HAL
open science

Some remarks on Smith predictors A geometric point of view

Irinel-Constantin Morarescu, Silviu-Iulian Niculescu, Keqin Gu

To cite this version:

Irinel-Constantin Morarescu, Silviu-Iulian Niculescu, Keqin Gu. Some remarks on Smith predictors A geometric point of view. 6th IFAC Workshop on Time Delay Systems, TDS 2006, Jul 2006, L'Aquila, Italy. pp.264-269. hal-02294066

HAL Id: hal-02294066

https://hal.science/hal-02294066v1

Submitted on 24 Sep 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

SOME REMARKS ON SMITH PREDICTORS: A GEOMETRIC POINT OF VIEW

Constantin-Irinel Morărescu [1] **Silviu-Iulian Niculescu** [2]
Keqin Gu [3]

Abstract: In this paper we develop a method to obtain the stability crossing curves of a Smith Predictor control scheme. More explicitly, we compute the crossing set, which consists of all frequencies corresponding to all points on the stability crossing curve, and we give their complete classification. Furthermore, the directions in which the zeros cross the imaginary axis are explicitly expressed.

Keywords: Smith Predictor, Delay uncertainty, Stability

1. INTRODUCTION

The stability and control of time-delay systems are subject of recurring interest since the *delay* is inherently present in various applications, from signal propagation in networks to population dynamics (see, for instance, [7, 2] for further references, and examples).

The aim of this paper is to present some *new interpretations* of the Smith predictors ([10]) subject to *delay uncertainty*. This problem was largely treated in the literature starting with the 80s (see, for instance, [8], [12], [11], [9], and the references therein), and it is reconsidered recently ([6]), mainly to the increasing interest of using such a methodology in applications, like control of congestions in high-speed networks (see, e.g. [5]).

The approach considered in the paper makes use of some simple geometric idea (triangles inequality), inspired by ([3]) and devoted to the characterization of the stability crossing curves for general systems with two delays. The novelty of the results lies in a *simple*, and *easy* to follow *classification* of all the situations where uncertainty on the nominal delay value will induce instabilities in the corresponding schemes, and also considers, and gives the corresponding characterizations of some of the *degenerate cases* mentioned, but not discussed by [3]. Furthermore, the approach under consideration completes the algebraic characterizations in ([7], [6]).

2. PROBLEM FORMULATION, AND PRELIMINARIES

As discussed in [7], the analysis of the effects induced by the delay uncertainty of the Smith predictor on the stability of the corresponding closed-loop schemes reduce to the analysis of the following characteristic equation:

$$P(s) + Q(s)\mathrm{e}^{-s\tau_1} - Q(s)\mathrm{e}^{-s(\tau_1+\delta)} = 0,$$

where τ_1 represents the nominal delay value, and $P(s), Q(s)$ are appropriate polynomials defined by the plant free of delay, and by corresponding controller.

[1] HeuDiaSyC (UMR CNRS 6599), Université de Technologie de Compiègne, Centre de Recherche de Royallieu, BP 20529, 60205, Compiègne, France. E-mail: `cmorares@hds.utc.fr`; The author is also with the Department of Mathematics, University "Politehnica" of Bucharest, Romania. The work of C.-I. MORĂRESCU was (partially) supported through a E. C. Marie Curie Fellowship and in the framework of the CTS, HPMT-CT-2001-00278

[2] HeuDiaSyC (UMR CNRS 6599), Université de Technologie de Compiègne, Centre de Recherche de Royallieu, BP 20529, 60205, Compiègne, France. E-mail: `niculescu@hds.utc.fr`; The work of S.-I. NICULESCU was partially funded by the CNRS-US Grant: *"Delays in interconnected dynamical systems: Analysis, and applications"* (2005-2007).

[3] Department of Mechanical and Industrial Engineering, Southern Illinois University at Edwardsville, Edwardsville, Illinois 62026-1805, USA. E-mail: `kgu@siue.edu`. The work of K. GU was partially funded by the CNRS-US Grant: *"Delays in interconnected dynamical systems: Analysis, and applications"* (2005-2007).

Let $\mathcal{G}$ be the set of all pairs $(x, y) \in \mathbb{R}_+^2$ such that $x < y$. It is obvious that replacing $\tau_2 = \tau_1 + \delta$ and taking $(\tau_1, \tau_2) \in \mathcal{G}$ we can consider the following equivalent equation:

$$D(s, \tau_1, \tau_2) = P(s) + Q(s)\mathrm{e}^{-s\tau_1} - Q(s)\mathrm{e}^{-s\tau_2} = 0. \tag{1}$$

More explicitly, we study the occurrence of any possible stability switch/reversal[4] resulting by increasing the time delay τ_1 or τ_2. In other words, we explicitly study the change of number of zeros of (1) on $\mathbb{C}_+$ as the delays (τ_1, τ_2) vary on $\mathcal{G}$. Since the main objective of this study is to identify the regions of (τ_1, τ_2) in $\mathcal{G}$ such that $D(s, \tau_1, \tau_2)$ is (asymptotically) stable, we will exclude some cases, and the following assumptions appear naturally:

Assumption 1. $\deg(Q) \leq \deg(P)$.

Assumption 2. $P(0) \neq 0$.

Assumption 3. P, Q do not have common zeros.

Assumption 4. P and Q are such that:

$$\lim_{s\to\infty} \left| \frac{Q(s)}{P(s)} \right| < \frac{1}{2}.$$

Remark 1. If the system is of retarded type then the assumption 4 is automatically satisfied since its left hand is zero. For neutral systems, let $c = \lim_{s\to\infty} \frac{Q(s)}{P(s)}$. Then it is well known that the stability of the system (1) is possible only if the difference equation

$$x(t) + cx(t - \tau_1) - cx(t - \tau_2) = 0 \tag{2}$$

is exponentially stable. Assumption 4 guarantees the stability of (2).

3. IDENTIFICATION OF THE CROSSING POINTS

Let $\mathcal{T}$ denote the set of all points $(\tau_1, \tau_2) \in \mathcal{G}$ such that $D(s)$ has at least one zero on the imaginary axis. Any $(\tau_1, \tau_2) \in \mathcal{T}$ is known as a crossing point. The set $\mathcal{T}$, which is the collection of all crossing points , is known as the *stability crossing curves*. Let $\mathcal{T}_\omega$ denote the set of all $(\tau_1, \tau_2) \in \mathcal{G}$ such that $D(s)$ has at least one zero for $s = j\omega$. Let Ω the set of all ω for which there exists a pair (τ_1, τ_2) such that $D(j\omega, \tau_1, \tau_2) = 0$. We will refer to Ω as the *crossing set*. Obviously

$$\mathcal{T} = \{\mathcal{T}_\omega | \omega \in \Omega\}.$$

Next, for the clarity of the presentation we will split our discussion in two parts. First we will consider only the case which satisfy the following non-degeneracy condition,

[4] We are using the same terminology as in Cooke and Grossman [1], that is a root of the characteristic equation *crossing* the imaginary axis, when some parameter is varying.

Assumption 5.

$$P(j\omega) \cdot Q(j\omega) \neq 0 \textit{ for all } \omega \in \Omega \tag{3}$$

and then we will discuss what happens in other cases.

3.1 Regular cases

In the sequel, we consider

$$h(s) = \frac{Q(s)}{P(s)}$$

and

$$H(s) = 1 + h(s)e^{-s\tau_1} - h(s)e^{-s\tau_2} \tag{4}$$

For given τ_1 and τ_2, as long as assumption 5 is satisfied, $D(s)$ and $H(s)$ share all the zeros in a neighborhood of the imaginary axis. Therefore, we may obtain all the crossing points and direction of crossing using $H(s) = 0$ instead of $D(s) = 0$. We may also consider the three terms in $H(s)$ as three vectors in the complex plane, with magnitudes 1, $|h(s)|$ and $|h(s)|$ respectively. So when we adjust the values of τ_1 and τ_2 in fact we adjust the directions of the vectors represented by the second and the third terms. Equation (4) means that if we put the first two vectors had to tail then we get the third vector. In other words they form an isosceles triangle and the existence condition reduces to: the sum of the equal sides exceed the other side. This allows us to give the following proposition.

Proposition 1. For some $(\tau_1, \tau_2) \in \mathcal{G}$, $H(s)$ has an imaginary zero $s = j\omega$, $\omega \neq 0$ if and only if

$$|h(j\omega)| \geq \frac{1}{2}. \tag{5}$$

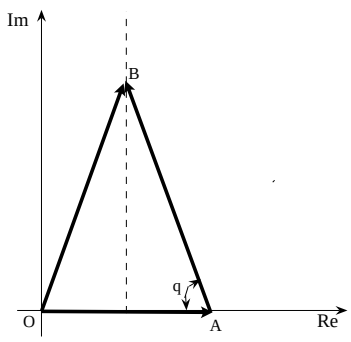

Fig. 1. Triangle formed by 1, $h(s)e^{-s\tau_1}$ and $h(s)e^{-s\tau_2}$

Due to the symmetry and assumption 2 we only need to consider positive ω. So Ω is the set of all $\omega > 0$ which satisfy (5). Also, for a given $\omega \in \Omega$ we may find all the pairs (τ_1, τ_2) satisfying $H(j\omega) = 0$ as follows:

$$\tau_1 = \tau_1^{u\pm}(\omega) = \frac{\angle h(j\omega) + (2u-1)\pi \pm q}{\omega}, \quad (6)$$
$$u = u_0^{\pm}, u_0^{\pm}+1, u_0^{\pm}+2, ...$$
$$\tau_2 = \tau_2^{v\pm}(\omega) = \frac{\angle h(j\omega) + 2v\pi \mp q}{\omega}, \quad (7)$$
$$v = v_0^{\pm}, v_0^{\pm}+1, v_0^{\pm}+2, ...$$

where $q \in [o, \pi]$ is the internal angle of triangle in Figure 1 which can be calculated by the cosine law as

$$q(j\omega) = \cos^{-1}\left(\frac{1}{2|h(\omega)|}\right) \quad (8)$$

and $u = u_0^+, u = u_0^-, v = v_0^+, v = v_0^-$ are the smallest integers (may depend on ω) such that the corresponding values $\tau_1^{u_0^++}$, $\tau_1^{u_0^--}$, $\tau_2^{u_0^++}$, $\tau_2^{u_0^--}$ are nonnegative and $\tau_2^{u_0^++} > \tau_1^{u_0^++}$, $\tau_2^{u_0^--} > \tau_1^{u_0^--}$. The position in Figure 1 corresponds to $(\tau_1^{u+}, \tau_2^{u+})$ and the mirror image about the real axis corresponds to $(\tau_1^{u-}, \tau_2^{u-})$. If we define $\mathcal{T}^+_{\omega,u,v}$ and $\mathcal{T}^-_{\omega,u,v}$ as the singletons $(\tau_1^{u+}(\omega), \tau_2^{u+}(\omega))$ and $(\tau_1^{u-}(\omega), \tau_2^{u-}(\omega))$ respectively, then we can characterize $\mathcal{T}_\omega$ as follows:

$$\mathcal{T}_\omega = \left(\bigcup_{u\geq u_0^+, v\geq v_0^+} \mathcal{T}^+_{\omega,u,v}\right) \bigcup \left(\bigcup_{u\geq u_0^-, v\geq v_0^-} \mathcal{T}^-_{\omega,u,v}\right)$$

Proposition 2. The crossing set Ω consists of a finite number of intervals of finite length including the cases which may violate assumption 5.

Proof: First one can observe easily that the number of points in Ω violating (3) is finite. So, we only need to show that the set of all points satisfying (5) consists of a finite number of intervals of finite length. Because

$$|h(j\omega)| = \frac{1}{2} \Leftrightarrow |P(j\omega)| = 2|Q(j\omega)|$$

is a polynomial equation of variable ω^2 it has a finite number of positive solutions. Therefore the solution of (5) consists of a finite number of intervals. Due to assumption 4, any sufficiently large ω violates (5). Therefore the lengths of all intervals are finite. In what follows we will denote these intervals as $\Omega_1, \Omega_2, ..., \Omega_N$ and without loss of generality we may suppose that the intervals are ordered such that for any $\omega_1 \in \Omega_{k_1}$, $\omega_2 \in \Omega_{k_2}$, $k_1 < k_2$ we have $\omega_1 < \omega_2$.

Remark 2. If (5) is satisfied for $\omega = 0$ and sufficiently small positive value of ω then we will take 0 as the left end of Ω_1. Considering ω_1^r the right end of Ω_1, according to assumption 2 we get $\Omega_1 = (0, \omega_1^r]$, so $0 \notin \Omega$.

Remark 3. From geometrical point of view an end point ω^* with $|P(j\omega^*)| = 2|Q(j\omega^*)|$, corresponds to the limit case where internal angle q of the triangle is 0. In this case we obtain $\overrightarrow{OB} = -\overrightarrow{AB}$ on the real axis.

We will not restrict $\angle h(j\omega)$ to be within the 2π range but make it a continuous function of ω within each Ω_k. Thus, for each fixed u, v and k, (6) and (7) are continuous curves denoted as $\mathcal{T}^{k+}_{u,v}$ respectively $\mathcal{T}^{k-}_{u,v}$. We should keep in mind that, for some u, v and k, part or entire curve $\mathcal{T}^{k+}_{u,v}$ (respectively $\mathcal{T}^{k-}_{u,v}$) may be outside of the range $\mathcal{G}$, and therefore, may not be physically meaningful. The collection of all the points in $\mathcal{T}$ corresponding to Ω_k may be expressed as

$$\mathcal{T}^k = \bigcup_{u=-\infty}^{\infty} \bigcup_{v=-\infty}^{\infty} \left[\left(\mathcal{T}^{k+}_{u,v} \cup \mathcal{T}^{k-}_{u,v}\right) \cap \mathcal{G}\right] = \bigcup_{\omega\in\Omega_k} \mathcal{T}_\omega$$
$$\mathcal{T} = \bigcup_{k=1}^{N} \mathcal{T}^k \quad (9)$$

Our previous discussions allow us to say that the ends of Ω_k must be in one of the following situation:

Type 1. It satisfies the equation $|h(x)| = \frac{1}{2}$.
Type 2. It equals 0.

If one end of Ω_k is of type 1 then $q = 0$ and $\mathcal{T}^{k+}_{u,v}$ is connected with $\mathcal{T}^{k-}_{u,v}$ at this end. So, if both ends of Ω_k are of type 1 we get $\mathcal{T}^k$ is a series of closed curves. Obviously just the left end of Ω_1 can be 0. In this case, as $\omega \to 0$, both τ_1 and τ_2 approach ∞. In fact $\mathcal{T}^{1+}_{u,v}$ and $\mathcal{T}^{1-}_{u,v}$ approach ∞ with asymptotes passing through the points $(\hat{h} \pm \hat{q}, \hat{h} \mp \hat{q})$ with slopes of

$$m^{\pm}_{u,v} = \frac{\tau_2^{v\pm}}{\tau_1^{u\pm}} = \frac{\angle h(0) + 2v\pi \mp q(0)}{\angle h(0) + (2u-1)\pi \pm q(0)},$$

where $q(0)$ is evaluated using (8) and

$$\hat{h} = \frac{\mathrm{d}}{\mathrm{d}\omega}[\angle h(j\omega)]\Big|_{\omega=0}, \ \hat{q} = \frac{\mathrm{d}}{\mathrm{d}\omega}[q(j\omega)]\Big|_{\omega=0}.$$

In the sequel we will say that an interval is of type 11 if both his end are of type 1, and Ω_1 is of type 21 if his left end is 0. Therefore, the crossing set Ω consists in a finite number of intervals of type 11, and eventually the first interval is of type 21. It is obvious that $\mathcal{T}^k$ consists in a series of curves belonging to one of the following categories:

A) A series of closed curves (Ω_k is of type 11)
B) A series of open ended curves with both ends approaching ∞ (Ω_k is of type 21)

We continue this section with some illustrative examples regarding the above characterization.

Example 1. ***(type 11)*** Consider a system with

$$h(s) = \frac{4s+1}{4(s^2+s+1)} \quad (10)$$

Figure 2 (up) plots $2|h(j\omega)|$ against ω. The crossing set can be easily identified from this figure, it contains one interval $\Omega_1 = [0.39, 2, 21]$

As an illustration of a series of *closed curves* we examine $\mathcal{T}^k$ corresponding to Ω_k of type 11. In this case, for a given u and v such that $\tau_2^{u\pm} > \tau_1^{u\pm} > 0$,

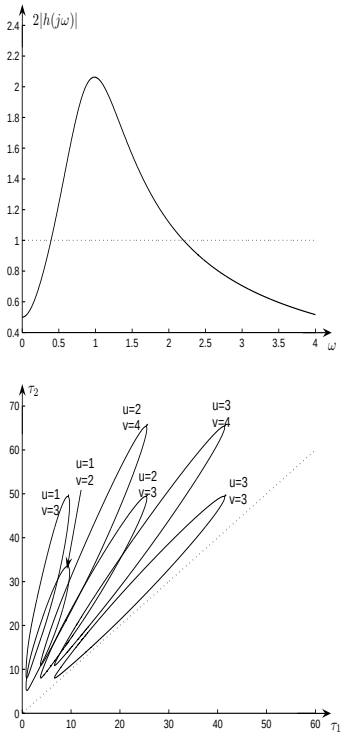

Fig. 2. The crossing set for the system (10) can be identified up and some crossing curves of this system are plotted down

we get $\mathcal{T}^{k+}_{u,v}$ and $\mathcal{T}^{k-}_{u,v}$ are connected on the both ends to form a closed curve. As u and v vary, we obtain a series of deformed versions of such closed curves situated above the first bisector. A suggestive image of a series of *closed curves* is given in Figure 2 (down) which show $\mathcal{T}$ of the system described in (10).

Example 2. ***(type 21)*** Consider a system with

$$h(s) = \frac{s + \sqrt{2}}{2s^3 + s^2 + 8s + 1} \tag{11}$$

Figure 3 (up) plots $|h(j\omega)|$ against ω. The crossing set Ω can be easily identified from the Figure 3, it contains two intervals: $\Omega_1 = (0, 0.364]$ of type 21, and $\Omega_2 = [1.673, 2.198]$ of type 11.

In the sequel, we consider $\Omega_k = [\omega^l_k, \omega^r_k]$. Obviously, the interval Ω_1 is open to the left if its left end is 0. To illustrate the case of *open ended curves* we consider $\mathcal{T}^1$ corresponding to Ω_1 of type 21. In this case $\Omega_1 = (0, \omega^r_1]$ and for a given u and v, $\mathcal{T}^{1+}_{u,v}$ and $\mathcal{T}^{1-}_{u,v}$ are connected at ω^r_1. The other end of $\mathcal{T}^{1-}_{u,v}$ extends to infinity with asymptotes passing through the points $(\hat{h} - \hat{q}, \hat{h} + \hat{q})$ with the slope $m^-_{u,v}$ and, the other end of $\mathcal{T}^{1+}_{u,v}$ extends to infinity with asymptotes passing through the points $(\hat{h} + \hat{q}, \hat{h} - \hat{q})$ with the slope $m^+_{u,v}$. Again, as u and v vary, we obtain a series of deformed versions of such open ended curves situated above the first bisector. Evidently, the slope is changing for different u and v. We can see a series of *open ended curves* in Figure 3 (right).

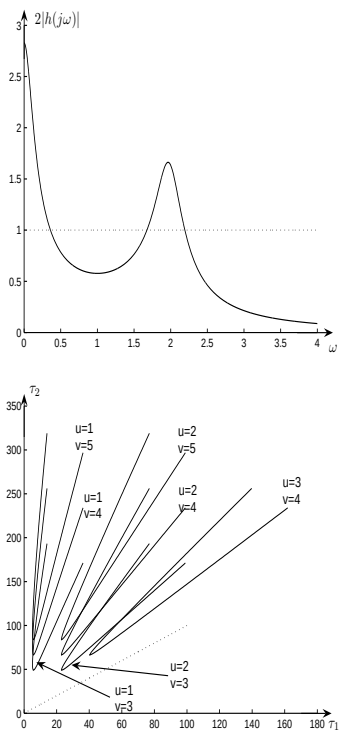

Fig. 3. The crossing set for the system (12) can be identified up and some crossing curves of this system are plotted down

3.2 Degenerate cases

In the sequel, we consider the cases violating assumption 5. Obviously, the interesting case is $P(j\omega) = 0$ has at least one positive solution. We can easily state the following:

Remark 4. . For $\omega^* \neq 0$ satisfying $Q(j\omega^*) = 0$ it is clear that increasing of τ_1 and/or τ_2 has no effect w.r.t. stability of the system. We note also that $h(j\omega^*) = 0$ imply that $h(j\omega) < \frac{1}{2}$, for all ω in a neighborhood of ω^*.

Next, we assume that $P(j\omega^*) = 0$ for $\omega^* \neq 0$. Using assumption 3 we get $Q(j\omega^*) \neq 0$ and therefore $\lim_{\omega \to \omega^*} |h(j\omega)| = \infty$. So that, $|h(j\omega)| > \frac{1}{2}$ for all ω in a neighborhood of ω^*. This mean that Ω contains one interval of type $[\omega^l, \omega^*]$ and one of type $[\omega^*, \omega^r]$. It is clear that the first interval is open to the left if $\omega^l = 0$.

Proposition 3. For $\omega^* \neq 0$ satisfying $P(j\omega^*) = 0$ we get $\mathcal{T}_{\omega^*}$ consists of the solutions of

$$\omega^* \tau_2 = \omega^* \tau_1 + 2m\pi,\ m \in \mathbb{Z} \text{ in } \mathcal{G}.$$

and

$$\lim_{\omega \to \omega^*} q(j\omega) = \frac{\pi}{2};$$
$$\lim_{\omega \to \omega^*} \tau_1(\omega) = \frac{2\angle Q(j\omega^*) + (4u - 2 \pm 1)\pi}{2\omega^*};$$
$$\lim_{\omega \to \omega^*} \tau_2(\omega) = \frac{2\angle Q(j\omega^*) + (4v \mp 1)\pi}{2\omega^*}.$$

Proof. Straightforward computations.

Remark 5. 1) $\mathcal{T}_{\omega^*}$ consists of an infinite number of straight lines of slope 1 of equal distance.
2) $\lim_{\omega\to\omega^*} \tau_2(\omega) - \lim_{\omega\to\omega^*} \tau_1(\omega) = 2m\pi,\ m \in \mathbb{Z}$

Let $\Omega_k = [\omega^l, \omega^*]$ and $\Omega_{k+1} = [\omega^*, \omega^r]$. In this case, using proposition 3 we get $\mathcal{T}_{u,v}^{k+}$ is connected with $\mathcal{T}_{u,v}^{(k+1)+}$ and $\mathcal{T}_{u,v}^{k-}$ is connected with $\mathcal{T}_{u,v}^{(k+1)-}$ at the end corresponding to ω^*. Using (9) and remark 5 we obtain that each crossing curve in $\mathcal{T}_{u,v}^{k\pm}$ consist of an union of one straight line of slope 1 and the curve corresponding to $\Omega_k \setminus \{\omega^*\}$. From Remark 5 we deduce that one end of the curve corresponding to $\Omega_k \setminus \{\omega^*\}$ is on the line in $\mathcal{T}_{\omega^*}$ which correspond to the pair (u, v). In the following if $P(j\omega^*) = 0$ we will say that ω^* is an end point of type 0.

Example 3. ***(type 20 and 01)*** Consider a system with

$$h(s) = \frac{s+2}{s^2+2} \tag{12}$$

Figure 3 (up) plots $\frac{1}{|h(j\omega)|}$ against ω. The crossing set Ω contains two intervals: $\Omega_1 = (0, \sqrt{2}]$ of type 20, and $\Omega_2 = [\sqrt{2}, 3.046]$ of type 01. Figure 4 (down) plots $\mathcal{T}_{3,4}^{2\pm}$ which are two curves of type 01.

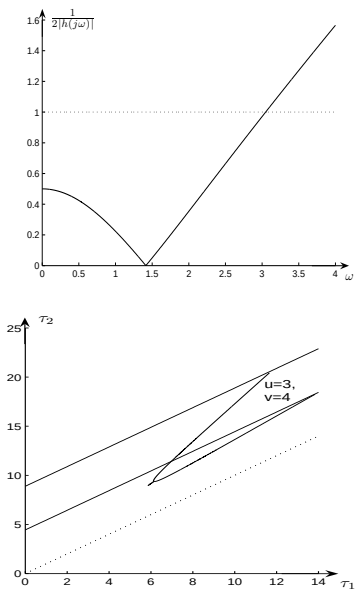

Fig. 4. The crossing set for the above system can be identified up and $\mathcal{T}_{3,4}^{2\pm}$ of this system are plotted down

4. CROSSING CURVES, CHARACTERISTIC ROOTS BEHAVIOR

4.1 Tangent and Smoothness

For a given k we will discuss the smoothness of the curves in $\mathcal{T}^k$ and thus $\mathcal{T} = \bigcup_{k=1}^{N} \mathcal{T}^k$. In this part we use an approach based on the implicit function theorem. For this purpose we consider τ_1 and τ_2 as implicit functions of $s = j\omega$ defined by (1).
For a given k, as s moves along the imaginary axis within Ω_k, $(\tau_1, \tau_2) = (\tau_1^{u\pm}(\omega), \tau_2^{v\pm}(\omega))$ moves along $\mathcal{T}^k$. For a given $\omega \in \Omega_k$, let

$$\begin{aligned}
R_0 &= Re\left(\frac{j}{s}\frac{\partial D(s,\tau_1,\tau_2)}{\partial s}\right)_{s=j\omega} \\
&= \frac{1}{\omega} Re\left\{[h'(j\omega) - \tau_1 h(j\omega)]\, e^{-j\omega\tau_1}\right. \\
&\quad \left. + [\tau_2 h(j\omega) - h'(j\omega)]\, e^{-j\omega\tau_2}\right\}, \\
I_0 &= Im\left(\frac{j}{s}\frac{\partial D(s,\tau_1,\tau_2)}{\partial s}\right)_{s=j\omega} \\
&= \frac{1}{\omega} Im\left\{[h'(j\omega) - \tau_1 h(j\omega)]\, e^{-j\omega\tau_1}\right. \\
&\quad \left. + [\tau_2 h(j\omega) - h'(j\omega)]\, e^{-j\omega\tau_2}\right\}
\end{aligned}$$

and

$$R_l = Re\left(\frac{1}{s}\frac{\partial D(s,\tau_1,\tau_2)}{\partial \tau_l}\right)_{s=j\omega} = (-1)^{l-1} Re\left(h(j\omega)e^{-j\omega\tau_l}\right),$$
$$I_l = Im\left(\frac{1}{s}\frac{\partial D(s,\tau_1,\tau_2)}{\partial \tau_l}\right)_{s=j\omega} = (-1)^{l-1} Im\left(h(j\omega)e^{-j\omega\tau_l}\right),$$

for l=1,2. Then, since $D(s, \tau_1, \tau_2)$ is an analytic function of s, τ_1 and τ_2, the implicit function theorem indicates that the tangent of $\mathcal{T}^k$ can be expressed as

$$\begin{pmatrix} \frac{\mathrm{d}T}{\mathrm{d}\omega} \\ \frac{\mathrm{d}\tau}{\mathrm{d}\omega} \end{pmatrix} = \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_0 I_2 - I_0 R_2 \\ I_0 R_1 - R_0 I_1 \end{pmatrix}, \tag{13}$$

provided that

$$R_1 I_2 - R_2 I_1 \neq 0. \tag{14}$$

It follows that $\mathcal{T}_k$ is smooth everywhere except possibly at the points where either (14) is not satisfied, or when

$$\frac{\mathrm{d}T}{\mathrm{d}\omega} = \frac{\mathrm{d}\tau}{\mathrm{d}\omega} = 0. \tag{15}$$

From the above discussions, we can conclude:

Proposition 4. The curve T^k is smooth everywhere except possibly at the points corresponding to $s = j\omega$ a multiple solution of (1).

4.2 Direction of crossing

Next, we will discuss the direction in which the solutions of (1) cross the imaginary axis as (τ_1, τ_2) deviates from the curve $\mathcal{T}^k$. We will call the direction of

the curve that corresponds to increasing ω the *positive direction*. We will also call the region on the left hand side as we head in the positive direction of the curve *the region on the left*.

To establish the direction of crossing we need to consider τ_1 and τ_2 as functions of $s = \sigma + j\omega$, i.e., functions of two real variables σ and ω, and partial derivative notation needs to be adopted. Since the tangent of $\mathcal{T}^k$ along the positive direction is $\left(\frac{\partial \tau_1}{\partial \omega}, \frac{\partial \tau_2}{\partial \omega}\right)$, the normal to $\mathcal{T}^k$ pointing to the left hand side of positive direction is $\left(-\frac{\partial \tau_2}{\partial \omega}, \frac{\partial \tau_1}{\partial \omega}\right)$. Corresponding to a pair of complex conjugate solutions of (1) crossing the imaginary axis along the horizontal direction, (τ_1, τ_2) moves along the direction $\left(\frac{\partial \tau_1}{\partial \sigma}, \frac{\partial \tau_2}{\partial \sigma}\right)$. So, if a pair of complex conjugate solutions of (1) cross the imaginary axis to the right half plane, then,

$$\left(\frac{\partial \tau_1}{\partial \omega}\frac{\partial \tau_2}{\partial \sigma} - \frac{\partial \tau_2}{\partial \omega}\frac{\partial \tau_1}{\partial \sigma}\right)_{s=j\omega} > 0, \qquad (16)$$

i.e. the region on the left of $\mathcal{T}^k$ gains two solutions on the right half plane. If the inequality (16) is reversed then the region on the left of $\mathcal{T}^k$ loses has two right half plane solutions. Similar to (13) we can express

$$\begin{pmatrix} \frac{\mathrm{d}\tau_1}{\mathrm{d}\sigma} \\ \frac{\mathrm{d}\tau_2}{\mathrm{d}\sigma} \end{pmatrix}_{s=j\omega} = \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_0 R_2 + I_0 I_2 \\ -R_0 R_1 - I_0 I_1 \end{pmatrix}.$$

Using this, we arrive to the following:

Proposition 5. Let $\omega \in (\omega_k^l, \omega_k^r)$ and $(\tau_1, \tau_2) \in T^k$ such that $j\omega$ is a simple solution of (1) and $D(j\omega', \tau_1, \tau_2) \neq 0$, $\forall \omega' > 0$, $\omega' \neq \omega$ (i.e. (τ_1, τ_2) is not an intersection point of two curves or different sections of a single curve of $\mathcal{T}$). Then a pair of solutions of (1) cross the imaginary axis to the right, through $s = \pm j\omega$ if $R_2 I_1 - R_1 I_2 > 0$. The crossing is to the left if the inequality is reversed.

5. ILLUSTRATIVE EXAMPLES

In this paragraph we reconsider one example already treated in the literature [6]. Consider:

$$P(s) = (k_1 k_2 + 1)s + (a + k_1), \quad Q(s) = k_1(k_2 s + 1).$$

[6] assume $a > 0$ and $(a + k_1)/(k_1 k_2 + 1) > 0$, which guarantees internal stability of the closed-loop system. The so-called "practical stability" criterion is given by assumption 4 which simply states $\left|\frac{k_1 k_2}{1 + k_1 k_2}\right| < \frac{1}{2} \Leftrightarrow -1/3 < k_1 k_2 < 1$. For $a = 1$, $k_1 = 2$, $k_2 = 1/4$ we get $\Omega = (0, 2.37]$, and, in conclusion, Ω consists of one interval of type 21. More precisely, we obtain a series of open ended curves with both ends approaching infinity, conclusion which is similar to the one in [6], but derived using a different (algebraic) approach.

6. CONCLUDING REMARKS

This paper focuses on the stability crossing curves for a class of delay systems controlled by a Smith predictor, subject to *uncertainty* in the delay. More precisely, the particular form of the closed-loop system allows an easy derivation of the stability crossing curves (crossing set characterization, direction of crossing, smoothness). Regular, and degenerate cases are both treated. Various examples complete the presentation.

REFERENCES

[1] Cooke, K. L. and Grossman, Z.: Discrete delay, distributed delay and stability switches. *J. Math. Anal. Appl.* **86** (1982) 592-627.

[2] Gu, K., Kharitonov, V.L. and Chen, J. (2003). *Stability and robust stability of time-delay systems*. (Birkhauser: Boston).

[3] Gu, K., Niculescu, S.-I., and Chen, J. (2005). On stability of crossing curves for general systems with two delays, *J. Math. Anal. Appl.* **311**, 231-253.

[4] Hale, J. K. and Huang, W. (1993). Global geometry of the stable regions for two delay differential equations. *J. Math. Anal. Appl.*, **178**, 344-362.

[5] Mascolo, S. (1997). Smith's principle for congestion control in high-speed data networks. *Proc. 37th IEEE Conf. Dec. Contr.*, San Diego, CA.

[6] Michiels, W., and Niculescu, S.-I. (2003). On the delay sensitivity of Smith Predictors, *Int.J. of Systems Science*, **34**, 543-551.

[7] Niculescu, S.-I. (2001). *Delay effects on stability. A robust control approach* (Springer: Heidelberg, LNCIS, vol. **269**).

[8] Palmor, Z. J. (1980). Stability properties of Smith dead-time compensator controllers. in *Int. J. Contr.* **32**, 937-949.

[9] Palmor, Z. J. (1996). Time-delay compensation – Smith predictor and its modifications, in *The Control Handbook* (W. S. Levine, Eds.), CRC Press, 224-237.

[10] Smith, O. J. M. (1959). A controller to overcome dead time. *Instrument Society of America Journal (ISA)* **6**, 28-33.

[11] Santacesaria, C. and Scattolini, R.: Easy tuning of Smith predictor in presence of delay uncertainty. in *Automatica* **29** (1993) 1595-1597.

[12] Yamanaka, K. and Shimemura, E. (1987). Effects of mismatched Smith controller on stability in systems with time-delay. in *Automatica* **23**, 787-791.