial}{\partial y_l}(Z_{kj}) \right\|_{L^{3p}(0,\infty;L^{3q}(\mathcal{F}))} \leq C \left(\|\partial_s\eta\|_{L^{3p}(0,\infty;L^{3q}(0,1))} + \|\partial_{ss}\eta\|_{L^{3p}(0,\infty;L^{3q}(0,1))} \right) \leq C\varepsilon.$$

Thus

$$\begin{aligned} &\left\| \nu e^{\beta(\cdot)} \sum_{j,k,l} \frac{\partial v_i}{\partial y_k} \frac{\partial}{\partial y_l}(Z_{kj}) Z_{lj} \right\|_{L^p(0,\infty;L^q(\mathcal{F}))} \\ &\leq C \sum_{j,k,l} \left\| \frac{\partial}{\partial y_l}(Z_{kj}) \right\|_{L^{3p}(0,\infty;L^{3q}(\mathcal{F}))} \left\| e^{\beta(\cdot)} \frac{\partial v_i}{\partial y_k} \right\|_{L^{3p/2}(0,\infty;L^{3q/2}(\mathcal{F}))} \leq C\varepsilon^2. \end{aligned}$$

The estimates of G lead to similar calculations, with moreover terms of the form $(\partial_{sss}\eta)v$ and $(\partial_{ts}\eta)v$ that can be handled as above.

Finally, the estimate for H is done as follows (the other terms are treated similarly):

$$\left\| e^{\beta(\cdot)} \nu \frac{\partial v_1}{\partial y_2}(\cdot, 0)(\partial_s\eta) \right\|_{L^p(0,\infty;L^q(0,1))} \leq C \|\partial_s\eta\|_{L^\infty(0,\infty;L^\infty(0,1))} \left\| e^{\beta(\cdot)}v \right\|_{L^p(0,\infty;W^{2,q}(\mathcal{F}))}.$$

□

Similarly as Proposition 7.2, we can prove the following result:

Proposition 7.3. *There exists a constant $C_{\text{Lip}} > 0$ depending only in p, q, β and \mathcal{F} such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for every $(v^j, \pi^j, \eta^j) \in \mathbf{B}_{\mathcal{S}}(\varepsilon)$, $j = 1, 2$, we have*

$$\begin{aligned} &\|e^{\beta(\cdot)}(F^1 - F^2)\|_{L^p(0,\infty;L^q(\mathcal{F}))} + \|e^{\beta(\cdot)}(G^1 - G^2)\|_{W_{p,q}^{1,2}((0,\infty)\times\mathcal{F})} + \|e^{\beta(\cdot)}(H^1 - H^2)\|_{L^p(0,\infty;L^q(0,1))} \\ &\leq C_{\text{Lip}}\varepsilon \left\| (v^1 - v^2, \pi^1 - \pi^2, \eta^1 - \eta^2) \right\|_{\mathcal{S}}, \end{aligned} \quad (7.8)$$

where

$$F^j = F(v^j, \pi^j, \eta^j), \quad G^j = G(v^j, \pi^j, \eta^j), \quad H^j = H(v^j, \pi^j, \eta^j).$$

We now prove Theorem 1.2 and Theorem 2.2.

Proof of Theorem 2.2. Let us set

$$\varepsilon_* = \min \left\{ \varepsilon_0, \frac{1}{2C_L C_N}, \frac{1}{2C_L C_{\text{Lip}}} \right\},$$

where ε_0 is defined in the previous section and C_L, C_N and C_{Lip} are the constants appearing in Theorem 5.2, Proposition 7.2 and Proposition 7.3.

Assume $(\eta_1^0, \eta_2^0, v^0)$ satisfies (2.16), (2.17), (2.18) and

$$\|\eta_1^0\|_{B_{q,p}^{2(2-1/p)}(0,1)} + \|\eta_2^0\|_{B_{q,p}^{2(1-1/p)}(0,1)} + \|v^0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F})} < \frac{\varepsilon}{2C_L}. \quad (7.9)$$

Let us consider the mapping

$$\mathcal{N} : \mathbf{B}_{\mathcal{S}}(\varepsilon) \longrightarrow \mathbf{B}_{\mathcal{S}}(\varepsilon), \quad (\widehat{v}, \widehat{\pi}, \widehat{\eta}) \longmapsto (v, \pi, \eta), \quad (7.10)$$

where (v, π, η) is the solution to

$$\left\{ \begin{array}{ll} \partial_t v - \operatorname{div} \mathbb{T}(v, \pi) = F(\widehat{v}, \widehat{\pi}, \widehat{\eta}) & t > 0, y \in \mathcal{F}, \\ \operatorname{div} v = \operatorname{div} G(\widehat{v}, \widehat{\pi}, \widehat{\eta}) & t > 0, y \in \mathcal{F}, \\ v(t, s, 0) = \partial_t \eta(t, s) e_2 & t > 0, s \in (0, 1), \\ v = 0 & t > 0, y \in \Gamma_0, \\ \partial_{tt} \eta + P_m(\Delta_s^2 \eta) - \Delta_s \partial_t \eta & \\ \quad = -P_m(\mathbb{T}(v, \pi)|_{\Gamma_S} e_2 \cdot e_2) + P_m(H(\widehat{v}, \widehat{\pi}, \widehat{\eta})) & t > 0, s \in (0, 1), \\ \eta = \partial_s \eta = 0 & t > 0, s \in \{0, 1\}, \\ \eta(0, \cdot) = \eta_1^0, \quad \partial_t \eta(0, \cdot) = \eta_2^0, \quad \text{in } (0, 1), \quad v(0) = v^0 \text{ in } \mathcal{F}. & \end{array} \right. \quad (7.11)$$

A solution of (2.10) is a fixed point to \mathcal{N} . Applying Theorem 5.2 and Proposition 7.2 to the system (7.11) and using (7.9), we obtain

$$\|(v, \pi, \eta)\|_{\mathcal{S}} \leq \frac{\varepsilon}{2} + C_L C_N \varepsilon^2 \leq \varepsilon,$$

with our choice of ε_* . Thus \mathcal{N} is well-defined. In a similar manner, using Theorem 5.2 and Proposition 7.3, one can check that \mathcal{N} is a strict contraction. This completes the proof. \square

Finally, we prove our main result

Proof of Theorem 1.2. Let (v, π, η) be the solution of the system (2.10)- (2.15) constructed in Theorem 2.2. Since X is a C^1 -diffeomorphism from \mathcal{F} to $\mathcal{F}(\eta(t))$, we set $Y(t, \cdot) = X^{-1}(t, \cdot)$ and for $x \in \mathcal{F}(\eta(t))$, $t \geq 0$

$$\widetilde{v}(t, x) = v(t, Y(t, x)), \quad \widetilde{\pi}(t, x) = \pi(t, Y(t, x)), \quad \widetilde{v}^0(x) = v^0(Y(0, x)).$$

One can easily verify that $(\widetilde{v}^0, \eta_1^0, \eta_2^0)$ satisfy the compatibility conditions (1.16). Moreover, $(\widetilde{v}, \widetilde{\pi}, \eta)$ satisfies the original system (1.13) satisfying the estimate (1.20). \square

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